Analytic Number Theory
Homework 3

1. Prove that the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ does not vanish when $\sigma = \Re(s) > 1$.

2. Recall that

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty (x - \lfloor x \rfloor) x^{-1-x}\,\mathrm{d}x$$

whenever $\sigma > 0$. Let

$$\frac{1}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n$$

be the Laurent series expansion of $\zeta(s)$ at $s = 1$. Prove that $a_0 = \gamma$, where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right).$$

The remaining exercises compute $\zeta(2k)$, where k is a positive integer, using residue calculus.
3. Let γ_N denote the square going counterclockwise through the four points $(N+\frac{1}{2})(-1-i)$, $(N+\frac{1}{2})(1-i)$, $(N+\frac{1}{2})(1+i)$, and $(N+\frac{1}{2})(1-i)$. Show that there is a constant $c > 0$ such that $|e^{2\pi i z} - 1| \geq c$ for all $N \geq 1$ and all z lying on γ_N.

4. If f is a meromorphic function on $\mathbf{C}$ having no zero or pole at an integer point $z = k$. Prove that

$$\frac{2\pi i f(z)}{e^{2\pi i z} - 1}$$

has a simple pole at $z = k$ with residue $f(k)$.

5. Define the *Bernoulli numbers* B_n to be such that

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

whenever z is sufficiently small (say $|z| < 2\pi$). Prove that $B_n \in \mathbf{Q}$. By integrating over γ_N, applying the residue theorem, and taking $N \to \infty$, show that

$$\zeta(2k) = \left[(-1)^{k-1} 2^{2k-1} \frac{B_{2k}}{(2k)!} \right] \pi^{2k}$$

for any positive integer k. Explicitly compute $\zeta(2)$, $\zeta(4)$, and $\zeta(6)$.