On the critical values of L-functions of tensor product of base change for Hilbert modular forms

Cristian Virdol
Department of Mathematics
Columbia University

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1 Introduction

Let F be a totally real number field and J_F be the set of all embeddings of F into $\mathbb{C}$. Let χ be a primitive system of eigenvalues of Hecke operators which occurs in the space of holomorphic Hilbert modular cusp forms on $\mathrm{GL}(2)/F$ of weight $k = (k(\tau))_{\tau \in J_F}$, where all $k(\tau)$ have the same parity and all $k(\tau) \geq 2$. Let f be the primitive cusp form of weight k which belongs to χ, let Π be the cuspidal automorphic representation of $\mathrm{GL}(2)/F$ of weight k generated by f, and let ρ_Π be the l-adic representation of $\Gamma_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F)$ attached to Π (for some prime l). In [S1], Shimura introduced an invariant $Q(\Pi, \delta) = Q(\chi, \delta) \in \mathbb{C}^\times$ (here Π is the cuspidal automorphic representation of $\mathrm{GL}(2)/F$ attached to χ by Jaquet-Langlands correspondence) for every subset δ of J_F when χ occurs in the space of holomorphic automorphic forms on a quaternion algebra over F of signature $(\delta, J_F \setminus \delta)$ and showed that this invariant appears in the critical values of the Rankin-Selberg convolution of two Hilbert modular forms. For $a, b \in \mathbb{C}$, we write $a \sim b$ if $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$. Then in [Y1], §6, $Q(\Pi, \delta) = Q(\chi, \delta)$ mod $\bar{\mathbb{Q}}^\times$ was defined assuming only $k(\tau) \geq 3$ for all $\tau \in \delta$ (i.e. when χ does not necessarily occur in the space of holomorphic automorphic forms on a quaternion algebra over F of signature $(\delta, J_F \setminus \delta)$; see §2 below for details).

In this paper we prove the following results:

Theorem 1.1. *Let F_1 and F_2 be totally real number fields. We fix two non-CM cuspidal automorphic representations Π_1 of GL(2)/F_1 of weight $k_1 = (k_1(\tau))_{\tau \in J_{F_1}}$, where all $k_1(\tau)$ have the same parity and all $k_1(\tau) \geq 2$, and Π_2 of GL(2)/F_2 of weight $k_2 = (k_2(\tau))_{\tau \in J_{F_2}}$, where all $k_2(\tau)$ have the same parity and all $k_2(\tau) \geq 2$. Let F' be a totally real number field containing F_1 and F_2. Assume that*

$$k_1(\tau|F_1) \geq 3,\ k_2(\tau|F_2) \geq 2 \quad if\ \tau \in \delta_1, \quad k_2(\tau|F_2) \geq 3,\ k_1(\tau|F_1) \geq 2 \quad if\ \tau \in \delta_2,$$

$$k_1(\tau|F_1) > k_2(\tau|F_2) \quad if\ \tau \in \delta_1, \quad k_2(\tau|F_2) > k_1(\tau|F_1) \quad if\ \tau \in \delta_2,$$

where $\delta_1 \cup \delta_2 = J_{F'}$ *and* $\delta_1 \cap \delta_2 = \emptyset$. *Let* $k_{10} = \max\{k_1(\tau)|\tau \in J_{F_1}\}$ *and* $k_{20} = \max\{k_2(\tau)|\tau \in J_{F_2}\}$. *Then if* $m \in \mathbb{Z}$ *such that*

$$(k_{10}+k_{20})/2 \leq m \leq (k_{10}+k_{20}+|k_1(\tau|F_1)-k_2(\tau|F_2)|)/2-1 \quad \textit{for every } \tau \in J_{F'},$$

we have

$$L(m, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}})$$
$$\sim \pi^{[F':\mathbb{Q}](2m+2-k_{10}-k_{20})} Q(\Pi_1, \delta_1|F_1)^{[F':F_1]} Q(\Pi_2, \delta_2|F_2)^{[F':F_2]}.$$

Theorem 1.1 is a generalization of theorem 6.8 of [Y1] (i.e. theorem 3.1 below), theorem 4.2 of [S] and theorem 5.3 of [S1] (the normalization of the invariant $Q(\Pi, \delta) = Q(\chi, \delta)$ in theorem 6.8 of [Y1] is a little different from ours).

Theorem 1.2. *For* $i = 1, \ldots, r$, *let* F_i *be a totally real number field, and for* $i = 1, \ldots, r$, *let* f_i *be a holomorphic non-CM newform of GL(2)/*F_i *of weight* $k_i \in \mathbb{Z}$, *with* $k_i \geq 3$. *Let* Π_i *be the cuspidal automorphic representation of GL(2)/*F_i *generated by* f_i. *Assume that no partial sum of the numbers* $(k_i - 1)$ *takes the value* $\frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$. *Let* F' *be a totally real number field containing* $F_1, \ldots, F_i$. *Let* $c_0 \in \mathbb{Z}$, *with* $c_0 < 1 + \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$ *such that* $[c_0, \sum_{i=1}^{i=r}(k_i - 1) - c_0 + 1]$ *is the critical strip for* $\rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{F'}}$ *which depends only on the weights* k_i, *for* $i = 1, \ldots, r$ *(see 3.3 of [B] for the exact definition of* c_0*). Assume that* $c_0 \leq \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$, *and that the conjecture of [B] (i.e. conjecture 3.2 below) is true for* $m \in \mathbb{Z}$, *with* $m \in [1 + \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1), \sum_{i=1}^{i=r}(k_i - 1) - c_0 + 1]$. *Then for* $m \in \mathbb{Z}$, *with* $m \in [1 + \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1), \sum_{i=1}^{i=r}(k_i - 1) - c_0 + 1]$, *we have*

$$L(m, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{F'}}) \sim \pi^{[F':\mathbb{Q}](2m - \sum_{i=1}^{i=r}(k_i - 1))2^{r-2}} \prod_{i=1}^{i=r} < f_i, f_i >^{n_i[F':F_i]},$$

where $< f_i, f_i >$ *is the inner product, and* n_i *appears in conjecture 3.2 below.*

Theorem 1.2 should hold for arbitrary weights k_i, i.e. not necessarily $k_i \in \mathbb{Z}$, and also for the critical values m of more general Langlands L-functions $L(s, \Pi, r)$ (see remark 5.1 below for the reason).

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2 Q-invariants

Let F be a totally real number field. Let J_F be the set of infinite places of F and let I_F be the free abelian group generated by J_F. We denote by $\mathbb{A}_{\mathbb{Q}}$ the adeles ring of $\mathbb{Q}$. Let B be a quaternion algebra over F of *signature* (δ, δ'), i. e. B splits (resp. ramifies) at the archimedian places $\tau \in \delta$ (resp. δ'). Assume that $\delta \neq \emptyset$. Set $G = \mathrm{Res}_{F/\mathbb{Q}}(B^\times)$ and denote by Z the center of G. Let $S_{k,\kappa}(B)$ (see [Y1], §6 for the definition) be the space of cusp forms of weight (k, κ) on $G(\mathbb{A}_{\mathbb{Q}})$, where $k = \sum_{\tau \in \delta} k(\tau)\tau$ and $\kappa = \sum_{\tau \in \delta'} \kappa(\tau)\tau \in I_F$. We denote by $S_{k,\kappa}(B, \bar{\mathbb{Q}})$ the set of all $\bar{\mathbb{Q}}$-rational elements in $S_{k,\kappa}(B)$.

If $f, g \in S_{k,\kappa}(B)$, define the inner product (the inner product defined in [Y] and [Y1] is a little different)

$$< f, g >= \pi^{\sum_{\tau\in\delta} k(\tau)} \int_{Z_{\infty+} G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})} {}^t\overline{f(x)} g(x) dx$$

where the invariant measure is normalized such that $\mathrm{vol}(Z_{\infty+} G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})) = 1$. If there exists some function $0 \neq f \in S_{k,\kappa}(B)$ and some Hecke character ψ of F of finite order such that

$$f|T(\wp) = \chi(\wp) f \text{ for almost all } \wp, \tag{2.1}$$

$$f(zx) = \psi(z) f(x),\ z \in Z(\mathbb{A}_{\mathbb{Q}}),\ x \in G(\mathbb{A}_{\mathbb{Q}}),$$

where $T(\wp)$ is the Hecke operator at the place $\wp$, we say that the system of eigenvalues of Hecke operators χ occurs in $S_{k,\kappa}(B)$ (it should be said actually, a system of eigenvalues of Hecke operators (χ, ψ), but for simplicity the character ψ is dropped from notation). For a given χ, we define

$$W(\chi, B) = \{f \in S_{k,\kappa}(B) | f|T(\wp) = \chi(\wp) f \quad \text{for almost all} \quad \wp,$$

$$\text{and} \quad f(zx) = \psi(z) f(x), \quad x \in Z(\mathbb{A}_{\mathbb{Q}}), \quad x \in G(\mathbb{A}_{\mathbb{Q}})\},$$

and let

$$W(\chi, B, \bar{\mathbb{Q}}) = W(\chi, B) \cap S_{k,\kappa}(B, \bar{\mathbb{Q}}).$$

We remark that if χ occurs in $S_{k,\kappa}(B)$, then by Jaquet-Langlands correspondence, χ also occurs in $S_{m,0}(M_2(F))$, where $m(\tau) = k(\tau)$ (resp, $\kappa(\tau) + 2$) if $\tau \in \delta$ (resp. $\tau \in \delta'$).

If χ occurs in $S_{k,0}(M_2(F))$, then in [Y1], §6 it is proved that

$$< f, f > \bmod \bar{\mathbb{Q}}^{\times} \quad \text{is independent of} \quad 0 \neq f \in W(\chi, B, \bar{\mathbb{Q}}),$$

and also that if B_1 and B_2 are of signature (δ, δ') and $k(\tau) \geq 2$ for all $\tau \in J_F$, then $< f, f > \sim < g, g >$ for $f \in W(\chi, B_1, \bar{\mathbb{Q}})$, $0 \neq g \in W(\chi, B_2, \bar{\mathbb{Q}})$.

If $W(\chi, B) \neq 0$ for some quaternion algebra B of signature (δ, δ'), we define

$$Q(\chi, \delta) = < f, f >$$

by taking some nonzero form $f \in W(\chi, B, \bar{\mathbb{Q}})$. Then by the above observations $Q(\chi, \delta)$ is well defined. Let F_1 be a totally real cyclic extension of degree l of F, and assume that we don't have $k(\tau) = 1$ for all $\tau \in J_F$. Then there exists a base change lift $\tilde{\chi}$ of χ which occurs in $S_{\tilde{k},0}(M_2(F_1))$ where $\tilde{k}(\tau) = k(\tau|F)$, $\tau \in J_{F_1}$. We have that

$$\text{if } \chi \text{ occurs in } S_{k,\kappa}(B), \text{ then } \tilde{\chi} \text{ occurs in } S_{\tilde{k},\tilde{\kappa}}(B \otimes_F F_1),$$

and

$$Q(\tilde{\chi}, \tilde{\delta}) = Q(\chi, \delta)^l \ \text{ if } \ k(\tau) \geq 3 \ \text{ for all } \quad \tau \in \delta, \tag{2.2}$$

where $\tilde{k}(\tau) = k(\tau|J_F)$, $\tilde{\kappa} = \kappa(\tau|J_F)$, $\tau \in J_{F_1}$ and $\tilde{\delta}$ is the full inverse image of δ under the restriction $J_{F_1} \to J_F$. One can use (2.2) to define $Q(\chi, \delta)$ when χ does not occur in any B of signature (δ, δ'), i.e. one can find F_1 and B_1 of signature $(\tilde{\delta}, \tilde{\delta}')$ such that $\tilde{\chi}$ occurs in $S_{\tilde{k},\tilde{\kappa}}(B_1)$ and set $Q(\chi, \delta) = Q(\tilde{\chi}, \tilde{\delta})^{1/l}$. Then $Q(\chi, \delta) \in \mathbb{C}^\times/\bar{\mathbb{Q}}^\times$ is well defined. We put $Q(\chi, \emptyset) = 1 \in \mathbb{C}^\times/\bar{\mathbb{Q}}^\times$. We denote by Π the cuspidal automorphic representation of $\mathrm{GL}(2)/F$ associated to f as above, and define $Q(\Pi, \delta) := Q(\chi, \delta)$.

3 Known and unknown results

Consider F a totally real number field. If Π is a cuspidal automorphic representation (discrete series at infinity) of weight $k = (k(\tau))_{\tau \in J_F}$ of $\mathrm{GL}(2)/F$, where all $k(\tau)$ have the same parity and all $k(\tau) \geq 2$, then there exists ([T]) a λ-adic representation

$$\rho_{\Pi,\lambda} : \Gamma_F \to GL_2(O_\lambda) \hookrightarrow GL_2(\overline{\mathbb{Q}}_l),$$

which is unramified outside the primes dividing $\mathbf{n}l$. Here O is the coefficients ring of Π and λ is a prime ideal of O above some prime number l, $\mathbf{n}$ is the level of Π. In order to simplify the notations we denote by ρ_Π the representation $\rho_{\Pi,\lambda}$ (by fixing an isomorphism $i : \bar{\mathbb{Q}}_l \to \mathbb{C}$ we can regard ρ_Π as a complex-valued representation).

We know (theorem 6.8 of [Y1]; the normalization of $Q(\Pi, \delta) = Q(\chi, \delta)$ in [Y1] is a little different, and the result in [Y1] is written for the Rankin-Selberg convolution of two Hilbert modular forms):

Theorem 3.1. *Let F be a totally real field, and Π_1 and Π_2 be two cuspidal automorphic representations of GL(2)/F of weight $k_1 = (k_1(\tau))_{\tau \in J_F}$ and $k_2 = (k_2(\tau))_{\tau \in J_F}$ respectively, such that all $k_1(\tau)$ have the same parity and all $k_1(\tau) \geq 2$, and all $k_2(\tau)$ have the same parity and all $k_2(\tau) \geq 2$. Assume that*

$$k_1(\tau) \geq 3,\ k_2(\tau) \geq 2 \quad if\ \tau \in \delta_1, \quad k_2(\tau) \geq 3,\ k_1(\tau) \geq 2 \quad if\ \tau \in \delta_2,$$

$$k_1(\tau) > k_2(\tau) \quad if\ \tau \in \delta_1, \quad k_2(\tau) > k_1(\tau) \quad if\ \tau \in \delta_2,$$

where $\delta_1 \cup \delta_2 = J_F$, and $\delta_1 \cap \delta_2 = \emptyset$. Let $k_{10} = \max\{k_1(\tau)|\tau \in J_F\}$ and $k_{20} = \max\{k_2(\tau)|\tau \in J_F\}$. Then if $m \in \mathbb{Z}$, with

$$(k_{10} + k_{20})/2 \leq m \leq (k_{10} + k_{20} + |k_1(\tau) - k_2(\tau)|)/2 - 1 \quad for\ every\ \tau \in J_F,$$

we have

$$L(m, \rho_{\Pi_1} \otimes \rho_{\Pi_2}) \sim \pi^{[F:\mathbb{Q}](2m+2-k_{10}-k_{20})} Q(\Pi_1, \delta_1) Q(\Pi_2, \delta_2).$$

For $i = 1, \ldots, r$, let $k_i \in \mathbb{Z}$ such that $k_i \geq 2$. Assume that no partial sum of the numbers $(k_i - 1)$ takes the value $\frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$. Let Σ be the set of all functions from $[1, \ldots, r]$ to $\{1, -1\}$ and let

$$\Sigma^+ = \{\sigma \in \Sigma | \sum_{i=1}^{i=r} \frac{1 - \sigma(i)}{2}(k_i - 1) < \frac{\sum_{i=1}^{i=r}(k_i - 1)}{2}\}.$$

Then Σ^+ has 2^{r-1} elements. Let m_i be the number of $\sigma \in \Sigma^+$ such that $\sigma(i) = -1$ and set $n_i = 2^{r-2} - m_i$.

Blasius (see the conjecture of [B] for $F = \mathbb{Q}$) conjectured that:

Conjecture 3.2. *Let F be a totally real number field, and for $i = 1, \ldots, r$, let f_i be a holomorphic newform of GL(2)/F of weight $k_i \in \mathbb{Z}$, with $k_i \geq 2$. Let Π_i be the cuspidal automorphic representation of GL(2)/F generated by f_i. Assume that no partial sum of the numbers $(k_i - 1)$ takes the value $\frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$. Then for $m \in \mathbb{Z}$, with $m \in [c_0, \sum_{i=1}^{i=r}(k_i - 1) - c_0 + 1]$, we have*

$$L(m, \rho_{\Pi_1} \otimes \ldots \otimes \rho_{\Pi_r}) \sim \pi^{[F:\mathbb{Q}](2m - \sum_{i=1}^{i=r}(k_i-1))2^{r-2}} \prod_{i=1}^{i=r} < f_i, f_i >^{n_i},$$

where n_i were defined above, and $c_0 \in \mathbb{Z}$, with $c_0 < 1 + \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$ and $[c_0, \sum_{i=1}^{i=r}(k_i - 1) - c_0 + 1]$ is the critical strip for $\rho_{\Pi_1} \otimes \ldots \otimes \rho_{\Pi_r}$ which depends only on the weights k_i, for $i = 1, \ldots, r$ (see [B], 3.3 for the exact definition of c_0).

Conjecture 3.2 was proved by Shimura (see theorem 1 of [S2]) when $r = 2$, and in this case this result is of course a particular case of theorem 3.1 above.

4 The proof of theorem 1.1

We know the following result (for $r = 1$ this is theorem 1.1 of [V] and for general r see the proof of theorem 1.1 of [V] and theorem 3.1 of [HSBT]):

Theorem 4.1. *For $i = 1, \ldots, r$, let F_i be a totally real number field, and for $i = 1, \ldots, r$, let Π_i be a cuspidal automorphic representation of weight $k_i = (k_i(\tau))_{\tau \in J_{F_i}}$ of GL(2)/F_i, where all $k_i(\tau)$ have the same parity and all $k_i(\tau) \geq 2$. Let F' be a totally real number field containing $F_1, \ldots, F_r$. Then there exists a totally real Galois extension F'' of $\mathbb{Q}$ containing F', such that for $i = 1, \ldots, r$, the representation $\rho_{\Pi_i}|_{\Gamma_{F''}}$ is modular i.e. for $i = 1, \ldots, r$, there exists an automorphic representation $\Pi_i^{''}$ of GL(2)/F'' of weight $k_i'' = (k_i''(\tau))_{\tau \in J_{F''}}$ of GL(2)/F'', where $k_i''(\tau) = k_i(\tau|F_i)$ for $\tau \in J_{F''}$, such that $\rho_{\Pi_i}|_{\Gamma_{F''}} \cong \rho_{\Pi_i''}$.*

For F_1 and F_2 totally real number fields, we fix two non-CM cuspidal automorphic representations Π_1 of GL(2)/F_1 of weight $k_1 = (k_1(\tau))_{\tau \in J_{F_1}}$, where all $k_1(\tau)$ have the same parity and all $k_1(\tau) \geq 2$, and Π_2 of GL(2)/F_2 of weight $k_2 = (k_2(\tau))_{\tau \in J_{F_2}}$, where all $k_2(\tau)$ have the same parity and all $k_2(\tau) \geq 2$. Let F' be a totally real number field containing F_1 and F_2. Then from theorem 4.1 we know that there exists a totally real Galois extension F'' of $\mathbb{Q}$ containing F' and two automorphic representations $\Pi_1^{''}$ and $\Pi_2^{''}$ of GL(2)/F'' such that $\rho_{\Pi_1}|_{\Gamma_{F''}} \cong \rho_{\Pi_1''}$ and $\rho_{\Pi_2}|_{\Gamma_{F''}} \cong \rho_{\Pi_2''}$. Then Π_1'' has weight $k_1'' = (k_1''(\tau))_{\tau \in J_{F''}}$, where $k_1''(\tau) = k_1(\tau|F_1)$ for $\tau \in J_{F''}$, and Π_2'' has weight $k_2'' = (k_2''(\tau))_{\tau \in J_{F''}}$, where $k_2''(\tau) = k_2(\tau|F_2)$ for $\tau \in J_{F''}$. Because Π_1 and Π_2 are non-CM, we know that Π_1'' and Π_2'' are cuspidal. Let $k_1' = (k_1'(\tau))_{\tau\in J_{F'}}$ such that $k_1'(\tau) = k_1(\tau|F_1)$ for $\tau \in J_{F'}$ and $k_2' = (k_2'(\tau))_{\tau\in J_{F'}}$ such that $k_2'(\tau) = k_2(\tau|F_2)$ for $\tau \in J_{F'}$. Assume that there exists a subset δ_1 of $J_{F'}$ such that $k_1'(\tau) \geq 3$, $k_2'(\tau) \geq 2$ if $\tau \in \delta_1$, and $k_2'(\tau) \geq 3$, $k_1'(\tau) \geq 2$ if $\tau \in J_{F'} \setminus \delta_1$, and $k_1'(\tau) > k_2'(\tau)$ if $\tau \in \delta_1$, and $k_2'(\tau) > k_1'(\tau)$ if $\tau \in J_{F'} \setminus \delta_1$. Define $\delta_2 := J_{F'} \setminus \delta_1$.

By Brauer's theorem (see theorems 16 and 19 of [SE]), we can find some subfields $M_j \subseteq F''$ such that $\mathrm{Gal}(F''/M_j)$ are solvable, some characters $\psi_j : \mathrm{Gal}(F''/M_j) \to \bar{\mathbb{Q}}^\times$ and some integers m_j, such that the trivial representation

$$1 : \mathrm{Gal}(F''/F') \to \bar{\mathbb{Q}}^\times,$$

can be written as $1 = \sum_{j=1}^{j=u} m_j \mathrm{Ind}_{\mathrm{Gal}(F''/M_j)}^{\mathrm{Gal}(F''/F')} \psi_j$ (a virtual sum). In particular we have $1 = \sum_{j=1}^{j=u} m_j [M_j : F']$. Then

$$L(s, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(s, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}} \otimes \mathrm{Ind}_{\Gamma_{M_j}}^{\Gamma_{F'}} \psi_j)^{m_j} =$$

$$\prod_{j=1}^{j=u} L(s, \mathrm{Ind}_{\Gamma_{M_j}}^{\Gamma_{F'}} (\rho_{\Pi_1}|_{\Gamma_{M_j}} \otimes \rho_{\Pi_2}|_{\Gamma_{M_j}} \otimes \psi_j))^{m_j} = \prod_{j=1}^{j=u} L(s, \rho_{\Pi_1}|_{\Gamma_{M_j}} \otimes \rho_{\Pi_2}|_{\Gamma_{M_j}} \otimes \psi_j)^{m_j}$$

Since $\rho_{\Pi_1}|_{\Gamma_{F''}}$ and $\rho_{\Pi_2}|_{\Gamma_{F''}}$ are modular and $\mathrm{Gal}(F''/M_j)$ is solvable, from Langlands base change for solvable extensions ([L]), one can deduce easily that the representations $\rho_{\Pi_1}|_{\Gamma_{M_j}}$ and $\rho_{\Pi_2}|_{\Gamma_{M_j}}$ are modular, and thus there exist two automorphic representations Π_{1j} of weight $k_{1j} = (k_{1j}(\tau))_{\tau\in J_{M_j}}$ of $GL(2)/M_j$, where $k_{1j}(\tau) = k_1(\tau|F_1)$ for $\tau \in J_{M_j}$, such that $\rho_{\Pi_1}|_{\Gamma_{M_j}} \cong \rho_{\Pi_{1j}}$, and Π_{2j} of weight $k_{2j} = (k_{2j}(\tau))_{\tau\in J_{M_j}}$ of $GL(2)/M_j$, where $k_{2j}(\tau) = k_2(\tau|F_2)$ for $\tau \in J_{M_j}$, such that $\rho_{\Pi_2}|_{\Gamma_{M_j}} \cong \rho_{\Pi_{2j}}$. Hence we get that:

$$L(s, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(s, \rho_{\Pi_{1j}} \otimes \rho_{\Pi_{2j}} \otimes \psi_j)^{m_j}.$$

Let $k_{10} = \max\{k_1(\tau)|\tau \in J_{F_1}\}$ and $k_{20} = \max\{k_2(\tau)|\tau \in J_{F_2}\}$. From proposition 5.2 of [S1], we know that $L(s, \rho_{\Pi_{1j}} \otimes \rho_{\Pi_{2j}} \otimes \psi_j)$ is holomorphic and non-zero for Re $s \geq (k_{10} + k_{20})/2$. Thus for each $m \in \mathbb{Z}$ which satisfies $(k_{10} + k_{20})/2 \leq m \leq (k_{10} + k_{20} + |k_1(\tau|F_1) - k_2(\tau|F_2)|)/2 - 1$ for every $\tau \in J_{F'}$, we get the identity

$$L(m, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(m, \rho_{\Pi_{1j}} \otimes \rho_{\Pi_{2j}} \otimes \psi_j)^{m_j}.$$

Recall that $1 = \sum_{j=1}^{j=u} m_j [M_j : F']$. Let δ_1'' be the full inverse image of δ_1 under the restriction $J_{F''} \to J_{F'}$, let δ_2'' be the full inverse image of δ_2 under the restriction $J_{F''} \to J_{F'}$, let δ_{1j} be the full inverse image of δ_1 under the restriction

$J_{M_j} \to J_{F'}$, and let δ_{2j} be the full inverse image of δ_2 under the restriction $J_{M_j} \to J_{F'}$. Then Π''_1 is the solvable base change of Π_{1j} to $\mathrm{GL}(2)/F''$, and Π''_2 is the solvable base change of $\Pi_{2j} \otimes \psi_j$ to $\mathrm{GL}(2)/F''$, and thus from (2.2) we get that $Q(\Pi''_1, \delta''_1) = Q(\Pi_{1j}, \delta_{1j})^{[F'':M_j]}$ and $Q(\Pi''_2, \delta''_2) = Q(\Pi_{2j} \otimes \psi_j, \delta_{2j})^{[F'':M_j]}$. Then from theorem 3.1 we get that

$$L(m, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(m, \rho_{\Pi_{1j}} \otimes \rho_{\Pi_{2j}} \otimes \psi_j)^{m_j}$$

$$\sim \prod_{j=1}^{j=u} \pi^{m_j [M_j:\mathbb{Q}](2m+2-k_{10}-k_{20})} (Q(\Pi_{1j}, \delta_{1j}) Q(\Pi_{2j}, \delta_{2j}))^{m_j}$$

$$= \pi^{\sum_{j=1}^{j=u} m_j [M_j:F'][F':\mathbb{Q}](2m+2-k_{10}-k_{20})} \prod_{j=1}^{j=u} (Q(\Pi''_1, \delta''_1) Q(\Pi''_2, \delta''_2))^{m_j/[F'':M_j]}$$

$$= \pi^{[F':\mathbb{Q}](2m+2-k_{10}-k_{20})} \prod_{j=1}^{j=u} (Q(\Pi''_1, \delta''_1) Q(\Pi''_2, \delta''_2))^{m_j [M_j:F']/[F'':F']}$$

$$= \pi^{[F':\mathbb{Q}](2m+2-k_{10}-k_{20})} (Q(\Pi''_1, \delta''_1) Q(\Pi''_2, \delta''_2))^{1/[F'':F']}.$$

But from the main theorem of [Y] (the normalization in [Y] is different) we know that

$$Q(\Pi''_1, \delta''_1) \sim \pi^{(k_{10}-1)|\delta''_1|} \prod_{\tau \in \delta''_1} c^+_\tau(\Pi''_1) c^-_\tau(\Pi''_1),$$

for some constants $c^\pm_\tau(\Pi''_1) \in \mathbb{C}^\times$ defined mod $\bar{\mathbb{Q}}^\times$ and also that

$$Q(\Pi_1, \delta_1|F_1) \sim \pi^{(k_{10}-1)|\delta_1|F_1|} \prod_{\tau \in \delta_1|F_1} c^+_\tau(\Pi_1) c^-_\tau(\Pi_1),$$

for some constants $c^\pm_\tau(\Pi_1) \in \mathbb{C}^\times$ defined mod $\bar{\mathbb{Q}}^\times$. But from theorem 1.3 of [V1] we know that $c^\pm_\tau(\Pi''_1) = c^\pm_{\tau|F_1}(\Pi_1)$ for every $\tau \in J_{F''}$.

Hence we get that

$$Q(\Pi''_1, \delta''_1)^{1/[F'':F_1]} = Q(\Pi_1, \delta_1|F_1),$$

and by symmetry also that

$$Q(\Pi''_2, \delta''_2)^{1/[F'':F_2]} = Q(\Pi_2, \delta_2|F_2),$$

and thus

$$L(m, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \rho_{\Pi_2}|_{\Gamma_{F'}})$$
$$\sim \pi^{([F':\mathbb{Q}](2m+2-k_{10}-k_{20})} Q(\Pi_1, \delta_1|F_1)^{[F':F_1]} Q(\Pi_2, \delta_2|F_2))^{[F':F_2]},$$

and hence theorem 1.1 is proved.

5 The proof of theorem 1.2

For $i = 1, \ldots, r$, let F_i be a totally real number field, and for $i = 1, \ldots, r$, let f_i be a holomorphic non-CM newform of $\mathrm{GL}(2)/F_i$ of weight $k_i \in \mathbb{Z}$, with $k_i \geq 3$. Let Π_i be the cuspidal automorphic representation of $\mathrm{GL}(2)/F_i$ generated by f_i. Assume that no partial sum of the numbers $(k_i - 1)$ takes the value $\frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$. Let F' be a totally real number field containing $F_1, \ldots, F_i$. Then from theorem 4.1 we know that there exists a totally real Galois extension F'' of $\mathbb{Q}$ containing F', and for $i = 1, \ldots, r$, an automorphic representation Π_i'' of $\mathrm{GL}(2)/F''$ of weight k_i, such that $\rho_{\Pi_i}|_{\Gamma_{F''}} \cong \rho_{\Pi_i''}$.

By Brauer's theorem (see theorems 16 and 19 of [SE]), we can find some subfields $M_j \subseteq F''$ such that $\mathrm{Gal}(F''/M_j)$ are solvable, some characters $\psi_j : \mathrm{Gal}(F''/M_j) \to \bar{\mathbb{Q}}^\times$ and some integers m_j, such that the trivial representation

$$1 : \mathrm{Gal}(F''/F') \to \bar{\mathbb{Q}}^\times,$$

can be written as $1 = \sum_{j=1}^{j=u} m_j \mathrm{Ind}_{\mathrm{Gal}(F''/M_j)}^{\mathrm{Gal}(F''/F')} \psi_j$ (a virtual sum). In particular we have $1 = \sum_{j=1}^{j=u} m_j [M_j : F']$. Then

$$L(s, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(s, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{F'}} \otimes \mathrm{Ind}_{\Gamma_{M_j}}^{\Gamma_{F'}} \psi_j)^{m_j}$$

$$= \prod_{j=1}^{j=u} L(s, \mathrm{Ind}_{\Gamma_{M_j}}^{\Gamma_{F'}} (\rho_{\Pi_1}|_{\Gamma_{M_j}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{M_j}} \otimes \psi_j))^{m_j}$$

$$= \prod_{j=1}^{j=u} L(s, \rho_{\Pi_1}|_{\Gamma_{M_j}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{M_j}} \otimes \psi_j)^{m_j}.$$

Since $\rho_{\Pi_1}|_{\Gamma_{F''}}, \ldots, \rho_{\Pi_r}|_{\Gamma_{F''}}$ are modular and $\mathrm{Gal}(F''/M_j)$ is solvable, from Langlands base change for solvable extensions ([L]), one can deduce easily that the representations $\rho_{\Pi_1}|_{\Gamma_{M_j}}, \ldots, \rho_{\Pi_r}|_{\Gamma_{M_j}}$ are modular, and thus there exist automorphic representations Π_{ij} of weight k_i of $GL(2)/M_j$, such that $\rho_{\Pi_i}|_{\Gamma_{M_j}} \cong \rho_{\Pi_{ij}}$. Let f_{ij} be the newform corresponding to Π_{ij}. Hence we get that:

$$L(s, \rho_{\Pi_1}|_{\Gamma_{F'}} \otimes \ldots \otimes \rho_{\Pi_r}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(s, \rho_{\Pi_{1j}} \otimes \ldots \otimes \rho_{\Pi_{rj}} \otimes \psi_j)^{m_j}.$$

Assume that the number $c_0 \in \mathbb{Z}$ which appears in conjecture 3.2 above verifies $c_0 \leq \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$, and that conjecture 3.2 is true for $m \in \mathbb{Z}$, with $m \in [1 + \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1), \sum_{i=1}^{i=r}(k_i - 1) - c_0 + 1]$. Recall that we have $1 = \sum_{j=1}^{j=u} m_j [M_j : F']$. The functions $L(s, \rho_{\Pi_{1j}} \otimes \ldots \otimes \rho_{\Pi_{rj}} \otimes \psi_j)$ are holomorphic and do not vanish for $\mathrm{Re}\ s \geq 1 + \frac{1}{2}\sum_{i=1}^{i=r}(k_i - 1)$, and thus for $m \in \mathbb{Z}$, with

$m \in [1+\frac{1}{2}\sum_{i=1}^{i=r}(k_i-1), \sum_{i=1}^{i=r}(k_i-1)-c_0+1]$ we obtain

$$L(m,\rho_{\Pi_1}|_{\Gamma_{F'}}\otimes\ldots\otimes\rho_{\Pi_r}|_{\Gamma_{F'}}) = \prod_{j=1}^{j=u} L(m,\rho_{\Pi_{1j}}\otimes\ldots\otimes\rho_{\Pi_{rj}}\otimes\psi_j)^{m_j}$$

$$\sim \prod_{j=1}^{j=u}\pi^{m_j[M_j:\mathbb{Q}](2m-\sum_{i=1}^{i=r}(k_i-1))2^{r-2}}\prod_{i=1}^{i=r} < f_{ij}, f_{ij} >^{n_i m_j}$$

$$\sim \prod_{j=1}^{j=u}\pi^{m_j[M_j:\mathbb{Q}](2m-\sum_{i=1}^{i=r}(k_i-1))2^{r-2}}\prod_{i=1}^{i=r} < f''_i, f''_i >^{n_i m_j 1/[F'':M_j]},$$

because Π''_i is a solvable base change of Π_{ij} or $\Pi_{ij}\otimes\psi_j$, and thus from (2.2) we know that

$$< f_{ij}, f_{ij} >=< f''_i, f''_i >^{1/[F'':M_j]},$$

and

$$< f_{ij_{\psi_j}}, f_{ij_{\psi_j}} >=< f''_i, f''_i >^{1/[F'':M_j]},$$

where $f_{ij_{\psi_j}}$ is the primitive cuspform which corresponds to $\Pi_{ij}\otimes\psi_j$.

But from the main theorem of [Y] we know that

$$< f''_i, f''_i >\sim \pi^{(k_i-1)|[F'':\mathbb{Q}]|}\prod_{\tau\in J_{F''}} c^+_\tau(\Pi''_i)c^-_\tau(\Pi''_i),$$

for some constants $c^{\pm}_\tau(\Pi''_i)\in\mathbb{C}^\times$ defined mod $\bar{\mathbb{Q}}^\times$ and also that

$$< f_i, f_i >\sim \pi^{(k_i-1)|[F_i:\mathbb{Q}]|}\prod_{\tau\in J_{F_i}} c^+_\tau(\Pi_i)c^-_\tau(\Pi_i),$$

for some constants $c^{\pm}_\tau(\Pi_i)\in\mathbb{C}^\times$ defined mod $\bar{\mathbb{Q}}^\times$. But from theorem 1.3 of [V1] we know that $c^{\pm}_\tau(\Pi''_i) = c^{\pm}_{\tau|F_i}(\Pi_i)$ for every $\tau\in J_{F''}$.

Hence we get that for $i=1,\ldots,r$, we have

$$< f''_i, f''_i >=< f_i, f_i >^{[F'':F_i]}$$

and thus

$$L(m,\rho_{\Pi_1}|_{\Gamma_{F'}}\otimes\ldots\otimes\rho_{\Pi_r}|_{\Gamma_{F'}})$$

$$\sim \prod_{j=1}^{j=u}\pi^{m_j[M_j:\mathbb{Q}](2m-\sum_{i=1}^{i=r}(k_i-1))2^{r-2}}\prod_{i=1}^{i=r} < f_i, f_i >^{n_i m_j[M_j:F_i]}$$

$$\sim \pi^{[F':\mathbb{Q}](2m-\sum_{i=1}^{i=r}(k_i-1))2^{r-2}}\prod_{i=1}^{i=r} < f_i, f_i >^{n_i[F':F_i]},$$

which proves theorem 1.2.

Remark 5.1. *Theorem 1.2 should hold for arbitrary weights k_i (i.e. not necessarily $k_i \in \mathbb{Z}$), and in this case in the statements of the theorem 1.2 and conjecture 3.2 above one has to replace the inner products $< f_i, f_i >$'s by some invariants $''Q(\Pi_i, \delta_i|F_i)''$'s as in the statement of theorem 1.1, and the proof is identical, since these invariants $''Q(\Pi_i, \delta_i|F_i)''$'s should also be products of some of the periods $c_\tau^\pm(\Pi_i)$, for $\tau \in J_{F_i}$, which as we have seen above (i.e theorem 1.3 of [V1]) are invariant under arbitrary base change to totally real number fields. Also, for the same reason, theorem 1.2 should be true for the critical values m of more general Langlands L-functions $L(s, \Pi, r)$, where Π is an automorphic representation of an algebraic group G (closely related to GL(2)) and r an automorphic representation of the the Langlands group ${}^LG$ (some progress has been made in this direction for some G and r, see [GH]).*

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