The Hanna Neumann Conjecture

Igor Mineyev, UIUC

うらやまし　美しう成て　散る紅葉

各務支考

How I envy maple leafage
which turns beautiful
then falls

Kagami Shikō

Theorem (Hanna Neumann 1956-1957)

Suppose Γ *is a free group and* A *and* B *are its* <u>*nontrivial*</u> *finitely generated subgroups. Then*

$$\operatorname{rk}(A \cap B) - 1 \leq 2\,(\operatorname{rk} A - 1)(\operatorname{rk} B - 1).$$

Conjecture (HNC, Hanna Neumann 1956-1957)

Suppose Γ *is a free group and* A *and* B *are its* <u>*nontrivial*</u> *finitely generated subgroups. Then*

$$\operatorname{rk}(A \cap B) - 1 \leq (\operatorname{rk} A - 1)(\operatorname{rk} B - 1).$$

The *reduced rank* of a free group, due to Walter Neumann:

$$\bar{r}(A) := \max\{0, \operatorname{rk} A - 1\} \geq 0.$$

HNC restated

Suppose Γ *is a free group and* A *and* B *are its finitely generated subgroups, trivial or not. Then* $\bar{r}(A \cap B) \leq \bar{r}(A) \cdot \bar{r}(B)$.

Conjecture (SHNC, Walter Neumann 1989-1990)

Suppose Γ *is a free group and* A *and* B *are its finitely generated subgroups. Then*

$$\sum_{z \in s(A \backslash \Gamma / B)} \bar{r}(A^z \cap B) \leq \bar{r}(A) \cdot \bar{r}(B).$$

This talk is based on the following preprints:

(1) *The topology and analysis of the Hanna Neumann Conjecture.* April 2010. 64 pages. **Proving HNC was not the goal** of this paper. It rather uses analysis to **generalize the statement** SHNC, and gives **approaches to those generalizations**.

(2) *Submultiplicativity and the Hanna Neumann Conjecture.* May 2011. 21 pages. The definition of submultiplicativity for complexes, a proof of submultiplicativity under additional assumptions, which implies SHNC. This is **"the first proof"**. It uses Hilbert modules and some graph theory.

(3) *Groups, graphs, and the Hanna Neumann Conjecture.* May 2011. 7 pages. A self-contained barebones proof of the original SHNC purely in terms of groups and graphs. This is **"the second proof"**.

- Exercise: What will be the length of the next preprint?

Feel free to ask questions!

Which of the two proofs is “better”?

It depends on your preferences, background, and goals. (Secret revealed: the two proofs are pretty much the same, just written in different languages.)

The first proof:

- Uses analysis: Hibert modules and Murray-von Neumann (!) dimension, plus some graph theory. Is intended for people with some background in analysis.
- Proves a more general result about complexes (submultiplicativity).
- Is less technical.
- Is easier to come up with.

The second proof:

- Is very explicit for the original SHNC.
- Does not require knowledge of Hilbert modules.
- Is intended for a “group-and-graph-theoretically-inclined” audience.

Joel Friedman has also recently announced a proof of SHNC. Preprints:

(1) “A Proof of the Strengthened Hanna Neumann Conjecture”, May 2009.
(2) “The Strengthened Hanna Neumann Conjecure I: A combinatorial proof”, March 2010.
(3) “The Strengthened Hanna Neumann Conjecure I: A combinatorial proof (revised July 6, 2010)”, July 2010.
(4) “Sheaves on Graphs and Their Homological Invariants”, April 2011.
(5) “Sheaves on Graphs and a Proof of the Hanna Neumann Conjecture”, April 2011.

Stallings' fiber product of graphs:

$\pi_1(X) = \Gamma$,
$\pi_1(Y) = A, \ \ \pi_1(Z) = B$.

$$\begin{array}{ccc} S & \xrightarrow{\nu} & Z \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\beta} \\ Y & \xrightarrow{\alpha} & X. \end{array}$$

The *reduced rank* of a finite graph Y:

$$\bar{r}(Y) := \sum_{K \in Comp(Y)} \max\{0, -\chi(K)\}.$$

If Y is nonempty,

$$\bar{r}(Y) = \sum_{K \in Comp(Y)} \bar{r}(\pi_1(K)).$$

A restatement of SHNC: $\quad \bar{r}(S) \overset{?}{\leq} \bar{r}(Y) \cdot \bar{r}(Z)$.
To prove SHNC, we want to describe $\bar{r}(S)$, $\bar{r}(Y)$, $\bar{r}(Z)$.

A tree

$\bar{r} = 0$

A flower

$\bar{r} = 0$

A forest

$\bar{r} = 0$

A garden

$\bar{r} = 0$

A graph Y.

An essential edge in Y.

An essential set in Y.

A maximal essential set in Y.

To repeat:

- An *essential edge* in Y: removing it decreases the reduced rank exactly by 1 (rather than by 0). *Inessential* otherwise.
- An *essential set* of edges, $E \subseteq E^Y$: one that decreases the reduced rank exactly by $\#E$, i.e. $\bar{r}(Y \setminus E) = \bar{r}(Y) - \#E$.
- A *maximal essential set*, $E \subseteq E^Y$, realizes $\bar{r}(Y)$, i.e. $\bar{r}(Y \setminus E) = \bar{r}(Y) - \#E = 0$.

$$\begin{array}{ccc} S & \xrightarrow{\nu} & Z \\ \mu \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\alpha} & X. \end{array}$$

How can one try to prove HNC, and fail?

In many ways!

For example, one can pick a maximal essential set in S, then project it to Y and to Z.

The problem is: those projections might not be essential sets in Y or in Z. (Exercise: find a counterexample.)

We want to make this idea work nevertheless.

A leafage is a map of cell complexes $\hat{S} \to \hat{Y}$ whose restriction to each component of $\hat{S}$ is injective. (Yes, it is a countable noun now.)

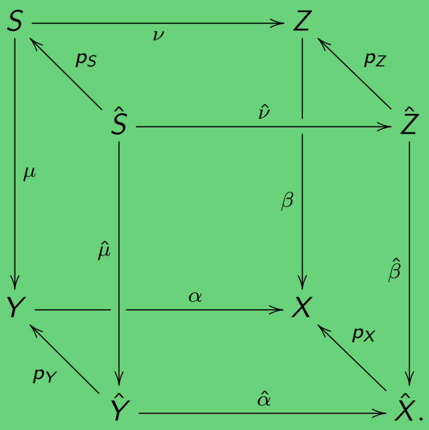
How do leafages arise?

$$\begin{array}{ccc} S & \xrightarrow{\nu} & Z \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\beta} \\ Y & \xrightarrow{\alpha} & X \end{array}$$

- Let $\alpha : Y \to X$ and $\beta : Z \to X$ be *immersions* of complexes, defined as maps that can be extended to covers of X. The main example: the Stallings' immersions of finite graphs.
- Let S be their fiber-product.
- Let $\hat{X}$ be a complex with a free Γ-action whose quotient is X, for example the universal cover of X. In the main example, Γ is a free group.
- Let $p_X : \hat{X} \to X$ be the quotient map.

Pull this whole diagram back by p_X.

The result is a *system* of complexes:



Γ acts freely on $\hat{X}$, $\hat{Y}$, $\hat{Z}$, $\hat{S}$,

with quotients X, Y, Z, S.

If $\hat{X}$ is simply connected (connected or not), then $\hat{\alpha}$ and $\hat{\beta}$ are leafages. Hence $\hat{\mu}$ and $\hat{\nu}$ are leafages as well.

Now assume Γ is left-orderable. For example, any free group is left-orderable.

Define a Γ-invariant total order $\leq$ on $E^{\hat{X}}$:

- Pick a Γ-transversal subset $\bar{E}^{\hat{X}}$ of $E^{\hat{X}}$. Then $E^{\hat{X}} \cong \Gamma \times \bar{E}^{\hat{X}}$.
- Put either lexicographic order on $E^{\hat{X}}$.

Put *pull-back orders* on $E^{\hat{Y}}$, $E^{\hat{Z}}$, $E^{\hat{S}}$. These are also Γ-invariant.

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\hat{\nu}} & \hat{Z} \\ {\scriptstyle \hat{\mu}}\downarrow & & \downarrow{\scriptstyle \hat{\beta}} \\ \hat{Y} & \xrightarrow{\hat{\alpha}} & \hat{X} \end{array}$$

The restrictions of each leafage to each component is strictly order-preserving (on edges).

Now we define *veins* in a graph $\hat{Y}$.

Let $\sigma \in E^{\hat{Y}}$,
$E \subseteq E^{\hat{Y}} \setminus \{\sigma\}$.

$\leftarrow$ *A finite vein* at σ in E.

$\leftarrow$ *An infinite vein* at σ in E.

If $\hat{Y}$ is a forest, all veins are infinite.

We say that σ *falls into* E if there is a vein at σ in E.

The most important definition:

An edge σ in $\hat{Y}$ is *order-essential* if it falls into

$$[E^{\hat{Y}} < \sigma] := \{\tau \in E^{\hat{Y}} \mid \tau < \sigma\}.$$

[An analytic comment: this is equivalent to $\partial\sigma \in \overline{\partial(\ell^2[E^{\hat{Y}} < \sigma])}$.]

Order-inessential otherwise.

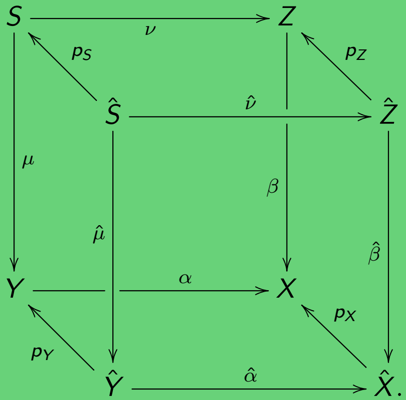
$\mathbb{E}^{\hat{Y}}$:= the set of order-essential edges in $\hat{Y}$,

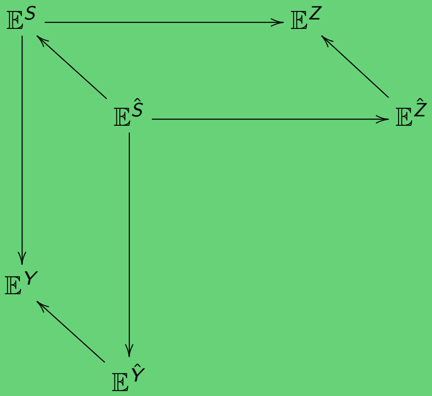
$\mathbb{I}^{\hat{Y}}$:= the set of order-inessential edges in $\hat{Y}$.

$\mathbb{E}^{\hat{Y}}$ and $\mathbb{I}^{\hat{Y}}$ are Γ-invariant. $\qquad \mathbb{E}^{Y} := \Gamma \backslash \mathbb{E}^{\hat{Y}}$.

The most important observation:

leafages map order-essential edges to order-essential edges.





Hence $\#\mathbb{E}^S \leq \#\mathbb{E}^Y \cdot \#\mathbb{E}^Z$.

It remains to show that $\#\mathbb{E}^Y = \bar{r}(Y)$.

Moreover, we can show that $\mathbb{E}^Y$ is a maximal essential set in Y. This is done in three steps.

The deep-fall property for graphs

If $\sigma \in \mathbb{E}^{\hat{Y}}$, then σ falls into $[\mathbb{I}^{\hat{Y}} < \sigma]$.

$$\mathcal{G} := V^{\hat{Y}} \sqcup \mathbb{I}^{\hat{Y}} = \hat{Y} \setminus \mathbb{E}^{\hat{Y}}.$$

Theorem

$\mathcal{G}$ is a forest and $\bar{r}(\Gamma \backslash \mathcal{G}) = 0$.

[The analytic version of this theorem is easier to prove.]

Theorem

If $\hat{Y}$ is a forest, then $\#\mathbb{E}^Y = \bar{r}(Y)$.

The analytic proof follows the same outline. Also, the analytic point of view allows generalizing the statement of SHNC.

Let $\hat{Y}$ be a *complex* with a free cocompact Γ-action.

$$a_i^{(2)}(\hat{Y},\Gamma) := \dim_\Gamma \mathrm{Ker}\left(\partial : \ell^2(\Sigma_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}})\right)$$

$$b_i^{(2)}(\hat{Y},\Gamma) := \dim_\Gamma H_i^{(2)}(\hat{Y}) \qquad \ell^2\text{-Betti number}$$

Here $\dim_\Gamma$ is the Murray-von Neumann dimension.

Submultiplicativity question.

(a) Under what conditions

$$a_i^{(2)}(\hat{Y}\hat{\square}\hat{Z},\Gamma) \leq a_i^{(2)}(\hat{Y},\Gamma)\cdot a_i^{(2)}(\hat{Z},\Gamma)\ ?$$

(b) Under what conditions

$$b_i^{(2)}(\hat{Y}\hat{\square}\hat{Z},\Gamma) \leq b_i^{(2)}(\hat{Y},\Gamma)\cdot b_i^{(2)}(\hat{Z},\Gamma)\ ?$$

These are generalizations of SHNC to complexes.


Q.E.D.