

Non-solvable base change for Hilbert modular representations and zeta functions of twisted quaternionic Shimura varieties

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1 Introduction

In the first part of this article we prove the following non-solvable base change for Hilbert modular representations:

Theorem 1.1. *Let π be a cuspidal automorphic representation of weight $k = (k_\tau)_{\tau \in J_F}$ of $GL(2)/F$, where F is a totally real field, J_F is the set of infinite places of F and all k_τ have the same parity and are ≥ 2 . If F_1 is a solvable extension of a totally real field containing F , then there exists a Galois extension F_2 of \mathbb{Q} , containing F_1 and which is a solvable extension of a totally real field, such that π admits a base change to $GL(2)/F_2$. If F_1 is a totally real field, then F_2 can be chosen to be a totally real field.*

To obtain this result, we follow some ideas of Taylor [T2] and then use R=T theorem of Fujiwara [F]. We recall that from Langlands [L] and Arthur-Clozel [AC], we know that if π is an automorphic representation of $GL(n)/F$, where F is a number field, and F_1 is a solvable extension of F , then π admits a base change to $GL(n)/F_1$.

In the second part of this article, we compute the zeta function of some "twisted" quaternionic Shimura varieties in terms of automorphic representations, and as an application of theorem 1.1 we prove that, under certain assumptions, the zeta function could be meromorphically continued to the entire complex plane and satisfies a functional equation. In [BL], Brylinski-Labesse computed the zeta function of quaternionic Shimura varieties associated to a totally indefinite quaternion algebra D over a totally real field F i.e. all the infinite places of F are unramified in D . In his book [R], Reimann generalized the result in [BL] and computed the semisimple zeta function of quaternionic Shimura varieties associated to indefinite quaternion algebras D . Then in [B], Blasius generalized the result in [R] and obtained the expression of the zeta function of quaternionic Shimura varieties at all places.

More exactly, in the second part of this article, we consider F a totally real field, $O := O_F$ the ring of integers of F and D an indefinite quaternion algebra over F . Let G be the algebraic group over F defined by the multiplicative group D^\times of D and let $\tilde{G} := \text{Res}_{F/\mathbb{Q}}(G)$. We fix a prime ideal \wp of O_F , such that $G(F_\wp)$ is isomorphic to $GL_2(F_\wp)$. Let $S_{\tilde{G}, \mathbf{K}} = S_{\mathbf{K}}$ be the canonical model of the quaternionic Shimura variety associated to an open compact subgroup $\mathbf{K} := K_\wp \times H$ of $\tilde{G}(\mathbb{A}_f)$, where K_\wp is the set of elements of $GL_2(O_\wp)$ which are congruent to 1 modulo \wp , H is an open compact subgroup of the restricted product of $(D \otimes_F F_{\mathfrak{p}})^\times$ where \mathfrak{p} runs over all the finite places of F , $\mathfrak{p} \neq \wp$ and \mathbb{A}_f is the finite part of the ring of adèles $\mathbb{A}_{\mathbb{Q}}$ of \mathbb{Q} . Then $S_{\mathbf{K}}$ is a quasi-projective variety defined over a totally real number field E called the canonical field of definition.

The variety $S_{\mathbf{K}}$ has a natural action of $GL_2(O/\wp)$ (see §3.2). For H sufficiently small this action is free. We fix such a small group H . If K is a number field, we denote $\Gamma_K := \text{Gal}(\mathbb{Q}/K)$. Consider a continuous Galois representation $\varphi : \Gamma_E \rightarrow GL_2(O/\wp)$ and let $S'_{\mathbf{K}}$ be the variety defined over E obtained from $S_{\mathbf{K}}$ via twisting by φ composed with the natural action of $GL_2(O/\wp)$ on $S_{\mathbf{K}}$ (see §3.2 for details).

From corollary 11.8 of [R] and theorem 3 of [B] (see also propositions 3.3 and 3.4 below), we know that the zeta function $L(s, S_{\mathbf{K}})$ of $S_{\mathbf{K}}$ is given by the formula (see §3.1 for notations):

$$L(s, S_{\mathbf{K}}) = \prod_{\pi} L(s - d'/2, \pi, r)^{m(\pi_\infty)m(\pi_f^{\mathbf{K}})}.$$

Here the product is taken over automorphic cohomological representations π of $\tilde{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, d' is the dimension of $S_{\mathbf{K}}$, $m(\pi_f^{\mathbf{K}})$ is the dimension of $\pi_f^{\mathbf{K}}$ ($\pi_f^{\mathbf{K}}$ denotes the subspace of \mathbf{K} -invariants of π_f), r is a well specified representation of the L -group ${}^L\tilde{G}|_{\Gamma_E}$ associated to \tilde{G} (see §3.1 for the definition of r) and $m(\pi_\infty)$ will be defined in §3.1.

In this article we obtain the following result (see §3.1 for notations and also proposition 3.5):

Theorem 1.2. *The zeta function $L(s, S'_{\mathbf{K}})$ of $S'_{\mathbf{K}}$ is given by the formula:*

$$L(s, S'_{\mathbf{K}}) = \prod_{\pi} L(s - d'/2, \pi, r \otimes (\pi_f^{\mathbf{K}} \circ \varphi))^{m(\pi_\infty)},$$

where the product is taken over automorphic cohomological representations π of $\tilde{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, such that $\pi_f^{\mathbf{K}} \neq 0$.

If $d' = 1$ or 2 and the field $L := \bar{\mathbb{Q}}^{Ker(\varphi)}$ is a solvable extension of a totally real field, then the zeta function $L(s, S'_{\mathbf{K}})$ can be meromorphically continued to the entire complex plane and satisfies a functional equation.

The first part of this theorem is proved in §3.3 by taking the injective limit of the representations of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology of Shimura varieties S_K and using some linear algebra. Here K is an open compact subgroup of $\tilde{G}(\mathbb{A}_f)$

and \mathbb{H}_K is the Hecke algebra of level K . In our argument the multiplicity one for \tilde{G} is important.

The second part of theorem 1.2 regarding the meromorphic continuation of the zeta function $L(s, S'_{\mathbf{K}})$ is proved in §4. We show using theorem 1.1 (see theorem 4.3) that if $d' = 1$ or 2 and π is a representation as in the product of theorem 1.2 and ω is an Artin representation of Γ_E such that the field $K := \tilde{\mathbb{Q}}^{K^{\text{er}(\omega)}}$ is a solvable extension of a totally real field, then the L -function $L(s, \pi, r \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation. We prove also the meromorphic continuation and functional equation of $L(s, S'_{\mathbf{K}})$ when $d' \geq 3$ and the field $L := \tilde{\mathbb{Q}}^{K^{\text{er}(\varphi)}}$ is a solvable extension of a totally real field, if we assume that some other Langlands L -functions can be meromorphically continued to the entire complex plane and satisfy a functional equation (see lemma 4.1).

We remark that when $D = M_2(F)$ the Shimura variety is not compact and in this case we use the l -adic intersection cohomologies of the Baily-Borel compactification of the Shimura variety.

In this article, if π is an automorphic representation of $\tilde{G}(\mathbb{A}_{\mathbb{Q}})$, we denote the automorphic representation of $GL_2(\mathbb{A}_F)$ (\mathbb{A}_F is the ring of adèles of F), obtained from π by Jacquet-Langlands correspondence (usually denoted $JL(\pi)$) by the same symbol π .

2 Non-solvable base change for Hilbert modular representations

Let π be a cuspidal automorphic representation of weight $k = (k_{\tau})_{\tau \in J_F}$ of $GL(2)/F$, where F is a totally real field, and all k_{τ} have the same parity and are ≥ 2 . Then from Taylor [T1], we know that there exists a λ -adic representation (for λ a prime of the field of coefficients \mathbf{O} of π , such that $\lambda | l$ for some rational prime l)

$$\rho_{\pi, \lambda} : \Gamma_F \rightarrow GL_2(\mathbf{O}_{\lambda}) \hookrightarrow GL_2(\bar{\mathbb{Q}}_l),$$

which is unramified outside the primes dividing $\mathbf{n}l$, where \mathbf{n} is the level of π .

We say that a representation $\rho : \Gamma_F \rightarrow GL_2(\bar{\mathbb{Q}}_l)$ is modular if $\rho \cong \rho_{\pi, \lambda}$, for some π and $\lambda | l$. Also we say that a representation $\rho : \Gamma_F \rightarrow GL_2(k)$, for some finite field k , is modular if $\rho \cong \bar{\rho}_{\pi, \lambda}$, for some π and λ , where we denote by $\bar{\rho}_{\pi, \lambda}$ the reduction of $\rho_{\pi, \lambda} : \Gamma_F \rightarrow GL_2(\mathbf{O}_{\lambda}) \pmod{\lambda}$.

In this section we prove the following result, which is equivalent to theorem 1.1 above:

Theorem 2.1. *If F is a totally real field, π is a cuspidal automorphic representation of $GL(2)/F$ of weight $k = (k_{\tau})_{\tau \in J_F}$ as above and F_1 is a solvable extension of a totally real field containing F , then there exists a Galois extension F_2 of \mathbb{Q} containing F_1 , such that F_2 is a solvable extension of a totally real field and there exists a prime λ of the field coefficients of π , such that $\rho_{\pi, \lambda}|_{\Gamma_{F_2}}$ is modular. If F_1 is a totally real field, then F_2 can be chosen to be a totally real field.*

For $F = \mathbb{Q}$ and $k = 2$ this is theorem 3.7 of [V1]. The proof in [V1] uses the positivity of the density of the set of ordinary primes which is known for cuspidal automorphic representations of $GL(2)/\mathbb{Q}$. This fact is not known for cuspidal automorphic representations of $GL(2)/F$ for general totally real field F . To prove the theorem for general totally real field F , one uses the argument below to generalize some results from [T2] and then apply theorem 2.9 (theorem R=T of [F]).

Now we start the proof of theorem 2.1.

We say that the automorphic representation π of $GL(2)/F$ is of CM-type if there exists some Galois character $\eta : I_F/F^\times \rightarrow \widehat{\mathbb{Q}}_l^\times$, where I_F denotes the idele group of F , with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. It is known (see theorem 7.11 of [G]) that if π is of CM-type, then $\rho_{\pi, \lambda}|_{\Gamma_L}$ is modular for every extension L/F and every prime λ of the field of coefficients of π . Thus in this case theorem 2.1 is proved.

From now on we assume that the representation π is non-CM. We can associate to π a Hilbert modular newform f of weight k and of level \mathbf{n} . We assume for later use, that the prime l splits completely in F . We consider a prime ideal λ above l of the field of coefficients O_f of f . We remark that by an extension if necessary, we may, and we do assume from now on that the field F satisfies $[F : \mathbb{Q}] = \text{even}$. Then, from Taylor [T1], we know that one can find a prime ideal λ_1 of O_F and a Hilbert modular newform g of weight k and of level $\mathbf{n}\lambda_1$ which is new at λ_1 , such that $f \equiv g \pmod{\lambda}$, in the sense that they have the same Hecke eigenvalues mod λ . Actually one can find a rational prime number l' and a Hilbert modular form g of level $\mathbf{n}l'$ and new at l' , such that $f \equiv g \pmod{\lambda}$ (see the final part of [T1]). The argument in [T1](final part) allows us to assume that l' splits completely in F (or in any fixed extension F_1 of F) and that $l' \equiv 1 \pmod{l}$. We assume these facts from now on.

Since g is new at l' , the automorphic representation generated by g is Steinberg at all $\lambda' | l'$, and we obtain:

Proposition 2.2. *For g as above i.e. new at l' , we have*

$$\rho_{g, \lambda'}|_{G_v} \cong \begin{pmatrix} \epsilon_{l'} \delta & * \\ 0 & \delta \end{pmatrix}$$

where G_v is a decomposition group at v , with $v | N\lambda'$, where $\epsilon_{l'}$ is the l' -adic cyclotomic character and δ is some character.

Let $\mathbb{F}_{l'}$ be the finite field of cardinal l' . The following result will be proved in §2.1:

Proposition 2.3. *We can choose the above form g and prime number l' , such that for all $\lambda' | l'$, the representation $\rho_{g, \lambda'}$ is full i.e. the image of the reduced representation $\bar{\rho}_{g, \lambda'}$ contains $SL_2(\mathbb{F}_{l'})$.*

Thus we can assume that the image of $\bar{\rho}_{g, \lambda'}$ contains $SL_2(\mathbb{F}_{l'})$.

Let F_1 be a totally real extension of F . As we said above, the argument in [T1](final part) allows us to assume that l' splits completely in F_1 , and from

now on we assume this fact. Since by our assumption the image of $\bar{\rho}_{g,\lambda'}$ contains $SL_2(\mathbb{F}_{l'})$ and F_1 is a totally real field, we know from [V1], proposition 3.5, that the image of $\bar{\rho}_{g,\lambda'}|_{\Gamma_{F_1}}$ contains $SL_2(\mathbb{F}_{l'})$ and thus the representation $\bar{\rho}_{g,\lambda'}|_{\Gamma_{F_1}}$ is irreducible. From proposition 2.2, after a twist by δ^{-1} , we obtain that the representation $\rho_{l'} := \bar{\rho}_{g,\lambda'}|_{\Gamma_{F_1}} : \Gamma_{F_1} \rightarrow GL_2(k)$ (here k is a finite field of characteristic l') satisfies the proprieties from Taylor's paper [T2] §1:

- i) is a continuous irreducible representation,
- ii) for every place v of F_1 above l' we have

$$\rho_{l'}|_{G_v} \sim \begin{pmatrix} \epsilon_{l'} & * \\ 0 & 1 \end{pmatrix}$$

where G_v is a decomposition group above v ,

- iii) the representation $\rho_{l'}$ is odd, i.e. for every complex conjugation c we have $\det \rho_{l'}(c) = -1$.

(we remark that Taylor treated the case when the representation $\rho_{l'}$ is irreducible and the image is not solvable, but as it is explained in [T2], the results for solvable image follow from [L], [TU], [W1], [TW], [DI] and [SW2]).

We want to use Taylor's argument from [T2] to find a totally real extension E_1/F_1 , with E_1 Galois over \mathbb{Q} , such that l splits completely in E_1 , and $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_1}}$ is modular, and then apply R=T theorem of [F] (see theorem 2.9 below) to prove theorem 2.1. In order to find a field E_1 having these proprieties, one can modify Taylor's argument from [T2] in the following way. Taylor used this theorem of Moret-Bailly [M]:

Theorem 2.4. *Let S be a finite set of places, K a number field and K_S/K a unique maximal extension inside a given algebraic closure of K , in which all the places of S split completely. If $X/\text{Spec}(K)$ is a geometrically irreducible smooth quasi-projective scheme and $X(K_v)$ is non-empty, then $X(K_S)$ is Zariski dense in X .*

We want to apply this theorem for S the set of places of F_1 above l, l', ∞ and another prime p which is considered in [T2].

Starting from the representation $\rho_{l'}$ one can define exactly as in [T2] §1 (one has only to replace the prime l from [T2] by l' , and the field F from [T2] by F_1) the field $N_0 = \mathbb{Q}(\zeta, \sqrt{1-4l'})$, where ζ is a root of unity of order $\#k^\times$, a place $\lambda_0|l'$ of N_0 , and an isomorphism $O_{N_0}/\lambda_0 \cong k$, elements $\beta_v \in O_{N_0}$ for each place $v|l'$ of F_1 , such that $\beta_v \equiv 1 \pmod{\lambda_0}$, a rational prime $p \nmid ll'n$ which splits completely in F_1 , such that p is coprime to $\beta_v - \beta_v^c$ for each $v|l'$ and such that $\rho_{l'}$ is unramified at each place $w|p$ of F_1 , and for each $w|p$ an element $\alpha_w \in O_{N_0}$ which is congruent modulo λ_0 to an eigenvalue of $\rho_{l'}(\phi_w)$.

We remind that $l' \equiv 1 \pmod{l}$. Thus $1-4l' \equiv -3 \pmod{l}$ and if we choose l such that $(\frac{-3}{l}) = 1$, then l splits in N_0 . We assume that l splits in N_0 from now on.

Then one can define a character ψ having similar proprieties as in [T2] §1. Thus, one can choose (see lemma 1.1 of [T2] for the proof) a quadratic extension

L of F_1 , a place \wp_0 of N_0 above p and a continuous character $\psi : \Gamma_L \rightarrow (\bar{N}_{0,\wp_0})^\times$, such that:

1. L is a totally imaginary field which is not contained in $F_1(\zeta_p)$, where ζ_p is the p th root of unity,
2. each place v of F_1 above l' splits as $v_1 v_1^c$ in L and $\psi|_{W_{L v_1}} = 1$ in $(\overline{O_{N_0}/\wp_0})$,
3. each place w of F_1 above p splits as $w_1 w_1^c$ in L and $\psi|_{G_{w_1}}$ is unramified and takes arithmetic Frobenius to a lift of $\alpha_w \in O_{N_0}/\wp_0$,
4. each place u of F_1 above l splits as $u_1 u_1^c$ in L ,
5. $\det \text{Ind}_{\Gamma_L}^{\Gamma_{F_1}} \psi = \epsilon_p$.

We consider $\bar{\psi} : \Gamma_L \rightarrow (\overline{O_{N_0}/\wp_0})^\times$ the reduction of ψ . We choose a Galois CM extension N/N_0 such that

- i) the primes above l' split in N/N_0 ,
- ii) the primes above p are unramified in N/N_0 ,
- iii) the primes above l are unramified in N/N_0 ,
- iv) there is a prime \wp above \wp_0 such that $\bar{\psi}$ has image in O_N/\wp .

Let λ' be a prime of O_N above λ_0 and λ_1 a prime of O_N above l . Let M be the maximal totally real subfield of N .

We know (this is lemma 1.2. of [T2]):

Proposition 2.5. *For each place v of F_1 above l' one can find an M -HBAV (A_v, i_v, j_v) over F_{1v} such that:*

1. A_v has potentially good reduction or potentially multiplicative reduction,
2. the action of G_v on $A_v[\lambda'|_M]$ is equivalent to $\rho_{\nu'}|_{G_v}$,
3. the action of G_v on $A_v[\wp|_M]$ is equivalent to $\bar{\psi}_{v_1} \oplus \bar{\psi}_{v_1^c}$.

The next two propositions are identical to lemma 1.3 of [T2].

Proposition 2.6. *For each place w of F_1 above p one can find an M -HBAV (A_w, i_w, j_w) over F_{1w} such that:*

1. A_w has potentially good reduction or potentially multiplicative reduction,
2. the action of G_w on $A_w[\lambda'|_M]$ is equivalent to $\rho_{\nu'}|_{G_w}$,
3. the action of G_w on $A_w[\wp|_M]$ is equivalent to $\bar{\psi}_{w_1} \oplus \bar{\psi}_{w_1^c}$,

Proposition 2.7. *For each place u of F_1 above l one can find an M -HBAV (A_u, i_u, j_u) over F_{1u} such that:*

1. A_u has potentially good reduction or potentially multiplicative reduction,
2. the action of G_u on $A_u[\lambda'|_M]$ is equivalent to $\rho_{\nu'}|_{G_u}$,
3. the action of G_u on $A_u[\wp|_M]$ is equivalent to $\bar{\psi}_{u_1} \oplus \bar{\psi}_{u_1^c}$,

Taylor proved the following result (see lemma 1.4 of [T2]):

Proposition 2.8. *For each infinite place x of F_1 there exists a M -HBAV (A_x, i_x, j_x) over F_{1x} .*

Then as in §1 of [T2], using theorem 2.4 above for S the set of places of F_1 above l, l', p and ∞ and propositions 2.5, 2.6, 2.7 and 2.8 above, we deduce that there exists a totally real extension E_1/F_1 , such that each place above l, l'

and p in F_1 splits completely in E_1 and a M -HBAV $(A, i, j)/E_1$, such that the representation of Γ_{E_1} on $A[\lambda']$ is equivalent to $\bar{\rho}_{g, \lambda'}|_{\Gamma_{E_1}}$ and the representation of Γ_{E_1} on $A[\varphi]$ is equivalent to $\text{Ind}_{\Gamma_L}^{\Gamma_{F_1}} \bar{\psi}|_{\Gamma_{E_1}}$. Also from [T2] §1, we know that the representation $\text{Ind}_{\Gamma_L}^{\Gamma_{F_1}} \bar{\psi}|_{\Gamma_{E_1}}$ is absolutely irreducible (because p is coprime to $\beta_v - \beta_v^c$ for each place $v|l'$ of F_1 , and each v splits completely in E_1 , and thus the restriction of $\bar{\psi}$ to the two places of LE_1 above any place $x|l'$ of E are different) and that A has good ordinary reduction at all primes of O_{E_1} above p .

Since the representation $\text{Ind}_{\Gamma_L}^{\Gamma_{F_1}} \bar{\psi}|_{\Gamma_{E_1}}$ is modular and irreducible and L does not contain $F_1(\zeta_p)$, by applying theorem 5.1 of [SW1], we deduce that $T_\varphi A$ is modular and we get that $T_{\lambda'} A$ is modular. Thus $\bar{\rho}_{g, \lambda'}|_{\Gamma_{E_1}}$ is modular and applying again theorem 5.1 of [SW1], we deduce that $\rho_{g, \lambda'}|_{\Gamma_{E_1}}$ is modular. From $f \equiv g \pmod{\lambda}$, we obtain that $\bar{\rho}_{f, \lambda}|_{\Gamma_{E_1}}$ is modular and also because we have assumed that l splits completely in F_1 , we get that l splits completely in E_1 .

Actually the field E_1 can be chosen to be Galois over \mathbb{Q} , because starting from an M -HBAV $(A, i, j)/E_1$ as above, one can consider by extensions of the scalars the M -HBAV $(A, i, j)/E_1^{gal}$, where E_1^{gal} is the Galois closure of E_1 , and since l, l' and p split completely in E_1 , we obtain that l, l' and p also split completely in E_1^{gal} and the above argument can be repeated for the M -HBAV $(A, i, j)/E_1^{gal}$. We assume from now on that E_1 is Galois over \mathbb{Q} .

Thus we proved that there exists a Galois totally real extension E_1 of \mathbb{Q} which contains F_1 such that l splits completely in E_1 and $\bar{\rho}_{f, \lambda}|_{\Gamma_{E_1}} \cong \bar{\rho}_{\pi', \gamma}$ for a cuspidal automorphic representation π' of $GL(2)/E_1$ and a prime $\gamma|l$.

We know the following result (see R=T theorem, definition 6.11, remark 6.12 and definition 3.14 of [F]):

Theorem 2.9. *Let F' be a totally real field, with $[F' : \mathbb{Q}] = \text{even}$, $l > 5$ a prime number and ζ_l a primitive l -th root of unity. An l -adic representation $\rho : \Gamma_{F'} \rightarrow GL_2(\bar{\mathbb{Q}}_l)$, with $\det \rho = \epsilon_l^{k-1} \chi$ for some finite order character χ , is modular if:*

1. $\bar{\rho} \cong \bar{\rho}_{\pi', \gamma}$ for a cuspidal automorphic representation π' of $GL(2)/F'$ and a prime $\gamma|l$,
2. the representation $\bar{\rho}|_{\Gamma_{F'(\zeta_l)}}$ is absolutely irreducible,
3. l splits completely in F' ,
4. for each $\ell|l$ we have that $\rho|_{G_{F'_\ell}}$ is a representation associated to a Barsotti-Tate group.
5. for each $\ell|l$ we have $\det \bar{\rho}|_{I_\ell} = \epsilon_l^{k-1}$, where I_ℓ is a inertia group at ℓ .

We want to apply this theorem to the representation $\rho_{f, \lambda}|_{\Gamma_{E_1}}$. As we remarked above, just before theorem 2.9, the conditions 1 and 3 are satisfied.

Since we have assumed that our π is non-CM, we can choose the prime number l such that $l > 5$ and the image of $\bar{\rho}_{f, \lambda}$ contains $SL_2(\mathbb{F}_l)$ (see proposition 3.8 of [D], where it is proved that if π is non-CM, then for all but a finite number of primes l , the image of $\bar{\rho}_{f, \lambda}$ contains $SL_2(\mathbb{F}_l)$). Since E_1 is totally real, the image of $\bar{\rho}_{f, \lambda}|_{\Gamma_{E_1}}$ contains $SL_2(\mathbb{F}_l)$ (see proposition 3.5 of [V1]). One can prove also that, for such a prime number l , the image of $\bar{\rho}_{f, \lambda}|_{\Gamma_{E_1(\zeta_l)}}$ contains

$SL_2(\mathbb{F}_l)$ and thus the representation $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_1(\zeta_l)}}$ is absolutely irreducible and the condition 2 of theorem 2.9 is satisfied.

From theorem 1.6 of [T3], we know that if $\bar{\rho}_{f,\lambda}$ (for λ prime to the level \mathbf{n} of f , which we have assumed) is irreducible (which is our case), then $\rho_{f,\lambda}|_{G_{F_\ell}}$ for each prime ℓ of F above l is a representation associated to a Barsotti-Tate group. Hence for each prime ℓ of E_1 above l , the representation $\rho_{f,\lambda}|_{G_{F_\ell}}$ is associated to a Barsotti-Tate group and in particular $\bar{\rho}_{f,\lambda}|_{G_{F_\ell}}$ is associated to a Barsotti-Tate group. Thus the condition 4 of theorem 2.9 is also satisfied.

From theorem 1.4 of [T3], we know that the representation $\bar{\rho}_{f,\lambda}$ is crystalline at each prime ℓ of F dividing l , for almost all l ($\lambda|l$) and we have $\det \bar{\rho}_{f,\lambda}|_{I_\ell} = \epsilon_l^{k-1}$ for each prime ℓ of F above l . Thus we could choose the prime number l from the very beginning such that $\det \bar{\rho}_{f,\lambda}|_{I_\ell} = \epsilon_l^{k-1}$ for each ℓ of F above l , and we get that $\det \bar{\rho}_{f,\lambda}|_{I_\ell} = \epsilon_l^{k-1}$ for each prime ℓ of E_1 above l .

Therefore all the conditions of theorem 2.9 are satisfied for our choice of the prime number l , and we obtain that $\rho_{f,\lambda}|_{\Gamma_{E_1}}$ is modular.

The totally real field E_1 is Galois over \mathbb{Q} , and thus we deduce theorem 2.1 for F_1 totally real.

Now we prove theorem 2.1 for F_1 a solvable extension of a totally real field F_0 containing F . From theorem 2.1 applied to the totally real extension F_0 of F , we deduce that there exists a totally real Galois extension F_2 of F_0 such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$ is modular. Then $F_1 F_2$ is a solvable extension of F_2 . Since F_2 is Galois over \mathbb{Q} , we deduce that the Galois closure F_3 of $F_1 F_2$ over \mathbb{Q} is a solvable extension of F_2 . Hence using the fact that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$ is modular, from Langlands base change for solvable extensions, we deduce that $\rho_{\pi,\lambda}|_{\Gamma_{F_3}}$ is modular and thus theorem 2.1 is proved in the general case. ■

2.1 The proof of proposition 2.3

More precisely, we prove that there exists a number N (from the proof below one could choose $N = \max\{5, N_1, \max S\}$, N_1 is defined in case 2. below, and where S is the set of all prime numbers which divide the product $(\prod_{\phi} (\text{the numerator of } L(-1, \phi)))N(\mathbf{n})$, where $N(\mathbf{n})$ is a number which depends on \mathbf{n} , and ϕ runs over all finite order Hecke characters of Γ_F which are unramified outside \mathbf{n}) such that if g is a Hilbert newform of $GL(2)/F$ which is new at λ' , with $\lambda'|l'$ and the representation $\rho_{g,\lambda'}$ is not full, then $l' \leq N$. Since we can choose the prime l' as big as we want, if we show this fact, proposition 2.3 is proved. To show this fact we use the following result:

Theorem 2.10. (Dickson) *If k is a finite field of characteristic l' then:*

(i) *An irreducible subgroup of $PGL_2(k)$ of order divisible by l' is conjugated to $PGL_2(\mathbb{F}_q)$ or $PSL_2(\mathbb{F}_q)$, for some q a power of l' .*

(ii) *An irreducible subgroup of $PGL_2(k)$ of order not divisible by l' is either dihedral or isomorphic to one of the groups A_4 , A_5 or S_4 .*

We denote by pr the canonical projection of $GL_2(k)$ to $PGL_2(k)$.

We distinguish three cases:

1. $\text{pr}(\bar{\rho}_{g,\lambda'}(\Gamma_F))$ is isomorphic to one of the groups A_4 , A_5 or S_4 . Since each element of A_4 , A_5 or S_4 has order at most 5, we get that each element of $\text{pr}(\bar{\rho}_{g,\lambda'}(\Gamma_F))$ has order at most 5. Thus from proposition 2.2, by considering the restriction $\bar{\epsilon}_{l'}|_{I_v}$ and the class field theory map $I_v \rightarrow O_v^\times$, we get that any generator $x \in \mathbb{F}_{l'^h}^\times$, where $|O/v| = l'^h$, belongs to $\bar{\epsilon}_{l'}(I_v) = \text{pr}(\bar{\rho}_{g,\lambda'}(I_v))$ and thus it has order at most 5. Hence $5 \geq l'^h - 1$, which implies that $5 \geq l' - 1$. Thus we proved that if $l' > 5$, then $\text{pr}(\bar{\rho}_{g,\lambda'}(\Gamma_F))$ is not isomorphic to one of the groups A_4 , A_5 or S_4 .

2. The representation $\bar{\rho}_{g,\lambda'}$ is reducible. We denote by $\bar{\rho}_{g,\lambda'}^{ss}$ the semisimplification of $\bar{\rho}_{g,\lambda'}$. Then $\bar{\rho}_{g,\lambda'}^{ss} = \phi'_1 \oplus \phi'_2$, for some characters $\phi'_1, \phi'_2 : \Gamma_F \rightarrow k^\times$ such that, because of proposition 2.2, ϕ'_1/ϕ'_2 is unramified outside $\mathfrak{n}l'$ and $(\phi'_1/\phi'_2)|_{G_v} = \epsilon_{l'}^\pm$ for all places v of F with $v|l'$.

Assume that for all $\emptyset \subsetneq J \subsetneq J_F$, there exists $\sigma \in O_+^\times$, $\sigma - 1 \in \mathfrak{n}$, where O_+^\times is the group of totally positive units of O_F , such that l' does not divide the non-zero integer $N_{F/\mathbb{Q}}(\sigma^{p(J)} - 1)$, where $p(J) = (\tau_j)_{j \in J}$ and $\tau_j = 1$ if $j \in J$ and $\tau_j = -1$ if $j \notin J$ (by choosing l' big enough, lets say $l' > N_1$, we can make this assumption).

Thus for every $\sigma \in O_+^\times$, $\sigma - 1 \in \mathfrak{n}$ (see §3.1 of [D] for details), we have the following equality in k :

$$1 = (\phi'_1/\phi'_2)(\sigma) = \prod_{v|l'} (\phi'_1/\phi'_2)_v(\sigma) = \prod_{v|l'} \epsilon_{l'}|_{I_v}^\pm(\sigma) = \sigma^{p(J)},$$

for some $J \subset J_F$. Thus we get that l' divides $N_{F/\mathbb{Q}}(\sigma^{p(J)} - 1)$, which by our assumption is impossible if $J \neq \emptyset$ or $J \neq J_F$. Thus we have that $J = \emptyset$ or $J = J(F)$, which implies that the signs which appear in the powers of $\epsilon_{l'}|_{I_v}^\pm$ are all equal. Hence $\phi'_2/\phi'_1 = \epsilon_{l'}^\pm \phi'$, where ϕ' is an unramified character outside \mathfrak{n} . But by changing the lattice if necessary, we may, and we do assume from now on that $\phi'_2/\phi'_1 = \epsilon_{l'} \phi'$, where ϕ' is an unramified character outside \mathfrak{n} .

Let $\phi : \Gamma_F \rightarrow \mathfrak{D}^\times$ (where \mathfrak{D} is the ring of integers of some local field) be the Teichmüller lift of ϕ' . Thus ϕ is a finite order Hecke character of conductor dividing \mathfrak{n} . There are only finitely many Hecke characters ϕ of finite order of conductor dividing \mathfrak{n} . But this is exactly the case which Wiles studied in [W3], and by the F -version of the Kummer's criterion (which follows from [W3]), we get that l' divides the numerator of $L(-1, \phi)$. Thus we get that $l' | \prod_{\phi} (\text{the numerator of } L(-1, \phi))$, where ϕ runs over all finite order Hecke characters of Γ_F which are unramified outside \mathfrak{n} .

Now we give another proof of this result. Using the same notations as above, we denote by ϕ_1 the Teichmüller lift of ϕ'_1 . Let g' be the cuspform obtained from g by twisting by ϕ_1^{-1} .

We want to prove now that λ' divides the numerator of $L(-1, \phi)$. Let $E(1, \phi)$ be the Eisenstein series of level \mathfrak{n} and weight 2 whose Mellin transform is the function $L(s, 1)L(s - 1, \phi)$ and $C(0, E(1, \phi))$ be the constant term of the q -expansion of the Eisenstein series $E(1, \phi)$ at some unramified cusp which we fix, and from now on all the q -expansions are done at this cusp.

Then $C(0, E(1, \phi)) \neq 0$ and we have that $C(0, E(1, \phi)) = 2^{-d}L(-1, \phi)$, where $d = [F : \mathbb{Q}]$.

Let $h := E(1, \phi) - g'$. Then, $h \equiv C(0, E(1, \phi)) \pmod{\lambda'}$ (congruence of coefficients of the q -expansions at our cusp; we remark that $1 + \phi(\text{Frob}_u) \equiv a(u) \pmod{\lambda'}$, for all places u of F outside \mathfrak{n} , where $a(u)$ is the eigenvalue of the Hecke operator T_u at u , but this congruence might be not true for $u|N\mathfrak{n}$. But after a solvable base change of F (which depends only on \mathfrak{n}), one can assume that g' is not supercuspidal at the places $u|N\mathfrak{n}$, and then one can modify $E(1, \phi)$ by $(\prod_{u|N\mathfrak{n}}(T_u - \alpha_u))E(1, \phi)$ for suitable α_u , such that the above congruence is true also for $u|N\mathfrak{n}$. We remark that the number of α_u is finite (it depends only on \mathfrak{n})).

If \mathfrak{m} is an ideal of O_F , then we denote by $S_\chi^{\text{ord}}(\mathfrak{m}; A)$ the space of cusp forms of $GL(2)/F$ of level \mathfrak{m} of weight $\chi \geq 2$ which are ordinary at l' , with coefficients in some ring A . We know (this is theorem 4.37 from [H]):

Proposition 2.11. *If \mathfrak{n} and l' are as above and $\chi \geq 3$, then*

$$S_\chi^{\text{ord}}(\mathfrak{n}'; \bar{\mathbb{F}}_{l'}) \cong S_\chi^{\text{ord}}(\mathfrak{n}; \bar{\mathbb{F}}_{l'}).$$

There exists a Hilbert modular form $E \in S_{(l'-1)^a}(\mathfrak{n}'; W)$ for some positive natural number a , where W is a local ring with residue field $\bar{\mathbb{F}}_{l'}$, such that $E \equiv 1 \pmod{l'}$. We get an injection

$$S_2^{\text{ord}}(\mathfrak{n}'; \bar{\mathbb{F}}_{l'}) \rightarrow S_{(l'-1)^{a+2}}^{\text{ord}}(\mathfrak{n}'; \bar{\mathbb{F}}_{l'}),$$

given by $f \mapsto fE$. Thus $hE \in S_{(l'-1)^{a+2}}^{\text{ord}}(\mathfrak{n}'; \bar{\mathbb{F}}_{l'}) \cong S_{(l'-1)^{a+2}}^{\text{ord}}(\mathfrak{n}; \bar{\mathbb{F}}_{l'})$ and $hE \equiv C(0, E(1, \phi_2)) \pmod{\lambda'}$.

If \mathfrak{m} is an ideal of O_F and k is some finite field of characteristic l' which contains isomorphic copies of the residue fields $\{k_{l'}\}_{l'|l'}$ of the prime ideals of O_F over l' , then we denote by $M_\chi(\mathfrak{m}; k)$ the space of Hilbert modular forms of level \mathfrak{m} corresponding to some weight $\chi \in \mathbb{X}_k$, where \mathbb{X}_k is the set of all weights of the space of Hilbert modular forms $M(\mathfrak{m}; k)$ of level \mathfrak{m} , defined over k .

From [AG], theorem 7.22 we know:

Proposition 2.12. *Consider the ideal of congruences*

$$I := \text{Ker}\{\oplus_{\chi \in \mathbb{X}_k} M_\chi(\mathfrak{n}; k) \rightarrow k[[q]]\}.$$

Then I is spanned by

$$\{h_\psi - 1 | \psi \in \mathbb{X}_k(1)^+\},$$

where $\mathbb{X}_k(1)^+$ is some subset of \mathbb{X}_k and h_ψ is a modular form of weight $l' - 1$.

Applying this proposition to $hE - C(0, E(1, \phi_2)) \equiv 0 \pmod{\lambda'}$, we get that $hE - C(0, E(1, \phi_2)) = \sum_{i=1}^{i=m} a_i(h_i - 1)$, for some $h_i \in \mathbb{X}_k(1)^+$, $i = 1, \dots, m$. But hE has weight $(l'-1)^{a+2}$ and each h_i has weight $l'-1$. Since $l'-1 \nmid (l'-1)^{a+2}$ for $l' > 3$, it is easy to see that the equality $hE - C(0, E(1, \phi_2)) = \sum_{i=1}^{i=m} a_i(h_i - 1)$ is impossible if $l' > 3$ and hE is not 0 and $\lambda' \nmid C(0, E(1, \phi_2))$. Thus, if $l' > 3$ we get that $\lambda' | C(0, E(1, \phi_2)) = 2^{-d}L(-1, \phi_2)$.

Thus we proved that if $l' > N_1$ and if $l' \nmid \prod_{\phi}(\text{the numerator of } L(-1, \phi))$, where ϕ runs over all finite order Hecke characters of Γ_F which are unramified outside \mathfrak{n} , then the representation $\bar{\rho}_{g, \lambda'}$ is irreducible.

3. $\text{pr}(\bar{\rho}_{g, \lambda'}(\Gamma_F))$ is the dihedral group D_{2m} , where $m \geq 3$ is an integer prime to l' . We denote by C_m the cyclic subgroup of order m of D_{2m} , and let $\epsilon : D_{2m} \rightarrow \{\pm\}$ be the signature map. Let $K = \bar{F}^{\ker(\epsilon \circ \text{pr} \circ \bar{\rho}_{l'})}$. Then K is a quadratic extension of F .

Let τ be the non-trivial element of $\text{Gal}(K/F)$. The representation $\bar{\rho}_{g, \lambda'}$ is absolutely irreducible, being odd and irreducible, but $\bar{\rho}_{g, \lambda'}|_{\Gamma_K}$ is not, and we get that there exists a character $\phi : \Gamma_K \rightarrow k^\times$ distinct from the Galois conjugate ϕ^τ , such that $\bar{\rho}_{g, \lambda'}|_{\Gamma_K} = \phi \oplus \phi^\tau$. If v_1 is a place of K above v , then from proposition 2.2, we get that

$$\bar{\rho}_{g, \lambda'}|_{G_{v_1}} \sim \begin{pmatrix} \bar{\epsilon}_{l'} \bar{\delta} & * \\ 0 & \bar{\delta} \end{pmatrix} \sim \begin{pmatrix} \phi^\tau & 0 \\ 0 & \phi \end{pmatrix}.$$

We can assume that $\bar{\delta} = \phi$ and $\bar{\epsilon}_{l'} \bar{\delta} = \phi^\tau$ as characters of G_{v_1} . Hence $\bar{\epsilon}_{l'} \phi = \phi^\tau$, and thus by conjugating by τ we get that $\bar{\epsilon}_{l'} \phi^\tau = \phi$ and from these two identities we get that $\bar{\epsilon}_{l'}^2|_{I_{v_1}}$ is the trivial character and as in the case 1. above (where each element of $\bar{\epsilon}_{l'}(I_v)$ had order at most 5), we deduce that $2 \geq l' - 1$. Thus we proved that if $l' > 5$, then $\text{pr}(\bar{\rho}_{g, \lambda'}(\Gamma_F))$ is not dihedral.

From theorem 2.10 and the cases 1, 2 and 3 treated above, we deduce proposition 2.3.

3 Quaternionic Shimura varieties

Consider a totally real number field F of degree d over \mathbb{Q} and let D be a quaternion algebra over F . Let \mathbb{A}_f be the finite part of the ring of adèles $\mathbb{A}_{\mathbb{Q}}$ of \mathbb{Q} . We denote by J_F the set of infinite places of F and we identify J_F as a $\Gamma_{\mathbb{Q}}$ -set with $\Gamma_F \backslash \Gamma_{\mathbb{Q}}$. Let J'_F be the subset of places of J_F where D is ramified. Let $d' :=$ the cardinal of $J_F - J'_F$. We assume that $d' > 0$, i.e. D is indefinite over F .

Let G be the algebraic group over F defined by the multiplicative group D^\times of D . Consider the algebraic group $\bar{G} := \text{Res}_{F/\mathbb{Q}}(G)$ over \mathbb{Q} defined by the propriety: $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$ for all \mathbb{Q} -algebras A . The L -group associated to \bar{G} is defined by the semidirect product:

$${}^L \bar{G} := {}^L \bar{G}^0 \rtimes \Gamma_{\mathbb{Q}},$$

where ${}^L \bar{G}^0$ is the product of d copies of $GL_2(\mathbb{C})$ indexed by elements $\sigma \in \Gamma_F \backslash \Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ acts on ${}^L \bar{G}^0$ by permuting the factors in the natural way. It is easy to see that $\bar{G}(\mathbb{R})$ is isomorphic to $GL_2(\mathbb{R})^{d'} \times \mathbf{H}^{\times(d-d')}$, where \mathbf{H} is the algebra of quaternions over \mathbb{R} .

For $v \in J_F - J'_F$, we fix an isomorphism of $G(F_v)$ with $GL_2(\mathbb{R})$. We have $\bar{G}(\mathbb{R}) = \prod_{v \in J_F} G(F_v)$. Let $J := (J_v) \in \bar{G}(\mathbb{R})$, where

$$J_v := \begin{cases} 1 & \text{for } v \in J'_F; \\ 1/\sqrt{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \text{for } v \in J_F - J'_F. \end{cases}$$

Let K_∞ be the centralizer of J in $\bar{G}(\mathbb{R})$. Put

$$X := \bar{G}(\mathbb{R})/K_\infty.$$

It is well known that X is complex analytically isomorphic to $(\mathfrak{H}_\pm)^{d'}$, where $\mathfrak{H}_\pm = \mathbb{C} - \mathbb{R}$. For each open compact subgroup $K \subset \bar{G}(\mathbb{A}_f)$ put

$$S_K(\mathbb{C}) := \bar{G}(\mathbb{Q}) \backslash X \times \bar{G}(\mathbb{A}_f)/K.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of complex points of a quasi-projective variety. In general $S_K(\mathbb{C})$ is not connected and is a finite disjoint union of quotients $\Gamma \backslash \mathfrak{H}_\pm^{d'}$, where $\Gamma \subset \bar{G}(\mathbb{Q})$ is a congruence subgroup. The subfield E of \mathbb{Q} having the propriety that Γ_E is the stabilizer of the subset $J'_F \subset \Gamma_F \backslash \Gamma_\mathbb{Q}$, for the natural right action of $\Gamma_\mathbb{Q}$ on $\Gamma_F \backslash \Gamma_\mathbb{Q}$, is called the canonical field of definition. It is known (see [D]) that $S_K(\mathbb{C})$ has a canonical model over E which is denoted by S_K . Then S_K is called a quaternionic Shimura variety. The dimension of S_K is equal to d' .

3.1 Zeta function of quaternionic Shimura varieties

In this section we introduce some notations and we shall expose the computation of the zeta function for quaternionic Shimura varieties following closely [RT].

Let π be an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q}) = G(\mathbb{A}_F)$. Then $\pi = \otimes \pi_v$, where the restricted tensor product is taken over all places v of F and π_v is a representation of $G(F_v)$, where F_v is the completion of F at v . An irreducible representation π_v of $GL_2(F_v)$ is called unramified if π_v contains a nonzero vector which is fixed under $GL_2(O_v)$, where O_v the completion of the ring of integers O_F at v . For almost all v , the representation π_v is unramified. We define $L(s, \pi) := \prod_v L(s, \pi_v)$, where if π_v is unramified

$$L(s, \pi_v) := \det(1 - Nv^{-s}g(\pi_v))^{-1},$$

and

$$g(\pi_v) = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

denotes the Langlands class of π_v .

For a continuous representation $r : {}^L\bar{G}^0 \rtimes \Gamma_E \rightarrow GL_n(\mathbb{C})$ and an automorphic representation $\pi = \otimes \pi_v$ of $\bar{G}(\mathbb{A}_\mathbb{Q}) = G(\mathbb{A}_F)$, where π_v denotes a representation of $G(F_v)$, one can define the L -function $L(s, \pi, r) := \prod_v L(s, \pi_v, r)$, where if π_v is unramified

$$L(s, \pi_v, r) := \det(1 - Nv^{-s}r(g(\pi_v)))^{-1}.$$

If $\omega : \Gamma_E \rightarrow GL_m(\mathbb{C})$ is an Artin representation, then we denote by the same symbol the representation of ${}^L\bar{G}^0 \rtimes \Gamma_E$ which extends ω and restricts to the trivial representation on ${}^L\bar{G}^0$. Then one can define as above the L -function $L(s, \pi, r \otimes \omega)$.

Consider ${}^L\bar{T}^0$ to be the subgroup of ${}^L\bar{G}^0$ of elements (t_σ) such that for all σ , t_σ is diagonal and let ν be the character of ${}^L\bar{T}^0$ defined by

$$\nu((t_\sigma)) := \prod \nu_\sigma(t_\sigma),$$

$$\nu_\sigma\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) := \begin{cases} a & \text{for } \sigma \in J_F - J'_F; \\ 1 & \text{for } \sigma \in J'_F. \end{cases}$$

Then Γ_E stabilizes the character ν .

We denote by r the finite dimensional representation of ${}^L\bar{G}^0$ whose highest weight with respect to the standard Borel subgroup is ν . Since Γ_E stabilizes ν , the representation r could be uniquely extended to ${}^L\bar{G}^0 \rtimes \Gamma_E$ such that Γ_E acts as the identity on the ν -weight space. Then, the dimension of r is $2^{d'}$. From now on in this paper we fix this representation r .

Let K be an open compact subgroup of $\bar{G}(\mathbb{A}_f)$. If l is a prime number, let \mathbb{H}_K be the Hecke algebra generated by the bi- K -invariant \mathbb{Q}_l -valued compactly supported functions on $\bar{G}(\mathbb{A}_f)$ under the convolution. If $\pi = \pi_f \otimes \pi_\infty$ is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote by π_f^K the space of K -invariants in π_f . The Hecke algebra \mathbb{H}_K acts on π_f^K .

We have an action of the Hecke algebra \mathbb{H}_K and an action of the Galois group Γ_E on the étale cohomology $H_{\text{ét}}^i(S_K, \mathbb{Q}_l)$ and these two actions commute. We say that the representation π is *cohomological* if $H^*(\mathfrak{g}, K_\infty, \pi_\infty) \neq 0$, where \mathfrak{g} is the Lie algebra of K_∞ (the cohomology is taken with respect to (\mathfrak{g}, K_∞) -module associated to π_∞). Then we know (see for example proposition 1.8 of [RT]):

Proposition 3.1. *The representation of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology $H_{\text{ét}}^i(S_K, \mathbb{Q}_l)$ is isomorphic to*

$$\bigoplus_{\pi} \sigma^i(\pi) \otimes \pi_f^K,$$

where $\sigma^i(\pi)$ is a representation of the Galois group Γ_E . The above sum is over weight 2 irreducible cohomological automorphic representations π of $\bar{G}(\mathbb{A}_\mathbb{Q})$, such that $\pi_f^K \neq 0$ and the \mathbb{H}_K -representations π_f^K are irreducible and mutually inequivalent.

The automorphic representations π which appear in proposition 3.1 are one-dimensional or cuspidal and infinite-dimensional and we know the following result (see propositions 1.5 and 1.8 of [RT]):

Proposition 3.2. *i) If π is infinite-dimensional, then*

$$\dim \sigma^i(\pi) = \begin{cases} 2^{d'} & \text{for } i = d', \\ 0 & \text{for } i \neq d'. \end{cases}$$

ii) If π is one-dimensional, then

$$\dim \sigma^i(\pi) = \begin{cases} \binom{d'}{i'} & \text{for } i = 2i', \\ 0 & \text{for } i \text{ odd.} \end{cases}$$

Fix an isomorphism $i_l : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$, and define the L -function

$$L(s, S_K) := \prod_{\pi} \prod_v \prod_i \det(1 - Nv^{-s} i_l(\sigma^i(\pi)(\phi_v)) | H_{et}^i(S_K, \bar{\mathbb{Q}}_l)^{I_v})^{-1^{i+1}},$$

where ϕ_v is a geometric Frobenius element at a finite place v of E and I_v is the inertia group at v (in order to define the local factors at the places of E dividing l one has to use actually the l' -adic cohomology for some $l' \neq l$ and theorem 3 of [B] which gives us the expression of the local factors of the zeta functions of quaternionic Shimura varieties).

For π cohomological, we define

$$m(\pi_{\infty}) = \begin{cases} (-1)^{d'} & \text{if } \pi_{\infty} \text{ is infinite-dimensional;} \\ 1 & \text{if } \pi_{\infty} \text{ is one-dimensional.} \end{cases}$$

We know (see corollary 1.10 of [RT]):

Proposition 3.3. *There exists a finite set S of primes of E , such that for all primes v of E not in S , and for π cohomological of weight 2:*

$$\prod_i \det(1 - Nv^{-s} i_l(\sigma^i(\pi)(\phi_v)))^{(-1)^{i+1}} = \det(1 - Nv^{-s+(d'/2)} r_v(g(\pi_v)))^{-m(\pi_{\infty})},$$

where r_v denotes the restriction of r to ${}^L\bar{G}^0 \rtimes G_v$, and G_v is a decomposition group at v .

3.2 Twisted quaternionic Shimura varieties

Let φ be a prime ideal of O_F such that $G(F_{\varphi})$ is isomorphic to $GL_2(F_{\varphi})$. Consider $\mathbf{K} := K_{\varphi} \times H$, where K_{φ} is the set of elements of $GL_2(O_{\varphi})$ which are congruent to 1 modulo φ and H is some open compact subgroup of the restricted product of $(D \otimes_F F_{\mathfrak{p}})^{\times}$, where \mathfrak{p} runs over all the finite places of F , with $\mathfrak{p} \neq \varphi$. Then it is well known (see for example [C], corollary 1.4.1.3) that for H sufficiently small, the group $GL_2(O/\varphi)$ acts freely on $S_{\mathbf{K}}$. We fix such a small H . Then the action of $GL_2(O/\varphi)$ on

$$S_{\mathbf{K}}(\mathbb{C}) = \bar{G}(\mathbb{Q}) \backslash X \times \bar{G}(\mathbb{A}_f) / \mathbf{K}$$

can be described in the following way : we have that $GL_2(O_{\varphi}) \hookrightarrow \bar{G}(\mathbb{A}_{\mathbb{Q}})$ by $\alpha \mapsto (1, \dots, \alpha, 1, \dots, 1)$, α at the φ component. Using the isomorphism $GL_2(O/\varphi) \cong GL_2(O_{\varphi})/K_{\varphi}$, the action of an element $g \in GL_2(O_{\varphi})$ is given by the right multiplication at the φ -component.

We fix a continuous representation

$$\varphi : \Gamma_E \rightarrow GL_2(O/\varphi).$$

Let L be the finite Galois extension of E defined by $L := (\bar{\mathbb{Q}})^{\text{Ker}(\varphi)}$.

Let

$$S' = S_{\mathbf{K}} \times_{\text{Spec}(E)} \text{Spec}(L).$$

The group $GL_2(O/\wp)$ acts on $S_{\mathbf{K}}$. Since $\varphi : \text{Gal}(L/E) \hookrightarrow GL_2(O/\wp)$, the group $\text{Gal}(L/E)$ acts on $S_{\mathbf{K}}$. We denote this action of $\text{Gal}(L/E)$ on $S_{\mathbf{K}}$ by φ' . The Galois group $\text{Gal}(L/E)$ has a natural action on $\text{Spec}(L)$ and we can descend via the quotient process S' to $S'_{\mathbf{K}}/\text{Spec}(E)$ using the diagonal action

$$\text{Gal}(L/E) \ni \sigma \rightarrow \varphi'(\sigma) \otimes \sigma$$

on S' . Thus, we obtain a quasi-projective variety $S'_{\mathbf{K}}/\text{Spec}(E)$. This is the twisted quaternionic Shimura variety which we mentioned in the title.

3.3 Computation of the zeta function of twisted quaternionic Shimura varieties

We consider the injective limit:

$$V^i := \varinjlim_K H_{et}^i(S_K, \bar{\mathbb{Q}}_l) \cong \varinjlim_K \oplus_{\pi} U^i(\pi) \otimes_{\bar{\mathbb{Q}}_l} \pi_f^K,$$

where $U^i(\pi)$ is the $\bar{\mathbb{Q}}_l$ -space which corresponds to $\sigma^i(\pi)$ (see proposition 3.1 for notations).

Using the multiplicity one for \bar{G} , we get that the π -component $V^i(\pi)$ of V^i is isomorphic to $\sigma^i(\pi) \otimes \pi_f$ as $\Gamma_E \times \mathbb{H}$ -module. Taking the \mathbf{K} -fixed vectors, we deduce that $V^i(\pi)^{\mathbf{K}}$ is isomorphic to $\sigma^i(\pi) \otimes \pi_f^{\mathbf{K}}$ as $\Gamma_E \times GL_2(O/\wp O)$ -module. Since the varieties $S_{\mathbf{K}}$ and $S'_{\mathbf{K}}$ become isomorphic over $\bar{\mathbb{Q}}$, we have the isomorphism $H_{et}^i(S_{\mathbf{K}}, \bar{\mathbb{Q}}_l) \cong H_{et}^i(S'_{\mathbf{K}}, \bar{\mathbb{Q}}_l)$. The actions of Γ_E on these cohomologies which give the expression of the zeta functions of these varieties are different. If we consider the component $V^{i'}(\pi)$ which corresponds to π of $H_{et}^i(S'_{\mathbf{K}}, \bar{\mathbb{Q}}_l)$ (see the decomposition of proposition 3.1), we get that $V^{i'}(\pi)$ is isomorphic to $\sigma^i(\pi) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)$ as Γ_E -module.

We consider a local field L of characteristic 0 and residue characteristic p . Let $W_L \subset \Gamma_L$ be the Weil group. The *Weil – Deligne* group WD_L of L is defined as the semidirect product of W_L with \mathbb{C} by the relation

$$\sigma z \sigma^{-1} = |\sigma| z$$

for all $\sigma \in W_L$ and $z \in \mathbb{C}$, where $|\cdot|$ is the norm map: $|\cdot| : W_L \rightarrow q^{\mathbb{Z}} \subset \mathbb{Q}^{\times}$, where q denotes the cardinality of the residue field of L , and $|\cdot| = 1$ on the inertia group $I_L \subset W_L$ and $|\Phi| = q$, where $\Phi \in W_L$ is an arithmetic Frobenius.

Fix a prime number l different from p . For a vector space V of finite dimension over \mathbb{Q}_l , let $\rho : \Gamma_L \rightarrow GL(V)$ be a continuous l -adic representation. We denote also by ρ its restriction to W_L . To the l -adic representation ρ , one can associate (see for example [B] for details) a pair (ρ^*, N) called *Frobenius semisimple parameter* of ρ , where $N \in \text{End}(V)$ is a nilpotent endomorphism and ρ^* is a representation of WD_L having the propriety that $\rho^*|_{W_L}$ is semisimple and for all $\sigma \in W_L$, $\rho^*(\sigma)$ is semisimple.

We know the following result which is a generalization of proposition 3.3 above (see theorem 3 of [B]):

Proposition 3.4. *Let l be a prime number and π be a cuspidal automorphic representation as in proposition 3.1. Then for each finite place v of E whose residue characteristic p is different from l , the isomorphism class of the Frobenius semisimple parameter $(\rho_{K,v}^*, N_v)$ of the Weil-Deligne group WD_{E_v} of E_v defined by the restriction of $\sigma^{d'}(\pi)$ to a decomposition group D_v at v coincides with the class*

$$((r \otimes | \cdot |^{d'/2}) \circ \sigma(\pi_v)), N_{K,v})$$

obtained by the restriction of $(r \otimes | \cdot |^{d'/2}) \circ \sigma$ to the decomposition group D_v , where $\sigma : \Gamma_E \hookrightarrow^L \bar{G}^o \rtimes \Gamma_E$ is the inclusion and $\sigma(\pi_v) : WD_{E_v} \rightarrow^L \bar{G}^o \rtimes \Gamma_E$ is the standard homomorphism.

Proof: The idea of the proof of the proposition 3.4 (for details see [B]) is that the representations $\sigma^{d'}(\pi)$ and $((r \otimes | \cdot |^{d'/2}) \circ \sigma)(\phi_v)$ satisfy (see §5.2 of [B]) the *Weight Monodromy Conjecture* (i.e. the eigenvalues of $\sigma^{d'}(\pi)(\phi_v)$ and $((r \otimes | \cdot |^{d'/2}) \circ \sigma)(\phi_v)$ are *Nv-Weil numbers of weight d'* , which means that the eigenvalues are algebraic integers α having the propriety that for each automorphism σ of \mathbb{Q} , we have $|\sigma(\alpha)| = |Nv|^{d'/2}$ and thus for each v their corresponding nilpotent data $N_{K,v}$ and N_v are uniquely determined (for details see §1.12 of [B]) by the semisimple representations $\rho_{K,v}^*$ and $(r \otimes | \cdot |^{d'/2}) \circ \sigma(\pi_v)$. Hence it is sufficient to prove that

$$\rho_{K,v}^* = (r \otimes | \cdot |^{d'/2}) \circ \sigma(\pi_v).$$

But we know that for almost all v , (i) $N_{K,v} = 0$ and $N_v = 0$, (ii) $\rho_{K,v}^*$ and $(r \otimes | \cdot |^{d'/2}) \circ \sigma(\pi_v)$ are unramified and from the computation of the unramified zeta function (see [R] and [BL], or proposition 3.3 above) shows that this formula is true. From Chebotarev theorem, we see that the semisimplification $\sigma^{d'}(\pi)^{ss}$ is isomorphic to $(r \otimes | \cdot |^{d'/2}) \circ \sigma$ and thus their restrictions to the decomposition group D_v for $v \nmid l$ are isomorphic and thus they give rise to isomorphic parameters $\rho_{K,v}^*$ and $(r \otimes | \cdot |^{d'/2}) \circ \sigma(\pi_v)$ and we conclude proposition 3.4. ■

We prove now the following result which implies the first part of theorem 1.2:

Proposition 3.5. *Let l be a prime number and π be a cuspidal automorphic representation as in proposition 3.1. Then for each finite place v of E whose residue characteristic is different from l , the isomorphism class of the Frobenius semisimple parameter $(\rho_{\mathbf{K},v}^*, N'_v)$ of the Weil-Deligne group WD_{E_v} of E_v defined by the restriction to the decomposition group D_v at v of $\sigma^{d'}(\pi) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)$ coincides with the class of*

$$(((r \otimes | \cdot |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v), N'_{K,v})$$

obtained by the restriction to the decomposition group D_v of $((r \otimes | \cdot |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma$.

Proof: From the proof of proposition 3.4, we know that $\sigma^{d'}(\pi)$ and $(r \otimes |^{d'/2}) \circ \sigma$ satisfy the *Weight Monodromy Conjecture*, and thus from Brauer's induction theorem, we get that the representations $\sigma^{d'}(\pi) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)$ and $((r \otimes |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma$ satisfy also the *Weight Monodromy Conjecture* and thus for each v their corresponding nilpotent data N'_v and $N'_{\mathbf{K},v}$ are uniquely determined by the semisimple representations $\rho'_{\mathbf{K},v}$ and $((r \otimes |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v)$. Thus it is sufficient to show that

$$\rho'_{\mathbf{K},v} = ((r \otimes |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v).$$

But again we know that for almost all v , (i) $N'_v = 0$ and $N'_{\mathbf{K},v} = 0$, (ii) $\rho'_{\mathbf{K},v}$ and $((r \otimes |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v)$ are unramified and proposition 3.3 combined with Brauer's induction theorem shows that this formula is true. From Cebotarev theorem, we see that the semisimplification $(\sigma^{d'}(\pi) \otimes (\pi_f^{\mathbf{K}} \circ \varphi))^{ss}$ is isomorphic to $((r \otimes |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma$ and thus their restrictions to the decomposition group D_v for $v \nmid l$ are isomorphic and thus they give rise to isomorphic parameters $\rho'_{\mathbf{K},v}$ and $((r \otimes |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v)$ and we conclude proposition 3.5. ■

4 Meromorphic continuation

As an application to theorem 1.1, in this section we prove the second part of theorem 1.2 regarding the meromorphic continuation of zeta functions of the twisted quaternionic Shimura varieties defined in §3.

We remark that the first part of theorem 1.2 remains true if we replace \wp by an ideal \mathfrak{m} of O_F such that if $\mathfrak{q}|\mathfrak{m}$, where \mathfrak{q} is a prime ideal of O_F , then $G(F_{\mathfrak{q}})$ is isomorphic to $GL_2(F_{\mathfrak{q}})$.

In this section, under some conditions, we continue meromorphically the zeta function $L(s, S'_{\mathbf{K}})$ to the entire complex plane and show that it satisfies also a functional equation.

Let $\omega = \pi_f^{\mathbf{K}} \circ \varphi$. We define $L := \bar{\mathbb{Q}}^{\text{Ker}(\varphi)}$ and $K := \bar{\mathbb{Q}}^{\text{Ker}(\omega)}$. From now on, in this paper, we assume that L is a solvable extension of a totally real field. Since $K \subseteq L$, the field K is also a solvable extension of a totally real field.

4.1 Definition of the representation $\rho(\pi)$

One can find a representation $\rho(\pi)$ of Γ_E ([BR] §7.2 and [R]) such that

$$L(s, \rho(\pi)) = L(s - d'/2, \pi, r).$$

We describe now the representation $\rho(\pi)$. Let G be a group and K and H be two subgroups of G . We consider a representation

$$\tau : H \rightarrow GL(W)$$

and a double coset $H\sigma K$ such that $d(\sigma) = |H \setminus H\sigma K| < \infty$. We define a representation $\tau_{H\sigma K}$ of K on the $d(\sigma)$ -fold tensor product $W^{\otimes d(\sigma)}$. Consider

the representatives $\{\sigma_1, \dots, \sigma_{d(\sigma)}\}$ such that $H\sigma K = \cup H\sigma_j$. If $\gamma \in K$, then there exists $\xi_j \in H$ and an index $\gamma(j)$ such that

$$\sigma_j \gamma = \xi_j \sigma_{\gamma(j)}.$$

We define the representation:

$$\tau_{H\sigma K}(\gamma)(\omega_1 \otimes \dots \otimes \omega_{d(\sigma)}) = \tau(\xi_1)\omega_{\gamma^{-1}(1)} \otimes \dots \otimes \tau(\xi_{d(\sigma)})\omega_{\gamma^{-1}(d(\sigma))}.$$

One can prove easily that the equivalence class of $\tau_{H\sigma K}$ is independent of the choice of the representatives $\sigma_1, \dots, \sigma_{d(\sigma)}$.

Let $J_F - J'_F = \{\delta_1, \dots, \delta_{d'}\}$, and $S := \cup \Gamma_F \delta_i$. We write S as a disjoint union of double cosets

$$S = \cup_{j=1}^k \Gamma_F \sigma_j \Gamma_E$$

and we denote by ρ_j the representation of Γ_E defined by $\rho_{\pi, \lambda}$, for $\lambda \mid l$, and the double coset $\Gamma_F \sigma_j \Gamma_E$. Then our representation $\rho(\pi)$ is isomorphic to $\rho_1 \otimes \dots \otimes \rho_k$. Thus

$$L(s - d'/2, \pi, r) = L(s, \rho(\pi)) = L(s, \rho_1 \otimes \dots \otimes \rho_k)$$

and we obtain also that

$$L(s - d'/2, \pi, r \otimes \omega) = L(s, \rho(\pi) \otimes \omega) = L(s, \rho_1 \otimes \dots \otimes \rho_k \otimes \omega).$$

4.2 Base change and Brauer's theorem

Lemma 4.1. *Let ϕ be an l -adic representation of Γ_E . Suppose that there exists a Galois extension F' of \mathbb{Q} , which contains the field $K := \mathbb{Q}^{\ker(\omega)}$ and that the L -function $L(s, \phi|_{\Gamma_{F''}} \otimes \chi)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation for any subfield F'' of F' containing E such that F' is a solvable extension of F'' and any finite order character χ of $\Gamma_{F''}$. Then $L(s, \phi \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation.*

Proof: By Brauer's theorem (see theorems 16 and 19 of [SE]), we can find some subfields $F_i \subset F'$ such that $\text{Gal}(F'/F_i)$ are solvable, some characters $\chi_i : \text{Gal}(F'/F_i) \rightarrow \bar{\mathbb{Q}}^\times$ and some integers m_i , such that the representation

$$\omega : \text{Gal}(F'/E) \rightarrow \text{Gal}(K/E) \rightarrow GL_N(\bar{\mathbb{Q}}_l),$$

can be written as $\omega = \sum_{i=1}^{i=k} m_i \text{Ind}_{\Gamma_{F_i}}^{\Gamma_E} \chi_i$ (a virtual sum). Then

$$\begin{aligned} L(s, \phi \otimes \omega) &= \prod_{i=1}^{i=k} L(s, \phi \otimes \text{Ind}_{\Gamma_{F_i}}^{\Gamma_E} \chi_i)^{m_i} \\ &= \prod_{i=1}^{i=k} L(s, \text{Ind}_{\Gamma_{F_i}}^{\Gamma_E} (\phi|_{\Gamma_{F_i}} \otimes \chi_i))^{m_i} = \prod_{i=1}^{i=k} L(s, \phi|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}. \end{aligned}$$

We know from our assumption that the L -functions $L(s, \phi|_{\Gamma_{F_i}} \otimes \chi_i)$ have a meromorphic continuation to the entire complex plane and verify a functional equation. Thus $L(s, \phi \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation. ■

4.3 Meromorphic continuation of the zeta functions for curves and surfaces

In this section we obtain the second part of theorem 1.2, which is a consequence of theorem 4.3 below.

It is known (theorem M of [RA1]) that:

Proposition 4.2. *If π_1 and π_2 are two cuspidal automorphic representations of $GL(2)/L$, where L is a number field, then $\pi_1 \otimes \pi_2$ is an automorphic (isobaric) representation of $GL(4)/L$.*

We prove now the following result:

Theorem 4.3. *If $K := \bar{\mathbb{Q}}^{Ker(\omega)}$ is a solvable extension of a totally real field and $d' = 1$ or $d' = 2$, then the function $L(s - d'/2, \pi, r \otimes \omega) = L(s, \rho(\pi) \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation.*

Proof: It is sufficient to show that there exists a Galois extension F' of \mathbb{Q} , which contains F and K , such that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of lemma 4.1.

We have two cases:

a) $d' = 1$. We assume for simplicity that $J_F - J'_F = \{1\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. In this case $E = F$ and $\rho(\pi) \cong \rho_{\pi, \lambda}$. From theorem 1.1, we deduce that one can find a field F' which is Galois over \mathbb{Q} , which contains F and K , such that $\rho_{\pi, \lambda}|_{\Gamma_{F'}}$ is modular. From Langlands base change for $GL(2)$ for solvable extensions ([L]), we obtain that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of lemma 4.1.

b) $d' = 2$. We assume for simplicity that $J_F - J'_F = \{1, c\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. We denote by the same symbol c the extension of c to $\bar{\mathbb{Q}}$. Then,

$$S = \Gamma_F \cup \Gamma_F c.$$

The stabilizer of F is Γ_E . It is easy to see that the stabilizer of S is equal to $(\Gamma_F c \cap c^{-1} \Gamma_F) \cup (\Gamma_F \cap c^{-1} \Gamma_F c)$. Thus we get

$$\Gamma_E = (\Gamma_F c \cap c^{-1} \Gamma_F) \cup (\Gamma_F \cap c^{-1} \Gamma_F c).$$

We distinguish two cases:

i) $\Gamma_F c \cap c^{-1} \Gamma_F = \emptyset$. Then, $\Gamma_E = \Gamma_F \cap c^{-1} \Gamma_F c$. Thus,

$$F \subset E \subset F^{gal}$$

where F^{gal} is the Galois closure of F . We have

$$S = \Gamma_F \cup \Gamma_F c = \Gamma_F 1 \Gamma_E \cup \Gamma_F c \Gamma_E.$$

If $\gamma \in \Gamma_E$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1) = \rho_{\pi, \lambda}(\gamma)(\omega_1)$$

and

$$\tau_{\Gamma_F c \Gamma_E}(\gamma)(\omega_1) = \rho_{\pi, \lambda}(c\gamma c^{-1})(\omega_1).$$

Thus

$$\rho(\pi) \cong \rho_{\pi, \lambda}|_{\Gamma_E} \otimes \rho_{\pi, \lambda}|_{\Gamma_E}^c,$$

where

$$\rho_{\pi, \lambda}|_{\Gamma_E}^c(\gamma) = \rho_{\pi, \lambda}(c\gamma c^{-1}).$$

The representation π is one-dimensional or cuspidal and infinite-dimensional. If π is one-dimensional, then $\pi(g) = \rho_{\pi}(N(g))|N(g)|^{1/2}$, where N is the reduced norm map and $|\cdot|$ denotes the ideles norm and ρ_{π} is a Hecke character.

From theorem 1.1, we deduce that one can find a field F' which is Galois over \mathbb{Q} , which contains F and K , such that $\rho_{\pi, \lambda}|_{\Gamma_{F'}}$ is modular. Thus

$$\rho(\pi)|_{\Gamma_{F'}} \cong \rho_{\pi, \lambda}|_{\Gamma_{F'}} \otimes \rho_{\pi, \lambda}|_{\Gamma_{F'}}^c$$

is a tensor product of two automorphic representations and from Langlands base change for $GL(2)$ for solvable extensions ([L]) and proposition 4.2, we obtain that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of lemma 4.1.

ii) $\Gamma_F c \cap c^{-1} \Gamma_F \neq \emptyset$. Let $\Gamma_{E_1} := \Gamma_F \cap c^{-1} \Gamma_F c$. Thus

$$F \subset E_1 \subset F^{gal}.$$

Since it is obvious now that $\Gamma_{E_1} \subset \Gamma_E$, $[\Gamma_E : \Gamma_{E_1}] = 2$ and $\Gamma_E \not\subseteq \Gamma_F$, we get $[E_1 : E] = 2$ and $F \not\subseteq E$. If $F_1 := E \cap F$, then $[F : F_1] = 2$ and we can easily see that c when restricted to F_1 is the trivial embedding. Hence c is the nontrivial automorphism of F over F_1 and we get that $\Gamma_{E_1} = \Gamma_F \cap c^{-1} \Gamma_F c = \Gamma_F$, which means that $E_1 = F$ and $E = F_1$ and therefore we have $[F : E] = 2$ and c is the nontrivial automorphism of F over E .

We have

$$S = \Gamma_F \cup \Gamma_F c = \Gamma_F 1 \Gamma_E.$$

If $\gamma \in \Gamma_F$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1 \otimes \omega_2) = \rho_{\pi, \lambda}(\gamma)\omega_1 \otimes \rho_{\pi, \lambda}(c\gamma c^{-1})\omega_2.$$

If $\gamma \in \Gamma_E - \Gamma_F$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1 \otimes \omega_2) = \rho_{\pi, \lambda}(\gamma c^{-1})\omega_2 \otimes \rho_{\pi, \lambda}(c\gamma)\omega_1.$$

Thus $\rho(\pi)$ is a subrepresentation of

$$\text{Ind}_{\Gamma_F}^{\Gamma_E}(\rho_{\pi, \lambda} \otimes \rho_{\pi, \lambda}^c)$$

which satisfies

$$\rho(\pi)|_{\Gamma_F} \cong \rho_{\pi,\lambda} \otimes \rho_{\pi,\lambda}^c,$$

which is actually automorphic (see theorem [D] of [RA2]).

From theorem 1.1, we deduce that one can find a field F' which is a Galois over \mathbb{Q} , which contains F and K , such that $\rho_{\pi,\lambda}|_{\Gamma_{F'}}$ is modular. We get

$$\rho(\pi)|_{\Gamma_{F'}} \cong \rho_{\pi,\lambda}|_{\Gamma_{F'}} \otimes \rho_{\pi,\lambda}|_{\Gamma_{F'}}^c.$$

Hence from Langlands base change for $GL(2)$ for solvable extensions, proposition 4.2 above, proposition 2.16 of [L] and §6 of [V2] and theorem [D] of [RA2], we deduce that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of lemma 4.1. ■

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