MATH 53 SECTION NOTES

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1. SEPTEMBER 26

[Notes more detailed than usual due to beginning-of-term section-switching, etc.]

1.1. **Miscellany.**

My office hours are on Tuesdays and Thursdays from 11:00 until 12:30 in room 380-M (in the basement). You can ask questions by email at `ebwarner@stanford.edu`. As far as I am concerned, your responsibilities as students are to check the course website (`http://math.stanford.edu/~brendle/math53.html`), to keep up with the textbook reading, to do your homework, and to ask any questions you have in section, in office hours, or by email.

1.2. **A trivial differential equation.**

It is probably good to keep in mind the most basic possible example of a differential equation you "know how to solve": an equation of the form

$$\frac{dy}{dx} = f(x) \tag{1}$$

where f is a known function, depending only on the independent variable x. To solve it, just integrate, getting

$$y(x) = \int f(x)\, dx + c \tag{2}$$

for some constant c. Of course, this requires that you know how to take an indefinite integral of the function f, which is certainly not always possible in closed form (i.e., in terms of functions we know and love like polynomials, sine and cosine, exponentials, etc.)! However, this is a lesser difficulty than solving your average differential equation in general, and for our purposes it's good enough. For an example of this kind of "trivial" differential equation (1), if we are given that

$$\frac{dy}{dx} = x^3 + \sin(x),$$

then we can immediately find the general solution

$$y(x) = \frac{1}{4}x^4 - \cos(x) + c$$

where c is a constant.

In general there are two types of questions you can ask about differential equations: *existence questions*, which ask if there are solutions to the equation, and *uniqueness questions*, which ask if there is only one solution, or if not, if we can explicitly determine all of the solutions (the "general solution"). We should ask ourselves whether our solution to the above example is the most general form possible or if there are others. In this case, we can settle the uniqueness question by using the following fact (from calculus): *if two functions have the same derivative, then they differ only by a constant.* We will use this over and over again. In other

words, taking a derivative preserves almost all of the information about a function – the only thing we lose is that overall constant. So if we want to integrate in greatest possible generality, then we'd better remember that we have flexibility exactly up to a constant. Therefore the solution (2) to equation (1) is the most general possible.

1.3. **A slightly nontrivial differential equation.** You are expected to know how to solve the following differential equation:

$$(3) \qquad \frac{dy}{dx} = ay + b,$$

where a and b are constants. To use fancy terminology, this is a *first-order linear ordinary differential equation with constant coefficients*. The word "first-order" means that only one derivative of y appears, the word "linear" means that the right hand side is linear in y, the word "ordinary" means that we have only one independent and dependent variable each (no partial derivatives!), and "constant coefficients" means that a and b are constants rather than functions of x.

The general solution can be found in the text (I believe on page 4, with slightly different notation). We will illustrate the technique by setting $a = b = 1$ for simplicity, considering

$$(4) \qquad \frac{dy}{dx} = y + 1$$

instead.

How to solve this? We can try what we did last time and just integrate everything (with respect to x, of course), getting

$$(5) \qquad y(x) = \int y(x) \, dx + x + c$$

for some constant c. But while last time we were given the function f that we had to take the indefinite integral of, this time we have to take the indefinite integral of y, the very function we're solving for! So this isn't actually any simpler than (4)! In fact, (5) is known as the *integral form* of the differential equation (4), and while it is useful for some things (particularly numerical computations), it really isn't any simpler and doesn't count as a solution for our purposes.

So let's try something else. We will first divide both sides of (4) by $y+1$, getting

$$\frac{1}{y+1} \cdot \frac{dy}{dx} = 1,$$

and then note that by the chain rule the left hand side is equal to $\frac{dy}{dx} \left[\log |y+1| \right]$. Therefore integrating both sides and using (some form of the) fundamental theorem of calculus, we arrive at the conclusion that

$$(6) \qquad \log |y+1| = x + c$$

for some constant c[1], which after some easy algebra rearranges to

$$y(x) = -1 \pm e^c \cdot e^x.$$

The range of the exponential function is all positive numbers, so $\pm e^c$ can take on any nonzero value, and we get the solution

$$y(x) = -1 + Ae^x, \quad A \neq 0.$$

But wait! We've forgotten something! If you stare at (4) for long enough, or if you were wise enough to draw a quick direction field, you should note that the equation $y = -1$ is also a solution to (4). This corresponds to $A = 0$, which is not covered by our manipulations above. The problem is that we divided by zero, at least potentially, by dividing by $y + 1$. If $y = -1$, then this step is clearly bogus, and we have to consider this possibility separately. In general, multiplying by an expression that could be zero can introduce erroneous solutions, while dividing by an expression that could be zero can destroy possible good solutions. In this problem, we only divided by $y + 1$ and did no multiplication, so we only killed the solution $y = -1$. Adding it back in, we get the actual general solution

$$y(x) = -1 + Ae^x$$

for any constant A.

1.4. **Direction fields.** For any first-order ordinary differential equation, we can consider its direction field. In symbols, this means we can draw a direction field for any equation of the form

$$\frac{dy}{dx} = f(x, y). \tag{7}$$

To do so, we plot the x-y plane, consider a lot of points (x, y), calculate $f(x, y)$, and draw a short line segment of slope $f(x, y)$ at each point (x, y). By (7), any solution to the equation passing through a given point (x, y) does in fact have derivative, or slope, $f(x, y)$, so if we were to plot any solution to (7) on the same axes we would find that each line segment was a tangent line to the solution curve. If you want fancier language, we call any such curve an *integral curve* of the direction field, so the solutions to (7) are precisely the integral curves to the direction field.

You'll get some good practice with direction fields on the first homework. The point is that, given a direction field, you can tell at a glance the qualitative properties of various solutions, so you know what to expect if you derive an exact solution. Additionally, there are differential equations of the form (7) that are not explicitly solvable (in some sense, most of them are not explicitly solvable), so the direction field will give you information you can't find elsewhere.

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[1]Instead of explicitly using the chain rule, it can be useful to think of "multiplying both sides by dx and then integrating;" i.e. starting from (4) we get

$$\frac{dy}{y + 1} = dx$$

and then integrate both sides to get (6). This way of thinking really emphasizes the power of Leibniz's notation: although there is no independent meaning to "dy" or "dx," we can often proceed as if there were. In this class, feel free to write things this way even though it is not formally correct, if you'd like. Just keep in mind that we're really using the chain rule behind the scenes!

1.5. **A modeling example.** This is one approach to section 2.2, question 4 in the text that gives me an excuse to talk about modeling problems and a little about integrating factors. Consider a 500 gallon tank containing 200 gallons of water with 100 pounds of salt mixed in at time zero. The inflow is three gallons of water per minute with 1 pound of salt per gallon in the water, while the outflow is two gallons per minute. We assume "well-mixedness"; i.e., the outflow contains precisely the concentration of salt found in the tank as a whole at each time. We want to know how much salt is in the tank when it overflows.

To set up, note that our independent variable is clearly time, which I will denote t. I choose the total amount of salt, S, as the dependent variable.

In this type of problem, the basic setup is

$$\frac{dS}{dt} = [\text{rate of salt in}] - [\text{rate of salt out}].$$

The rate in is clearly 3 pounds per minute (we have three gallons coming in each minute, each containing a pound of salt). To find the rate out, we note that

$$[\text{rate of salt out}] = [\text{rate of water out}] \cdot [\text{salt concentration}]$$

$$= [\text{rate of water out}] \cdot \frac{[\text{total amount of salt}]}{[\text{volume of water}]}.$$

The rate of water out is 2 gallons per minute and the total amount of salt is S. Noting that the volume of water starts at 200 gallons and increases by one gallon per minute until overflow at 300 minutes, the volume at time t is equal to 200 gallons plus t gallons per minute.[2] All in all, omitting units at the moment, we get

$$(8) \qquad \frac{dS}{dt} = 3 - \frac{2S}{200 + t}, \quad 0 \le t \le 300,$$

with the initial condition $S(0) = 100$.

That was the "modeling" part of the problem. To solve (8), I'm going to do something almost completely mysterious: multiply both sides of the equation by the function $(200 + t)^2$. As this function is only zero at $t = -200$, which is outside our range anyway, we aren't introducing any new solutions. We get the following:

$$(200 + t)^2 \frac{dS}{dt} + 2S(200 + t) = 3(200 + t)^2.$$

Now comes the magic of the choice of integrating factor: the right hand side is precisely of the form $f\frac{dS}{dt} + S\frac{df}{dt}$, where $f(t) = (200 + t)^2$. Therefore by the product rule[3], the right hand side is equal to $\frac{d}{dt}(fS)$; that is,

$$\frac{d}{dt}\left[(200 + t)^2 S\right] = 3 \cdot (200 + t)^2.$$

Now we can integrate both sides with respect to t using the fundamental theorem of calculus, getting

$$(200 + t)^2 S = t^3 + 600t^2 + 120000t + c$$

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[2]We're implicitly solving a little auxiliary differential equation here: if V denotes the volume, we know that $\frac{dV}{dt} = 1$ and $V(0) = 200$, which we can integrate easily to find that $V(t) = 200 + t$.

[3]Also known as Leibniz's Rule.

for some constant c. Plugging in the initial condition $t = 0$, $S = 100$, we get $c = 4000000$, so the solution is

$$S(t) = \frac{t^3 + 600t^2 + 120000t + 4000000}{(200 + t)^2}.$$

At overflow, $t = 300$, we therefore have 484 pounds of salt.

Let's check this solution for plausibility in two ways. First, the total amount of salt we have poured in over these 300 minutes is equal to $3 \cdot 300 = 900$ pounds. So far, so good, as our answer is less than 900 pounds. Second, let's try to be more clever. If we had an infinitely large tank and infinitely much time, the *concentration* of salt in the tank must approach the concentration of what we pour in; that is, one pound per gallon. In our situation, we start with a concentration of $\frac{1}{2}$ pound per gallon, so we will approach the limiting concentration from below. It therefore makes a lot of sense that our concentration at overflow is $\frac{484}{500}$ pounds per gallon; that is, slightly less than one. Our answer looks very plausible indeed.

We'll soon learn how to pick the integrating factor to solve this kind of differential equation in general, so my choice of $(200 + t)^2$ will become less mysterious.

2. OCTOBER 1

2.1. Miscellany.
Homework will be due at 5:00 on its due date. You can hand it in during section or in my mailbox on the first floor.

2.2. Separable equations.
Two general points to make about separable equations:

(1) Whenever we "divide by dx," we're really using the chain rule. For example: the usual way of solving $\frac{dy}{dx} = \frac{x^2}{y}$ is to write '$y\,dy = x^2\,dy$" and then integrate to get $\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$. This is technically a no-no, because dx and dy don't have independent meanings. What we're really doing is writing

$$y \cdot \frac{dy}{dx} = x^2$$

and noticing that the left hand side, by the chain rule, is equal to $\frac{d}{dx}\left[\frac{1}{2}y^2\right]$. Therefore the fundamental theorem of calculus tells us that $\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$, as desired. Again, feel free to use this notation, but remember that the chain rule is what makes it work!

(2) Separable equations are easy to solve, but that statement glosses over the fact that we still have to be able to integrate functions and then solve for y. In general, this will not be achievable in terms of functions we know and love ("in closed form"). If you're given a separable equation as a homework or test problem, you can probably expect that your functions have indefinite integrals, but sometimes it's best to not try to solve for y, leaving your answer as an *implicit solution*.

Examples of separable equations: $\frac{dy}{dx} = \frac{x\sin(y)}{y}$, $\frac{dy}{dx} = \frac{1}{xy} - \frac{x}{y}$. Example of an equation that is not separable: $\frac{dy}{dx} = \frac{1}{x+y}$.

There is a final class of equations that I want to mention: the so-called *homogeneous* equations (this terminology is unfortunate; it has nothing to do with the

other use of the word homogeneous, where some right hand side is equal to zero).
Let's illustrate with an example: say

$$(9) \qquad \frac{dy}{dx} = \frac{y - 3x}{x - y}.$$

This equation is not separable, but I claim there is an easy change of variables that
will make it separable. If we set $v = \frac{y}{x}$, then by the product rule $\frac{dy}{dx} = x\frac{dv}{dx} + v$, so
(9) becomes

$$x\frac{dv}{dx} + v = \frac{v - 3}{1 - v}.$$

After some simple algebra, it is clear that this is a separable equation. Therefore
we can easily solve for $v(x)$, and plug in $y = \frac{v}{x}$ to find a solution in terms of y. This
change of variables will always work as long as the right hand side is a function of
$\frac{y}{x}$ alone, and therefore can be a rather powerful technique.

2.3. **Integrating factors.** Whenever we have a first-order linear ODE, we can
use an integrating factor (remember, "linear" means "linear in the dependent vari-
able"). Following the text, let's write the general equation as

$$(10) \qquad \frac{dy}{dx} + f(x)y = g(x).$$

If we recall, the trick here is to multiply the whole equation by some $\mu(x)$ so that
the resulting left hand side is in the form given by the product rule. So let's find
what μ can be! Upon multiplying, we get the left hand side $\mu(x)\frac{dy}{dx} + \mu(x)f(x)y$,
which we note can be written as $\frac{d}{dx}[\mu(x)y]$ if and only if the following auxiliary
differential equation holds:

$$\frac{d\mu}{dx} = \mu \cdot f(x).$$

This is clearly separable, with the general solution $\mu(x) = \exp\left[\int_{x_0}^x f(t)\,dt + C\right]$.
Picking $C = 0$ for simplicity (remember, we just want some μ, we don't care about
all possibilities), we get

$$\mu(x) = \exp\left[\int_{x_0}^x f(t)\,dt\right],$$

where x_0 can be chosen for convenience.

From here, it's easy: the previous work shows that (10) becomes

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)g(x),$$

which upon integrating and dividing by $\mu(x)$ (which is nonzero, by construction!)
shows that

$$y(x) = \frac{1}{\mu(x)}\left[\int_{x_1}^x \mu(s)g(s)\,ds + C\right]$$

is our solution.

We should check, just for safety's sake, that our choices of x_0 and x_1 don't affect
the form of the solution. With x_1 this is obvious; any change just gets absorbed
into C. If we change x_0, then μ changes by an overall multiplicative constant, and
therefore as long as C changes by the reciprocal of this multiplicative constant the
solution stays in the same form. Great!

Here's one example, just to prove that integrating factors aren't really particularly scary. Consider the equation

$$(11) \qquad \frac{dy}{dx} - y = e^{2x}.$$

Clearly not separable, clearly linear in y, so integrating factors are our best bet. We choose

$$\mu(x) = e^{\int_0^x (-1)\, dt} = e^{-x},$$

and upon multiplication (11) becomes

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^x.$$

By the product rule, this is equivalent to

$$\frac{d}{dx}\left[e^{-x}y\right] = e^x.$$

Integrating,

$$e^{-x}y = e^x + C.$$

Solving for y, we find the general solution

$$y = e^{2x} + Ce^x.$$

Easy!

3. October 3

3.1. **Autonomous equations.** An *autonomous* differential equation is one which does not depend on the independent variable in any way (except via the derivatives, of course). Since the independent variable is often time, we also call these *time-independent* equations. A first-order autonomous equation can be written as

$$\frac{dy}{dx} = f(y)$$

for some function f. This equation is clearly separable, so all first-order autonomous equations are (in principle, up to being able to integrate things) explicitly solvable.

3.2. **Practice problems.** We have an exam in a week. Here are the initial value problems (IVPs) from class, with solutions and very brief explanations. Throughout all of these problems, watch for solutions dropped due to dividing by zero! Usually you don't have to worry about them for initial value problems because it's unlikely your initial value will happen to fall right on the weirdest solution, but if the question asked for the general solution instead you'd be on the hook for them.

A.

$$\frac{dy}{dx} = -2y + 6, \quad y(0) = 1.$$

This equation is linear, separable, and autonomous! Separation of variables is generally easier, so let's do that: dividing both sides by $-2y + 6$, integrating, solving for the constant, and solving for y (those last two steps can be done in either order), we get

$$y(x) = 3 - 2e^{-2x}.$$

B.

$$\frac{dy}{dx} = xy + 4y, \quad y(2) = 1.$$

This equation is again linear and separable (though not autonomous). Separation of variables is still easier, so we rearrange to get

$$\frac{dy}{y} = (x + 4) \, dx,$$

integrate, and then solve for the constant and for y to get

$$y(x) = e^{\frac{1}{2}x^2 + 4x - 10}.$$

C.

$$\frac{dy}{dx} = \frac{2y}{x} + x^3 e^{-x}, \quad y(1) = 0.$$

This equation is linear but not separable, so we want to use an integrating factor. Shoving the first factor on the right hand side to the left so the equation is in the form we know and love, we want to choose

$$\mu(x) = e^{\int \frac{-2}{x} \, dx} = e^{-2\ln(x)} = \frac{1}{x^2}.$$

Multiplying the original equation by μ and doing the usual product-rule manipulation, we get

$$\frac{d}{dx}\left[y \cdot \frac{1}{x^2} \right] = xe^{-x}.$$

Integrating both sides yields

$$y \cdot \frac{1}{x^2} = -(x + 1)e^{-x} + C$$

(integration by parts!), and solving for C and y gives us

$$y(x) = -x^2(x + 1)e^{-x} + \frac{2}{e}x.$$

D.

$$\frac{dy}{dx} + y^2 \sin(x) = 0, \quad y(0) = \frac{1}{4}.$$

This equation is separable but not linear, so we separate variables to get

$$\frac{dy}{y^2} = -\sin(x) \, dx,$$

integrate, and solve for the constant and for y to get

$$y(x) = \frac{1}{-\cos(x) - 5}.$$

E.

$$x\frac{dy}{dx} = y - x^2 e^{-x}, \quad y(1) = 0.$$

This equation is linear and not separable. To get it into the right form to use an integrating factor, divide both sides by x and move the y term over:

$$\frac{dy}{dx} - \frac{y}{x} = -xe^{-x}.$$

Thus our integrating factor should be

$$\mu(x) = e^{\int \frac{-1}{x}\, dx} = e^{-\ln(x)} = \frac{1}{x}.$$

Multiplying through by μ and simplifying via the product rule, we get

$$\frac{d}{dx}\left[y\frac{1}{x}\right] = -e^{-x},$$

which integrates to

$$\frac{y}{x} = e^{-x} + C.$$

Solving for C and y gives us

$$y(x) = xe^{-x} - \frac{x}{e}.$$

F.

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}, \quad y(1) = 1.$$

This equation is neither linear nor separable (nor autonomous). It is *homogeneous*, with the meaning of that word mentioned in last section; that is, we can write the right-hand side as a function of y/x alone. A substitution will then make this equation separable, hence solvable. Let's do this: multiplying numerator and denominator by $1/x^2$, we have

$$\frac{dy}{dx} = \frac{1 + 3(y/x)^2}{2y/x}.$$

Set $v = y/x$. Then by the product rule, $\frac{dy}{dx} = v + x\frac{dv}{x}$, so in terms of x and v our equation is

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}.$$

This simplifies algebraically to

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v},$$

so we can separate variables, getting

$$\frac{2v\, dv}{1 + v^2} = \frac{dx}{x}.$$

Integrating,

$$\ln(v^2 + 1) = \ln(x) + C,$$

so

$$v^2 + 1 = e^C x,$$

which implies that

$$y^2 + x^2 = e^C x^3.$$

Solving for the constant gives

$$y^2 + x^2 = 2x^3.$$

If we want an explicit solution rather than an implicit one, we can write this as

$$y(x) = \pm\sqrt{2x^3 - x^2} = \pm x\sqrt{2x - 1}.$$

4. October 8

4.1. **Miscellany.** The first midterm exam is on Thursday at 7:00 p.m. in Cubberley Auditorium. It is closed book, closed notes, two hours. Questions should be very similar to what you have seen on homework (including Homework 3). Everything covered in lecture is fair game.

Types of (first-order, ordinary) differential equations we know how to solve:

- Separable equations, including autonomous equations
- Linear equations (use integrating factors!)
- Homogeneous equations
- Exact equations

Other things you should be familiar with for the exam: equilibrium solutions, direction fields, "modelling" problems (i.e., word problems).

4.2. **Homogeneous equation example.** We went over problem F from yesterday. Solution is above, in last section's notes.

4.3. **Exact equations.** Exact equations are another (rare) class of equations that we can actually solve. Assume that we have an equation of the form

$$(12) \qquad\qquad M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

By definition, this equation is *exact* if there exists a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$.[4] Why is this a good notion? Well, if we had such a ψ, then our equation would be

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0.$$

By the multivariable calculus chain rule (remember, ψ depends on x and y, while y depends on x), this is equivalent to the simpler equation

$$\frac{d\psi}{dx} = 0$$

that involves the total derivative. Integrating with respect to x, we find that the solutions to (12) are given implicitly by

$$(13) \qquad\qquad \psi(x, y) = c$$

where c is an arbitrary constant. That is, the equation (12) is exact, its general solution is really easy to write down: it is just (13).

As a quick example: if we are given

$$(2x + 3) + (2y - 2)\frac{dy}{dx} = 0$$

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[4]I will use the "subscript means partial derivative" notation without comment, because it saves a lot of space and is pretty intuitive.

and happen to notice that $\psi(x, y) = x^2 + y^2 + 3x - 2y$ is such that $\psi_x = M$ and $\psi_y = N$ (M and N being notated as above), then we immediately can conclude that the general solution is

$$x^2 + y^2 + 3x - 2y = c.$$

We should, at this point, have two questions. First, how do we know whether ψ exists? Second, if it does, how do we find it?

On the subject of the first question, we have the following:

Theorem 4.1. *With notation as above, (12) is exact if and only if $M_y = N_x$.*

This is Theorem 2.5.1 in the text. One direction is easy: if (12) is exact, then we have a ψ such that $\psi_x = M$ and $\psi_y = N$, so $M_y = \psi_{xy} = \psi_{yx} = N_x$ by the equality of mixed partials from multivariable calculus.[5] The other direction is more complicated, but not that bad. Essentially, given that $M_y = N_x$ you explicitly write down some integrals to define ψ, very similarly to what we do in practice, below. You should read about it in the textbook.

This is the answer to the first question. We can check immediately if $M_y = N_x$, and if so, the equation is exact; if not, it isn't.

For the second question, we know that we should have $\psi_x = M$ and $\psi_y = N$. Integrating the first equation in x gives us

$$(14) \qquad \psi(x, y) = \int M(x, y)\, dx + g(y),$$

where g is an arbitrary function. Remember, when we integrate we need to leave room for a constant, and from the perspective of x, y is a constant. So the "constant" of the above integration is actually allowed to vary with y. Similarly, integrating the second equation in y gives us

$$(15) \qquad \psi(x, y) = \int N(x, y)\, dy + h(x)$$

for an arbitrary function h. These two equations are enough to determine ψ up to a constant, if it exists.

To see this in action, consider the following example:

$$\frac{y}{x} + 6x + (\ln x - 2)\frac{dy}{dx} = 0.$$

First we check that it is exact: this is easy; we have $M_y = N_x = \frac{1}{x}$. Then (14) tells us that $\psi(x, y) = y \ln x + 3x^2 + g(y)$, while (15) tells us that $\psi(x, y) = y \ln x - 2y + h(x)$. Therefore we should take $g(y) = -2y$ and $h(x) = 3x^2$, so

$$\psi(x, y) = y \ln x - 2y + 3x^2,$$

and our general solution is therefore

$$y \ln x - 2y + 3x^2 = c$$

for an arbitrary constant c.

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[5]There is a continuity condition, but I'm ignoring it in the name of simplicity.

4.4. **Integrating factors for exact equations.** Sometimes $M + N\frac{dy}{dx} = 0$ itself isn't exact, but we could imagine that it could become exact if we multiplied it by some integrating factor $\mu(x, y)$. Let's see where this leads. If

$$\mu M + \mu N \frac{dy}{dx} = 0$$

were exact, then by Theorem 4.1 we would have

$$(\mu M)_y = (\mu N)_x,$$

or expanding by the product rule,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Unfortunately, as an equation for μ, this is even more complicated than the equation we started with! In particular, it is a partial differential equation, involving both μ_x and μ_y. So in general, we've hit a dead end.

Fortunately, sometimes things simplify a bunch, and you should be familiar with the idea. For example, assume that

(16) $\frac{M_y - N_x}{N}$ is a function only of x.

Then we can assume that μ depends on x only as well, and therefore $\mu_y = 0$ and we have the equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

This is separable, with solution

$$\mu(x) = \exp\left[\int \frac{M_y - N_x}{N}\,dx\right].$$

So if (16) is true, we can multiply our equation by this integrating factor, and it will become exact, so we can solve it.

Similarly, if

$$\frac{N_x - M_y}{M} \text{ is a function only of } y$$

holds, then multiplying by an integrating factor of

$$\mu(y) = \exp\left[\int \frac{N_x - M_y}{M}\,dy\right]$$

will make our equation exact. If you're interested in another example of this phenomenon, do Problem 24 of section 2.5 in the text.

As an example, consider

$$y + e^{-x} + \frac{dy}{dx} = 0.$$

(This equation is linear, but forget about that for the moment: I want to use this method.) This is not exact, as $M_y = 1 \neq 0 = N_x$. However, we have

$$\frac{M_y - N_x}{N} = 1,$$

so an integrating factor of

$$\mu(x) = \exp\left[\int 1\,dx\right] = e^x$$

will work. Multiplying by μ gives us

$$ye^x + 1 + e^x \frac{dy}{dx} = 0.$$

This equation is exact, for now $M_y = e^x = N_x$. We have

$$\psi(x, y) = \int (ye^x + 1) \, dx = ye^x + x + g(y)$$

and also

$$\psi(x, y) = \int e^x \, dx = ye^x + h(x),$$

so we can choose $\psi(x, y) = ye^x + x$, and our general solution is

$$ye^x + x = c$$

for an arbitrary constant c.

5. October 10

5.1. **Miscellany.** Exam today! Unrelatedly, you have issues with the grading of the first homework assignment, see me.

5.2. **Practice problems.**

1. This problem intentionally left blank because I messed up writing it down.

2.

$$e^t y' + e^{(t+1)e^{-t}+t} = ty.$$

This equation is linear (despite the complexity in t); in standard form we can write it as

$$y' - te^{-t}y = -e^{(t+1)e^{-t}}.$$

The integrating factor will therefore be

$$\mu(t) = \exp\left[-\int te^{-t} \, dt\right] = \exp\left[-(1+t)e^{-t}\right],$$

and the general solution will be

$$y(t) = \frac{-t + C}{\exp\left[-(1+t)e^{-t}\right]}$$

for any constant C.

3.

$$\frac{dy}{dx} - 2y^2 = xy^2.$$

This equation can be written in the form

$$\frac{dy}{dx} = (2 + x)y^2$$

and is therefore clearly separable. The general solution is

$$y(y) = \frac{1}{-\frac{1}{2}x^2 - 2x + C} \quad \text{or} \quad y = 0,$$

for any constant C. The second solution $y = 0$ comes because we divide by y^2 when we separate the variables, and an easy check shows that $y = 0$ is a valid solution not already accounted for.

4.

$$e^t(2ty - 9t^2) + e^t(2y + t^2 + 1)\frac{dy}{dt} = 0.$$

Upon dividing by e^t, this is an exact equation. The general (implicit) solution is

$$y^2 + (x^2 + 1)y - 3x^3 = C$$

for any constant C.

5.

$$\frac{dy}{dx} = \frac{y}{x} + \csc\frac{y}{x}.$$

This is a homogeneous equation. By making the change of variables $v = y/x$, we get

$$\frac{dv}{dx}x + v = v + \csc v,$$

which simplifies to

$$\sin v\frac{dv}{dx} = \frac{1}{x}.$$

This separable equation is readily solved; the general (implicit) solution is

$$e^{-\cos\frac{y}{x}} = Ax,$$

where $A \neq 0$.

6.

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}.$$

This equation is linear and already in the desired form. Using the integrating factor

$$\mu(x) = \exp\left[\int\frac{2}{x}\,dx\right] = x^2,$$

we find the general solution

$$y(x) = \frac{\sin x + C}{x^2}$$

for an arbitrary constant C.

7.

$$2xy^2 + 4 = 2(3 - x^2y)\frac{dy}{dx}.$$

This is an exact equation (at least when we move the right hand side to the left). The general (implicit) solution is

$$x^2y^2 - 6y + 4x = C$$

for an arbitrary constant C.

8.
$$\frac{dy}{dx} = \frac{x + 3y}{x - y}.$$

This is a homogeneous equation: we can write it as

$$\frac{dy}{dx} = \frac{1 + 3(y/x)}{1 - (y/x)}.$$

Making the change of variables $v = y/x$, we get

$$\frac{dv}{dx}x + v = \frac{1 + 3v}{1 - v}.$$

Upon separating variables and solving, we find the general solution to be

$$\frac{2x}{x + y} + \ln|x + y| = C \quad \text{or} \quad y = -x.$$

As usual, the second solution comes from division-by-zero issues.

9.
$$t^2 \sin t \, dt + ty^2 \, dy = 0.$$

This equation is separable; we immediately put it in the form

$$t \sin t \, dt = -y^2 \, dy.$$

The general solution is thus

$$y(t) = \sqrt[3]{3t \cos t - 3 \sin t + C}$$

for some arbitrary constant C.

10.
$$y \, dx + (2xy - e^{-2y}) \, dy = 0.$$

This is not an exact equation, but it can be made into one with an appropriate integrating factor. With the usual notation, we note that

$$\frac{N_x - M_y}{M} = 2 - \frac{1}{y}$$

is a function only of y. Therefore the integrating factor

$$\mu(y) = \exp\left[\int \frac{N_x - M_y}{M} \, dy\right] = \exp\left[2y - \ln y\right] = \frac{1}{y}e^{2y}$$

should work. We indeed verify that

$$e^{2y} \, dx + \left(2xe^{2y} - \frac{1}{y}\right) \, dy = 0$$

is exact, and has the general solution

$$xe^{2y} - \ln|y| = C \quad \text{or} \quad y = 0.$$

The second solution comes from the fact that we divided by y upon multiplying by the integrating factor, and a quick check shows that $y = 0$ is in fact a valid solution to the original differential equation.

6. October 15

6.1. **Miscellany.** As usual, the homework assignment is due today at 5:00. The first midterm exam will be handed back in lecture tomorrow.

As a look ahead, if you are not totally comfortable with, for example, the words "determinant" and "eigenvalue" and "eigenvector," you should look over that. We are about to study systems of differential equations, whose solution requires elementary techniques from linear algebra. Section 3.1 of the text goes over 2 by 2 matrices in detail, while Appendix A has the general case.

6.2. **An integral.** The single least understood piece of the exam seemed to boil down to the following integral:

$$\int \frac{t}{1+t} \, dt.$$

The standard way to do this is to substitute $u = 1 + t$. Then $du = dt$, so the integral is equal to

$$\int \frac{u-1}{u} \, du = \int \, du - \int \frac{du}{u} = u - \ln|u| + C = 1 + t - \ln|1+t| + C.$$

Alternatively, you could notice that

$$\frac{t}{t+1} = 1 - \frac{1}{t+1}$$

and proceed from there. To my surprise, there were two people who successfully integrated by parts (twice) and derived the correct result after a considerable amount of algebra; there were many more who attempted to integrate by parts and gave up.

6.3. **Existence and uniqueness for first-order ODEs.** We went over a sampler of theorems that one would prove about first-order ODEs in a less computational course.

6.3.1. *A theorem for linear equations.*

Theorem 6.1. *Given the initial value problem $y' + p(t)y = q(t)$, $y(t_0) = y_0$, there exists a unique solution on the largest open interval containing t_0 such that p and q are defined and continuous.*

This theorem tells us that existence and uniqueness hold on the largest interval that they could reasonably be expected to hold for linear equations. The theorem is stated and proved in section 2.3 of the textbook. The reason it can be proved so easily is that we know how to solve first-order linear equations, more or less explicitly.

As an example, consider the linear IVP

$$y' + \sin(t)y = t^2, \quad y(0) = 1.$$

From Theorem 6.1, we conclude that a solution exists and is unique on the whole real line, because $\sin(t)$ and t^2 are defined and continuous on the whole real line.

As another example, consider the linear IVP

$$y' = \frac{y}{t} + \frac{1}{t-2}, \quad y(1) = 5.$$

The largest interval containing 1 on which $1/t$ and $1/(t-2)$ are both defined and continuous is $(0, 2)$, so Theorem 6.1 tells us that there exists a unique solution to this IVP on $(0, 2)$.

6.3.2. *Two theorems for nonlinear equations.*

Theorem 6.2 (Picard-Lindelöf)**.** *Given the initial value problem* $y' = f(t, y(t))$, $y(t_0) = y_0$, *assume* f *is continuous in* t *and Lipschitz continuous in* y *on some interval around* t_0. *Then there exists a unique solution on some (possibly very small) interval* $(t_0 - \epsilon, t_0 + \epsilon)$, *where* $\epsilon > 0$.

This theorem tells us that for a nonlinear equation, if we assume some continuity conditions on f, then we still can conclude existence and uniqueness of the solution, but only for a very short time interval. Its proof is beyond the scope of this course.

What, exactly, is the continuity condition on f? Lipschitz continuity for a function g, by definition, means that there exists a constant K such that for every pair of points t_0, t_1 we have the inequality

$$|g(t_1) - g(t_0)| \leq K|t_1 - t_0|.$$

If we recall, continuity of g means that as t_0 and t_1 tend towards each other, $g(t_1)$ and $g(t_0)$ do as well. That is, as the left hand side of the above equation tends to zero, so does the right hand side. It is therefore clear that Lipschitz continuity, for any K, implies continuity: Lipschitz continuity is a stronger notion than continuity.

In more geometric terms, the Lipschitz condition means exactly that there is a double cone with some slope (K) that, as its vertex is translated along the graph of g, always stays away from the graph of g (that is, the graph of g never touches the inside of the double cone).

How can Lipschitz continuity fail? There are essentially two ways. First, consider the function

$$g(t) = t^2.$$

No matter what cone we draw at the origin, it will always intersect this parabola somewhere. The slope just increases too fast for the function to possess a global Lipschitz constant. Theorem 6.2, however, only hypothesizes a Lipchitz constant on some interval ("locally Lipschitz"), so this kind of example is not actually a problem for us in this context.

The more serious way that Lipschitz continuity can fail for a continuous function is for that function to possess a vertical tangent line at a point. For example, consider the function

$$g(t) = t^{1/3}.$$

This function is defined and continuous for all real numbers, but at the origin it has a vertical tangent (and therefore an undefined derivative). No cone that we draw at the origin will fail to intersect the graph of g.

A key fact: any continuously differentiable function (that is, a function with a derivative that is itself continuous) is locally Lipschitz. Therefore, we can conclude that if $f(t, y)$ is continuously differentiable, we can apply Theorem (6.2).

As an example, consider the IVP

$$y' = \sin(y + t)^2, \quad y(0) = 1.$$

This is not a linear ODE, so we cannot apply Theorem 6.1. As a function of y and t, $\sin(y + t)^2$ is continuously differentiable, therefore locally Lipschitz continuous,

so we can apply Theorem 6.2 to conclude that there is some $\epsilon > 0$ such that there exists a unique solution to this ODE on $(-\epsilon, \epsilon)$.

If we give up Lipschitz continuity, we also have to give up uniqueness, but we still have the following:

Theorem 6.3 (Peano)**.** *Given the initial value problem $y' = f(t, y(t))$, $y(t_0) = y_0$, assume that f is continuous in y and t. Then there exists at least one solution on some (possibly very small) interval $(t_0 - \epsilon, t_0 + \epsilon)$, where $\epsilon > 0$.*

The hypotheses here can actually be relaxed a small amount: we can assume only local integrability in t, as long as we relax the concept of "solution" a little bit. This generalization is due to Carathéodory. We won't worry about it.

The Peano theorem tells us that initial value problems involving equations like $y' = y^{1/3}$ do always have solutions, but those solutions might be very non-unique. To illustrate this phenomenon, let's actually solve the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0.$$

This equation is clearly separable, so we can divide by $y^{1/3}$ (keeping in mind that $y = 0$ might be a solution too, so we have to check it separately), integrate, and solve for the constant to get

$$y(t) = \left(\frac{2t}{3}\right)^{3/2} \quad \text{or} \quad y(t) = 0.$$

Already, we see non-uniqueness: these are two different but equally valid solutions. Furthermore, the first is actually two different solutions: in taking the square root, we can pick the positive or negative root, so we really should write the first alternative as

$$y(t) = \pm \left(\frac{2t}{3}\right)^{3/2}.$$

Even worse: these three don't exhaust all the possibilities. I claim that, for any $t_0 > 0$, the following piecewise-defined function is a valid solution:

$$y(t) = \begin{cases} 0 & \text{if } t < t_0, \\ \pm \left[\frac{2}{3}(t - t_0)\right]^{3/2} & \text{if } t \geq t_0. \end{cases}$$

This function is zero for a while, then behaves like our first solution. In dividing by $y^{1/3}$, not only did we kill the solution $y = 0$, we also killed any solution that started out like $y = 0$ on some interval! You should check that this function is differentiable everywhere and satisfies our IVP.

The lesson here is that if Lipschitz continuity fails, uniqueness can fail, and if uniqueness fails, it can fail rather badly.

6.3.3. *Practice examples.* Let's look at how to apply these theorems with several examples.

What can we say about the IVP

$$y' = \frac{y}{\sin(t)}, \quad y\left(\frac{\pi}{2}\right) = 3?$$

It's linear, so we can apply Theorem 6.1 to conclude that there exists a unique solution to this problem on $(0, \pi)$, which is the largest interval on which $\frac{1}{\sin(t)}$ is defined and continuous.

What can we say about

$$y' = \exp[y + t^2 - \cos(y)t], \quad y(0) = 1?$$

It's horribly nonlinear, so we can't apply the first theorem, but we can apply Theorem 6.2 to conclude that there is an $\epsilon > 0$ for which there exists a unique solution on the interval $(-\epsilon, \epsilon)$.

What can we say about

$$y' = g(y) \cdot t, \quad y(1) = 1,$$

where g is a continuous function such that $g(1) = 1$ and g has a vertical tangent at that point? Well, we care about whether there is a Lipschitz constant near $t = 1, y = 1$, and that will not be the case for the function $f(t, y) = g(y) \cdot t$ due to the vertical tangent. Therefore we can only apply Theorem 6.3 to conclude that there exists an ϵ such that there exists at least one solution to this initial value problem on $(1 - \epsilon, 1 + \epsilon)$.

What can we say about

$$y' = \cos(y), \quad y(8) = 4000?$$

It's not linear, but Theorem 6.2 implies immediately that there exists a unique solution locally (that is, on $(8 - \epsilon, 8 + \epsilon)$ for some $\epsilon > 0$).

What can we say about

$$t^2 y' = y + e^t, \quad y(1) = 5?$$

It's linear once we divide by t^2, so Theorem 6.1 implies that there exists a unique solution on $(0, \infty)$, which is the largest interval on which $1/t^2$ and e^t/t^2 are defined and continuous.

7. OCTOBER 17

7.1. **Systems of equations: terminology.** We now graduate from ordinary differential equations to systems of ordinary differential equations, meaning more than one dependent variable (though only one independent variable still). To ease the transition, we will consider linear equations first. With one dependent variable, a first-order linear equation must take the form

$$\frac{dx}{dt} = p(t)x + q(t),$$

where $p(t)$ and $q(t)$ are functions of t. A system of first-order linear equations is just this, "promoted" to vector notation. That is, a first-order linear equation takes the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{q}(t),$$

where $\mathbf{x}$ is the vector consisting of all the dependent variables, $\mathbf{P}(t)$ is a square matrix whose entries are functions of t, and $\mathbf{q}(t)$ is a vector whose functions are entries of t. In the case of two dependent variables, this means the same as the system of two equations

$$\frac{dx_1}{dt} = P_{11}(t)x_1 + P_{12}(t)x_2 + q_1(t),$$
$$\frac{dx_2}{dt} = P_{21}(t)x_1 + P_{22}(t)x_2 + q_2(t).$$

We use vector notation for the obvious reason: it is much cleaner and uses much less lead/ink/chalk.

Unlike first-order linear equations by themselves, there is no algorithm to solve a general system of first-order linear equations. To start, we will consider a very special case: *constant-coefficient* linear equations, in which $\mathbf{P}(t)$ and $\mathrm{q}(t)$ are replaced by constants $\mathbf{A}$ and $\mathbf{b}$. We will eventually see that we can solve all such systems. For now, though, we will make a further restriction: let $\mathbf{b} = 0$. This is known as the *homogeneous* case, in analogy to linear algebra. Just like in linear algebra, there will turn out to be close connections between the case when $\mathbf{b} = 0$ and the case when $\mathbf{b} \neq 0$, the *inhomogeneous* case. In a couple of weeks, we will use the method of variation of parameters to solve such inhomogeneous equations. For now, we stick to $\mathbf{b} = 0$.

7.2. **Linear, constant-coefficient, homogeneous systems of equations.** After these simplifications, we are left with the equation

$$(17) \qquad\qquad \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}.$$

Let's note the following fact, known as the *principle of superposition*: if we have any two solutions to (17), then any linear combination of those two solutions will also be a solution. By "linear combination," I mean that we are allowed to multiply by constants and add things together at will. Let's check this: if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions to (17), then for any constants C_1 and C_2 we have

$$\begin{aligned}
\frac{d}{dt}\left(C_1\mathbf{x}_1 + C_2\mathbf{x}_2\right) &= C_1\frac{d\mathbf{x}_1}{dt} + C_2\frac{d\mathbf{x}_2}{dt} \\
&= C_1\mathbf{A}\mathbf{x}_1 + C_2\mathbf{A}\mathbf{x}_2 \\
&= \mathbf{A}(C_1\mathbf{x}_1 + C_2\mathbf{x}_2).
\end{aligned}$$

Therefore $C_1\mathbf{x}_1 + C_2\mathbf{x}_2$ is also a solution to (17). Another way of saying this is that the space of solutions to (17) is a subspace of the space of all linear functions. Actually, this property characterizes homogeneous linear systems of equations: if superposition holds, then the system is homogeneous and linear, as well as vice-versa (which we've checked).

So now we know that the general solution to (17) will be some linear subspace of functions. What functions? And what is the dimension of this subspace? Well, by analogy to the case of one dependent variable, we expect exponential solutions. And we expect to have one degree of freedom – that is, one choice of a constant – for each dependent variable: after all, a system of equations is just a bunch of first-order equations, each of which should give us one constant of integration. These guesses both turn out to be largely correct.

Let's try solutions of the form

$$(18) \qquad\qquad \mathbf{x} = e^{\lambda t}\mathbf{v},$$

where λ is some constant scalar and $\mathbf{v}$ is some constant vector. Plugging into (17), we get

$$\lambda e^{\lambda t}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v}.$$

Canceling the exponents, we get

$$\lambda\mathbf{v} = \mathbf{A}\mathbf{v},$$

which means precisely that λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is a corresponding eigenvector! That is, we have shown that if our solution is of the form (18), then λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ a corresponding eigenvector, and conversely if λ is an eigenvalue with corresponding eigenvector $\mathbf{v}$, then (18) is a solution of (17).

Since in general an n-by-n matrix has n distinct eigenvalues and we expect an n-dimensional solution space, this type of solution should give us everything we want.

7.3. Examples.
Let's find the general solution to the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

(This example is intended to be deliberately simple.) To find eigenvalues, we find the roots of the characteristic polynomial

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} 2 - \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} = (2 - \lambda)(1 - \lambda),$$

which in this case are clearly $\lambda_1 = 1$ and $\lambda_2 = 2$. To find eigenvectors, look at the defining equation: we want a nonzero $\mathbf{v}_1$ such that

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = 0 \iff \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Clearly, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a good choice. With the same procedure, we find $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Therefore the general solution to this system of equations is given by

$$\mathbf{x} = C_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants.

For a second example, consider the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}.$$

Here the characteristic polynomial is

$$\det \begin{pmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{pmatrix} = (3 - \lambda)(-2 - \lambda) - (-2)2$$
$$= \lambda^2 - \lambda - 2$$
$$= (\lambda + 1)(\lambda - 2).$$

Therefore our eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$. The corresponding eigenvectors, as is easily verified, can be chosen to be $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the general solution to this system of equations is

$$\mathbf{x} = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants.

7.4. **Possible issues.** Two problems could come up when solving this sort of equa-
tion. First, eigenvalues, being the roots of an essentially arbitrary polynomial, are
not in general going to take real values. Fortunately, as we will see soon, we can
deal with this issue basically by ignoring it, then simplifying the result.

Here's an example: say we are given the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

The matrix in question has characteristic polynomial $\lambda^2 + 1$, which has roots $\pm i$,
where i is a square root of -1. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$
and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Ignoring any nagging doubts we may have, let's plug this in and
find our general solution: it is

$$\mathbf{x} = C_1 e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Now we really have to let C_1 and C_2 be complex – secretly we may have been doing
this all along, but now we have to be explicit about it. What does e^{it} mean? Well,
the easiest way to find out is probably via the exponential function's power series:
if the exponential function has a meaning in the complex domain, this should be
it. In fact, it does: everything goes through beautifully, and one can conclude the
famous Euler formula:

$$e^{it} = \cos t + i \sin t.$$

Writing things this way lets us ignore complex numbers altogether, after some
algebraic manipulation. We'll deal with this in more detail later.

The second problem that could occur is more serious. What if $\lambda_1 = \lambda_2$? If
we remember our linear algebra, we recall that there are two possibilities in this
situation: either the eigenspace associated with this eigenvalue has dimension two,
or it has dimension one. In the former case, we have no problem: we can just pick
two linearly independent eigenvalues and proceed with our lives. As an example,
consider the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

The matrix in question has a single eigenvalue, $\lambda = 1$, but with a two-dimensional
eigenspace (that is, *any* nonzero vector is an eigenvector for this eigenvalue). So
we can pick any basis for the plane as our eigenvectors – say, for simplicity, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ – and conclude that we have the general solution

$$\mathbf{x} = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In other words, $x_1 = C_1 e^t$ and $x_2 = C_2 e^t$, which is of course what we expect.

There is a problem when the eigenspace has dimension one. Consider the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which has the single eigenvalue $\lambda = 0$. Its corresponding eigenspace is one-dimensional: we can only choose its eigenvector to be some nonzero multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So we only get one independent solution to the corresponding differential equation, which isn't enough: we still expect two solutions. We'll learn how to get around this problem in a couple of weeks by the method of *variation of parameters*. To spoil the punch-line, we end up having to consider functions of the form $te^{\lambda t}$ as well as $e^{\lambda t}$. We'll worry about this more later.

7.5. **A trick with second-order equations.** Consider the following problem: find the general solution to the *second-order* constant-coefficient linear differential equation

$$(19) \qquad \frac{d^2 y}{dt^2} + 2\frac{dy}{dt} - 3y = 0.$$

At first glance, this is something totally new. But there is an easy trick to reduce this problem to a system of two first-order constant-coefficient linear differential equations: let $x_1 = y$ and $x_2 = \frac{dy}{dt}$, both functions of t. Then by definition $\frac{dx_1}{dt} = x_2$, and from equation (19) we know that

$$\frac{dx_2}{dt} = \frac{d^2 y}{dt^2} = -2\frac{dy}{dt} + 3y = -2x_2 + 3x_1.$$

So we in fact have the following system of equations:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \mathbf{x}.$$

This can readily be solved in the usual manner; we get as a general solution

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We're really interested in $y = x_1$, which is the first row of the above expression, so we have the following general solution for (19):

$$y = C_1 e^{-3t} + C_2 e^t.$$

(Note that I have redefined C_1 to be its negative; since it's an arbitrary constant anyway I can get away with that.) This also illustrates the idea that a second-order differential equation should have a general solution with two degrees of freedom. This trick can be readily extended to higher-order equations.

8. November 5

8.1. **Constant-coefficient linear homogeneous ODEs.** Let's consider the following equation:

$$(20) \qquad y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0.$$

We use the usual notation that $y^{(m)}$ means the mth derivative of y, and we specify that each a_m is a constant. The equation (20) is therefore an *nth order constant-coefficient linear homogeneous ODE*.

We know how to solve this already, as it happens, by turning it into a system of first-order ODEs. Let $x_1 = y$, $x_2 = y'$, etc., all the way to $x_n = y^{(n-1)}$. Then we get the system of equations

$$x_1' = x_2,$$
$$x_2' = x_3,$$
$$\ldots$$
$$x_{n-2}' = x_{n-1},$$
$$x_{n-1}' = -a_0 x_1 - a_1 x_2 - \ldots - x_{n-1} x_n.$$

In matrix notation,

$$\mathbf{x}' = A\mathbf{x},$$

where A is the n-by-n matrix consisting of 1s on the upper diagonal and zeroes elsewhere except the bottom row, which consists of the entries $-a_0, -a_1, \ldots, -a_{n-1}$. It takes a bit of work (proof by induction on dimension!), but one can show that

$$\det(A - \lambda I) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0).$$

For the purposes of finding the roots, the leading $(-1)^n$ is irrelevant, so we will ignore it. Look at what we've shown, though: the characteristic polynomial associated to the differential equation (20) is really easy to write down: all we do is put λ^m wherever we had $y^{(m)}$ in the original equation.[6] That is, the eigenvalues associated to (20) are just the roots of the polynomial

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0.$$

If the roots are all distinct (say, $\lambda_1, \ldots, \lambda_n$), we can conclude that we have solutions $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$, and therefore the general solution

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}.$$

(Recall that as this is an nth degree equation, we expect an n-dimensional space of solutions. In order to specify a single solution, for example as in an initial value problem, we would need to specify n parameters, such as $y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)$.)

If we have repeated roots, the above procedure doesn't give us enough solutions. In general, solving the system $\mathbf{x}' = A\mathbf{x}$ where A has repeated eigenvalues involves finding generalized eigenvalues (or an equivalent procedure). But in this case, things aren't so bad: because we're really only interested in $x_1 = y$, instead of looking for solutions of the form $\mathbf{v}e^{\lambda t} + \mathbf{w}te^{\lambda t}$ we only care about the first entry in each. And by linearity, the first entry can be specified arbitrarily: the other entries will then be determined, but we don't care what they are. So we can just set them to equal one, taking as fundamental solutions $e^{\lambda t}$ and $te^{\lambda t}$. Our procedure for repeated eigenvalues is therefore to add factors of t until the multiplicity is exhausted, which is in practice very easy to do.

Let's see some examples. For example, let's find the general solution to $y'''' - 1 = 0$. The corresponding characteristic polynomial is $\lambda^4 - 1$, which can be factored as $(\lambda + 1)(\lambda - 1)(\lambda^2 + 1)$ and therefore has roots $\pm 1, \pm i$. We therefore get the solutions $e^t, e^{-t}, e^{it}, e^{-it}$. The latter two are complex conjugate pairs, so as before we can use

---

[6]Actually, this version of the concept of "characteristic polynomial" came first, historically. Matrices were invented by Cayley in the mid-19th century, but Euler in the 18th century knew how to solve differential equations like (20) by essentially the method I've just described.

as fundamental solutions $\Re(e^{it}) = \cos t$ and $\Im(e^{it}) = \sin t$. Therefore our general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t,$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

As another example, let's find the general solution of $y''' - y'' - y' + y = 0$. The corresponding characteristic polynomial is $\lambda^3 - \lambda^2 - \lambda + 1$, which can be factored into $(\lambda - 1)^2(\lambda + 1)$. The eigenvalues are therefore 1, with multiplicity 2, and -1, and by the above discussion the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t}.$$

It really is that easy!

A caveat: although this method reduces the solution of (20) to, essentially, finding the roots of a polynomial, it is not always so easy to find the roots of a polynomial. In degree two we have the quadratic equation, and there also exist corresponding cubic and quartic equations. However, it is a major theorem of early 19th century mathematics that no such equations exist for polynomials of degree ≥ 5.

8.2. The matrix exponential.

This subject won't help with computations at all, but it does have theoretical importance. Consider, again, our favorite system of differential equations:

$$(21) \qquad\qquad \mathbf{x}' = A\mathbf{x}.$$

In one dimension (so A is a scalar), the solutions are given by $x(t) = x_0 e^{At}$. We can ask the following naïve question: can we express the solutions to (21) in a similar manner? Surprisingly, the answer is yes.

Recall that the exponential function can be defined by the power series

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots + \frac{a^n}{n!} + \ldots.$$

We will do the same thing for matrices. Define, for a square matrix A,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots + \frac{A^n}{n!} + \ldots.$$

Here $A^2 = A \cdot A$ (matrix multiplication) and so on.

It is fairly easy to prove that e^A converges no matter what A is (in fact, this is proven essentially the same way that one proves that e^x has a power series that converges for all x. The rapid increase of the factorial function ensures it.)

Now to the main point.

Theorem 8.1. *The vector $e^{At} \cdot \mathbf{x}_0$ is the unique solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.*

Proof. If $\mathbf{x} = e^{At} \cdot \mathbf{x}_0$ is a solution, then it is certainly unique by the standard uniqueness theorems for ODEs. The initial condition is easy to verify: we have

$$\mathbf{x}(0) = e^{A \cdot 0} \cdot \mathbf{x}_0 = e^{\mathbf{0}} \cdot \mathbf{x}_0 = I \cdot \mathbf{x}_0 = \mathbf{x}_0,$$

because $e^{\mathbf{0}} = I + 0 + 0 + 0 + \ldots = I$. To verify that $\mathbf{x}$ satisfies (21), we will ignore questions about manipulating infinite sums, which are easy enough to make

rigorous in this case anyway. We have

$$
\begin{aligned}
\mathbf{x}' &= \left(e^{At} \cdot \mathbf{x}_0\right)' \\
&= \left(e^{At}\right)' \cdot \mathbf{x}_0 \\
&= \left(I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots\right)' \cdot \mathbf{x}_0 \\
&= \left(\mathbf{0} + A + tA^2 + \frac{t^2 A^3}{2!} + \cdots\right) \cdot \mathbf{x}_0 \\
&= A \cdot \left(I + tA + \frac{t^2 A^2}{2!} + \cdots\right) \cdot \mathbf{x}_0 \\
&= A \cdot e^{tA} \cdot \mathbf{x}_0 \\
&= A \cdot \mathbf{x}. \hspace{4cm} \square
\end{aligned}
$$

This is not so good for calculations with paper and pencil: to calculate e^{At}, in principle you have to add infinitely many terms. It can be proven, however, that if λ is an eigenvalue of A, then e^λ is an eigenvalue of e^A with the same eigenvector and same multiplicity, so it is at least possible to find a lot of information about the matrix e^A fairly quickly. One also has the interesting formula

$$
\det e^A = e^{\operatorname{tr} A}
$$

which follows from the above remark about eigenvalues but can also be proven independently of it.

9. November 7

9.1. **Springs.** The spring system is a good application of constant-coefficient linear ODEs, so we'll look at it in some detail.

Consider the following physical situation. We have a spring. One end is attached to a wall. The other is attached to a mass m which is allowed to move, frictionlessly, in one dimension (extension and contraction of the spring). We further assume that this is an ideal (*Hookean*) spring: there exists some equilibrium position, and the force exerted by the spring is directly proportional to the distance the end of the spring is to this equilibrium position. In symbols, let y be the position of the mass, with $y = 0$ denoting the equilibrium position. Then the force exerted by the spring is ky, where $k > 0$ is some constant (the "spring constant"). The other force involved comes from the acceleration of the mass, in the opposite direction, and is equal to $-my''$ by Newton's law. Equating the forces therefore gives the differential equation

$$
my'' + ky = 0.
$$

To solve this equation, consider the corresponding characteristic polynomial $m\lambda^2 + k = 0$, which has roots

$$
\lambda = \pm i\sqrt{\frac{k}{m}}
$$

(mass is positive). This is a complex conjugate pair, so the general solution via Euler's formula is

$$
y(t) = c_1 \sin\left(\frac{k}{m}t\right) + c_2 \cos\left(\frac{k}{m}t\right).
$$

This makes sense: we have a mass on a spring, so it should move back and forth periodically.

Now let's add friction. The simplest kind of friction is modeled by a force that acts against the direction of motion with a magnitude proportional to the velocity. Checking signs to make sure things work out all right, we find that our equation should be

$$my'' + \gamma y' + ky = 0$$

where $\gamma \geq 0$ is the friction coefficient. The eigenvalues corresponding to this equation are

$$\frac{\gamma}{2m} \left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}} \right).$$

It is easy enough to see that so long as m, γ, k are always positive, then the real parts of these eigenvalues are always negative, so the solutions are exponentially decaying (possibly with some oscillation along the way). Again, this makes physical sense: if we have nonzero friction, then the spring should eventually come to rest as energy leaves the system.

We have three qualitatively different possibilities for our solution. If $\gamma^2 - 4km < 0$, then the two eigenvalues are complex, and we get some oscillation on the way to rest. This is called the *underdamped* case. If $\gamma^2 - 4km > 0$, then the two eigenvalues are real, so we have simple exponential decay. This is called the *overdamped* case. Finally, if $\gamma^2 - 4km = 0$, then the eigenvalues are equal, so we get the solutions $e^{\lambda t}, te^{\lambda t}$. The function $te^{\lambda t}$ goes to zero as $t \to \infty$ more gently then the exponential by itself, so in this *critical damping* case we get a particularly smooth passage to rest. This is relevant for engineering, for instance, if we have a system that we want to respond especially smoothly to external disruptions.

Finally, we can generalize our equation by adding an external force $F(y)$, so the equation becomes

$$my'' + \gamma y' + ky = F(y).$$

Often, the external force will be sinusoidal, or at least periodic.

9.2. **The method of undetermined coefficients.** This is a fancy term for a simple idea: we know that in order to find the solution to an inhomogeneous ODE, it suffices to solve the corresponding homogeneous ODE and find one particular solution. Therefore if we can cleverly guess the form of this solution and solve for the coefficients, we'll be well on our way.

Here's a first example. Consider the ODE

(22) $$y'' - 2y' - 3y = 3.$$

If we want to find a particular solution, we should think of finding a solution that looks like the right hand side and becomes simpler, or at least maintains its form, under differentiation. So let's guess that $y(t) = A$, a constant: this certainly maintains its form under differentiation, and we have a right hand side of this form. We calculate $y''(t) = y'(t) = 0$, so according to (22), we know that $-3A = 3$, and therefore that $A = -1$. A particular solution is $y(t) = -1$. The homogeneous solutions are easily found: the roots of the characteristic polynomial $\lambda^2 - 2\lambda - 3 = 0$ are $\lambda = -1, 3$, so we have the general solution

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - 1$$

for arbitrary constants c_1, c_2.

This example was pretty silly, because the particular solution in question was so obvious. Here's a less obvious example:

$$y'' - 2y' - 3y = 3e^{2t}.$$

We already know the homogeneous solutions from the above example. Now we make the guess $y(t) = Ae^{2t}$, because this form reproduces itself upon differentiation. We calculate $y'(t) = 2Ae^{2t}$ and $y'' = 4Ae^{2t}$, so plugging back into the original equation gives us

$$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t}.$$

Canceling the e^{2t} terms and solving for A, we get $A = -1$. Therefore the general solution is

$$y(t) = c_1 e^{3t} + c^2 e^{-t} - e^{2t}.$$

In general, if the inhomogeneous part (right hand side) is any function that tends to repeat itself or simplify itself upon differentiation, we can apply this method. For example, let's look at

$$y'' + 2y' + y = 4\sin(2t).$$

The particular solution would be found by guessing that $y(t) = A\sin(2t) + B\cos(2t)$ and solving for A and B. This should work because taking derivatives will give another expression of the same form, just with different coefficients. As another example, if we have an inhomogeneous part of the form $e^{3t}\cos(t)$ we should guess $y(t) = Ae^{3t}\cos(t) + Be^{3t}\sin(t)$ for a particular solution. If we have an inhomogeneous part consisting of a sum of terms, we can guess a particular solution that is just the sum of the appropriate terms: for example, if the inhomogeneous part is $3 + \sin(4t)$, then we would guess $y(t) = A + B\sin(4t) + C\sin(4t)$ and solve for A, B, and C. This is another example of the phenomenon of superposition.

This method can encounter one major problem. Consider

$$y'' - y = 4e^{-t}.$$

The homogeneous solutions are of the form $c_1 e^t + c_2 e^{-t}$. According to the above, we should guess $y(t) = Ae^{-t}$ as a particular solution. We calculate that $y''(t) = Ae^{-t}$, and upon plugging in we get

$$Ae^{-t} - Ae^{-t} = 2e^{-t} \implies 0 = 4.$$

Oops! Clearly our guess was wrong. What's happening here is that the homogeneous solutions are interfering with our particular solution: all solutions Ae^{-t} are already solutions to the corresponding homogeneous equation! To solve it, we have to make a new guess, and the general rule for this sort of situation is to multiply our previous guess by t. So now if we try $y(t) = Ate^{-t}$, then $y'(t) = Ae^{-t} - Ate^{-t}$ and $y''(t) = -2Ae^{-t} + Ate^{-t}$, so plugging in we get

$$-2Ae^{-t} + Ate^{-t} - Ate^{-t} = 4e^{-t} \implies A = -2.$$

Our particular solution is $y(t) = -2te^{-t}$ and our general solution is thus

$$y(t) = c_1 e^t + c_2 e^{-t} - 2te^{-t}$$

for arbitrary constants c_1, c_2.

In general for an nth order equation we may have to multiply our guesses by up to t^n, depending on how many eigenvalues of the characteristic polynomial are interfering with our guess.

Here are two last examples of guessing the correct form. If our inhomogeneous part is $te^t \sin(2t)$, assuming no interference we should guess

$$y(t) = Ate^t \sin(2t) + Be^t \sin(2t) + Cte^t \cos(2t) + De^t \cos(2t).$$

If our inhomogeneous part is $4t^2 + e^t + e^{-2t} \cos t$ and there is no interference, we should guess

$$y(t) = At^2 + Bt + C + De^t + Ee^{-2t} \cos t + Fe^{-2t} \sin t.$$

10. November 12

10.1. **Miscellany.** If you haven't picked up your exams, talk to Professor Brendle; he has them. The average for the second midterm was between 49 and 50 and the median was 52, both considerably higher than for the first midterm.

10.2. **Resonance.** Let's go back to our spring ODE with an external sinusoidal force of period ω and amplitude A:

$$(23) \qquad\qquad my'' + \gamma y' + ky = Ae^{i\omega t}.$$

We should think of this system (which also appears in basic circuit analysis) as having an input (the external force) and an output (the steady-state solution to the ODE). Then we can ask questions about how we can vary the input to affect the output.

For this equation in particular, as we have already discussed, all homogeneous solutions are exponentially decaying (possibly with some oscillation). We call them *transient*. Therefore we will for the time being ignore them, and concentrate only on the particular (steady-state) solution.

To find the particular solution, let's employ the method of undetermined coefficients, with the guess $y(t) = Ce^{i\omega t}$. Then $y'(t) = Ci\omega e^{i\omega t}$ and $y''(t) = -C\omega^2 e^{i\omega t}$, so plugging pack into (23) and doing a little algebra yields

$$C = \frac{A}{k - m\omega^2 + i\gamma\omega}.$$

The standard way of keeping track of this coefficient is by defining

$$G(i\omega) = \frac{1}{k - m\omega^2 + i\gamma\omega},$$

the *gain function* of the system. Then the steady-state solution to (23) is

$$y(t) = AG(i\omega)e^{i\omega t}.$$

It is clear that $|G(i\omega)|$ is the ratio of the amplitude of the output to the amplitude of the input. The argument of $G(i\omega)$ – that is, the angle made by the complex number $G(i\omega)$ with the real axis – is the amount that the phase of the output is shifted relative to the phase of the input. *Resonance* is the phenomenon whereby the output amplitude may for some value of ω be unexpectedly large; that is, when $|G(i\omega)|$ is very large. We can easily calculate the maximum value of $|G(i\omega)|$ by taking the derivative in ω and setting it equal to zero; the textbook does this in some detail. Resonance is extremely important in engineering applications, usually carefully avoided but occasionally used to magnify a signal. The less friction, the more of an effect, although for a narrower range of ω values; for a more precise discussion of this see section 4.6 in the textbook.

10.3. **Laplace transforms introduction.** The *Laplace transform* of a function $f(t)$ is, where it exists, defined to be the following function of s:

$$[\mathcal{L}(f)](s) = \int_0^\infty e^{-st} f(t) \; dt.$$

The transform $\mathcal{L}$ is thus a map taking functions to other functions.[7] We define the *inverse Laplace transform* of a function to be the inverse map $\mathcal{L}^{-1}$. In order to justify that this makes sense, we have to prove that $\mathcal{L}$ is one-to-one, for if it were not it would not possess an inverse. Fortunately, this is a true theorem, except for small complications that we will ignore.[8]

Let's calculate a few Laplace transforms. If A is a constant, then

$$[\mathcal{L}(A)](s) = \int_0^\infty A e^{-st} \; dt = \left[-\frac{A e^{-st}}{s} \right]_{t=0}^{t=\infty} = \frac{A}{s} \text{ if } s > 0.$$

Note that the integral does not converge if $s \leq 0$, so the Laplace transform is a function that is only defined for a certain half-infinite interval of s. This is a general phenomenon.

As another example, if $f(t) = e^{At}$,

$$[\mathcal{L}(f)](s) = \int_0^\infty e^{At} e^{-st} \; dt = \int_0^\infty e^{-(s-A)t} \; dt = \left[\frac{e^{-(s-A)t}}{A-s} \right]_0^\infty = \frac{1}{s-A} \text{ if } s > A.$$

We should note that this computation is totally valid if A is a complex number, too! In that case, our convergence condition is that $s > \Re(A)$.

The basic idea in using Laplace transforms to solve initial value problems is the following three-step process:

(1) Take the Laplace transform of the ODE, get an algebraic equation,
(2) Solve the algebraic equation,
(3) Take the inverse Laplace transform of the algebraic solution to get the solution of the IVP.

The Laplace transform is somewhat magical to a first (or second, or third) glance. As with most magic, it is probably better to see an example of it in action before talking about why and how it works. So let's take out our handy page of Laplace transforms (on pg. 328 of the textbook) and solve the IVP

$$(24) \qquad\qquad y' + 2y = \sin(4t), \quad y(0) = 1.$$

Of course, we could solve this by using undetermined coefficients or variation of parameters, but we'll ignore that for now.

Steps one and two: apply $\mathcal{L}$ and solve. By consulting our table, we find that if $\mathcal{L}(y) = Y$, then $\mathcal{L}(y') = sY(s) - y(0)$ and $\mathcal{L}(\sin(4t)) = \frac{4}{s^2+16}$. Therefore upon applying $\mathcal{L}$, (24) becomes

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s^2 + 16}.$$

---

[7]The Laplace transform is a basic example of a more general class of maps from functions to other functions called *integral transforms*, of which the Fourier transform and the Mellin transform are other important examples. Here the *kernel* of the integral is e^{-st}.

[8]There is an explicit formula for the inverse Laplace transform called the Bromwich integral, but its definition involves some familiarity with complex contour integration.

We don't know what Y is yet, of course, but we're getting there. Plugging in $y(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^2 + 20}{(s+2)(s^2+16)}.$$

Step three: take an inverse Laplace transform. In order to do this, we need to simplify our above expression, and we can do this via partial fraction decomposition. We get

$$Y(s) = \frac{6}{5} \frac{1}{s+2} - \frac{1}{5} \frac{s}{s^2+16} + \frac{1}{10} \frac{4}{s^2+16}.$$

By looking at our table of Laplace transforms, we see that $\mathcal{L}(e^{At}) = \frac{1}{s-a}$, $\mathcal{L}(\cos(At)) = \frac{s}{s^2+A^2}$, and $\mathcal{L}(\sin(At)) = \frac{A}{s^2+A^2}$. Therefore, using linearity of the Laplace transform (the Laplace transform of a sum is the sum of the Laplace transforms, and we can pull out constants at will), we can apply $\mathcal{L}^{-1}$ and get

$$y(t) = \frac{6}{5} e^{-2t} - \frac{1}{5} \cos(4t) + \frac{1}{10} \sin(4t),$$

which is our desired solution to (24).

Three morals of this example: tables of Laplace transforms will be our friends, linearity is similarly useful, and we'll have to remember how to do partial fraction decompositions at some point.

11. November 14

Today we're just going to keep filling in our table of Laplace transforms, with a few examples.

11.1. Sines and cosines. We could try to calculate $\mathcal{L}(\sin(At))$ and $\mathcal{L}(\cos(At))$ directly from the definition (integrating by parts, etc.), but instead we will use Euler's formula to reduce these to the case of exponential functions, which we have already considered. Euler's formula says

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Plugging in $-\theta$ and keeping in mind that the cosine is an even function and the sine is an odd function, we get

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding these two equations together gives us

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \implies \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtracting one equation from the other gives us

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \implies \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Both of these equations are also known as Euler's formula, and express the sine and cosine in terms of the complex exponential function. With them in hand, we can

calculate our Laplace transforms quickly:

$$
\begin{aligned}
\mathcal{L}(\sin(At)) &= \mathcal{L}\left(\frac{e^{iAt} - e^{-iAt}}{2i}\right) && \text{by Euler's formula} \\
&= \frac{1}{2i}\left[\mathcal{L}\left(e^{iAt}\right) - \mathcal{L}\left(e^{-iAt}\right)\right] && \text{by linearity} \\
&= \frac{1}{2i}\left[\frac{1}{s - iA} - \frac{1}{s + iA}\right] \text{ if } s > 0 && \text{by previous calculation} \\
&= \frac{1}{2i}\frac{s + iA - s + iA}{s^2 + A^2} \text{ if } s > 0 && \text{by rationalizing denominators} \\
&= \frac{A}{s^2 + A^2} \text{ if } s > 0.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}(\cos(At)) &= \mathcal{L}\left(\frac{e^{iAt} + e^{-iAt}}{2}\right) && \text{by Euler's formula} \\
&= \frac{1}{2}\left[\mathcal{L}\left(e^{iAt}\right) + \mathcal{L}\left(e^{-iAt}\right)\right] && \text{by linearity} \\
&= \frac{1}{2}\left[\frac{1}{s - iA} + \frac{1}{s + iA}\right] \text{ if } s > 0 && \text{by previous calculation} \\
&= \frac{1}{2}\frac{s + iA + s - iA}{s^2 + A^2} \text{ if } s > 0 && \text{by rationalizing denominators} \\
&= \frac{s}{s^2 + A^2} \text{ if } s > 0.
\end{aligned}
$$

We can therefore add these rules to our table, below.

11.2. **Multiplying by exponentials, shifting, derivatives, and multiplying by t.** The plan going forward is to assume that we know that, say, $\mathcal{L}(y) = Y$, and see what changes to y we can make that produce reasonable changes to Y upon taking the Laplace transform. In the first place, we can multiply y by an exponential, e^{At}:

$$
\mathcal{L}[e^{At}y(t)](s) = \int_0^\infty e^{-st}e^{At}y(t)\, dt = \int_0^\infty e^{-(s-A)t}y(t)\, dt = Y(s - A).
$$

Thus multiplying by an exponential causes the Laplace to shift. It is clear that if $Y(s)$ is defined for $s > c$, then $Y(s - A)$ is defined for $s > c + A$.

As an example, let's quickly calculate $\mathcal{L}(e^t \sin(2t))$. We know from above that $\mathcal{L}(\sin(2t)) = \frac{2}{s^2+4}$ where $s > 0$, and the e^t shifts the result by one unit, so

$$
\mathcal{L}(e^t \sin(2t)) = \frac{2}{(s-1)^2 + 4} \text{ if } s > 0.
$$

Next let's say we know that $\mathcal{L}(y) = Y$ and we want to calculate $\mathcal{L}(y')$. This is obviously extremely important to the use we are putting the Laplace transform in

solving ODEs. We integrate by parts, as follows:

$$[\mathcal{L}(y')](s) = \int_0^\infty e^{-st} y'(t) \, dt$$

$$= \left[y(t)e^{-st}\right]_{t=0}^{t=\infty} + s \int_0^\infty y(t)e^{-st} \, dt$$

$$= 0 - y(0) + s[\mathcal{L}(y)](s)$$

$$= sY(s) - y(0).$$

To justify this calculation we have to check that the bracketed function does in fact tend to zero as $t \to \infty$. This occurs whenever $y(t)$ has *exponential order* less than s, and by a theorem in the text the Laplace transform of a function is defined exactly for s greater than the function's exponential order. Thus if $Y(s)$ is defined on $s > c$, $\mathcal{L}(y')$ will be defined on the same domain.

As an example, let's find $\mathcal{L}\left[\frac{d}{dt}(e^t \sin(2t))\right]$. We could take the derivative and calculate things directly, but we can also use the above result and the previous calculation of $\mathcal{L}(e^t \sin(2t))$ to conclude that

$$\mathcal{L}\left[\frac{d}{dt}(e^t \sin(2t))\right] = s\frac{s}{(s-1)^2 + 4} - e^0 \sin 0 = \frac{s^2}{(s-1)^2 + 4} \text{ if } s > 1.$$

Finally, we can build up the stockpile of functions just a bit more by figuring out $\mathcal{L}(t \cdot y)$ (pointwise multiplication by t) whenever we know $\mathcal{L}(y)$. By repeating this and using linearity, it is clear that we will then be able to multiply functions by arbitrary polynomials in t and figure out their Laplace transforms. The calculation requires a bit of cleverness, sometimes known as "differentiating under the integral sign":

$$\mathcal{L}(ty) = \int_0^\infty te^{-st} y(t) \, dt$$

$$= \int_0^\infty \frac{d}{ds}\left[-e^{-st}\right] y(t) \, dt$$

$$= -\frac{d}{ds} \int_0^\infty e^{-st} y(t) \, dt$$

$$= -\frac{d}{ds} Y(s).$$

The cleverness is in recognizing that $te^{-st} = -\frac{d}{ds}[e^{-st}]$ and then interchanging the derivative in s and the integral in t. There is a theorem governing such actions, and similarly to interchanging integrals (Fubini's theorem) and interchanging partial derivatives (Clairaut's theorem), in most cases of practical interest these action are permissible. We therefore have another entry for our table of Laplace transforms. The domain of definition for $\mathcal{L}(ty)$ is the same as that of $\mathcal{L}(y)$.

As an example, let us calculate the following:

$$\mathcal{L}(te^t \sin(2t)) = -\frac{d}{ds}\left[\frac{2}{(s-1)^2 + 4}\right] = \frac{4(s-1)}{((s-1)^2 + 4)^2} \text{ if } s > 1.$$

11.3. **Putting it all together.** The actual point of all this is taking Laplace transforms of ODEs. As an example, let us find the Laplace transform Y of the function y that satisfies the initial value problem

$$y'' + 2y' - y = te^t, \quad y(0) = 1, \quad y'(0) = 1.$$

By the above, we have $\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 1$. Applying the same rule again, we get $\mathcal{L}(y'') = s(sY(s) - 1) - y'(0) = s^2Y(s) - s - 1$. Finally, for the inhomogeneous part, we have

$$\mathcal{L}(te^t) = -\frac{d}{ds}\mathcal{L}(e^t) = -\frac{d}{ds}\frac{1}{s-1} = \frac{1}{(s-1)^2} \text{ if } s > 1.$$

In all, we get

$$s^2Y(s) - s - 1 + 2sY(s) - 2 - Y(s) = \frac{1}{(s-1)^2} \text{ if } s > 1,$$

which after some simple algebra reduces to

$$Y(s) = \frac{1 + (s-1)^2(s+3)}{(s-1)^2(s^2 + 2s - 1)} \text{ if } s > 1.$$

If we could apply an inverse Laplace transform to Y, we would find our desired solution y. We'll talk more about this later, but our table of Laplace transforms so far does indicate that we should be able to take inverse Laplace transforms of rational functions after performing a partial fraction decomposition.

As an aside, one could ask what happens if we start with a nonlinear ODE? For example, let's consider the IVP

$$y'' + ty' + y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Then $\mathcal{L}(y') = sY(s) - y(0) = sY(s)$ so $\mathcal{L}(ty') = -\frac{d}{ds}[sY(s)] = -Y(s) - sY'(s)$ and $\mathcal{L}(y'') = s^2Y(s) - y'(0) = s^2Y(s) - 1$. Putting this together and rearranging we get

$$-sY'(s) + s^2Y(s) = 1;$$

that is, a differential equation for Y! In this case, it's a bit simpler, because it is first-order rather than second-order, but still highly nonlinear. In any event, we can't hope to solve nonlinear equations by Laplace transforms alone.[9]

11.4. **Table of Laplace transforms.** Assume that $[\mathcal{L}(y)](s) = Y(s)$, etc., as usual, and that $Y(s)$ is defined for $s > c$.

f	$\mathcal{L}(f)$
1	$\frac{1}{s}$, if $s > 0$
e^{At}	$\frac{1}{s-A}$, if $s > \Re(A)$
$c_1y_1(t) + c_2y_2(t)$	$c_1Y_1(s) + c_2Y_2(s)$ (linearity)
$\sin(At)$	$\frac{A}{s^2+A^2}$, if $s > 0$
$\cos(At)$	$\frac{s}{s^2+A^2}$, if $s > 0$
$e^{At} \cdot y(t)$	$Y(s - A)$, if $s > c + A$
$y'(t)$	$sY(s) - y(0)$, if $s > c$
$t \cdot y(t)$	$-Y'(s)$, if $s > c$

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[9]As Adam pointed out in the 10:00 section, we could try taking *another* Laplace transform of the equation for Y. Unfortunately, if we try this we get the same equation back again (try it yourself!). Does this always happen? How about for particularly simple nonlinear equations (homogeneous with coefficients that are polynomials in t)?

12. NOVEMBER 19

12.1. **Inverse Laplace transforms.** Because the Laplace transform, hence the inverse Laplace transform, is linear, we can calculate piece by piece. To split up the rational functions that we get, we use partial fractions.

For example, let's calculate

$$\mathcal{L}^{-1}\left\{\frac{s-18}{(s+2)(s-3)}\right\}.$$

We try the partial fraction decomposition

$$\frac{s-18}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3}.$$

Clearing denominators, we get

$$s - 18 = A(s-3) + B(s+2).$$

Equating coefficients and solving, we get $A = -3$ and $B = 4$, so

$$\mathcal{L}^{-1}\left\{\frac{s-18}{(s+2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s-3}\right\} = 4e^{-2t} - 3e^{3t}.$$

As a slightly less trivial example, let's take the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{4s}{(s-1)^2}\right\}.$$

Now we have to posit a decomposition of the form

$$\frac{4s}{(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2}.$$

We have $4s = A(s-1) + B$, so $A = 4$ and $B = 4$, and

$$\mathcal{L}^{-1}\left\{\frac{4s}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s-1} + \frac{4}{(s-1)^2}\right\} = 4e^t + 4te^t.$$

Normally when taking a partial fraction decomposition we split into linear and irreducible (over the real numbers) quadratic factors. However, here's a trick: we can just as well perform a partial fraction decomposition over the complex numbers, and then *everything* decomposes into linear factors. Here's an example: let's say we're calculating

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\}$$

and we forget that we know that this rational function is the Laplace transform of $2\cos t$. We write

$$\frac{2s}{s^2+1} = \frac{A}{s-i} + \frac{B}{s+i}$$

and solve, getting $A = B = 1$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+i}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-i}\right\}$$
$$= e^{-it} + e^{it}$$
$$= \cos t - i\sin t + \cos t + i\sin t$$
$$= 2\cos t,$$

as we expected. Note that we used the rule $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, which works for all complex a.

12.2. **Discontinuous functions and the Laplace transform.** We can take Laplace transforms of discontinuous functions. Note that the transform will not necessarily be unique for discontinuous functions, because the arbitrary change of a finite number of points will not affect the result of an integration.

Let's consider the Heaviside step function $u_c(t)$, defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c, \\ 1 & \text{if } t \geq c. \end{cases}$$

It doesn't really matter how we define u_c at the point c itself.

We can calculate the Laplace transform:

$$\mathcal{L}(u_c) = \int_0^\infty u_c(t)e^{-st}\,dt = \int_c^\infty e^{-st}\,dt = \frac{e^{-sc}}{s}.$$

More importantly, $u_c(t)$ plays very nicely with respect to multiplication. If we know that $\mathcal{L}\{f\} = F$, then we can calculate

$$\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty u_c(t)f(t-c)e^{-st}\,dt$$

$$= \int_c^\infty f(t-c)e^{-st}\,dt$$

$$= \int_0^\infty f(t)e^{-s(t+c)}\,dt \qquad \text{where we change variables } t \mapsto t+c$$

$$= e^{-cs}\int_0^\infty f(t)e^{-st}\,dt$$

$$= e^{-cs}F(s).$$

That is, if we need to take the Laplace transform of the Heaviside step function multiplied by some other function that we know how to deal with, we can do it. For an example, let's solve

$$y'' + y = 1 - u_{\pi/2}(t)$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = -1$. The Laplace transform of the left hand side is $s^2 Y(s) + Y(s) - 1$, while that of the right hand side is $\frac{1}{s} - \frac{e^{-\pi s/2}}{s}$. Solving, we have

$$Y(s) = \frac{1}{s(s^2+1)} - \frac{e^{-\pi s/2}}{s(s^2+1)} + \frac{1}{s^2+1}.$$

In the usual way (partial fractions), we can determine that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t,$$

and of course

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

From the above calculation of $\mathcal{L}(u_c(t)f(t-c))$, we conclude that

$$\mathcal{L}^{-1}\left\{-\frac{e^{-\pi s/2}}{s(s^2+1)}\right\} = -u_{\pi/2}(t)(1 - \cos(t + \pi/2)) = u_{\pi/2}(t)(1 - \sin t).$$

Adding everything together,

$$y(t) = 1 + \sin t - \cos t - u_{\pi/2}(t)(1 - \sin t).$$

If you graph this, you see that the function is smooth to first order. This is an example of the smoothing properties of integration: we started with the discontinuous function $1 - u_{\pi/2}(t)$, and we ended up with a once continuously differentiable solution.

13. November 21

13.1. **The delta function.** Consider the function

$$\delta_\epsilon(t) = \frac{u_0(t) - u_\epsilon(t)}{\epsilon},$$

whose graph looks like a long narrow rectangle, ϵ wide and $1/\epsilon$ tall, and consider the differential equation

$$y'' + y = I_0 \delta_\epsilon(t).$$

This represents a quickly applied force (for ϵ time) of impulse I_0. We can solve the equation with the same methods as in the above section, getting the solution

$$y_\epsilon(t) = \frac{I_0}{\epsilon} \left[u_0(t)(1 - \cos t) - u_\epsilon(t)[1 - \cos(t - \epsilon)] \right].$$

We're interested in what happens when there's a *really* abrupt force, so let's take the (pointwise) limit as $\epsilon \to 0$. For any $t \geq 0$, as long as $\epsilon < t$ we have $u_\epsilon(t) = 1$, so in the limit we can replace $u_\epsilon(t)$ with 1. Thus

$$\lim_{\epsilon \to 0} \frac{I_0 \left[u_0(t)(1 - \cos t) - u_\epsilon(t)[1 - \cos(t - \epsilon)] \right]}{\epsilon} = I_0 \lim_{\epsilon \to 0} \frac{\cos(t) - \cos(t - \epsilon)}{\epsilon}.$$

Using L'Hospital's rule, the limit evaluates to $\sin(t)$ (the derivative of $-\cos t$), so we get

$$\lim_{\epsilon \to 0} y_\epsilon(t) = I_0 \sin t,$$

a particularly nice result. Thus our "hammer blow" at $t = 0$ caused an immediate sine wave, starting at $t = 0$.

There's a way of deriving this answer very quickly. We define something called the "delta function" $\delta(t)$, which is a sort of limit of $\delta_\epsilon(t)$ as $\epsilon \to 0$. Unfortunately, $\delta(t)$ is not really a function: equal to zero when $t \neq 0$, equal to infinity when $t = 0$, and somehow integrating to one. Its defining feature is that

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) \, dt = f(t_0).$$

Formally, we can take its Laplace transform: if $t_0 \geq 0$, then

$$\mathcal{L}(\delta(t - t_0)) = \int_0^{\infty} \delta(t - t_0) e^{-st} \, dt = e^{-st_0}.$$

In particular, $\mathcal{L}(\delta) = 1$. Let's apply this to the (quasi-)differential equation

$$y'' + y = I_0 \delta(t).$$

Taking the Laplace transform, we get

$$(s^2 + 1)Y(s) = I_0 \implies Y(s) = \frac{I_0}{s^2 + 1},$$

so taking the inverse Laplace transform,

$$y(t) = I_0 \sin t.$$

We have therefore derived the same result as above with much less effort, although our methods are a bit sketchy. Note that we again see the smoothing effect of integration: the delta function, which is discontinuous enough to not even be a function, yields a continuous (though not differentiable at $t = 0$) solution.

The delta function can be put on a firm mathematical foundation with something called the theory of distributions. Suffice it to say that all of this sketchiness can be cleared up, but we will not do it in this class.

14. December 3

14.1. **Stability in almost-linear systems.** Let's consider the equation

(25) $$\mathbf{x}' = f(\mathbf{x}),$$

a (in general nonlinear) system of first-order ODEs. We can't solve this in general, but we can usually do something more qualitative: around each critical point, we can figure out approximately what the solutions look like: are they saddle points, nodes, or what? And even more importantly than what the solutions look like near critical points, we can determine their *stability*: do solutions that start close stay close, or do they fly away?

To this end, let's recall that a critical point of (25) is defined to be a point $\mathbf{x}_0$ such that $f(\mathbf{x}_0) = 0$ (so, in particular, to solve for the critical points means to solve a system of ordinary equations). With this in mind, make the following definitions: a critical point $\mathbf{x}_0$ is *stable* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that every solution $\mathbf{x}(t)$ that satisfies

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta$$

also satisfies

$$||\mathbf{x}(t) - \mathbf{x}_0|| < \epsilon$$

for all time $t \geq 0$. Here $||\cdot||$ is the usual Euclidean norm. In other words, for every little ball we draw around the critical point, there exists an even smaller ball such that if we start in the smaller ball, we'll never leave the bigger ball. An example of a stable solution is a center, which is periodic.

A critical point is *unstable* if it is not stable. So every critical point is either stable or unstable.

A critical point $\mathbf{x}_0$ is *asymptotically stable* if it is stable and also there exists a $\delta > 0$ such that for every solution $\mathbf{x}(t)$ that satisfies

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta$$

also satisfies

$$\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}_0.$$

In other words, if we start off close enough to the critical point, not only do we not get too far away, we actually go to the critical point in the limit as time gets large. Note that an asymptotically stable critical point is stable by definition.[10]

Consider the old standby linear system of equations $\mathbf{x}' = A\mathbf{x}$, and assume $\det A \neq 0$ to ensure that $\mathbf{x}_0 = \mathbf{0}$ is an isolated critical point. We know the types

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[10]It is possible to construct critical points that satisfy the limit condition but are not stable, but they are a bit exotic. Don't worry about this unless you want to.

of critical point we can get here, and it is a simple matter to categorize them according to stability: nodal sinks and spiral points that spiral inwards (so, the eigenvalues have a negative real part) are asymptotically stable, centers are stable but not asymptotically stable, nodal sources, saddles, and spiral points that spiral outwards are unstable.

It turns out that the nonlinear equation (25) can be analyzed in largely the same way, by first linearizing and then seeing what happens when we perturb the linear system slightly, which in practice means perturbing the matrix coefficients. We'll go over how to linearize next section. In the meantime, let's define an *almost linear* system at a critical point $\mathbf{x}_0$ to be one of the form (25) that can be written as

$$\mathbf{x}' = A(\mathbf{x} - \mathbf{x}_0) + g(\mathbf{x} - \mathbf{x}_0),$$

where $||g(\mathbf{x} - \mathbf{x}_0)||/||\mathbf{x} - \mathbf{x}_0|| \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. We view the first term on the right hand side as the main term and the second as a remainder (small) term. By Taylor expanding, we see that if $f(\mathbf{x})$ has continuous second partials the system is automatically almost linear.

Let's take an almost linear system with linearization $\mathbf{x}' = A\mathbf{x}$ (assuming that $\mathbf{x}_0 = 0$, as we are free to do). The effect of perturbing A slightly depends on the original eigenvalues. For example, if $\lambda_1 < \lambda_2 < 0$, then perturbing the entries of A by a small enough amount will lead to eigenvalues $\lambda_1' < \lambda_2' < 0$; that is, we will still have a nodal sink at the origin. The same goes for a nodal source or saddle point, or a spiral point (either sink or source). The tricky cases are when we have repeated eigenvalues or when we have a center. In the former case, an arbitrarily small nonzero perturbation may perturb the eigenvalues so that they become distinct. Thus, an improper node may become proper. The stability type, however, will be unchanged. In the latter case, an arbitrarily small nonzero perturbation may cause the real part of the eigenvalues to become positive, become negative, or remain zero. Thus, a center may become a spiral point sink or source, or remain a center. *Therefore, if the linearization of an almost linear system has a center at a critical point, the stability of the almost linear system cannot be determined.* It may be unstable, stable, or asymptotically stable. In all other cases, we can determine the stability.

As an example of this, consider

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

This is a linear system with a critical point at the origin; the eigenvalues are $\pm i$ so the origin is a center (thus stable). Let's perturb this system in the following way: consider

$$\mathbf{x}' = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix} \mathbf{x}$$

where ϵ is an arbitrarily small quantity. Then we find that the characteristic polynomial is $(\epsilon - \lambda)(\epsilon - \lambda) + 1 = \lambda^2 - 2\epsilon\lambda + 1 + \epsilon^2$, so the eigenvalues are $\lambda = \epsilon \pm i$. The stability of this new system is therefore undetermined, depending on the sign of ϵ.

Here's another example. If

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

then the eigenvectors are $\lambda = -1$ with multiplicity 2, so we have an asymptotically stable node (in fact, it is an improper node). If we perturb slightly to get the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ \epsilon & -1 \end{pmatrix},$$

then (do the calculations yourself!) if $\epsilon > 0$ we get a spiral point that's asymptotically stable, and if $\epsilon < 0$ then we get a node that's asymptotically stable. So no matter what, we get asymptotic stability, but the precise type is undetermined.

15. December 5

15.1. **Linearization.** If we're given an almost linear system $\mathbf{x}' = f(\mathbf{x})$, which we write as $x' = F(x, y)$ and $y' = G(x, y)$, we are interested in the corresponding linear system at a critical point $\mathbf{x}_0$, which is given (check this with a two-dimensional Taylor expansion!) by

$$\mathbf{u}' = \begin{pmatrix} F_x(\mathbf{x}_0) & F_y(\mathbf{x}_0) \\ G_x(\mathbf{x}_0) & G_y(\mathbf{x}_0) \end{pmatrix} \mathbf{u},$$

where $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$ (a new variable which moves $\mathbf{x}_0$ to the origin). This matrix is called the *Jacobian matrix*. The point is that we are capturing the behavior of the system up to first order derivatives, and ignoring all higher derivatives. We know that this is reasonable because we have good error estimates for approximating Taylor series by Taylor polynomials.

Here's an example. Let's say we are given the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2x + y \\ x^2 - y \end{pmatrix}$$

and we are asked to find all critical points, and the type and stability of each. To find the critical points, we solve the system $-2x + y = x^2 - y = 0$, which gives the two points $(0, 0)$ and $(2, 4)$. Calculating with the above formula, the Jacobian of this system is

$$\begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}.$$

At the point $(0, 0)$, the linearization is therefore

$$\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{u},$$

which has eigenvalues $\lambda = -1, -2$ and is therefore an asymptotically stable node. At the point $(2, 4)$, the linearization is

$$\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix} \mathbf{u},$$

which has eigenvalues $\lambda = \frac{-3 \pm \sqrt{17}}{2}$ and is therefore an unstable saddle point.

15.2. **A tricky inverse Laplace transform.** Some people had difficulty with a similar problem on the homework, so I thought I'd go over this. Let's say we need to calculate

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 1)^2} \right\}.$$

There are two ways of doing this. First, we could hit it with a hammer: we know how to do a partial fraction decomposition over the complex numbers, and we could do it here. Explicitly, we can write

$$\frac{1}{(s^2+1)^2} = \frac{A}{s+i} + \frac{B}{(s+i)^2} + \frac{C}{s-i} + \frac{D}{(s-i)^2}$$

and solve for A, B, C, D. We know it has this form because $\pm i$ are the roots of $(s^2+1)^2$, each having multiplicity 2. After we solve for the coefficients, we know the inverse Laplace transforms will be constants multiplied by e^{-it}, te^{-it}, e^{it}, and te^{it}, and we can use Euler's formula to simplify.

Second, we can be clever. Let's calculate

$$\mathcal{L}(t\cos t) = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$$

and notice that this second term is very close to what we want. The first term is just the Laplace transform of $\sin t$, so let's subtract it off: we get

$$\mathcal{L}(t\cos t - \sin t) = -\frac{2}{(s^2+1)^2}.$$

Finally, we can pull out a factor of $-\frac{1}{2}$, getting

$$\mathcal{L}\left(\frac{1}{2}\sin t - \frac{1}{2}t\cos t\right) = \frac{1}{(s^2+1)^2},$$

or equivalently,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}\sin t - \frac{1}{2}t\cos t,$$

which is what we want. A procedure like this will always work to find this type of inverse Laplace transform; we find that in general

$$\mathcal{L}^{-1}\left\{\frac{1}{((s-a)^2+b^2)^2}\right\} = e^{at}\left(\frac{1}{2}\sin(bt) - \frac{1}{2}t\cos(bt)\right).$$

15.3. **Some review topics.** In class, we did a few scattershot review examples until we ran out of time. Here is a (probably incomplete) list of topics to make sure you understand for the final exam:

- Types of first-order equations: separable, linear (so use integrating factors!), exact, homogeneous (in the sense that you change variables $v = y/x$),
- 2-dimensional systems of first-order linear constant-coefficient equations (i.e., $\mathbf{x}' = A\mathbf{x}$) in all cases: real and distinct eigenvalues, complex eigenvalues, repeated eigenvalues,
- Second-order constant-coefficient homogeneous equations,
- Dealing with inhomogeneous parts: variation of parameters and undetermined coefficients/guess and check/ansatzes,
- Laplace transforms, including knowing the table and knowing how to do partial fraction decompositions,
- Linearization,
- "Modelling" problems (e.g. vats with salt water),
- The occasional 3-dimensional system,
- Wronskians (what are they and why do we care?),
- Basic existence and uniqueness theorems (linear vs. nonlinear cases),
- What it means for a system to be Hamiltonian,

- Qualitative behavior of solutions near critical points.

I wish you all productive studying, a smooth final exam, and an excellent winter break!