Tate classes and L-functions on a product of a quaternionic Shimura surface and a Picard modular surface

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1 Introduction

Let X be a smooth projective variety defined over a number field F and let

$$\bar{X} = X \times_F \overline{\mathbb{Q}}.$$

For a prime number l, let $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ be the l-adic cohomology of $\bar{X}$. If K is a number field, we denote $\Gamma_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. The Galois group Γ_F acts on $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ by a representation $\rho_{i,l}$. For any $j \in \mathbb{Z}$, let $H^i_{et}(X, \overline{\mathbb{Q}}_l)(j)$ denote the representation of Γ_F on $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ defined by $\rho_{i,l} \otimes \xi_l^j$, where ξ_l is the l-adic cyclotomic character. The elements of $V^i(X, E) := (H^{2i}_{et}(X, \overline{\mathbb{Q}}_l)(i))^{\Gamma_E}$ are called *Tate classes* on X defined over E. The union

$$V^i(X) := \cup_E V^i(X, E)$$

is the space of all *Tate classes* on X.

To each algebraic subvariety Y of X of codimension i defined over a finite extension E of F, one can associate a cohomology class $c(Y) \in (H^{2i}_{et}(X, \overline{\mathbb{Q}}_l)(i))^{\Gamma_E}$ by Poincaré duality. A cohomology class obtained in this way is called *algebraic*. The first part of the Tate's conjecture states that every Tate class is algebraic.

The L-function $L^{2i}(s, X_{/F})$ (more exactly the Euler product) attached to the representation $\rho_{2i,l}$ converges for $\mathrm{Re}(s) > i+1$. The second part of the Tate conjecture [TA] states that the L-function $L^{2i}(s, X_{/E})$ has a meromorphic continuation to the complex plane and has a pole at $s = i+1$ of order equal to

$$\dim_{\overline{\mathbb{Q}}_l} V^i(X, E).$$

In their work [HLR], Harder, Langlands and Rapoport had proved the first part of the Tate conjecture for Hilbert modular surfaces for non-CM submotives. In [K] and [MR] it was proved the first part of the Tate conjecture for

Hilbert modular surfaces for CM sub-motives and thus using the two results, one gets the full first part of the Tate conjecture asserting the algebraicity of all the Tate classes of Hilbert modular surfaces over an arbitrary number field. The first part of the Tate's conjecture for Picard modular surfaces was proved in [BR]. This problem was studied in [MP] and [K1] where it was computed the space of Tate classes on the product of two Hilbert modular surfaces and on the product of two Picard modular surfaces in terms of automorphic representations including the exact determination of their fields of definition, but it was not proved that all these Tate classes are algebraic.

In this paper we consider a totally real field F and a quaternion algebra D over F which is unramified at exactly 2 infinite places of F. Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D and let $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$. Let $S_K := S_{\bar{G},K}$ be the canonical model of the quaternionic Shimura surface associated to an open compact subgroup K of $\bar{G}(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the finite part of the ring of adeles of $\mathbb{Q}$. Then S_K is a quasi-projective surface defined over a totally real finite extension $E/\mathbb{Q}$ called the canonical field of definition.

Let L be a quadratic imaginary extension of $\mathbb{Q}$ and fix a Hermitian inner product on L^3 of signature $(2,1)$. Let GU be the associated quasi-split unitary similitude group over $\mathbb{Q}$. For each open compact subgroup of $\mathbf{K} \subseteq \mathrm{GU}(\mathbb{A}_f)$ let $\mathbf{S_K} := \mathbf{S}_{\mathrm{GU},\mathbf{K}}$ be the associated compactified Picard modular surface (see §5 for details). Then $\mathbf{S_K}$ is defined over L.

As we mentioned above the first part of the Tate conjecture is known for Hilbert modular surfaces [HLR], [K], [MR] and for Picard modular surfaces [BR]. Also the first part of the Tate conjecture is known in the non-CM case for the quaternionic Shimura surfaces treated in [L], corresponding to a quadratic real field F and to a quaternion algebra $D = B \otimes_{\mathbb{Q}} F$, where B is a quaternion algebra over $\mathbb{Q}$, such that D splits at the real places and F splits over the places where B ramifies.

The second part of the Tate conjecture for Hilbert modular surfaces was proved in [HLR], [K] and [MR] for solvable number fields. This result was generalized in [V1] for Tate classes of quaternionic Shimura surfaces defined over an arbitrary solvable extension of a totally real field that contains the canonical field of definition of that variety. Also the second part of the Tate conjecture for the Picard modular modular surfaces $\mathbf{S_K}$ was proved in [BR] for solvable extensions of the field L.

In this article we want to compute the space of Tate classes on a product of a quaternionic Shimura surface and a Picard modular surface in terms of automorphic representations (or more exactly in terms of Galois representations associated to automorphic representations) including the determination of their fields of definition (see Theorems 6.2 and 6.3 for details). The algebraicity of some of these Tate classes is obtained from the fact that they are a product of Tate classes of the two factors, which are known to be algebraic in some cases (see above). More precisely, we compute the Tate classes on the product $S_{K/M} \times \mathbf{S}_{\mathbf{K}/M}$ (see above), where $M := EL$. We also prove in some special cases (see Theorem 7.1 for details), that the function $L^4(s, S_{K/k} \times \mathbf{S}_{\mathbf{K}/k})$ has a pole at $s = 3$ of order equal to

$$\dim_{\overline{\mathbb{Q}}_l} V^2(S_{K/k} \times \mathbf{S}_{\mathbf{K}/k}, k),$$

where k is a finite extension of M (the same result could be obtained for the L^i-part of the L-function, where $i \neq 4$, but the proof is easier (for details see the beginning of §6)) .

2 Quaternionic Shimura surfaces

Let F be a totally real field of degree d over $\mathbb{Q}$ and $O := O_F$ be its the ring of integers. Let $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \mathbb{A}_f$ be the adeles ring of $\mathbb{Q}$ and $\mathbb{A}_F$ the adeles ring of F. We denote by $I_\mathbb{Q}$ and I_F the ideles groups of $\mathbb{Q}$ and F, respectively.

We consider a quaternion algebra D over F which is unramified at exactly 2 infinite places of F. We denote by S_∞ the set of infinite places of F and we identify S_∞ as a $\Gamma_\mathbb{Q}$-set with $\Gamma_F \setminus \Gamma_\mathbb{Q}$. Let S'_∞ be the subset of S_∞ at which D is ramified. Thus the cardinal of $S_\infty - S'_\infty$ is equal to 2.

Let G be the algebraic group over F defined by the multiplicative group $D^\times$. By restricting the scalars, we obtain the algebraic group $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$ over $\mathbb{Q}$ defined by the property: $\bar{G}(A) = G(A \otimes_\mathbb{Q} F)$ for all $\mathbb{Q}$-algebras A. It is easy to see that $\bar{G}(\mathbb{R})$ is isomorphic to $GL_2(\mathbb{R})^2 \times \mathbf{H}^{*(d-2)}$, where $\mathbf{H}$ is the algebra of quaternions over $\mathbb{R}$.

For $v \in S_\infty - S'_\infty$, we fix an isomorphism of $G(F_v)$ with $GL_2(\mathbb{R})$. We have $\bar{G}(\mathbb{R}) = \prod_{v \in S_\infty} G(F_v)$. Let $J = (J_v) \in \bar{G}(\mathbb{R})$, where

$$J_v = \begin{cases} 1 & \text{for } v \in S'_\infty; \\ 1/\sqrt{2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) & \text{for } v \in S_\infty - S'_\infty. \end{cases}$$

Let K_∞ be the centralizer of J in $\bar{G}(\mathbb{R})$. Set

$$X = \bar{G}(\mathbb{R})/K_\infty.$$

It is well known that X is complex analytically isomorphic to $(\mathfrak{H}_\pm)^2$ where $\mathfrak{H}_\pm = \mathbb{C} - \mathbb{R}$. For each open compact subgroup $K \subseteq \bar{G}(\mathbb{A}_f)$ set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/K.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of complex points of a quasi projective variety. The canonical field of definition of S_K is by definition the subfield E of $\overline{\mathbb{Q}}$ such that Γ_E is the stabilizer of $S'_\infty \subseteq \Gamma_F \setminus \Gamma_\mathbb{Q}$. It is known (see [D]) that S_K has a canonical model over E which is denoted by S_K. The dimension of S_K is equal to 2.

3 Cohomology for quaternionic Shimura surfaces

From now on, if π is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote the automorphic representation of $GL_2(\mathbb{A}_F)$, obtained from π by Jacquet-Langlands correspondence (usually denoted $JL(\pi)$) by the same symbol π.

If l is a prime number, we fix an isomorphism $i : \overline{\mathbb{Q}}_l \to \mathbb{C}$, and from now on we identify these two fields. If π is an cuspidal automorphic representation of weight 2 of $GL(2)/F$, then there exists ([T]) a λ-adic representation for $\lambda \nmid \mathbf{n}$ ($\mathbf{n}$ is the level of π)

$$\rho_{\pi,\lambda} : \Gamma_F \to GL_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L_v(s,\pi) = L_v(s,\rho_{\pi,\lambda})$ for almost all finite places v of F and is unramified outside the primes dividing $\mathbf{n}l$. Here λ (with $\lambda|l$) is a prime ideal of the ring of coefficients O of π and if $\rho_{\pi,\lambda}$ is unramified at v, then

$$L_v(s,\rho_{\pi,\lambda}) = \det(1 - i(\rho_{\pi,\lambda}(\mathrm{Frob}_v))Nv^{-s})^{-1},$$

where Frob_v is a geometric Frobenius. In order to simplify the notations we denote by ρ_π the representation $\rho_{\pi,\lambda}$.

We assume that $K = \Pi_{v<\infty} K_v$ where K_v is open compact in $G(F_v)$ and $K_v = GL_2(O_v)$ for almost all v, where O_v is the ring of integers of F_v. Let $\mathbb{H}_K$ be the Hecke algebra of complex valued of bi-K-invariant compactly supported functions on $\bar{G}(\mathbb{A}_f)$. If $\pi = \pi_\infty \otimes \pi_f$ is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote by π_f^K the space of K invariants in π_f. The Hecke algebra $\mathbb{H}_K$ acts on π_f^K.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group Γ_E on the étale cohomology $H^2_{et}(S_K, \overline{\mathbb{Q}}_l)$ and these two actions commute (we remark that when $D = M_2(F)$, the Shimura variety S_K is not compact and in this case, one should replace the étale cohomology by the intersection cohomology of the Baily-Borel compactification of S_K). We say that the representation π is *cohomological* if $H^2(\mathfrak{g}, K_\infty, \pi_\infty) \neq 0$, where $\mathfrak{g}$ is the Lie algebra of K_∞ (the cohomology is taken with respect to $(\mathfrak{g}, K_\infty)$-module associated to π_∞). Then we know (see for example [RT], Proposition 1.8):

Proposition 3.1. *The representation of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology $H^2_{et}(S_K, \overline{\mathbb{Q}}_l)(1)$ is isomorphic to*

$$\oplus_{\pi_f} \rho(\pi_f) \otimes \pi_f^K,$$

where $\rho(\pi_f)$ is a representation of the Galois group Γ_E. The above sum is over cohomological automorphic representations π of $\bar{G}(\mathbb{A}_\mathbb{Q})$ and the $\mathbb{H}_K$-representations π_f^K are irreducible and mutually inequivalent, i.e. the decomposition is isotypic with respect to the action of $\mathbb{H}_K$.

The irreducible unitary automorphic representations that appear in Proposition 3.1 are one-dimensional or cuspidal of weight 2 and infinite-dimensional. If π is one-dimensional then $\rho(\pi_f)$ has dimension two and if π is infinite-dimensional, then $\rho(\pi_f)$ has dimension four. Let $H(\pi_f)(1) = V(\pi_f) \otimes \pi_f^K$ be the space corresponding to $\rho(\pi_f) \otimes \pi_f^K$ in the above decomposition.

We assume for simplicity that $S_\infty - S_\infty^{'} = \{1, \tau\}$, where 1 is the trivial embedding of F in $\overline{\mathbb{Q}}$. We denote by the same symbol τ an extension of τ to $\overline{\mathbb{Q}}$. Consider

$$S = \Gamma_F \cup \Gamma_F \tau.$$

The stabilizer of S is Γ_E. It is easy to check that the stabilizer of S is equal to $(\Gamma_F\tau \cap \tau^{-1}\Gamma_F) \cup (\Gamma_F \cap \tau^{-1}\Gamma_F\tau)$. Thus we get

$$\Gamma_E = (\Gamma_F\tau \cap \tau^{-1}\Gamma_F) \cup (\Gamma_F \cap \tau^{-1}\Gamma_F\tau).$$

Now we describe the representation $\rho(\pi_f)$ which is semisimple (the proof of the semisimplicity of $\rho(\pi_f)$ is the same as in the case of Hilbert modular surfaces, see [HLR], §4 or [G], Corollary 3.8).

We distinguish two cases:

i) $\Gamma_F\tau \cap \tau^{-1}\Gamma_F = \emptyset$. Then, $\Gamma_E = \Gamma_F \cap \tau^{-1}\Gamma_F\tau$. Thus,

$$F \subset E \subset F^{gal}$$

where F^{gal} is the Galois closure of F.

If π is an infinite-dimensional cuspidal automorphic representation, we denote for simplicity $\rho_\pi := \rho_{\pi,\lambda}$. Then we have (see for example [V] 4.3):

$$\rho(\pi_f) \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E}^{\tau},$$

where

$$\rho_\pi|_{\Gamma_E}^{\tau}(\gamma) = \rho_\pi(\tau\gamma\tau^{-1}).$$

If π is one-dimensional, then $\pi(g) = \rho_\pi(N(g))$, where N is the reduced norm map and ρ_π is a Hecke character. We denote also by ρ_π the λ-adic representation associated to ρ_π. Then

$$\rho(\pi_f) \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E}^{\tau}.$$

ii) $\Gamma_F\tau \cap \tau^{-1}\Gamma_F \neq \emptyset$. Let $\Gamma_{E_1} := \Gamma_F \cap \tau^{-1}\Gamma_F\tau$. Thus

$$F \subset E_1 \subset F^{gal}.$$

Since it is obvious now that $\Gamma_{E_1} \subset \Gamma_E$, $[\Gamma_E : \Gamma_{E_1}] = 2$ and $\Gamma_E \not\subseteq \Gamma_F$, we get $[E_1 : E] = 2$ and $F \not\subseteq E$. If $F_1 := E \cap F$, then $[F : F_1] = 2$ and we can easily see that τ when restricted to F_1 is the trivial embedding. Hence τ is the nontrivial automorphism of F over F_1 and we get that $\Gamma_{E_1} = \Gamma_F \cap \tau^{-1}\Gamma_F\tau = \Gamma_F$, which means that $E_1 = F$ and $E = F_1$ and therefore we have $[F : E] = 2$ and τ is the nontrivial automorphism of F over E.

If π is infinite-dimensional cuspidal automorphic, then we know that (see for example [V] 4.3) $\rho(\pi_f)$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_E}(\rho_\pi \otimes \rho_\pi^{\tau}),$$

which verifies

$$\rho(\pi_f)|_{\Gamma_F} = \rho_\pi \otimes \rho_\pi^{\tau}.$$

If π is one-dimensional, then $\pi(g) = \rho_\pi(N(g))$ and we have (see for example [G], Proposition 2.7)

$$\rho(\pi_f) \cong \rho_\pi|_{I_E} \oplus \rho_\pi|_{I_E} \cdot \omega_{F/E},$$

where $\omega_{F/E}$ is the quadratic character corresponding to F/E.

4 Known results

It is known that (see for example [HLR] Proposition 4.5.4):

Proposition 4.1. *If π is a cuspidal automorphic representation of weight 2 of $GL(2)/F$, where F is a totally real field, then one of the following two statements holds:*

(i) $\rho_\pi|_{\Gamma_L}$ is irreducible for each finite extension L/F.

(ii) There exists a quadratic extension L/F and an algebraic Hecke character ψ of L such that $\rho_\pi \cong \mathrm{Ind}_L^F(\psi)$.

We say that a representation ρ of a group G is *dihedral* if there exists a normal subgroup N of index 2 in G and a character $\chi : N \to \mathbb{C}^\times$ such that $\rho = \mathrm{Ind}_N^G \chi$.

We say that an automorphic representation π of $GL(2)/L$ for some number field L is of CM *type* if there exists some quadratic Galois character $\eta : I_L/L^\times \to \overline{\mathbb{Q}}_l^\times$, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. If π is an automorphic representation of weight 2 of $GL(2)/L$, then π is of CM type if and only if ρ_π is a dihedral representation.

We know the following result (Theorem 2.1 of [MP]):

Proposition 4.2. *The tensor product of two 2 dimensional irreducible complex representations of a group is reducible only if either both representations are dihedral or they are the twist of each other by a character.*

We know (Proposition 4.1 of [MP]):

Proposition 4.3. *Suppose that π is a cuspidal, non-*CM *automorphic representation of $GL(2)/K$ for some finite extension $K/\mathbb{Q}$. Suppose that K is a quadratic extension of k and τ is the automorphism of K over k. If $\pi^\tau \cong \pi \otimes \chi$ for a Hecke character χ of K, then χ is trivial when restricted to the ideles of k.*

We know (Corollary 2.6 of [MP]):

Proposition 4.4. *Let ρ be a 2-dimensional irreducible representation of a group G. Then $Sym^2(\rho)$ is reducible if and only if ρ is dihedral.*

It is known ([RA], Theorem M) that:

Proposition 4.5. *If π_1 and π_2 are two cuspidal unitary automorphic representations of $GL(2)/L$, where L is a number field, then $\pi_1 \times \pi_2$ is an automorphic (isobaric) representation of $GL(4)/L$.*

We know ([JPSS]):

Proposition 4.6. *If π_1 and π_2 are two cuspidal unitary automorphic representations of $GL(n)/L$ and $GL(m)/L$, where L is a number field, then the function $L(s,\pi_1\times\pi_2)$ verifies a functional equation and is meromorphic with possible poles only at $s=0$ and 1. The function $L(s,\pi_1\times\pi_2)$ is holomorphic iff $\pi_1 \ncong \pi_2^*$ and if $\pi_1\cong\pi_2^*$, then it has a pole of order 1 at $s=1$.*

We know (Theorem 1.1 of [V]):

Proposition 4.7. *If F is a totally real field, π is a cuspidal automorphic representation of weight 2 of $GL(2)/F$ and F_1 is a solvable extension of a totally real field containing F, then there exists a Galois extension F_2 of $\mathbb{Q}$ containing F_1, such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$ is modular i.e. there exists a cuspidal automorphic representation π_1 of $GL(2)/F_2$ and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}\cong\rho_{\pi_1,\beta}$.*

5 Picard modular surfaces

Consider a quadratic imaginary extension L of $\mathbb{Q}$ and a Hermitian inner product on L^3 of signature $(2,1)$. Let GU be the associated quasi-split unitary similitude group over $\mathbb{Q}$. Then $\mathrm{GU}_\infty=\mathrm{GU}(\mathbb{R})$ is isomorphic to the real Lie group $\mathrm{GU}(2,1)$. Set

$$B=\mathrm{GU}(\mathbb{R})/K_\infty Z_\infty,$$

where K_∞ is the maximal compact subgroup of $\mathrm{GU}(\mathbb{R})$ and Z is the center of GU. Then B is complex analytically isomorphic to the unit ball in $\mathbb{C}^2$. For sufficiently small open compact subgroup of $\mathbf{K}\subseteq\mathrm{GU}(\mathbb{A}_f)$ let $\mathbf{S_K}:=\mathbf{S}_{\mathrm{GU},\mathbf{K}}$ be the associated compactified Shimura variety. Then $\mathbf{S_K}$ is defined over L, has dimension 2 and $\mathbf{S_K}(\mathbb{C})$ is the compactification of

$$\mathrm{GU}(\mathbb{Q})\setminus B\times\mathrm{GU}(\mathbb{A}_f)/\mathbf{K}$$

which is a disjoint union of arithmetic quotients of B.

We have an action of the Hecke algebra $\mathbb{H}_\mathbf{K}$ and an action of the Galois group Γ_L on the étale cohomology $H^2_{et}(\mathbf{S_K},\overline{\mathbb{Q}}_l)$ and these two actions commute. We say that the representation Π of $\mathrm{GU}(\mathbb{A}_\mathbb{Q})$ is *cohomological* if $H^2(\mathrm{Lie}(\mathrm{GU}_\infty),K'_\infty,\Pi_\infty)\neq 0$, where K'_∞ is the centralizer of the center of K_∞ in GU_∞.

Proposition 5.1. *The representation of $\Gamma_L\times\mathbb{H}_\mathbf{K}$ on the étale cohomology $H^2_{et}(\mathbf{S_K},\overline{\mathbb{Q}}_l)(1)$ is isomorphic to*

$$\oplus_{\Pi_f}\phi(\Pi_f)\otimes\Pi_f^{\mathbf{K}},$$

where $\phi(\Pi_f)$ is a representation of the Galois group Γ_L. The above sum is over Π_f such that Π is a cohomological automorphic representation of $\mathrm{GU}(\mathbb{A}_\mathbb{Q})$ that *occur in the discrete spectrum of* $\mathrm{GU}(\mathbb{A}_{\mathbb{Q}})$ *and the* $\mathbb{H}_{\mathbf{K}}$*-representations* $\Pi_f^{\mathbf{K}}$ *are irreducible and mutually inequivalent, i.e. the decomposition is isotypic with respect to the action of* $\mathbb{H}_{\mathbf{K}}$.

The irreducible automorphic representations Π that appear in Proposition 4.1 are one-dimensional or cuspidal and infinite-dimensional and $\phi(\Pi_f)$ has dimension $d(\Pi_f) \leqslant 3$. The representation Π is cohomological if and only if $\Pi_\infty \in \{triv, D^+, D, D^-\}$, where $triv$ is the trivial representation and D^+, D, D^- are the lowest holomorphic, non-holomorphic and anti-holomorphic discrete series representations of GU_∞ with trivial central character.

Fix an embedding $i : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Then $\phi(\Pi_f)$ is unramified at almost all places v of L and the local L-factor at at such unramified place v is defined by

$$L_v(s, \phi(\Pi_f)) = \det(1 - i(\phi(\Pi_f)(\mathrm{Frob}_v))Nv^{-s})^{-1},$$

where Frob_v is a geometric Frobenius.

There exists an automorphic representation σ_Π of $GL_{d(\Pi_f)}(\mathbb{A}_L)$ such that for almost all finite places v of L we have

$$L_v(s, \phi(\Pi_f)) = L_v(s, \sigma_\Pi).$$

We say that Π is AI if σ_Π is automorphically induced from a Hecke character of some field L_1 of degree $d(\Pi_f)$ over L.

We know (see [BR], Theorem 2.2.1):

Proposition 5.2. *If* Π *and* $\phi(\Pi_f)$ *are as above, then one of the following two statements holds:*

(i) $\phi(\Pi_f)|_{\Gamma_{L_1}}$ *is irreducible for each finite extension* L_1/L.

(ii) There exists an extension L_1/L *of degree* $d(\Pi_f)$ *and an algebraic Hecke character* Ψ *of* L_1 *such that* $\phi(\Pi_f) \cong \mathrm{Ind}_{L_1}^{L}(\Psi)$. *If* $d(\Pi_f) \geqslant 2$, *then the infinity type of* Ψ *is not trivial.*

The second case occurs iff Π is AI.

Let $H(\Pi_f)(1) = V(\pi_f) \otimes \Pi_f^{\mathbf{K}}$ denote the space corresponding to $\phi(\Pi_f) \otimes \Pi_f^{\mathbf{K}}$ in the decomposition of Proposition 5.1. We know (see [BR], Proposition 3.2.1):

Proposition 5.3. *If* $H^T(\Pi_f)$ *denotes the space of Tate classes in* $H(\Pi_f)(1)$, *then*

$$H^T(\Pi_f) = \begin{cases} H(\Pi_f)(1) & \text{if } d(\Pi_f) = 1 \text{ and } \Pi_\infty = D \text{ or } triv, \\ \{0\} & \text{otherwise.} \end{cases}$$

All these Tate classes are defined over abelian extensions of L.

We remark that in the case of quaternionic Shimura surfaces the space of Tate classes $H^T(\pi_f)$ is defined over abelian extensions of E if π is cuspidal non-CM or one-dimensional and not necessarily over abelian extensions of E if π is cuspidal CM. This could be seen easily from §6 below.

6 Tate classes

Let $M := EL$ and $S_1 := S_{K/M}$ be a quaternionic Shimura surface over M associated to some sufficiently small open compact subgroup K of $\bar{G}(\mathbb{A}_f)$ and $S_2 := \mathbf{S}_{\mathbf{K}/M}$ be a Picard modular surface over M associated to some sufficiently small open compact subgroup $\mathbf{K}$ of $\mathrm{GU}(\mathbb{A}_f)$. By the Künneth formula we have

$$H^4_{et}(S_1 \times S_2, \overline{\mathbb{Q}}_l)(2) = \oplus_{i+j=4} H^i_{et}(S_1, \overline{\mathbb{Q}}_l) \otimes H^j_{et}(S_2, \overline{\mathbb{Q}}_l)(2).$$

The essential part of this decomposition is

$$H^2_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes H^2_{et}(S_2, \overline{\mathbb{Q}}_l)(1),$$

because $H^1_{et}(S_1, \overline{\mathbb{Q}}_l) = \{0\}$ and $H^3_{et}(S_1, \overline{\mathbb{Q}}_l) = \{0\}$ and on H^0_{et} and H^4_{et} the Galois representations are abelian. More exactly we have $H^{2i}_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \cong \oplus_{\pi_f} \rho^i(\pi_f) \otimes \pi_f^K$, for $i = 0$ or 2, where the Galois representation $\rho^i(\pi_f)$ is trivial if π is infinite-dimensional, and is one-dimensional and equal to $\rho_\pi | \ |^{-i}$, where $| \ |$ is the ideles norm, if $\pi(g) = \rho_\pi(N(g))$ and we have a similar decomposition for $H^{2i}_{et}(S_2, \overline{\mathbb{Q}}_l)(1)$, where $i = 0$ or 2. Thus the computation of the Tate classes is trivial for the remaining parts i.e. $H^0_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes H^4_{et}(S_2, \overline{\mathbb{Q}}_l)(1)$, and $H^4_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes H^0_{et}(S_2, \overline{\mathbb{Q}}_l)(1)$. For the same reason the computations of the Tate classes are trivial for $H^{2i}_{et}(S_1 \times S_2, \overline{\mathbb{Q}}_l)(i)$, where $i \neq 2$.

From the Propositions 3.1 and 5.1 we obtain

$$\begin{aligned} & H^2_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes H^2_{et}(S_2, \overline{\mathbb{Q}}_l)(1) \\ &= (\oplus_{\pi_f} V(\pi_f) \otimes \pi_f^K) \otimes (\oplus_{\Pi_f} V(\pi_f) \otimes \Pi_f^{\mathbf{K}}) \\ &= \oplus_{\pi_f, \Pi_f} (V(\pi_f) \otimes V(\Pi_f)) \otimes (\pi_f^K \otimes \Pi_f^{\mathbf{K}}), \end{aligned}$$

where π and Π run over a finite set of automorphic representations of $\bar{G}(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GU}(\mathbb{A}_\mathbb{Q})$ respectively. The group Γ_M acts on each summand above by $\rho(\pi_f) \otimes \phi(\Pi_f) \otimes 1$.

For an extension k of M we must compute the Γ_k-invariant subspace of $V(\pi_f) \otimes V(\Pi_f)$ which is isomorphic to

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f)^*)$$

which is isomorphic (see the computation of $\rho(\pi_f)$ from §3) to

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f^*)).$$

For notational convenience we replace Π^* by Π and thus we have to determine

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f)).$$

We know (see [K1], Lemma 2):

Proposition 6.1. *Let σ and τ two n-dimensional representations of a group G over $\overline{\mathbb{Q}}_l$ and assume that H is an open normal subgroup of G and $\tau|_H$ is irreducible. Then $\sigma|_H \cong \tau|_H$ iff $\sigma \cong \tau \otimes \varphi$ for some $\varphi : G \to \overline{\mathbb{Q}}_l^*$, which is trivial on H.*

6.1 Non-AI case

Let k/M be a finite extension. In this section we assume that the representation Π is non-AI. Then from Proposition 5.2, we know that $\phi(\Pi_f)|_{\Gamma_k}$ is irreducible of dimension 2 or 3. By Shur's lemma the dimension of the $\overline{\mathbb{Q}}_l$-space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ is equal to the multiplicity of $\phi(\Pi_f)|_{\Gamma_k}$ in $\rho(\pi_f)|_{\Gamma_k}$. But for π not one-dimensional, $\rho(\pi_f)|_{\Gamma_k}$ has dimension 4. Thus if π is not one-dimensional and the multiplicity is not 0, then we get that $\rho(\pi_f)|_{\Gamma_k}$ is reducible.

We consider three cases:

A) The representation π is cuspidal non-CM.
In the case i) of §3, we know that

$$\rho(\pi_f)|_{\Gamma_k} \cong \rho_\pi|_{\Gamma_k} \otimes \rho_\pi|^{\tau}_{\Gamma_k}.$$

Since π is non-CM, from Proposition 4.1 we deduce that $\rho_\pi|_{\Gamma_k}$ is irreducible and non-dihedral. Assume that $\rho(\pi_f)|_{\Gamma_k}$ is reducible. Applying Propositions 4.2 and 6.1, we get that $\rho^{\tau}_\pi|_{\Gamma_E} \cong \rho_\pi|_{\Gamma_E} \otimes \chi$ for some Hecke character χ of E. Therefore:

$$\rho(\pi_f)|_{\Gamma_k} \cong \rho_\pi|_{\Gamma_k} \otimes \rho_\pi|_{\Gamma_k} \otimes \chi \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_k}) \cdot \chi \oplus \wedge^2(\rho_\pi|_{\Gamma_k}) \cdot \chi.$$

Since $\rho_\pi|_{\Gamma_k}$ is irreducible and non-dihedral, from Proposition 4.5, we know that $\mathrm{Sym}^2(\rho_\pi|_{\Gamma_k})$ is irreducible. We obtain that the dimension of the space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ could be 0 or 1, and it is equal to 1 exactly when $\phi(\Pi_f)|_{\Gamma_k} \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_k}) \cdot \chi$. If $\phi(\Pi_f)|_{\Gamma_k} \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_k}) \cdot \chi$ and k is a Galois over M (or one can replace k by $\tilde{k}$, the Galois closure of k over M in this argument), then from Proposition 6.1, we deduce that $d(\Pi_f) = 3$ and $\phi(\Pi_f)|_{\Gamma_M} \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_M}) \cdot \varphi$ for some finite order character φ of Γ_M which satisfies $\varphi|_{\Gamma_k} = \chi|_{\Gamma_k}$. In this case the Tate classes obtained are defined, by class field theory, over the finite abelian extension of M defined by $\varphi\chi^{-1}$ (i.e over $\overline{\mathbb{Q}}^{\ker(\varphi\chi^{-1})}$).

In the case ii) of §3, we know that $[F : E] = 2$, τ is the nontrivial automorphism of F over E and $\rho(\pi_f)$ is a subrepresentation of

$$\mathrm{Ind}^{\Gamma_E}_{\Gamma_F}(\rho_\pi \otimes \rho^{\tau}_\pi),$$

which verifies

$$\rho(\pi_f)|_{\Gamma_F} \cong \rho_\pi \otimes \rho^{\tau}_\pi.$$

Since π is non-CM, from Proposition 4.1, we deduce that $\rho_\pi|_{\Gamma_{Fk}}$ is irreducible and non-dihedral. Assume that $\rho(\pi_f)|_{\Gamma_k}$ is reducible. Thus, in particular $\rho(\pi_f)|_{\Gamma_{Fk}} \cong \rho_\pi|_{\Gamma_{Fk}} \otimes \rho_\pi|^{\tau}_{\Gamma_{Fk}}$ is reducible. Applying Propositions 4.1 and 4.2, we get that $\rho^{\tau}_\pi \cong \rho_\pi \otimes \alpha$ for some Hecke character α of F. Hence, from Proposition 4.3, we know that α is a Hecke character of I_F which is trivial on

I_E. Therefore α can be written as $\alpha = \chi^\tau/\chi$ for some Hecke character χ of I_F. Hence

$$(\pi \otimes \chi^{-1})^\tau \cong \pi \otimes \chi^{-1}.$$

So $\pi \cong \pi_{0/F} \otimes \chi$, where $\pi_{0/F}$ is the base change to F of some automorphic representation π_0 of $GL(2)/E$.

Then from the properties of $\rho(\pi_f)$ (see for example [MP]), we have:

$$\rho(\pi_f) \cong (\mathrm{Sym}^2\rho_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/E}) \otimes \chi|_{I_E},$$

where ω_{π_0} is the central character of π_0 and $\omega_{F/E}$ is the quadratic character that corresponds to F/E.

Thus we get

$$\rho(\pi_f)|_{\Gamma_k} \cong (\mathrm{Sym}^2\rho_{\pi_0}|_{\Gamma_k} \otimes \chi|_{I_E}|_{\Gamma_k}) \oplus (\omega_{\pi_0}|_{\Gamma_k} \cdot \omega_{Fk/k} \cdot \chi|_{I_E}|_{\Gamma_k}).$$

Since π is non-CM, the representation π_0 is non-CM, from Proposition 4.1, we know that the representation $\rho_{\pi_0}|_{\Gamma_k}$ is irreducible and non-dihedral and from Proposition 4.4, we deduce that $\mathrm{Sym}^2(\rho_{\pi_0}|_{\Gamma_k})$ is irreducible. We obtain that the dimension of $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ could be 0 or 1, and it is equal to 1 precisely when $\phi(\Pi_f)|_{\Gamma_k} \cong \mathrm{Sym}^2(\rho_{\pi_0}|_{\Gamma_k}) \cdot \chi|_{I_E}|_{\Gamma_k}$. If $\phi(\Pi_f)|_{\Gamma_k} \cong \mathrm{Sym}^2(\rho_{\pi_0}|_{\Gamma_k}) \cdot \chi|_{I_E}|_{\Gamma_k}$ and k is Galois over M, then from Proposition 6.1, we deduce that $\phi(\Pi_f)|_{\Gamma_M} \cong \mathrm{Sym}^2(\rho_{\pi_0}|_{\Gamma_M}) \cdot \varphi$, for some character φ of Γ_M which verifies $\varphi|_{\Gamma_k} = \chi|_{I_E}|_{\Gamma_k}$. In this case the Tate classes obtained are defined, by class field theory, over the finite abelian extension of M defined by $\varphi \cdot \chi|_{I_E}|_{\Gamma_M}^{-1}$.

B) The representation π is cuspidal CM. Thus there exists a Hecke character χ of some quadratic CM-extension N of F such that $\rho_\pi = \mathrm{Ind}_{\Gamma_N}^{\Gamma_F}\chi$. Then from the proprieties of $\rho(\pi_f)$ described in §3, we deduce that

$$\rho(\pi_f)|_{\Gamma_{kNN^\tau}} \cong (\chi \oplus \bar{\chi}) \otimes (\chi^\tau \oplus \bar{\chi}^\tau)$$

$$\cong \chi\chi^\tau \oplus \chi\bar{\chi}^\tau \oplus \bar{\chi}\chi^\tau \oplus \bar{\chi}\bar{\chi}^\tau,$$

where $\bar{\chi}$ is the complex conjugate of χ. But Π is non-AI, and from Proposition 5.2, we know that $\phi(\Pi_f)|_{\Gamma_{kNN^\tau}}$ is irreducible of dimension 2 or 3. By Shur's lemma we obtain that $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_{kNN^\tau}]}(V(\pi_f), V(\Pi_f))$ has dimension 0 and thus in particular our space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ has dimension 0.

C) The representation π is one-dimensional. Then from §3, we know that $\rho(\pi_f)$ is a direct sum of one-dimensional representations and as in case B), we obtain that our space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ has dimension 0.

From A), B) and C) we get the following result:

Theorem 6.2. *Assume that Π is cuspidal non-AI. Then:*

1) If π is one-dimensional or cuspidal CM, *then $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ contains no Tate classes.*

*2) Assume that π is cuspidal non-*CM *and that we are in the case i) of §3, then the space $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ contains a Tate class if and only if $\rho_\pi^\tau|_{\Gamma_E} \cong \rho_\pi|_{\Gamma_E} \otimes \chi$ for some finite order character χ of Γ_E and $\phi(\Pi_f^*)|_{\Gamma_M} \cong Sym^2(\rho_\pi|_{\Gamma_M}) \cdot \varphi$, for some finite order character φ of Γ_M. In this case, the subspace of Tate classes has the same dimension as $\pi_f^K \otimes \Pi_f^{\mathbf{K}}$, and all such Tate classes are defined over the abelian extension of M defined by $\varphi \cdot \chi^{-1}$.*

*3) Assume that π is cuspidal non-*CM *and that we are in the case ii) of §3, then the space $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ contains a Tate class if and only if $\pi \cong \pi_{0/F} \otimes \chi$, where $\pi_{0/F}$ is the base change to F of some automorphic representation π_0 of $GL(2)/E$ and χ is a finite order Hecke character of F and $\phi(\Pi_f^*)|_{\Gamma_M} \cong Sym^2(\rho_{\pi_0}|_{\Gamma_M}) \cdot \varphi$, for some finite order character φ of Γ_M. In this case, the subspace of Tate classes has the same dimension as $\pi_f^K \otimes \Pi_f^{\mathbf{K}}$, and all such Tate classes are defined over the abelian extension of M defined by $\varphi \cdot \chi|_{I_E}|_{\Gamma_M}^{-1}$.*

All these Tate classes of $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ do not come as a product of Tate classes from individual factors.

6.2 AI case

In this section we assume that the representation Π is AI. Then $\phi(\Pi_f) = \mathrm{Ind}_{\Gamma_{L_1}}^{\Gamma_L} \eta$ for some extension L_1 of L, with $[L_1 : L] \leqslant 3$ and some Hecke character η of Γ_{L_1}. As at the beginning of section 6.1, we deduce that if the space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ has dimension bigger than 0, and π is not one dimensional, then $\rho(\pi_f)|_{\Gamma_k}$ is reducible.

Again, we consider three cases:

A) The representation π is cuspidal non-CM.

With the same notations as in section 6.1, in the case i), we get

$$\rho(\pi_f)|_{\Gamma_k} \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_k}) \cdot \chi \oplus \wedge^2(\rho_\pi|_{\Gamma_k}) \cdot \chi.$$

The representation $\mathrm{Sym}^2(\rho_\pi|_{\Gamma_k})$ is irreducible. If k contains the Galois closure $\tilde{L}_1$ of L_1 over L, then we have

$$\phi(\Pi_f)|_{\Gamma_k} = \begin{cases} \eta \oplus \eta^\epsilon & \text{if } [L_1 : L] = 2, \\ \eta \oplus \eta^\epsilon \oplus \eta^{\epsilon^2} & \text{if } [L_1 : L] = 3. \end{cases}$$

where $\eta^\epsilon(\sigma) = \eta(\epsilon^{-1}\sigma\epsilon)$ and $\epsilon \in \Gamma_L - \Gamma_{L_1}$. We cannot have $\mathrm{Sym}^2(\rho_\pi|_{\Gamma_k}) \cong \phi(\Pi_f)|_{\Gamma_k}$ because for an extension k' of k that contains $\tilde{L}_1$, the representation $\mathrm{Sym}^2(\rho_\pi|_{\Gamma_{k'}})$ is irreducible and the representation $\phi(\Pi_f)|_{\Gamma_{k'}}$ is one-dimensional or is reducible. Thus if $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ has dimension bigger than 0, then we must have $\wedge^2(\rho_\pi|_{\Gamma_{k\tilde{L}_1}}) \cdot \chi \cong \eta$ or η^ϵ or η^{ϵ^2}. If $d(\Pi_f) \geqslant 2$, this is impossible because in this case the infinity type of $\wedge^2(\rho_\pi|_{\Gamma_{k\tilde{L}_1}}) \cdot \chi$ is trivial, while from Proposition 5.2, we know that the infinity type of η or η^ϵ or η^{ϵ^2} is not trivial.

We obtain that if $d(\Pi_f) \geqslant 2$, then the dimension of $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ is 0. If $d(\Pi_f) = 1$, then η has trivial infinity type and thus η and $\wedge^2(\rho_\pi)\cdot\chi$ become isomorphic, by class field theory, after restriction to the absolute Galois group of the abelian extension of M defined by $\eta^{-1}\cdot\wedge^2(\rho_\pi)\cdot\chi$, and hence contribute to Tate classes defined over that field.

Keeping the same notations as in the section 6.1, in the case ii), we get

$$\rho(\pi_f)|_{\Gamma_k} \cong (\mathrm{Sym}^2\rho_{\pi_0}|_{\Gamma_k} \otimes \chi|_{I_E}|_{\Gamma_k}) \oplus (\omega_{\pi_0}|_{\Gamma_k} \cdot \omega_{Fk/k} \cdot \chi|_{I_E}|_{\Gamma_k}).$$

Using the same argument as in case i) above, we deduce that if the space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ has dimension bigger than 0, then we must have $\omega_{\pi_0}\cdot\omega_{F/E}\cdot\chi|_{I_E}|_{\Gamma_{k\tilde{L}_1}} \cong \eta$ or η^ϵ or η^{ϵ^2}. But this is again impossible if $d(\Pi_f) \geqslant 2$, because in this case the infinity type of $\omega_{\pi_0}\cdot\omega_{F/E}\cdot\chi|_{I_E}|_{\Gamma_{k\tilde{L}_1}}$ is trivial, while the infinity type of η or η^ϵ or η^{ϵ^2} is not trivial. Thus if $d(\Pi_f) \geqslant 2$, then the dimension of $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ is 0. For $d(\Pi_f) = 1$, the characters $\omega_{\pi_0}\cdot\omega_{F/E}\cdot\chi|_{I_E}$ and η have trivial infinity type and thus they become isomorphic, by class field theory, after restriction to the absolute Galois group of the abelian extension of M defined by $\eta^{-1}\cdot\omega_{\pi_0}\cdot\omega_{F/E}\cdot\chi|_{I_E}$, and hence contribute to Tate classes defined over that field.

B) The representation π is cuspidal CM. Thus there exists a Hecke character χ of some quadratic CM-extension N of F such that $\rho_\pi = \mathrm{Ind}_{\Gamma_N}^{\Gamma_F}\chi$. Then from the proprieties of $\rho(\pi_f)$ described in §3, we deduce that

$$\begin{aligned}\rho(\pi_f)|_{\Gamma_{kNN^\tau\tilde{L}_1}} &\cong (\chi\oplus\bar{\chi})\otimes(\chi^\tau\oplus\bar{\chi}^\tau)\\ &\cong \chi\chi^\tau\oplus\chi\bar{\chi}^\tau\oplus\bar{\chi}\chi^\tau\oplus\bar{\chi}\bar{\chi}^\tau,\end{aligned}$$

where $\bar{\chi}$ is the complex conjugate of χ.

As above, we have

$$\phi(\Pi_f)|_{\Gamma_{kNN^\tau\tilde{L}_1}} = \begin{cases}\eta\oplus\eta^\epsilon & \text{if } [L_1:L]=2,\\ \eta\oplus\eta^\epsilon\oplus\eta^{\epsilon^2} & \text{if } [L_1:L]=3.\end{cases}$$

Thus if the dimension of $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ is bigger than 0, we must have that $\chi\chi^\tau$ or $\chi\bar{\chi}^\tau$ or $\bar{\chi}\chi^\tau$ or $\bar{\chi}\bar{\chi}^\tau$ is isomorphic to η or η^ϵ or η^{ϵ^2} as characters of $\Gamma_{kNN^\tau\tilde{L}_1}$. If two of these characters have infinity types equal, then they become isomorphic, by class field theory, after restriction to the absolute Galois group of some extension of M, and hence contribute to Tate classes defined over that field. This extension of M is not necessarily abelian over M, but is abelian over $MNN^\tau\tilde{L}_1$.

C) The representation π is one-dimensional.

In case i), from §3, we know that

$$\rho(\pi_f)|_{\Gamma_k} \cong \rho_\pi|_{\Gamma_k}\otimes\rho_\pi|_{\Gamma_k}^\tau,$$

where ρ_π is a Hecke character of F and has dimension 1. The infinity type of $\rho_\pi|_{\Gamma_k}$ and $\rho_\pi|^\tau_{\Gamma_k}$ is trivial and as in A) above, we obtain that if $d(\Pi_f) \geqslant 2$, then the dimension of $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ is 0. If $d(\Pi_f) = 1$, the infinity type of the characters η and ρ_π or ρ_π^τ is trivial, and thus they become isomorphic after restriction to the absolute Galois group of the abelian extension of M defined by $\eta^{-1} \cdot \rho_\pi$ or $\eta^{-1} \cdot \rho_\pi^\tau$, and hence contribute to Tate classes defined over that fields.

In case ii), from §3, we know that $[F : E] = 2$ and

$$\rho(\pi_f) \cong \rho_\pi|_{I_E} \oplus \rho_\pi|_{I_E} \cdot \omega_{F/E},$$

where ρ_π is a Hecke character of F of dimension 1 and $\omega_{F/E}$ is the quadratic character corresponding to F/E. The infinity type of $\rho_\pi|_{I_E}|_{\Gamma_k}$ or $\rho_\pi|_{I_E}|^\tau_{\Gamma_k}$ is trivial and as above, we get that if $d(\Pi_f) \geqslant 2$, then the dimension of the space $\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_f), V(\Pi_f))$ is 0. If $d(\Pi_f) = 1$ the characters η and $\rho_\pi|_{I_E}$ or $\rho_\pi|_{I_E} \cdot \omega_{F/E}$ become isomorphic after restriction to the absolute Galois group of the abelian extension of M defined by $\eta^{-1} \cdot \rho_\pi|_{I_E}$ or $\eta^{-1} \cdot \rho_\pi|_{I_E} \cdot \omega_{F/E}$, and hence contribute to Tate classes defined over that fields.

From A), B) and C) we get the following result (replacing Π by Π^* is equivalent to replacing η by η^{-1}):

Theorem 6.3. *Assume that Π is cuspidal AI and that $\phi(\Pi_f) = Ind_{\Gamma_{L_1}}^{\Gamma_L} \eta$ for some algebraic Hecke character η of the field L_1 of degree $d(\Pi_f)$ over L. Then:*

*1) Assume that π is cuspidal non-*CM *and that we are in the case i) of §3, then the space $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ contains a Tate class if and only if $\rho_\pi^\tau|_{\Gamma_E} \cong \rho_\pi|_{\Gamma_E} \otimes \chi$ for some finite order character χ of Γ_E and $d(\Pi_f) = 1$. In this case, the subspace of Tate classes has the same dimension as $\pi_f^K \otimes \Pi_f^{\mathbf{K}}$, and all such Tate classes are defined over the abelian extension of M defined by $\eta \cdot \wedge^2(\rho_\pi) \cdot \chi$.*

*2) Assume that π is cuspidal non-*CM *and that we are in the case ii) of §3, then the space $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ contains a Tate class if and only if $\pi \cong \pi_{0/F} \otimes \chi$, where $\pi_{0/F}$ is the base change to F of some automorphic representation π_0 of $GL(2)/E$ and χ is a finite order Hecke character of F and $d(\Pi_f) = 1$. In this case, the subspace of Tate classes has the same dimension as $\pi_f^K \otimes \Pi_f^{\mathbf{K}}$, and all such Tate classes are defined over the abelian extension of M defined by $\eta \cdot \omega_{\pi_0} \cdot \omega_{F/E} \cdot \chi|_{I_E}$.*

3) Assume that π is cuspidal CM *and $\rho_\pi = \mathrm{Ind}_{\Gamma_N}^{\Gamma_F} \chi$, where N is some quadratic* CM*-extension of F and χ is an algebraic Hecke character of N, then the dimension of the subspace of Tate classes is equal to $a \cdot \dim_{\overline{\mathbb{Q}}_l}(\pi_f^K \otimes \Pi_f^{\mathbf{K}})$, where a is equal to the number of pairs between the set $\{\chi\chi^\tau, \chi\bar{\chi}^\tau, \bar{\chi}\chi^\tau, \bar{\chi}\bar{\chi}^\tau\}$ and $\{\eta, \eta^\epsilon, \eta^\epsilon\}$, where $\epsilon \in \Gamma_L - \Gamma_{L_1}$, that have characters whose product has trivial infinity type and all such Tate classes are defined over an extension of M that is abelian over $MNN^\tau \tilde{L}_1$, where $\tilde{L}_1$ is the Galois closure of L_1 over L.*

4) Assume that π is one-dimensional and that we are in the case i) of §3, then the space $H(\pi_f)(1) \otimes H(\Pi_f)(1)$ contains a Tate class if and only if $d(\Pi_f) = 1$. In this case, the dimension of the subspace of Tate classes is equal to $2 \cdot \dim_{\overline{\mathbb{Q}}_l}(\pi_f^K \otimes$

$\Pi_f^{\mathbf{K}}$), *and all such Tate classes are defined over the abelian extension of* M *equal to the composition field of the abelian extensions of* M *defined by* $\eta\cdot\rho_\pi$ *and* $\eta\cdot\rho_\pi^\tau$.

5) Assume that π *is one-dimensional and that we are in the case ii) of* §3, *then the space* $H(\pi_f)(1)\otimes H(\Pi_f)(1)$ *contains a Tate class if and only if* $d(\Pi_f)=1$. *In this case, the dimension of the subspace of Tate classes is equal to* $2\cdot\dim_{\overline{\mathbb{Q}}_l}(\pi_f^K\otimes\Pi_f^{\mathbf{K}})$, *and all such Tate classes are defined over the abelian extension of* M *defined by* $\eta\cdot\rho_\pi|_{I_E}$.

In 1), 2), 4) and 5) all these Tate classes of $H(\pi_f)(1)\otimes H(\Pi_f)(1)$ *come as a product of Tate classes from individual factors and in 3) this is not necessarily true.*

7 Poles of L-functions

For k a finite extension of M, define:

$$\mathbf{V}(\pi_f,\Pi_f,k):=\{x\in V(\Pi_f)(1)\otimes V(\Pi_f)(1)|\rho(\pi_f)\otimes\phi(\Pi_f)(a)x=x,\ \text{for all } a\in\Gamma_k\}.$$

In this section we prove the following result:

Theorem 7.1. *Let* k *be a finite extension of* M. *Then:*

1) If k *is a solvable extension of* $\mathbb{Q}$ *and in the case ii) of* §3, k *contains the field* F, *then the order of the pole at* $s=1$ *of* $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ *is equal to* $\dim_{\overline{\mathbb{Q}}_l}\mathbf{V}(\pi_f,\Pi_f,k)$.

2) If π *is cuspidal* CM *or one-dimensional and* Π *is AI, then the order of the pole at* $s=1$ *of* $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ *is equal to* $\dim_{\overline{\mathbb{Q}}_l}\mathbf{V}(\pi_f,\Pi_f,k)$.

3) If Π *is AI and* k *is a solvable extension of a totally real field, then the order of the pole at* $s=1$ *of* $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ *is equal to* $\dim_{\overline{\mathbb{Q}}_l}\mathbf{V}(\pi_f,\Pi_f,k)$.

4) If π *is cuspidal* CM *or one dimensional,* $d(\Pi_f)=2$ *and* k *is a solvable extension of a totally real field, then the order of the pole at* $s=1$ *of* $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ *is equal to* $\dim_{\overline{\mathbb{Q}}_l}\mathbf{V}(\pi_f,\Pi_f,k)$.

5) If π *is cuspidal* CM *or one-dimensional,* $d(\Pi_f)=3$ *and* k *is a solvable extension of* $\mathbb{Q}$, *then the order of the pole at* $s=1$ *of* $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ *is equal to* $\dim_{\overline{\mathbb{Q}}_l}\mathbf{V}(\pi_f,\Pi_f,k)$.

Proof:

1) Assume that k is a solvable extension of $\mathbb{Q}$ and that k contains F. Then from §3, we obtain

$$\rho(\pi_f)\otimes\phi(\Pi_f)|_{\Gamma_k}\cong\rho_\pi|_{\Gamma_k}\otimes\rho_\pi|_{\Gamma_k}^\tau\otimes\phi(\Pi_f)|_{\Gamma_k}.$$

Since k is solvable, from the base change for solvable extensions [L],[AC], we get that $\rho_\pi|_{\Gamma_k}$, $\rho_\pi|_{\Gamma_k}^\tau$ and $\phi(\Pi_f)|_{\Gamma_k}$ are automorphic. But from Proposition 4.5, we deduce that the Galois representation $\rho_\pi|_{\Gamma_k}\otimes\rho_\pi|_{\Gamma_k}^\tau$ is automorphic i.e. corresponds to an automorphic representation of $GL(4)/k$, and from Proposition 4.6 and the results of §6, one could prove easily the part 1) of Theorem 7.1 (as usually the order of the pole at $s=1$ of $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ will be equal to the multiplicity of the trivial representation of Γ_k in $(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k}$ because one could decompose $\rho(\pi_f)|_{\Gamma_k}$ and $\phi(\Pi_f)|_{\Gamma_k}$ as sums of automorphic irreducible representations and from Proposition 4.6, Theorem 6.2 and Theorem 6.3 the result follows).

2) We assume first that π is cuspidal CM and Π is AI. Thus there exists a quadratic extension N/F and an algebraic Hecke character χ of N such that $\rho_\pi = \mathrm{Ind}_{\Gamma_N}^{\Gamma_F}\chi$.

In the case i) of §3 we know that:

$$\rho(\pi_f) \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi^\tau|_{\Gamma_E}.$$

In the case ii) of §3, from the proprieties of $\rho(\pi_f)$, we have that (see for example [MR] 6.3):

$$\Lambda^2(\mathrm{Ind}_{\Gamma_N}^{\Gamma_E}\chi) \cong \rho(\pi_f) \oplus \mathrm{Ind}_{\Gamma_F}^{\Gamma_E}(\omega_\pi),$$

where ω_π is the central character of π.

From these identities, we get that $\rho(\pi_f)|_{\Gamma_k}$ is a virtual sum of monomial representations of Γ_k. Here a monomial representation of Γ_k is a representation which is induced from a one-dimensional representation of an open subgroup.

Since $\rho(\pi_f)|_{\Gamma_k}$ and $\phi(\Pi_f)|_{\Gamma_k}$ are sums of monomial representations (because Π is AI), we obtain that $\rho(\pi_f)\otimes\phi(\Pi_f)|_{\Gamma_k}$ is a sum of monomial representations and it is obvious that the order of the pole at $s=1$ of $L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k})$ is equal to the dimension of the space of Tate classes $\mathbf{V}(\pi_f,\Pi_f,k)$.

The same argument works when π is one-dimensional and Π is AI, since in this case, we know from §3 that $\rho(\pi_f)$ is a sum of one-dimensional representations.

3) Assume that Π is AI and k is a solvable extension of a totally real field. Then $\phi(\Pi_f) = \mathrm{Ind}_{\Gamma_{L_1}}^{\Gamma_L}\eta$ for some algebraic Hecke character η of the field L_1 of degree $d(\Pi_f)$ over L. From Proposition 4.7, we deduce that there exists a Galois extension F_2 of $\mathbb{Q}$ containing Fk and an automorphic representation π_1 of $GL(2)/F_2$ and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}} \cong \rho_{\pi_1,\beta}$.

By Brauer's Theorem (see [SE], Theorems 16 and 19), we can find some subfields $F_i \subset F_2$ such that $\mathrm{Gal}(F_2/F_i)$ are solvable, some characters $\chi_i : \mathrm{Gal}(F_2/F_i) \to \overline{\mathbb{Q}}^\times$ and some integers m_i, such that the trivial representation

$$1|_{\Gamma_k} : \mathrm{Gal}(F_2/k) \to \overline{\mathbb{Q}}_l^\times,$$

can be written as $1|_{\Gamma_k} = \sum_{i=1}^{i=n} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k}\chi_i$ (a virtual sum). Then

$$L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k}\otimes 1|_{\Gamma_k}) = \prod_{i=1}^{i=n} L(s,(\rho(\pi_f)\otimes\phi(\Pi_f))|_{\Gamma_k}\otimes\mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k}\chi_i)^{m_i}$$

$$= \prod_{i=1}^{i=n} L(s, \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k}((\rho(\pi_f) \otimes \phi(\Pi_f))|_{\Gamma_{F_i}} \otimes \chi_i))^{m_i}$$

$$= \prod_{i=1}^{i=n} L(s, (\rho(\pi_f) \otimes \phi(\Pi_f))|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

Since $\phi(\Pi_f) = \mathrm{Ind}_{\Gamma_{L_1}}^{\Gamma_L} \eta$, we deduce that $\phi(\Pi_f)|_{\Gamma_{F_i}} = \sum_{j=1}^{j=m} \mathrm{Ind}_{\Gamma_{F_i^j}}^{\Gamma_{F_i}} \varphi_i^j$ (direct sum), where $[F_i^j : F_i] \leqslant 3$, and φ_i^j is the restriction of η, η^ϵ or η^{ϵ^2} to $\Gamma_{F_i^j}$, where $\epsilon \in \Gamma_L - \Gamma_{L_1}$. We get that

$$L(s, (\rho(\pi_f) \otimes \phi(\Pi_f))|_{\Gamma_k} \otimes 1|_{\Gamma_k}) = \prod_{i=1}^{i=n} \prod_{j=1}^{j=m} L(s, (\rho(\pi_f)|_{\Gamma_{F_i^j}} \otimes \varphi_i^j \otimes \chi_i|_{\Gamma_{F_i^j}}))^{m_i}.$$

Since $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$ is automorphic and F_2 is a solvable extension of F_i, if $F \subseteq F_i$ we obtain that $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$ is automorphic and because $[F_i^j : F_i] \leqslant 3$ and the base change for $GL(2)$ is known for cubic extensions not necessarily normal (see [JPSS1]), we get that if $F \subseteq F_i$, then $\rho_{\pi,\lambda}|_{\Gamma_{F_i^j}}$ is automorphic. Then one could check easily that the pole at $s = 1$ of $L(s, (\rho(\pi_f)|_{\Gamma_{F_i^j}} \otimes \varphi_i^j \otimes \chi_i|_{\Gamma_{F_i^j}}))$ is equal to the space of Tate classes corresponding to the representation $(\rho(\pi_f)|_{\Gamma_{F_i^j}} \otimes \varphi_i^j \otimes \chi_i|_{\Gamma_{F_i^j}})$ (for details see [V1] §6, or one could use the fact proved in [RA1], Theorem D that $\rho(\pi_f)|_{\Gamma_{F_i^j}}$ is automorphic, being the Asai representation associated to an automorphic representation or a tensor product of 2 automorphic representations of dimension 2).

4) We assume that π is cuspidal CM or one-dimensional, $d(\Pi_f) = 2$ and k is a solvable extension of a totally real field.

From [BR], Lemma 1.12.2, we know that $\sigma_\Pi = \sigma_E \otimes \eta$, where σ_E is a base change to $GL(2)_{/E}$ of some cuspidal representation of $GL(2)_{/\mathbb{Q}}$ and η is a Hecke character of E. Thus, from Proposition 4.7, we deduce that there exists a Galois extension F_2 of $\mathbb{Q}$ containing Fk and a cuspidal automorphic representation π_1 of $GL(2)/F_2$ and a prime β of the field of coefficients of π_1 such that $\phi(\Pi_f)|_{\Gamma_{F_2}} \cong \rho_{\pi_1,\beta}$. From 2), we know that $\rho(\pi_f)$ is a sum of monomial representations that are induced from representations of Hecke characters of solvable extensions of E. Thus $\rho(\pi_f)|_{\Gamma_k} = \sum_{i=1}^{i=n} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$ for some subfields $F_i \subset F_2$ such that $\mathrm{Gal}(F_2/F_i)$ are solvable, some characters $\chi_i : \mathrm{Gal}(F_2/F_i) \to \overline{\mathbb{Q}}^\times$ and some integers m_i. Then

$$L(s, (\rho(\pi_f) \otimes \phi(\Pi_f))|_{\Gamma_k}) = \prod_{i=1}^{i=n} L(s, \phi(\Pi_f)|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

Since $\phi(\Pi_f)|_{\Gamma_{F_2}}$ is automorphic and F_2 is a solvable extension of F_i, we obtain that $\phi(\Pi_f)|_{\Gamma_{F_i}}$ is automorphic and hence we deduce the part 4) of Theorem 7.1.

5) We assume that π is cuspidal CM or one-dimensional and $d(\Pi_f)=3$ and that k is a solvable extension of $\mathbb{Q}$. From 2), we know that $\rho(\pi_f)$ is a sum of monomial representations that are induced from representations of Hecke characters of solvable extensions of E. Thus there exists a solvable extension F_2 of k, some subfields $F_i \subset F_2$, some characters $\chi_i : \mathrm{Gal}(F_2/F_i) \to \overline{\mathbb{Q}}^{\times}$ and some integers m_i such that $\rho(\pi_f)|_{\Gamma_k} = \sum_{i=1}^{i=n} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$. Then

$$L(s, (\rho(\pi_f) \otimes \phi(\Pi_f))|_{\Gamma_k}) = \prod_{i=1}^{i=n} L(s, \phi(\Pi_f)|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

Since F_2 is a solvable extension of $\mathbb{Q}$, from base change for $GL(n)$ for solvable extensions [AC], we deduce that $\phi(\Pi_f)|_{\Gamma_{F_2}}$ is automorphic and thus because F_2 is a solvable extension of F_i, we get that $\phi(\Pi_f)|_{\Gamma_{F_i}}$ is automorhic and we deduce the part 5) of Theorem 7.1.

References

[AC] J.Arthur, L.Clozel, *Simple algebras, base change and the advanced theory of the trace formula*, Annals of Math. Studies, Princeton University Press, 1989.

[BR] D.Blasius, J,Rogawski, *Tate classes and arithmetic quotients of the two-ball*, in [LR].

[D] P. Deligne, *Travaux de Shimura*, Sém. Bourbaki Féb. 71, Exposé 389, Lectures Notes in Math. vol. 244. Berlin-heidelberg-New York; Springer 1971.

[HLR] G.Harder, R.P.Langlands, M.Rapoport, *Algebraische Zycklen auf Hilbert-Blumenthal-Flächen*, J. Reine Angew. Math. 366, 53-120 (1986).

[G] G. van der Geer, *Hilbert modular surfaces*, Springer-Verlag 1988.

[GJ] S.Gelbart, H.Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, Ann. Sci. École Norm. Sup. 11(1979),471-542.

[JPSS] H.Jacquet, I.I.Piatetski-Shapiro, J.A.Shalika, *Rankin-Selberg convolutions*, American Journal of Mathematics, 105, nr. 2 (1983), 367-464.

[JPSS1] H.Jacquet, I.I.Piatetski-Shapiro, J.A.Shalika, *Relevement cubique non normal*, C. R. Acad. Sci. paris 292 (1981), 567-579.

[K] C.Klingenberg, *Die Tate-Vermutungen für Hilbert-Blumenthal-Flächen*, Invent. Math. 89, 291-317(1987).

[K1] A.Knightly, *Tate classes on a Product of two Picard modular surfaces*, J. Number Theory, 107 (2004), 335-344.

[L] K.F.Lai, *Algebraic cycles on compact Shimura surface*, Math. Z. 189, 593-602 (1985).

[LR] R.Langlands, D.Ramakrishnan, *Zeta functions of Picard modular surfaces*, U. Montreal Press, 1992.

[MP] V.K.Murty, D.Prasad, *Tate cycles on a product of two Hilbert modular surfaces*, J. Number Theory 80(1)(2000) 25-43.

[MR] V.K.Murty, D.Ramakrishnan, *Period relations and the Tate conjecture for Hilbert modular surfaces*, Invent. Math. 89, 319-345(1987).

[RA] D.Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2)*, Ann. of Math., 152(2000), 45-111.

[RA1] D.Ramakrishnan, *Algebraic cycles on Hilbert modular fourfolds and poles of L-functions*, Proceedings of the International Conference on Algebraic Groups and Arithmetic, 271-274 (2005).

[RT] J.D.Rogawski, J.B.Tunnell, *On Artin L-functions associated to Hilbert modular forms of weight one*, Inventiones Mathematicae, 74, 1983, 1-43.

[SE] J.-P.Serre, *Linear representations of finite groups*, Springer 1977.

[T] R.Taylor, *On Galois representations associated to Hilbert modular forms*, Inventiones Matematicae, 98, 1989,265-280.

[TA] J.Tate, *Algebraic cycles and poles of zeta functions*, In: Schilling, O.D.G. (ed.), Arithmetical algebraic geometry, New York: Harper and Row, 1966.

[V] C.Virdol, *Non-solvable base change and zeta functions of twisted quaternionic Shimura varieties*, preprint; see my web page: http://www.math.columbia.edu/ virdol/.

[V1] C.Virdol, *Tate classes and poles of L-functions of twisted quaternionic Shimura surfaces*, Journal of Number Theory 123 (2007), no. 2, 315-328.