Hyperbolic volume, Mahler measure, and homology growth

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Outline

1. Homology Growth and volume
2. Torsion and Determinant
3. $L^2$-Torsion
4. Approximation by finite groups
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3. $L^2$-Torsion
4. Approximation by finite groups
Finite covering of knot complement

\( K \) is a knot in \( S^3 \), \( X = S^3 \setminus K \), \( \pi = \pi_1(X) \).
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\[ [\pi : G_k] < \infty, \quad \cap_k G_k = \{1\} . \]
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If \( [\pi : G] < \infty \), let \( X_G = G\text{-covering of } X \)

\( X_G^{\text{br}} = \text{branched } G\text{-covering of } S^3 \)
Finite covering of knot complement

$K$ is a knot in $S^3$, $X = S^3 \setminus K$, $\pi = \pi_1(X)$.

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If $[\pi : G] < \infty$, let $X_G = G$-covering of $X$

$$X_G^{br} = \text{branched } G\text{-covering of } S^3$$

Want: Asymptotics of $H_1(X_G^{br}, \mathbb{Z})$ as $k \to \infty$. 
Growth and Volume

(Kazhdan-Lück) \[
\lim_{k \to \infty} \frac{b_1(X^{\text{br}}_{G_k})}{[\pi : G_k]} = 0 \quad (= L^2 - \text{Betti number}).
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\[t(K, G) := |\text{Tor}H_1(X_G^\text{br}, \mathbb{Z})|.\]
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Definition of \(\text{Vol}(K)\): \(X = S^3 \setminus K\) is Haken.

\[X \setminus (\sqcup \text{tori}) = \sqcup \text{pieces}\]

each piece is either hyperbolic or Seifert-fibered.

\[\text{Vol}(K) := \frac{1}{6\pi} \sum \text{Vol}(\text{hyperbolic pieces}) = C(\text{Gromov norm of } X).\]
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Theorem

\[
\limsup_{k \to \infty} t(K, G_k)^{1/[\pi:G_k]} \leq \exp(\text{Vol}(K)).
\]
Knots with 0 volumes

As a corollary, when $\text{Vol}(K) = 0$, we have

$$\lim_{k \to \infty} \frac{t(K, G_k)}{[\pi : G_k]} = \exp(\text{Vol}(K)) = 1.$$
Knots with 0 volumes

As a corollary, when $\text{Vol}(K) = 0$, we have

$$\lim_{k \to \infty} t(K, G_k)^{1/[\pi:G_k]} = \exp(\text{Vol}(K)) = 1.$$ 

$\text{Vol}(K) = 0$ if and only if $K$ is in the class

i) containing torus knots

ii) closed under connected sum and cabling.
More general limit: limit as $G \to \infty$

$\pi$: a countable group.
$S$: a finite symmetric set of generators, i.e. $g \in S \Rightarrow g^{-1} \in S$. 
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- The length of $x \in \pi$:

$$\ell_S(x) = \text{smallest length of words representing } x$$

It follows that

$$\lim_{n \to \infty} \ell_S(x_n) = \infty \iff \lim_{n \to \infty} \ell_{S'}(x_n) = \infty$$
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- $S'$: another symmetric set of generators. Then $\exists k_1, k_2 > 0$ s.t.
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  \forall x \in \pi, \quad k_1 \ell_S(x) < \ell_{S'}(x) < k_2 \ell_S(x).
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\( f \): a function defined on a set of finite index normal subgroups of \( \pi \).

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\lim_{\text{diam} G \to \infty} f(G) = L
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means there is \( S \) such that

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**Remark:** If $\lim_{k \to \infty} \text{diam}G = \infty$ then $\cap G_k = \{1\}$ (co-final).
Homology Growth and Volume

Conjecture

("volume conjecture") For every knot $K \subset S^3$, 

$$\limsup_{G \to \infty} t(K, G)^{1/[\pi:G]} = \exp(\text{Vol}(K)).$$
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• True: \( LHS \leq RHS \). True for knots with \( \text{Vol} = 0 \).
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- True: $LHS \leq RHS$. True for knots with $\text{Vol} = 0$.
- To prove the conjecture one needs to find $\{G_k\}$ — finite index normal subgroups of $\pi$ s. t. $\lim_k \text{diam}(G_k) = \infty$ and

$$\lim_{k \to \infty} t(K, G_k)^{1/[\pi:G_k]} = \exp(\text{Vol}(K)).$$

\text{(\textit{*})}
Homology Growth and Volume

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\(\text{(*)}\)

It is unlikely that for any sequence $G_k$ of normal subgroups s.t. $\lim \text{diam}G_k = \infty$ one has (\(\ast\)). Which $\{G_k\}$ should we choose?
Long-Lubotzky-Reid (2007): \( \forall \) hyperbolic knot, \( \exists \{ G_k \} \) — finite index normal subgroups, such that \( \pi \) has property \( \tau \) w.r.t. \( \{ G_k \} \).
Expander family

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\( \iff \) Cayley graphs of \( \pi / G_k \) w.r.t. a fixed symmetric set of generators form a family of expanders

Based on deep results of Bourgain-Gamburg (2007) on expanders from \( \text{SL}(2, \mathbb{F}_p) \).

Conjecture (*) holds for the Long-Lubotzky-Reid sequence \( \{ G_k \} \).

Justification: to follow.
Expander family

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⇔ the least non-zero eigenvalue of the Laplacian of the Cayley graphs of \pi/G_k is ≥ a fixed ε > 0.
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Reidemeister Torsion

- $\mathcal{C}$: Chain complex of finite dimensional $\mathbb{C}$-modules (vector spaces).

$$0 \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0.$$

Suppose $\mathcal{C}$ is acyclic and based. Then the torsion $\tau(\mathcal{C})$ is defined.
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$c_i$: base of $C_i$. Each $\partial_i$ is given by a matrix.
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\[
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\]

\[
\tau(C) = \left[ \frac{\partial_2(c_2) \partial^{-1} c_0}{c_1} \right]
\]

Here $[a/b]$ is the determinant of the change matrix from $b$ to $a$. 
Torsion of chain of Hilbert spaces

$C$: complex of finite dimensional Hilbert spaces over $\mathbb{C}$; acyclic.
Choose orthonormal base $c_i$ for each $C_i$, define $\tau(C, c)$.

Change of base: $\tau(C) := |\tau(C, c)|$ is well-defined.
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$C$: complex of Hilbert spaces over $\mathbb{C}[\pi]$. Want to define $\tau(C)$.

0 $\to$ $C_n$ $\xrightarrow{\partial_n}$ $C_{n-1}$ $\xrightarrow{\partial_{n-1}}$ $\cdots$ $C_1$ $\xrightarrow{\partial_1}$ $C_0$ $\to$ 0.

More specifically,

$C_i = \mathbb{Z}[\pi]^{n_i}$, free $\mathbb{Z}[\pi]$ module, or $C_i = \ell^2(\pi)^{n_i}$

$\partial_i \in \text{Mat}(n_i \times n_{i-1}, \mathbb{Z}[\pi])$, acting on the right.
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Need to define what is the determinant of a matrix $A \in \text{Mat}(m \times n, \mathbb{Z}[\pi])$. 


Trace on $\mathbb{C}[\pi]$

For square matrix $A$ with complex entries: $\log \det A = \text{tr} \log A$.
One can define a good theory of determinant of there is a good trace.
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Regular representation: $\mathbb{C}[\pi]$ acts on the right on the Hilbert space

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- Adjoint operator: $x = \sum c_g g \in \mathbb{C}[\pi]$, then $x^* = \sum \bar{c}_g g^{-1}$. 
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- Adjoint operator: $x = \sum c_g g \in \mathbb{C}[\pi]$, then $x^* = \sum \bar{c}_g g^{-1}$.

- Similarly to the finite group case, define $\forall g \in \pi$,

  $$\text{tr}(g) = \delta_{g,1}$$

  $\forall x \in \mathbb{C}[\pi], \text{tr}(x) = \langle x, 1 \rangle = \text{coeff. of 1 in } x$. 
Trace

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- $A \in \text{Mat}(n \times n, \mathbb{C}[\pi])$. Define

$$\text{tr}(A) := \sum_{i=1}^{n} \text{tr}(A_{ii}).$$

Convergence of the RHS?
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(not rigorous) Define $\det(A)$ using

$$\log \det A = \text{tr} \log A$$

$$= -\text{tr} \sum_{p=1}^{\infty} (I - A)^p / p$$

$$= -\sum \frac{\text{tr}[(I - A)^p]}{p}.$$
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- Convergence of the RHS?
Fuglede-Kadison-Lück determinant for
\( A \in \text{Mat}(m \times n, \mathbb{C}[\pi]) \)

\[ B := A^* A, \text{ where } (A^*)_{ij} := (A_{ji})^*. \quad \ker(B) = \ker A, \quad B \geq 0. \]
Fuglede-Kadison-Lück determinant for $A \in \text{Mat}(m \times n, \mathbb{C}[[\pi]])$

- $B := A^*A$, where $(A^*)_{ij} := (A_{ji})^*$. $\ker(B) = \ker A$, $B \geq 0$.
- Choose $k > \|B\|$. Let $C = B/k$. $I \geq I - C \geq 0$, and $(I - C)^p \geq (I - C)^{p+1} \geq 0$. 

$b = b(A)$ depends only on $A$, equal to the Von-Neumann dimension of $\ker A$. 

$log \det \pi C = -\sum 1_p (tr(I - C)^p - b) = \text{finite or } -\infty$. 

$B = kC$, $\det \pi B = k^n - b \det C \in \mathbb{R} \geq 0$, $\det \pi A = \sqrt{\det \pi B}$. 

Most interesting case: $A$ is injective ($b = 0$), $m = n$, but not invertible.
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- The sequence $\text{tr}[(I - C)^p]$ is decreasing $\Rightarrow \lim \text{tr}[(I - C)^p] = b \geq 0$. $b = b(A)$ depends only on $A$, equal to the Von-Neumann dimension of $\ker A$. 
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- Use $b$ as the correction term in the log series to define $\det_\pi C$:

$$\log \det_\pi C = - \sum \frac{1}{p} (\text{tr}[(I - C)^p] - b) = \text{finite or } -\infty.$$
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- \( B = kC \), \( \det_\pi B = k^{n-b} \det C \in \mathbb{R}_{\geq 0} \), \( \det_\pi A = \sqrt{\det_\pi B} \).
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- Use $b$ as the correction term in the log series to define $\det_\pi C$:

  \[
  \log \det_\pi C = - \sum \frac{1}{p} (\text{tr}[(I - C)^p] - b) = \text{finite or } -\infty.
  \]

- $B = kC$, $\det_\pi B = k^{\overline{n-b}} \det C \in \mathbb{R}_{\geq 0}$, \(\det_\pi A = \sqrt{\det_\pi B}.\)

Most interesting case: $A$ is injective ($b = 0$), $m = n$, but not invertible.
FKL determinant – Example: Finite group

Let $D \in \text{Mat}(n \times n, \mathbb{C})$. Let $p(\lambda) = \det(\lambda I + D)$.

$$\det' D := \text{coeff. of smallest degree of } p = \prod_{\lambda \text{ eigenvalue } \neq 0} \lambda.$$
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- $\pi = \{1\}$, $A \in \text{Mat}(m \times n, \mathbb{C})$. Then in general $\det_{\{1\}} A \neq \det A$.

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- $|\pi| < \infty$, $A \in \text{Mat}(m \times n, \mathbb{C}[\pi])$. Then $A$ is given by a matrix $D \in \text{Mat}(m|\pi| \times n|\pi|, \mathbb{C})$.

  $$\det_{\pi} A = \left(\det'(D^* D)\right)^{1/2|\pi|}.$$
FKL determinant— Example: \( \Pi = \mathbb{Z}^{\mu} \)

- \( f(t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}) \in \mathbb{C}[\mathbb{Z}^{\mu}] \equiv \mathbb{C}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \).
  
  Assume \( f \neq 0 \). \( f \): 1 \times 1 \) matrix.
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- It is known that (Lück) $\det_{\mathbb{Z}^\mu}(f)$ is the Mahler measure:

$$\det_{\mathbb{Z}^\mu} f = M(f) := \exp \left( \int_{\mathbb{T}^\mu} \log |f| d\sigma \right)$$

where $\mathbb{T}^\mu = \{(z_1, \ldots, z_\mu) \in \mathbb{C}^\mu \mid |z_i| = 1\}$, the $\mu$-torus. $d\sigma$: the invariant measure normalized so that $\int_{\mathbb{T}^\mu} d\sigma = 1$. 
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  \( d\sigma \): the invariant measure normalized so that \( \int_{\mathbb{T}^\mu} d\sigma = 1 \).

- \( f(t) \in \mathbb{Z}[t^{\pm 1}] \), \( f(t) = a_0 \prod_{j=1}^n (t - z_j) \), \( z_j \in \mathbb{C} \). Then

  \[
  M(f) = a_0 \prod_{|z_j| > 1} |z_j|.
  \]
Outline

1. Homology Growth and volume
2. Torsion and Determinant
3. $L^2$-Torsion
4. Approximation by finite groups
$L^2$-Torsion, $L^2$-homology of $\mathbb{C}[\pi]$- complex

\[ C : \quad 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0. \]

$C_i = \ell^2(\pi)^{n_i}$, \quad $\partial_i \in \text{Mat}(n_i \times n_{i-1}, \mathbb{C}[\pi])$. 
$L^2$-Torsion, $L^2$-homology of $\mathbb{C}[\pi]$- complex

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$C_i = \ell^2(\pi)^{n_i}, \quad \partial_i \in \text{Mat}(n_i \times n_{i-1}, \mathbb{C}[\pi]).$

$C$ is of det-class if $\det_\pi(\partial_i) \neq 0 \forall i$. In that case

$$\tau^{(2)}(C) := \frac{\det_\pi(\partial_1) \det_\pi(\partial_3) \det_\pi(\partial_5)\ldots}{\det_\pi(\partial_2) \det_\pi(\partial_4)\ldots}.$$
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- $L^2$-homology (no need to be of det-class)

  $$H_i^{(2)} := \ker \partial_i / \text{Im}(\partial_{i-1}).$$
$L^2$-Torsion, $L^2$-homology of $\mathbb{C}[\pi]$- complex

\[ \mathcal{C} : \quad 0 \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0. \]

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- $L^2$-homology (no need to be of det-class)

\[ H_i^{(2)} := \ker \partial_i / \text{Im}(\partial_{i-1}). \]

- \( \mathcal{C} \) is $L^2$-acyclic if \( H_i^{(2)} = 0 \ \forall i \).
\( L^2 \)-Torsion of manifolds: Definition

- \( \tilde{X} \) is a \( \pi \)-space such that \( p : \tilde{X} \to X := \tilde{X}/\pi \) is a regular covering. 
- \( \tilde{X}, X \) manifold.
\( L^2 \)-Torsion of manifolds: Definition

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- Finite triangulation of \( X \): \( C(\tilde{X}) \) becomes a complex of free \( \mathbb{Z}[\pi] \)-modules.
  If \( C(\tilde{X}) \) is of det-class, then \( L^2 \)-torsion, denoted by \( \tau(2)(\tilde{X}) \), can be defined. Depends on the triangulation.
$\tilde{X}$ is a $\pi$-space such that $p : \tilde{X} \rightarrow X := \tilde{X}/\pi$ is a regular covering. $\tilde{X}, X$ manifold.

Finite triangulation of $X$: $C(\tilde{X})$ becomes a complex of free $\mathbb{Z}[\pi]$-modules. If $C(\tilde{X})$ is of det-class, then $L^2$-torsion, denoted by $\tau^{(2)}(\tilde{X})$, can be defined. Depends on the triangulation.

If $C(\tilde{X})$ is acyclic and of det-class for one triangulation, then it is acyclic and of det-class for any other triangulation, and $\tau^{(2)}(\tilde{X})$ of the two triangulations are the same: we can define $\tau^{(2)}(\tilde{X})$. 

$L^2$-Torsion of manifolds: Definition
$L^2$-Torsion of knots: universal covering

- $K$ a knot in $S^3$. $X = S^3 - K$, $\tilde{X}$: universal covering. $\pi = \pi_1(X)$. Then $\tilde{X}$ is a $\pi$-space with quotient $X$. 

Theorem (Lück-Schick) \[ \log \tau(2)(K) = -\text{Vol}(K) \]

based on results of Burghelea-Friedlander-Kappeler-McDonald, Lott, and Mathai.
$L^2$-Torsion of knots: universal covering

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$$\tau^{(2)}(K) := \tau^{(2)}(\tilde{X}).$$
\( L^2 \)-Torsion of knots: universal covering

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$L^2$-Torsion of knots: computing using knot group

$\pi = \pi_1(S^3 \setminus K)$.

$\pi = \langle a_1, \ldots, a_{n+1} | r_1, \ldots, r_n \rangle$. 
$L^2$-Torsion of knots: computing using knot group

- $\pi = \pi_1(S^3 \setminus K)$.

- $\pi = \langle a_1, \ldots, a_{n+1} | r_1, \ldots, r_n \rangle$.

- $Y$: 2-CW complex associated with this presentation. $X$ and $Y$ are homotopic.
  
  $Y$ has 1 0-cell, $(n+1)$ 1-cells, and $n$ 2-cells. $\tilde{Y}$: universal covering.

- $C(\tilde{Y}) : 0 \to \mathbb{Z}[\pi]^n \xrightarrow{\partial_2} \mathbb{Z}[\pi]^{n+1} \xrightarrow{\partial_1} \mathbb{Z}[\pi] \to 0$. 
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C(\tilde{Y}) : \quad 0 \rightarrow \mathbb{Z}[\pi]^n \xrightarrow{\partial_2} \mathbb{Z}[\pi]^{n+1} \xrightarrow{\partial_1} \mathbb{Z}[\pi] \rightarrow 0.
\]

\[
\partial_1 = \begin{pmatrix}
a_1 - 1 \\
a_2 - 1 \\
\vdots \\
a_{n+1} - 1
\end{pmatrix}, \quad \partial_2 = \left( \frac{\partial r_i}{\partial a_j} \right) \in \text{Mat}(n \times (n + 1), \mathbb{Z}[\pi])
\]
By definition

$$\tau^{(2)}(K) = \frac{\det_\pi \partial_1}{\det_\pi \partial_2}$$

$L^2$-torsion of knots: computing using knot group

Lück showed that

$$\tau^{(2)}(K) = 1 \cdot \det_\pi \partial'$$

It follows that

$$\log \det_\pi (\partial') = \text{Vol}(K).$$
By definition

\[ \tau^{(2)}(K) = \frac{\det_\pi \partial_1}{\det_\pi \partial_2} \]

Let

\[ \partial'_2 := \left( \frac{\partial r_i}{\partial a_j} \right)_{i,j=1}^n \in \text{Mat}(n \times n, \mathbb{Z}[\pi]). \]

Lück showed that

\[ \tau^{(2)}(K) = \frac{1}{\det_\pi \partial'_2} \]
\( L^2 \)-Torsion of knots: computing using knot group

By definition

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Lück showed that

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\]

It follows that

\[
\log \det_\pi (\partial'_2) = \text{Vol}(K).
\]
$L^2$-Torsion of knots: Figure 8 knot

\[ \pi = \langle a, b | ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b \rangle. \]

\[ \partial'_2 = \frac{\partial r}{\partial a} = 1 - ab^{-1}a^{-1} + ab^{-1}a^{-1}b - b - bab^{-1}a^{-1}. \]
$L^2$-Torsion of knots: Figure 8 knot

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Then

\[ \log \det_\pi(\frac{\partial r}{\partial a}) = \text{Vol}(K). \]
**$L^2$-Torsion: free abelian group $\pi = \mathbb{Z}^\mu$**

$C : \quad 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots C_1 \xrightarrow{\partial_1} C_0 \to 0.$

$C_i = \mathbb{Z}[[\mathbb{Z}^\mu]]^{n_i}, \quad \partial_i \in \text{Mat}(n_i \times n_{i-1}, \mathbb{Z}[[\mathbb{Z}^\mu]]).$
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$C \otimes F$: complex over $F$ – fractional field of $\mathbb{Z}[\mathbb{Z}^\mu] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}].$
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If $C$ is $F$-acyclic $\implies$ Reidemeister torsion $\tau^R(C)$ can be defined. Milnor-Turaev formula to calculate Reidemeister torsion. In this case, $\tau^R(C) \in \mathbb{Z}(t_1^{\pm 1}, \ldots, t_\mu^{\pm \mu})$, a rational function.
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For $C$: $L^2$-acyclic $\iff$ $F$-acyclic (Lück, Elek).
**$L^2$-Torsion: free abelian group \( \pi = \mathbb{Z}^\mu \)**

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For \( C \): \( L^2 \)-acyclic \( \iff \) \( F \)-acyclic (Lück, Elek).

**Theorem**

If \( C \) is \( F \)-acyclic, then

\[ \tau^{(2)}(C) = M(\tau^R(C)). \]
$L^2$-Torsion for abelian covering of links

$L$ a link of $\mu$ components. $X = S^3 \setminus L$.

$$\pi = \pi_1(X).$$

Abelianization map $ab : \pi \rightarrow \mathbb{Z}^\mu$.

$\tilde{X}^{ab}$: abelian covering corresponding to ker(ab), $\mathbb{Z}^\mu$-space.

Let $\Delta_0(L)$ be the (first) Alexander polynomial.

Let $\Delta_0(L) \neq 0$ always.
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Let $\Delta_0(L)$ be the (first) Alexander polynomial.

**Proposition**

$C(\tilde{X}^{\text{ab}})$ is of det-class. $C(\tilde{X}^{\text{ab}})$ is acyclic if and only if $\Delta_0(L) \neq 0$. If $\Delta_0(L) \neq 0$

$$\tau^{(2)}(\tilde{X}^{\text{ab}}) = \frac{1}{M(\Delta_0(L))}.$$ 

If $\mu = 1$, then $\Delta_0 \neq 0$ always.
Finite quotient

$C$: $\mathbb{Z}[\pi]$-complex, free finite rank. $G$ a normal subgroup, $\pi \to \Gamma = \pi / G$. 

$$C_G := C \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\Gamma].$$
Finite quotient

$C$: $\mathbb{Z}[\pi]$-complex, free finite rank. $G$ a normal subgroup, $\pi \to \Gamma = \pi / G$.

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- If $\Gamma$ is finite, then $C_G$ is a $\mathbb{Z}$-complex of free finite rank $\mathbb{Z}$-modules.

$C_G$ may not be acyclic even when $C$ is. But the Betti numbers of $C_G$ are “small” compared to $[\pi : G]$. 

Question

When $\lim_{\text{diam } G \to \infty} t(C_G) = \tau(C)$?
Finite quotient

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- If $C_G$ is acyclic, then $\tau^R(C_G) = t(C, G)$ (Milnor-Turaev formula), where

  $$t(C, G) := \frac{|\text{Tor}H_0(C_G, \mathbb{Z})| |\text{Tor}H_2(C_G, \mathbb{Z})| \cdots}{|\text{Tor}H_1(C_G, \mathbb{Z})| |\text{Tor}H_3(C_G, \mathbb{Z})|}.$$
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\]

- In general, \[\lim_{\text{diam}G \rightarrow \infty} \text{tr}_{\pi / G}(x) = \text{tr}_\pi(x)\].
Finite quotient

\(C: \mathbb{Z}[\pi]\)-complex, free finite rank. \(G\) a normal subgroup, \(\pi \to \Gamma = \pi/G\).

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- If \(\Gamma\) is finite, then \(C_G\) is a \(\mathbb{Z}\)-complex of free finite rank \(\mathbb{Z}\)-modules. \(C_G\) may not be acyclic even when \(C\) is. But the Betti numbers of \(C_G\) are “small” compared to \([\pi: G]\).
- If \(C_G\) is acyclic, then \(\tau^R(C_G) = t(C, G)\) (Milnor-Turaev formula), where
  \[t(C, G) := \frac{|\text{Tor}H_0(C_G, \mathbb{Z})|}{|\text{Tor}H_1(C_G, \mathbb{Z})|} \frac{|\text{Tor}H_2(C_G, \mathbb{Z})|}{|\text{Tor}H_3(C_G, \mathbb{Z})|} \cdots.\]
- In general,
  \[\lim_{\text{diam}G \to \infty} \text{tr}_{\pi/G}(x) = \text{tr}_{\pi}(x).\]

**Question** When

\[\lim_{\text{diam}G \to \infty} t(C, G)^{1/[\pi: G]} = \tau^{(2)}C?\]
Theorem

\[ \pi = \mathbb{Z}. \quad G_k = k\mathbb{Z} \subset \mathbb{Z}. \]

\[ \lim_{k \to \infty} t(C, G_k)^{1/k} = \tau(2)C. \]
Full result for $\pi = \mathbb{Z}$

**Theorem**

$\pi = \mathbb{Z}$. \quad \mathcal{G}_k = k\mathbb{Z} \subset \mathbb{Z}$.

$$\lim_{k \to \infty} t(\mathcal{C}, \mathcal{G}_k)^{1/k} = \tau^{(2)}\mathcal{C}.$$ 

- Proof of theorem used a special case, a result of Lück (Riley, Gonzalez-Acuna, and Short) based on Gelfond-Baker theory of diophantine approximation): $f \in \mathbb{Q}[\mathbb{Z}]$, then

$$\det_{\mathbb{Z}} f = \lim_{n \to \infty} \det_{\mathbb{Z}/k}(f_{\mathbb{Z}/k})$$

and a result relating $\det_{\mathbb{Z}_k}$ to $|\text{Tor}|$. 
Partial result $\pi = \mathbb{Z}^\mu$

Consider only lattice $G < \mathbb{Z}^\mu$ such that $\text{rk } G = \mu$. 
Partial result $\pi = \mathbb{Z}^\mu$

Consider only lattice $G < \mathbb{Z}^\mu$ such that $\text{rk } G = \mu$.

Theorem
$A \in \text{Mat}(m \times n, \mathbb{C}[\mathbb{Z}^\mu])$. Then

$$\det_{\mathbb{Z}^\mu} A = \limsup_{\text{diam } G \to \infty} \det_{\mathbb{Z}^\mu} / G(A_G).$$
Application: Link case

$L$: $\mu$-component link in $S^3$. Assume $\Delta_0(L) \neq 0$ (always the case if $\mu = 1$).

$G$ a lattice in $\mathbb{Z}^\mu$ of rank $\mu$. $X_G^{br}$: branched $G$-covering of $X = S^3 \setminus L$.

$$t(L, G) = |\text{Tor}H_1(X_G^{br}, \mathbb{Z})|.$$
Application: Link case

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\[ t(L, G) = |\text{Tor}H_1(X_G^{br}, \mathbb{Z})|. \]

Corollary

(Silver-Williams)

\[ M(\Delta_0(L)) = \limsup_{\text{diam } G \to \infty} t(L, G)^{1/[\mathbb{Z}^\mu : G]}. \]

If $\mu = 1$, then $\limsup$ can be replaced by $\lim$.

was proved by Silver and Williams using tools from symbolic dynamics.
Application: Link case

$L$: $\mu$-component link in $S^3$. Assume $\Delta_0(L) \neq 0$ (always the case if $\mu = 1$).

$G$ a lattice in $\mathbb{Z}^\mu$ of rank $\mu$. $X^\text{br}_G$: branched $G$-covering of $X = S^3 \setminus L$.

$$t(L, G) = |\text{Tor}H_1(X^\text{br}_G, \mathbb{Z})|.$$  

Corollary

(Silver-Williams)

$$M(\Delta_0(L)) = \limsup_{\text{diam} G \to \infty} t(L, G)^{1/[\mathbb{Z}^\mu : G]}.$$  

If $\mu = 1$, then $\limsup$ can be replaced by $\lim$.

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- For knots: Question of Gordon, answered by Riley and by Gonzalez-Acuna and Short.
When $\Delta_0 = 0$, it’s natural to take $\Delta(L) = \Delta_s(L)$, the smallest $s$ such that $\Delta_s(L) \neq 0$. 
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**Conjecture (Silver and Williams):**

$$\limsup_{\text{diam} G \to \infty} t(L, G) \left( \frac{1}{[\mathbb{Z}^\mu : G]} \right) = M(\Delta(L)).$$
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**Proposition**  
\[ \limsup_{\text{diam} G \to \infty} t(L, G)^{1/[\mathbb{Z}^\mu : G]} \geq M(\Delta(L)). \]

Knot case: Expander family

$$0 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0,$$

$$\tau^{(2)} = \frac{\det_\pi \partial_1}{\det_\pi \partial_2}$$
Knot case: Expander family

\[ 0 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0, \]

One can prove the volume conjecture

\[ \exp(\text{Vol}(K)) = \limsup_{\text{diam}G \to \infty} t(K, G)^{1/[\pi:G]} \]

if one can approximate both \( \det_{\pi} \partial_1, \det_{\pi} \partial_2 \) by finite quotients.

For expander family, requirements of Lück criterion are satisfied trivially for \( \partial_1 \): \( \partial_1 \) can be approximated by finite quotients (from expander family). Same for \( \partial_2 ? \) Yess \( \Rightarrow \) 'volume conjecture" for hyperbolic knots.
Knot case: Expander family

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A convergence criterion of Lück: For \( A \in \text{Mat}(m \times n, \mathbb{Z}[\pi]) \), \( B = A^* A \), if the eigenvalues of the \( B_G \) near 0 "behaves well", then

\[ \det_\pi A = \lim_{G \rightarrow \infty} \det_\pi/G A_G. \]
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Same for \( \partial_2 \)? Yes \( \implies \) ‘volume conjecture” for hyperbolic knots.
THANK YOU!