

MATH 21 SECTION NOTES

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1. MARCH 29

1.1. **Administrative miscellany.** These weekly sections will be for some review and many example problems, in general. Attendance will be taken as per class policy. We will be using playing cards for immediate feedback; black means yes or true, red means no or false, and the back of the card means I have no idea (but this is a cop-out!).

In general for administrative things the course website, <http://web.stanford.edu/class/math21/>, is quite comprehensive. These notes will be posted at <http://math.stanford.edu/~ebwarner/>.

1.2. **Sequences.** For our purposes, a sequence is just a list of numbers, which we notate with something like $\{a_n\}$, or if we want to be pedantic about indices, $\{a_n\}_{n=1}^{\infty}$. (The starting index, usually zero or one, is just a matter of convention and is usually chosen so as to make the notation in any particular situation as simple as possible.)

Often sequences are given by explicit formulas. For example, $a_n = n$, $n \geq 1$ defines the sequence $1, 2, 3, 4, 5, \dots$, while $a_n = \frac{1}{n^2}$, $n \geq 1$ defines the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$. Even better than writing out the first few terms, we can graph sequences just as we graph functions (the functions that we are familiar with are just maps from the real numbers to the real numbers; sequences are maps from the natural numbers to the real numbers so here the “ x -axis” is just a bunch of points going off to the right).

There’s no reason a sequence has to be given by an explicit formula (whatever that means). Another way of uniquely specifying a sequence is by a recurrence relation; the usual example is the Fibonacci numbers, which are defined as follows: $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_n$ for all $n \geq 2$. That is, we fix the first two values and declare that each subsequent value is the sum of the previous two. This clearly specifies the sequence $1, 1, 2, 3, 5, 8, 13, \dots$ (In fact, in this case one can without too much difficulty derive an explicit formula for the Fibonacci sequence, but with more complicated recurrence relations this need not be the case. And for some purposes a recurrence relation may be more useful than an explicit formula.)

A sequence $\{a_n\}$ has a *limit* C if as n gets larger and larger, a_n gets closer and closer to C . (A more precise definition is in section 8.2 of the text, but right now we’re just building intuition.) We write

$$\lim_{n \rightarrow \infty} a_n$$

to denote this limit, if it exists. Of course it may not, which makes the above notation occasionally slightly frustrating.

Here are some examples:

- $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$ because as n gets larger and larger the square of its reciprocal gets closer and closer to zero.
- $\lim_{n \rightarrow \infty} n$ does not exist because there is no real number such that as n gets larger and larger it gets closer and closer to that number. However, this limit does fail to exist in a particular way: as n gets larger and larger, the sequence eventually grows larger than any value (and stays there), so in this situation we write $\lim_{n \rightarrow \infty} n = \infty$.
- $\lim_{n \rightarrow \infty} (-1)^n$ does not exist: it oscillates forever between the values -1 and 1 , and so never gets close to any one value.
- $\lim_{n \rightarrow \infty} \sin(2\pi n)$ does exist; if we evaluate the sequence on any n we get zero and the limit of the zero sequence is zero! So while the real-valued function $\sin(2\pi x)$ oscillates and consequently does not have a limit as $x \rightarrow \infty$, the sequence $\sin(2\pi n)$ does have a limit as $n \rightarrow \infty$ because it just so happens that all of the oscillation happens away from the integers. (Note the confusing notation, which doesn't properly distinguish between sequences and functions! Usually the use of a letter like i, j, k, ℓ, m, n is a signal that we have a sequence, while a letter like t, u, v, x, y, z is a signal that we are considering a real-valued function.)

1.3. **Series.** We use the above concept of a limit to *define* what we mean by the sum of infinitely many real numbers, which is not *a priori* at all well-defined. Namely, we define the sum of the sequence $\{a_n\}_{n=1}^{\infty}$, which is usually notated

$$\sum_{n=1}^{\infty} a_n,$$

to be the limit of the *sequence of partial sums*. The sequence of partial sums is the sequence $a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \dots$; it is the sequence whose n th entry is the sum of the first n terms (which is well-defined; we certainly know how to add finitely many numbers together!). An infinite sum like this is called a *series*.

Because limits of sequences might not exist, the value of a series might not exist. Here are some examples:

- $\sum_{n=1}^{\infty} 0$ is, by definition, the limit of the sequence of partial sums, which is $0, 0 + 0, 0 + 0 + 0, 0 + 0 + 0 + 0, \dots$. Obviously all terms of this sequence are zero, so the limit is also zero. Thus under our definition of the sum of a sequence, infinitely many zeroes sum to zero.
- $\sum_{n=1}^{\infty} 1$ is the limit of the sequence $1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, \dots$, which is just the sequence $1, 2, 3, 4, \dots$, which has no limit (we say goes to infinity). Therefore this sum is not defined (and we could say it is infinite).
- $\sum_{n=1}^{\infty} (-1)^n$ has partial sums $-1, -1 + 1 = 0, -1 + 1 - 1 = -1$, and so on. That is, the sequence of partial sums oscillates forever between -1 and 0 , and consequently has no limit, so this sum is not defined.

In the 9:30 section, we had time to sketch an argument that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does exist, even if we can't easily figure out what the precise value is. We will come back to this in more detail later.

2. APRIL 5

2.1. Series and decimals. We should now be moderately familiar with geometric series, which are one of the only kinds of series we can evaluate exactly. As an example, let's calculate

$$S = \sum_{n=1}^{\infty} \frac{3}{10^n}.$$

The usual formula for the sum of a geometric series (proved in lecture, and worth reviewing!) is

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1.$$

If $|r| \geq 1$ it is easy to see that the series does not converge. Here we want to apply it to a series starting at $n = 1$, so we peel off the first term, which is $r^0 = 1$:

$$\sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n,$$

so

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} - 1 = \frac{r}{1-r}.$$

Back to our example: we can pull out the constant multiple of 3 (for the same reason we can pull out a constant multiple from an integral: both are defined in terms of limits, and we can pull out constants from limits in the same way). So, using the formula we derived above,

$$S = 3 \sum_{n=1}^{\infty} \frac{1}{10^n} = 3 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 3 \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 3 \cdot \frac{1}{9} = \frac{1}{3}.$$

Great! This sort of calculation should get to the point where it is quick and easy.

But here's an even quicker way to see what the sum is. Let's write the partial sums in decimal notation. We have $S_1 = 0.3$, $S_2 = 0.33$, $S_3 = 0.333$, and so on. Clearly, in the limit we should have

$$S = 0.\bar{3} = \frac{1}{3},$$

where we recognize the decimal expansion in question as precisely one third.

The underlying lesson here is that "decimals" really are a way of writing a real number as a very particular kind of series! A lot of the intuition we have for decimals is really intuition about series. For example,

$$\pi = 3.14159\dots,$$

which is the same as writing the series

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \dots$$

Of course this series converges, because its sum is π ! We can't write down a *formula* for the terms of this series, because the digits of π are highly irregular, but it is a convergent series nonetheless. If the decimal is repeating, then we can write down the series explicitly and it will be a geometric series.

Example: calculate $S = \sum_{n=1}^{\infty} \frac{9}{100^n}$. We can solve this using the geometric series formula, getting $\frac{1}{11}$, or we can do the following: in decimal notation, the number is

$$S = 0.\overline{09}.$$

How can we recover a fraction from a repeating decimal? (Last time we just recognized $0.\overline{3}$ as one third because it is so common.) In general, here's a trick: notice that

$$\frac{S}{100} = 0.00\overline{09},$$

so

$$S - \frac{S}{100} = 0.09 = \frac{9}{100}$$

exactly (all other digits cancel). Solving for S , we have

$$\frac{99}{100}S = \frac{9}{100} \implies S = \frac{9}{99} = \frac{1}{11}.$$

In fact in general a repeating decimal with n repeating digits is equal to the fraction with those n digits as numerator and $10^n - 1$ in the denominator. For example, $0.\overline{16} = \frac{16}{99}$ and $0.\overline{153} = \frac{153}{999}$.

Two last facts about decimals. One, there is of course nothing special about decimals, as opposed to expansions in any other base (binary, ternary, etc.); they are all still series in disguise. Two, decimal expansions are slightly nonunique, which causes a lot of confusion when people realize that $0.\overline{9} = 1$ if they think that real numbers are defined as decimals. They're not! Decimals are merely a (sometimes) convenient way of writing down or approximating real numbers. In fact $0.\overline{9} = 1$ is "essentially" the only sort of non-uniqueness you get in decimal expansions, and it is entirely harmless.

2.2. Telescoping series. Here's another rare form of series that we can calculate exactly. Consider

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

This is certainly not a geometric series, so to see what is going on we have no choice but to start writing down partial sums. We get

$$\begin{aligned} S_1 &= \left(1 - \frac{1}{2} \right), \\ S_2 &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right), \\ S_3 &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right), \end{aligned}$$

and so on. But notice that if we "regroup" the terms, the $\frac{1}{2}$ terms cancel, and the $\frac{1}{3}$ terms cancel, and clearly this will continue as we take more and more terms. In fact for each partial sum the middle terms all cancel, leaving exactly

$$S_N = 1 - \frac{1}{N+1}.$$

By definition, the sum is the limit of the partial sums, which is

$$\lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1} \right) = 1.$$

So not only does this series converge, it has a particularly simple sum!

As another example, consider

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Here (check!) all of the terms in each partial sum except four cancel, and we have

$$S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}.$$

Taking the limit as $N \rightarrow \infty$, the sum is therefore $\frac{3}{2}$.

Telescoping sums are rare, so you probably won't meet any "in the wild."

2.3. A conceptual question. True or false:

$$\text{If } \sum_{n=0}^{\infty} a_n \text{ converges, then } \lim_{n \rightarrow \infty} a_n = 0.$$

(In other words, if a series converges, then its terms must go to zero.)

This is true; the heuristic thinking should be something like "if the partial sums are converging to a particular value, then the terms with very large index had better not be very big." And the proof is not so difficult: we write each term as a difference of partial sums as follows:

$$a_n = S_n - S_{n-1}$$

(why is this true?). Then taking the limit as $n \rightarrow \infty$, we find that $\lim_{n \rightarrow \infty} S_n = S$ (because the series converges) and $\lim_{n \rightarrow \infty} S_{n-1} = S$ (because this is the same sequence, just indexed slightly differently). So

$$\lim_{n \rightarrow \infty} a_n = S - S = 0,$$

as desired.

As we will learn shortly, the converse is very much false! That is, there are plenty of sequences a_n whose terms go to zero but such that $\sum_{n=0}^{\infty} a_n$ does not converge. The simplest example is given by $a_n = \frac{1}{n}$ (the harmonic series).

3. APRIL 12

3.1. The tests we know so far. Here are some of the tests we know so far:

- Divergence test: if the limit of the terms is not zero, the series diverges.
- Integral test: a (nonnegative, monotonically decreasing) series converges if and only if the "obvious" corresponding integral does.
- 'Geometric series' test: if the series is geometric, then it converges if and only if the ratio of terms is less than 1 in absolute value (and we can even calculate the sum!).
- p -test: if the series is of the form $\sum_{n=N}^{\infty} \frac{1}{n^p}$, it converges if and only if $p > 1$.
- Comparison test: if a series with nonnegative terms is bounded termwise by a convergent series, it is convergent. Similarly, if a series is greater termwise than a divergent series with nonnegative terms, it is divergent.

- Limit comparison test.

Since the limit comparison test is so new, let's go over it again. It states that if a_n and b_n are all nonnegative terms and

$$0 < \lim_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$$

(so in particular the limit in question exists), then $\sum a_n$ and $\sum b_n$ either both converge or both diverge. The limit need not be 1, although it often will be (and you can always arrange it to be such by multiplying one of the series by a constant, of course); the important thing is that it exists and is not zero. We tend to use this test if there is a series that “looks like” one we know how to deal with except for a smaller extra term.

For example, take

$$\sum_{n=1}^{\infty} \frac{1}{n + \log n}.$$

We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and we hope that the extra logarithm is irrelevant. So we apply the limit convergence test to $a_n = \frac{1}{n + \log n}$ and $b_n = \frac{1}{n}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n + \log n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n + \log n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{\log n}{n}} = 1.$$

So the limit comparison test applies, and since we already know that $\sum b_n$ diverges, we conclude that $\sum a_n$ does as well. Note that we could not have used the comparison test in any straightforward way here, because $a_n < b_n$ for all n so knowing that $\sum b_n$ diverges is not helpful.

Notice that the two comparison tests don't tell you anything unless you already know the convergence of another series; we use them to bootstrap from knowledge about a simple series to knowledge about a more complicated one.

3.2. Examples. These examples alternate between easy and hard.

- $\sum_{n=1}^{\infty} \frac{1}{n^\pi}$. This series converges by the p -test because $\pi > 1$.
- $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. We use the integral test because $\frac{1}{x \log x}$ has an elementary antiderivative, using the substitution $u = \log x$:

$$\int \frac{dx}{x \log x} = \int \frac{du}{u} = \log u = \log \log x,$$

so

$$\int_2^{\infty} \frac{dx}{x \log x} = [\log \log x]_{x=2}^{\infty} = \infty.$$

The integral does not converge, so the sum does not converge either.

- $\sum_{n=1}^{\infty} \frac{1}{n - \frac{1}{2}}$. We have $\frac{1}{n - \frac{1}{2}} > \frac{1}{n}$ for each n , and the series $\sum \frac{1}{n}$ diverges, so by the comparison test so does this one.
- $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$. Here we should notice first that we're evaluating sine on values that grow closer and closer to zero, so the convergence of the sum should only depend on what the sine function does very close to zero. And in fact $\sin x < x$ for every nonnegative x , so $\sin \frac{1}{n^2} < \frac{1}{n^2}$, and by the comparison

test the series converges. Alternatively, we could use the limit comparison test with $\sum \frac{1}{n^2}$ if we remember that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, so certainly

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}} = 1,$$

and the limit comparison test applies.

- $\sum_{n=2}^{\infty} \frac{1}{\log n}$. Here we can just use the straightforward comparison test with the harmonic series $\frac{1}{n}$; certainly termwise the given sum dominates the harmonic series so because the harmonic series diverges so does this one.

4. APRIL 19

4.1. Ratio test. The only new tool we have for determining convergence of series is the ratio test, which is useful when you see terms like $k!$ or 2^k because then the ratios $\frac{a_{n+1}}{a_n}$ will be particularly simple. The rule itself is simple: calculate

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, the series converges. If $L > 1$ (including if the limit goes to ∞), the series diverges. If $L = 1$ or does not exist in another way, the ratio test doesn't tell you anything. This makes a lot of sense, given what we know about geometric series (where the ratio is constant); "morally" the ratio test is kind of like a limit comparison test applied to a geometric series.

Example: determine the convergence of

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}.$$

This sum is tailor-made for the ratio test. We calculate

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{2^{n+1}(n+1)!}}{\frac{(2n)!}{2^n n!}} = \frac{(2n+2)!}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{(2n)!} = \frac{(2n+2)(2n+1)}{2(n+1)} = 2n+1.$$

Clearly the limit is infinite, so by the ratio test this series diverges.

Bad example: determine the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

using the ratio test. We get

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2}.$$

The limit as $n \rightarrow \infty$ is 1, which is unhelpful: the ratio test doesn't tell us anything. In fact the ratio test will fail for any series whose n th term is a ratio of polynomials in n in precisely the same way. Fortunately, we know that this series converges by other methods (p -test, integral test).

4.2. **Some exercises.** What test should I use to determine the convergence of the following series?

- $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.
- $\sum_{n=1}^{\infty} \frac{4^{n+1}}{(3n+1)!}$
- $\sum_{n=1}^{\infty} \frac{4^{n+1}}{(3n+1)! - n}$
- $\sum_{n=2}^{\infty} \frac{1}{\log(n)^{\log(n)}}$

Answers, respectively: the integral test (since we have a u -substitution we can use!), the ratio test, either the ratio test directly or the limit comparison test with the previous example, and direct comparison. To see the last one, the hardest part is the following algebraic manipulation:

$$\log(n)^{\log(n)} = \left[e^{\log(\log n)} \right]^{\log(n)} = \left[e^{\log(n)} \right]^{\log(\log n)} = n^{\log(\log n)}.$$

Now you can compare the terms of the sum to, say, $\frac{1}{n^2}$.

As an easy-ish exercise for yourself, carry out the tests as indicated above to see whether the series converge or not!

5. APRIL 26

5.1. **Exam recap.** The majority of both sections was spent going over the trickier problems from the exam, so see the exam solutions for any lingering questions.

5.2. **Alternating series.** Alternating series are quite easy; you should be happy if you see one! For our purposes, an alternating series is a series of the form

$$\sum_{n=A}^{\infty} (-1)^n a_n,$$

where $a_n \geq 0$ for all n . There is a quite general and (in practice) easy to apply convergence test for such series: they converge if (1) $\lim_{n \rightarrow \infty} a_n = 0$ and (2) the sequence of terms $\{a_n\}$ is eventually monotonically decreasing. We could state (and prove) more general results for more interesting sign patterns than $+ - + - \dots$, if we wanted, but we won't.

Example: does $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{0.0001}}$ converge? Yes! Its terms, when we ignore the $(-1)^n$, go to zero (albeit rather slowly), and are monotonically decreasing. Note that without the $(-1)^n$, this series would diverge due to the p -test, so in some sense "alternatingness" helps us converge.

This is a general fact: if $\sum |a_n|$ converges, then $\sum a_n$ does too. If we have a series for which the sum of the absolute value of the terms converges, we call it *absolutely convergent*; if a series is convergent but not absolutely convergent (like the example above) we call it *conditionally convergent* (as in, its convergence is conditional on the fact that it has both positive and negative terms).

So instead of just having two possibilities for the convergence of a series (convergence or divergence) we now have three (absolute convergence, conditional convergence, or divergence). This is actually sometimes important: there are theorems that hold for absolutely convergent series that do not hold for conditionally convergent series. As an example, any rearrangement of the terms of an absolutely convergent series yields the same sum (as one would hope!) but this dramatically fails to be true for conditionally convergent series.

As a postscript, a word about condition (2) in the alternating series test. In basically all examples that anyone is going to throw at you, it will be easily satisfied, but it is actually necessary. Consider the series $\sum (-1)^n a_n$, where $a_n = 2/n$ if n is even and $a_n = 1/n$ if n is odd. We have $\lim a_n = 0$, but the sequence isn't eventually monotonic (it bounces up and down forever as it approaches zero). And in fact I've cooked this series up in such a way that it does not converge: every even term contributes a lot in the positive direction and every odd term contributes not so much in the negative direction, and this systematic bias leads to divergence (as a good exercise, prove this!). The monotonicity condition (2) rules out this sort of bias.