Analytic Number Theory
Homework 2

1. Recall the "big O" notation: if S is a set and f and g are functions $S \to \mathbf{R}$, we say that $f = O(g)$ if there exists an $M > 0$ such that $|f(s)| \leq Mg(s)$ for all $s \in S$. Usually the set S is taken to be the interval $[a, \infty)$ for some sufficiently large $a \in \mathbf{R}$ that is left unstated, or an open interval like $(1, 2]$ if we care about asymptotic behavior near 1.
a) Prove the following basic facts: if $f = O(g)$ and $g = O(h)$ then $f = O(h)$. If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 f_2 = O(g_1 g_2)$ and $f_1 + f_2 = O(g_1 + g_2) = O(\max(g_1, g_2))$. If g is a fixed positive function, the set of f such that $f = O(g)$ is a vector space.
b) Suppose that $S = [a, \infty)$ and $f = O(g)$ with f and g integrable function. Prove that $\int_a^x f = O(\int_a^x g)$. Give an example to show that if f and g are differentiable with $f = O(g)$, it is not necessarily the case that $f' = O(g')$.

2. We proved in class that

$$s \int_1^\infty \pi(y) y^{-1-s} \, \mathrm{d}y = \sum_p p^{-s} = \log \frac{1}{s-1} + O(1).$$

a) We don't know the PNT at this point, but we can show that the above estimate is consistent with it: let

$$I(s) = \int_2^\infty \frac{y^{-s}}{\log y} \, \mathrm{d}y.$$

By differentiating in s, estimating, and then integrating (using part b) of exercise 1), show that

$$I(s) = \log \frac{1}{s-1} + O(1)$$

for, say, $s \in (1, 2]$.
b) Use this computation to show that for each $c < 1$ there are arbitrarily large x such that $\pi(x) > cx/\log x$ and for each $1 < C$ there are arbitrarily large x' such that $\pi(x') < Cx'/\log x'$.

3. A natural number is *square-free* if no square numbers divide it (equivalently, is prime factorization has no exponents greater than one). Define the *Möbius function* $\mu : \mathbf{Z}_{>0} \to \mathbf{R}$ given by setting

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free and has an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free and has an odd number of prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\sum_{p \leq x} \log p = \sum_{k=1}^\infty \mu(k) \psi(x^{1/k}),$$

where

$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n)$$

is the Chebyshev function as defined in class.