Algebraic Topology
Homework 12

Reading associated with the Hurewicz theorem: Hatcher, pp. 166-8 for an elementary proof of the Hurewicz theorem if \(n = 1\), pp. 363-5 for some preliminary homotopy calculations, and pp. 366-373 for the Hurewicz theorem, including a more general statement than the one we proved in class (allowing a pair \((X, A)\) with \(A\) not simply connected); this version uses the action of \(\pi_1\) on \(\pi_n\) that we have not discussed. Reading associated with the Künneth formula in homology: pp. 261-280, referring to the occasional earlier algebra lemma. We will prove the Künneth formula directly and deduce the universal coefficient theorem, while Hatcher takes a more meaning route to get there.


2. (Double points) Let \(X\) be an \((n−1)\)-connected CW complex with \(n > 1\).
   a) By looking at the long exact sequences for \((X, X^{n+1})\), prove that the natural map \(\pi_{n+1}(X^{n+1}) \to \pi_{n+1}(X)\) is surjective, and similarly for \(H_{n+1}\).
   b) By looking at the Hurewicz morphism between the long exact sequences for the pair \((X^{n+1}, X^n)\), prove that the Hurewicz map \(\pi_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1})\) is surjective.
   c) Deduce from the above that \(\pi_{n+1}(X) \to H_{n+1}(X)\) is surjective. This extends the Hurewicz theorem slightly to say something about dimension \(n + 1\).
   d) Show that if \(n = 1\) the corresponding statement is false. [HINT: you can take \(X\) to be a surface.]

3. In class we defined a boundary map \(\partial\) on \(C \otimes C'\), where \(C\) and \(C'\) are chain complexes (say, of abelian groups). Verify that \(\partial^2 = 0\), so \(C \otimes C'\) is in fact a chain complex. Notice how the sign comes into play.

4. Let \(F\) be a field and \(X\) a space such that \(H_i(X; F)\) has finite dimension for all \(i\) (e.g., a CW complex with finitely many cells in each dimension). Define the Poincaré series \(p_X\) to be the formal power series
   \[
   p_X(t) = \sum_i (\dim_F H_i(X; F)) t^i.
   \]
   (We will see that this does not depend on \(F\) via the universal coefficient theorem.) Write down (formulas for) the Poincaré series of \(S^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{R}P^\infty, \mathbb{C}P^\infty\), and the orientable surface \(M_g\) of genus \(g\). If \(X\) and \(Y\) are spaces with well-defined Poincaré series, show that \(p_{X \amalg Y}(t) = p_X(t) + p_Y(t)\), and if \(X\) and \(Y\) are additionally path-connected, show that \(p_{X \vee Y}(t) = p_X(t) + p_Y(t) - 1\).

5. When we learn what the Tor groups are, we’ll discover that they vanish when the base ring is a field. Therefore if \(F\) is a field, the Künneth formula reduces to a (natural!) isomorphism
   \[
   H_n(X \times Y; F) \cong \bigoplus_{p+q=n} H_p(X; F) \otimes_F H_q(Y; F).
   \]
Use this to prove that if $X$ and $Y$ are spaces with well-defined Poincaré series, then $p_{X \times Y}(t) = p_X(t)p_Y(t)$. 