Algebraic Topology
Homework 3

Reading associated with this assignment: Hatcher, pp. 40-50 (if you haven’t already read it) and pp. 162-165 (ignore any examples that don’t make sense yet). We have covered certain extra examples (the fundamental groupoid) and concepts (diagrams, adjoints, equivalences, limits and colimits) in the lectures. Definitions of these and much more information can be found in, e.g., Categories for the Working Mathematician (Mac Lane), and also of course on wikipedia.

1. Hatcher, p. 52, exercise 3.


5. Which of the following are categories? In each example, composition of morphisms is function composition (the obvious choice).
   a) Objects are finite sets, morphisms are injective maps of sets.
   b) Objects are sets, morphisms are surjective maps of sets.
   c) Objects are abelian groups, morphisms are isomorphisms of abelian groups.
   d) Objects are sets, morphisms are maps of sets which are not surjective.
   e) Objects are topological spaces, morphisms are homeomorphisms.

6. Give an example of a category with
   a) One object and four morphisms.
   b) Two objects and five morphisms.

7. We discussed the notion of a colimit in lecture. The simplest possible category is the empty category $\mathbf{0}$, consisting of no objects and no morphisms. Given another category $\mathbf{C}$ there is a unique functor $F : \mathbf{0} \rightarrow \mathbf{C}$, taking nothing nowhere. By definition, $\text{colim} F$ is called an initial object of $\mathbf{C}$, if it exists.
   a) Show that an object $X$ is initial if and only if for any other object $Y \in \text{Ob}(\mathbf{C})$, there exists a unique morphism $X \rightarrow Y$ (this is just untangling the definition).
   b) Show directly (without appealing to general facts about colimits) that any two initial objects in a category are uniquely isomorphic.
   c) Which of the following categories have initial objects, and what are they: $\textbf{Set}$, $\textbf{Gp}$, $\textbf{Top}$, $\textbf{Top}_*$, the category of fields with field homomorphisms, the category of infinite-dimensional vector spaces
over a given field with linear maps, the category of small categories with functors \textbf{Cat}? Briefly justify your answers.

d) By analogy, define the notion of a \textit{terminal object} in a category. Which of the categories above admit terminal objects, and what are they? Briefly justify your answers.

8. A \textit{natural isomorphism} is a natural transformation \( \alpha \), say between two functors \( F, G : \mathcal{C} \to \mathcal{D} \), such that \( \alpha_X \) is an isomorphism for each \( X \in \text{Ob}(\mathcal{C}) \). Two categories \( \mathcal{C} \) and \( \mathcal{D} \) are \textit{equivalent} if there exist functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) such that there are natural isomorphisms \( \epsilon : FG \to \text{id}_\mathcal{D} \) and \( \eta : \text{id}_\mathcal{C} \to GF \). Let \( X \) be a topological space and \( x \in X \). Regard \( \pi_1(X, x) \), a group, as a category with one element \( x \). Let \( \Pi(X) \) be the fundamental groupoid of \( X \) as defined in class: its objects are points of \( X \) and its morphisms from \( x \) to \( y \) are homotopy classes of paths (with fixed endpoint) from \( x \) to \( y \). There is an evident “inclusion functor” \( J : \pi_1(X, x) \to \Pi(X) \). Show that if \( X \) is path-connected, \( J \) is an equivalence of categories. [HINT: The hard part is to construct a functor in the opposite direction. You will need to make some arbitrary choices. The point of this problem is that, if we are willing to work systematically with groupoids instead of groups, we can avoid picking a basepoint and still recover the “same” data as that held by the fundamental group. If \( X \) is not path-connected, \( \Pi(X) \) is the “better” object.]