Algebraic cycles on compact quaternionic Shimura fourfolds and poles of L-functions

Cristian Virdol
Department of Mathematics
Columbia University

October 21, 2009

1 Introduction

Let X be a smooth projective variety of dimension n defined over a number field F and let

$$\bar{X} = X \times_F \overline{\mathbb{Q}}.$$

For a prime number l, let $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ be the étale cohomology of $\bar{X}$. If K is a number field, we denote $\Gamma_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. The Galois group Γ_F acts on $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ by a representation $\phi_{i,l}$. For any $j \in \mathbb{Z}$, let $H^i_{et}(X, \overline{\mathbb{Q}}_l)(j)$ denote the representation of Γ_F on $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ defined by $\phi_{i,l} \otimes \xi_l^j$, where ξ_l is the l-adic cyclotomic character. For any finite extension E/F the elements of $V^i(X, E) := H^{2i}_{et}(X, \overline{\mathbb{Q}}_l)(i)^{\Gamma_E}$ are called *Tate cycles* on X defined over E. The union

$$V^i(X) := \cup_E V^i(X, E)$$

is the space of all *Tate cycles* on X.

To each algebraic subvariety Y of X of codimension i, one can associate a cohomology class

$$[Y] \in H_{2n-2i}(X(\mathbb{C}), \mathbb{Q}) \cong H^{2i}_B(X(\mathbb{C}), \mathbb{Q})(i),$$

where $H^{2i}_B(X(\mathbb{C}), \mathbb{Q})$ is the Betti cohomology. Then using the isomorphism

$$H^{2q}_B(X(\mathbb{C}), \mathbb{Q})(i) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H^{2i}_{et}(X, \mathbb{Q}_l)(i),$$

we obtain a class $[Y] \in H^{2i}_{et}(X, \mathbb{Q}_l)(i)$. A cohomology class $[Y]$ obtained in this way is called *algebraic*. If Y is defined over a finite extension E of F, then we obtain a class $[Y] \in H^{2i}_{et}(X, \mathbb{Q}_l)(i)^{\Gamma_E}$. Let $U^i(X, E)$ the space of algebraic cycles defined over E. Then $U^i(X, E) \subseteq V^i(X, E)$ and the first part of the Tate conjecture [TA] states that for any finite extension E/F we have

$$U^i(X, E) = V^i(X, E),$$

i.e. every Tate cycle is algebraic.

The L-function $L^{2i}(s, X_{/F})$ (more exactly the Euler product) attached to the representation $\phi_{2i,l}$ converges for $\mathrm{Re}(s) > i+1$. The second part of the Tate conjecture [TA] states that for any finite extension E/F the L-function $L^{2i}(s, X_{/E})$ has a meromorphic continuation to the entire complex plane and has a pole at $s = i+1$ of order equal to

$$\dim_{\overline{\mathbb{Q}}_l} V^i(X, E).$$

We consider a quartic totally real number field F containing a quadratic subfield. Let B be a quaternion division algebra over $\mathbb{Q}$ and let $D := B \otimes_{\mathbb{Q}} F$. We assume that D is a quaternion division algebra over F which splits at the real places. Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D and let $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$. We denote by $S_K := S_{\bar{G},K}$ the canonical model of the quaternionic Shimura variety associated to an open compact subgroup K of $\bar{G}(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. Then S_K is a 4-dimensional proper smooth variety defined over $\mathbb{Q}$.

In this paper we prove the first part of the Tate conjecture for S_K for non-CM submotives if we assume that the field F is Galois over $\mathbb{Q}$. We prove the second part of the Tate conjecture for S_K, without assuming that F is Galois over $\mathbb{Q}$, but only for solvable number fields (see theorem 8.2 for details). We remark that similar results were obtained by Ramakrishnan [R] in the case of Hilbert modular fourfolds and by Harder, Langlands, Rapoport [HLR], Murty, Ramakrishnan [MR], Klingenberg [K], Lai [L] and Flicker, Hakim [FH] in the case of Hilbert modular surfaces and compact quaternionic Shimura surfaces.

2 Quaternionic Shimura fourfolds and surfaces

Let F be a totally real field of degree 4 over $\mathbb{Q}$ such that F contains a quadratic number field F_0. We consider a quaternion division algebra B over $\mathbb{Q}$ and let $D := B \otimes_{\mathbb{Q}} F$. Assume that D is a quaternion division algebra over F which splits at the real places (we remark that given a quaternion division algebra D over F which splits at the real places, there exists a quaternion division algebra B over $\mathbb{Q}$ such that $D := B \otimes_{\mathbb{Q}} F$ if and only if for each rational prime p we have $\sum_{v|p} \mathrm{inv}_v D_v = 0$, where v runs over the places of F dividing p, and inv_v denotes the invariant of D at v). Let G be the algebraic group over F defined by the multiplicative group $D^\times$. By restricting the scalars, we obtain the algebraic group $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$ over $\mathbb{Q}$ defined by the propriety: $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$ for all $\mathbb{Q}$-algebras A.

Then $\bar{G}(\mathbb{R})$ is isomorphic to $\mathrm{GL}_2(\mathbb{R})^4$. Let $h : \mathbb{C}^* \to \bar{G}(\mathbb{R})$ be defined by $a + bi \mapsto \delta(\begin{pmatrix} a & b \\ -b & a \end{pmatrix})$, where δ denotes the diagonal embedding of $\mathrm{GL}(2,\mathbb{R})$ in $\bar{G}(\mathbb{R})$. Let K_∞ be the centralizer of h in $\bar{G}(\mathbb{R})$. For each open compact subgroup $K \subset \bar{G}(\mathbb{A}_f)$ set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus \bar{G}(\mathbb{A}_{\mathbb{Q}})/KK_\infty.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of the complex points of a proper smooth 4-dimensional variety S_K defined over $\mathbb{Q}$, which is called a compact quaternionic Shimura fourfold.

Let D_0 be a quaternion algebra over F_0 which splits at the real places such that $D = D_0 \otimes_{F_0} F$ (we remark that $B \otimes_{\mathbb{Q}} F_0$ is a quaternionic division algebra over F_0 which has this propriety). Let G_0 be the algebraic group over F_0 defined by the multiplicative group $D_0^\times$. As above by restricting the scalars, we obtain the algebraic group $\bar{G}_0 = \mathrm{Res}_{F_0/\mathbb{Q}}(G_0)$. Then $\bar{G}_0(\mathbb{R})$ is isomorphic to $GL_2(\mathbb{R})^2$. Let $h_0 : \mathbb{C}^* \to \bar{G}_0(\mathbb{R})$ be defined by $a + bi \mapsto \delta_0(\begin{pmatrix} a & b \\ -b & a \end{pmatrix})$, where δ_0 denotes the diagonal embedding of $\mathrm{GL}(2, \mathbb{R})$ in $\bar{G}_0(\mathbb{R})$. Let L_∞ be the centralizer of h_0 in $\bar{G}_0(\mathbb{R})$. For each open compact subgroup $L \subset \bar{G}_0(\mathbb{A}_f)$ set

$$S_{0L}(\mathbb{C}) = \bar{G}_0(\mathbb{Q}) \setminus \bar{G}_0(\mathbb{A}_{\mathbb{Q}})/LL_\infty.$$

For L sufficiently small, $S_{0L}(\mathbb{C})$ is a complex manifold which is the set of the complex points of a proper smooth 2-dimensional variety S_{0L} defined over $\mathbb{Q}$, which is called a compact quaternionic Shimura surface.

3 Cohomologies for quaternionic Shimura fourfolds

Let K be a sufficiently small open compact subgroup of $\bar{G}(\mathbb{A}_f)$.

If l is a prime number, let $\mathbb{H}_K$ be the Hecke algebra generated by the bi-K-invariant $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $\bar{G}(\mathbb{A}_f)$ under the convolution. If $\pi' = \pi'_f \otimes \pi'_\infty$ is an automorphic representation of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$, we denote by $\pi_f^{'K}$ the space of K-invariants in π'_f. The Hecke algebra $\mathbb{H}_K$ acts on $\pi_f^{'K}$.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group $\Gamma_{\mathbb{Q}}$ on the étale cohomology $H^4_{et}(S_K, \bar{\mathbb{Q}}_l)$ and these two actions commute. We say that the representation π' is *cohomological* if $H^*(\mathfrak{g}, K_\infty, \pi'_\infty) \neq 0$, where $\mathfrak{g}$ is the Lie algebra of K_∞ (the cohomology is taken with respect to $(\mathfrak{g}, K_\infty)$-module associated to π'_∞).

Proposition 3.1. *The representation of $\Gamma_{\mathbb{Q}} \times \mathbb{H}_K$ on the étale cohomology $H^4_{et}(S_K, \bar{\mathbb{Q}}_l)(2)$ is isomorphic to*

$$\oplus_{\pi'} \rho(\pi') \otimes \pi_f^{'K},$$

where $\rho(\pi')$ is a representation of the Galois group $\Gamma_{\mathbb{Q}}$. The above sum is over weight 2 cohomological automorphic representations π' of $G(\mathbb{A}_{\mathbb{Q}})$, such that $\pi_f^{'K} \neq 0$, and the $\mathbb{H}_K$-representations $\pi_f^{'K}$ are irreducible and mutually inequivalent, i.e. the decomposition is isotypic with respect to the action of $\mathbb{H}_K$.

The representations π' that appear in the proposition 3.1 are one-dimensional or cuspidal and infinite-dimensional. If π' is one-dimensional, then $\rho(\pi')$ is

6-dimensional and if π' is cuspidal and infinite-dimensional, then $\rho(\pi')$ is 16-dimensional. From now on in this paper we assume that π' is cuspidal and infinite-dimensional, because for π' one-dimesional the algebraicity of the Tate cycles corresponding to the π'-component of $H^4_{et}(S_K, \bar{\mathbb{Q}}_l)$ (see proposition 3.1) could be proved in the same way as in proposition 4.11 of [R], and second part of the Tate conjecture in this case is also trivial (see [R]). We denote by π the cuspidal automorphic representation of $\mathrm{GL}(2)_{/F}$ corresponding to π' by Jaquet-Langlands correspondence.

Let $\rho_{\pi'} = \rho_\pi$ be the l-adic 2-dimensional semisimple representation of $\Gamma_{\mathbb{Q}}$ associated to π' or to π (see [C], [T]). Then the representation $\rho(\pi')$ is semisimple (see §7 of [R1]) and $\rho(\pi') = \mathrm{As}_{F/\mathbb{Q}}\rho_{\pi'}$ (see §7.2 of [BR]), where $\mathrm{As}_{F/\mathbb{Q}}\rho_{\pi'}$ is the Asai (or tensor induction) representation (see §6 of [R]).

We fix an isomorphism $j : \bar{\mathbb{Q}}_l \to \mathbb{C}$ and define the L-function

$$L^4(s, S_K) := \prod_{\pi'} \prod_{q} \det(1 - q^{-s+2} j(\rho(\pi')(\mathrm{Frob}_q)) | H^4_{et}(S_K, \bar{\mathbb{Q}}_l)(2)^{I_q})^{-1},$$

where Frob_q is a geometric Frobenius element at a rational prime q and I_q is a inertia group at q (in order to define the local factor at l one has to use actually the $l^{'}$-adic cohomology for some $l^{'} \neq l$ and theorem 3 of [B] which gives us the expression of the local factors of the zeta functions of quaternionic Shimura varieties).

We have the canonical isomorphisms:

$$\Phi_K : H^4_B(S_K(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \to H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l),$$

and

$$\Phi : H^4_B(S(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \to H^4_{\mathrm{et}}(S, \bar{\mathbb{Q}}_l),$$

where $H^4_B(S_K(\mathbb{C}), \mathbb{Q})$ is the Betti cohomology, and $S := \varprojlim S_K$. We denote by $V(\pi')$ the π'-component of $H^4_{\mathrm{et}}(S, \bar{\mathbb{Q}}_l)(2)$ in the decomposition of proposition 3.1 and by $V_B(\pi')$ the corresponding π'-component of $H^4_B(S(\mathbb{C}), \mathbb{Q})(2)$. Thus

$$V_B(\pi') \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \cong V(\pi').$$

Since $\rho(\pi') = \mathrm{As}_{F/\mathbb{Q}}\rho_{\pi'}$, for each finite order Hecke character ν of F we have $\rho(\pi') \otimes \nu|_{I_{\mathbb{Q}}} \cong \rho(\pi' \otimes \nu)$, where $I_{\mathbb{Q}}$ is the idele group of $\mathbb{Q}$, i.e. $V(\pi') \otimes \nu|_{I_{\mathbb{Q}}} \cong V(\pi' \otimes \nu)$ as $\Gamma_{\mathbb{Q}}$-modules.

4 Meromorphic continuation

For π' a cuspidal representation as in proposition 3.1, we denote by $As_{F/F_0}(\pi')$ the isobaric automorphic representation of $\mathrm{GL}(4, \mathbb{A}_{F_0})$ defined in theorem D of [R2]. Let

$$\rho_{As_{F/F_0}(\pi')} : \Gamma_{F_0} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_l),$$

be the l-adic representation associated to $As_{F/F_0}(\pi')$. Then $\rho_{As_{F/F_0}(\pi')} = As_{F/F_0}\rho_{\pi'}$ and $L(s, \rho_{As_{F/F_0}(\pi')}) = L(s, As_{F/F_0}(\pi'))$. From the proprieties of the Asai representations we know that $\rho(\pi') = As_{F/\mathbb{Q}}\rho_{\pi'} = As_{F_0/\mathbb{Q}}(As_{F/F_0}\rho_{\pi'})$, and because $As_{F/F_0}\rho_{\pi'}$ is automorphic, from theorem 6.11 of [R], and using the solvable base change for GL(2) (see [LA]) and the main theorem of [JPSS], one obtains easily that (see also [B]):

Proposition 4.1. *If $k/\mathbb{Q}$ is solvable, then the function $L(s, \rho(\pi')|_{\Gamma_k})$, has a meromorphic continuation to the entire complex plane and satisfies a functional equation $s \leftrightarrow 1-s$.*

5 Some definitions

For k a number field, define:

$$\mathbf{V}(\pi', k) := \{x \in V(\pi') | \rho(\pi')(a)x = x, \text{ for all } a \in \Gamma_k\},$$

and

$$\mathbf{V}(\pi', \bar{\mathbb{Q}}) := \cup_k \mathbf{V}(\pi', k),$$

where $V(\pi')$ is the space corresponding to $\rho(\pi')$. The elements of $\mathbf{V}(\pi', k)$ are called *Tate cycles* defined over k, and the elements of $\mathbf{V}(\pi', \bar{\mathbb{Q}})$ are called *Tate cycles*. We denote by $\mathbf{U}(\pi', k) \subseteq \mathbf{V}(\pi', k)$ the subspace of *algebraic cycles* defined over k.

We denote by $r_{alg}(\pi', k) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{U}(\pi', k)$, by $r_l(\pi', k) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', k)$, by $r_l(\pi', \bar{\mathbb{Q}}) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', \bar{\mathbb{Q}})$, and for k solvable number filed by $r_{an}(\pi', k)$ the order of the pole of $L(s, \rho(\pi')|_{\Gamma_k})$ at $s = 1$. Then $r_{alg}(\pi', k) \leq r_l(\pi', k)$.

For ν a finite order character of $\Gamma_{\mathbb{Q}}$, define:

$$\mathbf{V}(\pi'; \nu) := \{x \in V(\pi') | \rho(\pi')(a)x = \nu^{-1}(a)x, \text{ for all } a \in \Gamma_{\mathbb{Q}}\},$$

and

$$\mathbf{V}(\pi', \mathbb{Q}^{ab}) := \cup_\nu \mathbf{V}(\pi'; \nu).$$

Let $\mathbf{U}(\pi'; \nu) \subseteq \mathbf{V}(\pi'; \nu)$ and $\mathbf{U}(\pi', \mathbb{Q}^{ab}) \subseteq \mathbf{V}(\pi', \mathbb{Q}^{ab})$ be the subspaces of *algebraic cycles*. Then $\mathbf{V}(\pi'; \nu) = \mathbf{V}(\pi' \otimes \nu|_{\Gamma_F}; 1)$ and $\mathbf{U}(\pi'; \nu) = \mathbf{U}(\pi' \otimes \nu|_{\Gamma_F}; 1)$. We remark that when π' is non-CM, for k sufficiently large we have $\mathbf{V}(\pi', k) = \mathbf{V}(\pi', \mathbb{Q}^{ab})$, i.e. all the Tate cycles are defined over abelian extensions of $\mathbb{Q}$. For π' of CM type, it is possible to have for all k that $\mathbf{V}(\pi', k) \neq \mathbf{V}(\pi', \mathbb{Q}^{ab})$.

We denote by $r_{alg}(\pi'; \nu) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{U}(\pi', \nu)$, by $r_l(\pi'; \nu) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', \nu)$, by $r_{an}(\pi', \nu)$ the order of the pole of $L(s, \rho(\pi') \otimes \nu)$ at $s = 1$. Then $r_{alg}(\pi'; \nu) \leq r_l(\pi'; \nu)$, $r_l(\pi', \mathbb{Q}^{ab}) \leq r_l(\pi', \bar{\mathbb{Q}})$, $r_{alg}(\pi'; \nu) = r_{alg}(\pi' \otimes \nu|_{\Gamma_F}; 1)$, $r_l(\pi'; \nu) = r_l(\pi' \otimes \nu|_{\Gamma_F}; 1)$ and $r_{an}(\pi'; \nu) = r_{an}(\pi' \otimes \nu|_{\Gamma_F}; 1)$.

6 Matching Tate cycles and poles

We say that an automorphic representation π of $GL(2)/F$ for some number field F is of CM *type* if there exists some quadratic Galois character $\eta : I_F/F^\times \to \bar{\mathbb{Q}}_l^\times$, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. We say that a representation ρ of a group G is *dihedral* if there exists a normal subgroup N of index 2 in G and a character $\chi : N \to \mathbb{C}^\times$ such that $\rho = \mathrm{Ind}_N^G \chi$. If π is an automorphic representation of $GL(2)/F$ as in proposition 3.1, then π is of CM type if and only if ρ_π is a dihedral representation. If π corresponds to an automorphic representation π' of $\bar{G}(\mathbb{A}_f)$ by Jacquet-Langlands correspondence, then we say that π' is CM if π is CM.

Using in particular the decomposition (in some cases) of $\rho(\pi')$ as a sum of automorphic representations, more exactly as a direct sum of l-adic representations associated Hecke characters and to twists by Hecke characters of $\mathrm{Sym}^2\pi''$ and $\mathrm{Sym}^4\pi''$ for non-CM representations π'' of GL(2), which from [JG] and [K] we know that are cuspidal and irreducible, in [R] (propositions 8.6 and 8.8) are proved the following two lemmas:

Lemma 6.1. *For π' non-CM, all the Tate classes in $V(\pi')$ are rational over an abelian number field k, with*

$$r_l(\pi', k) \leq 2,$$

hence

$$r_l(\pi', \mathbb{Q}^{ab}) = r_l(\pi', \bar{\mathbb{Q}}) \leq 2.$$

Lemma 6.2. *Let $F/\mathbb{Q}$ be Galois, and π' non-CM. Then*
(a) $r_l(\pi', \mathbb{Q}^{ab}) \neq 0$ iff a twist of π is a base change from a quadratic subextension of F.
(b) $r_l(\pi', \mathbb{Q}^{ab}) = 2$ iff a twist of π is a base change from $\mathbb{Q}$.
(c) The following are equivalent:
(i) $r_l(\pi'; \nu) = 2$ for some ν.
(ii) A twist of π is a base change from $\mathbb{Q}$, and F is biquadratic.

From lemma 6.1 above and §8 of [R] we know that:

Proposition 6.3. *For π' non-CM we have:*

$$r_l(\pi'; \nu) = r_{an}(\pi'; \nu) \leq 2,$$

and thus because $r_{alg}(\pi'; \nu) \leq r_l(\pi'; \nu)$, we have

$$r_{alg}(\pi'; \nu) \leq r_l(\pi'; \nu) = r_{an}(\pi'; \nu) \leq 2.$$

We also know that (see proposition 8.5 of [R]):

Proposition 6.4. *If π' is of CM type, we have for any k,*

$$r_l(\pi', k) = r_{an}(\pi', k).$$

7 Twisted Hirzebruch-Zagier cycles

We use the same notations as in §2, i.e. we consider a quaternion division algebra D_0 over some quadratic subextension F_0 of F such that $D = D_0 \otimes_{F_0} F$. Then the map h factors through the map h_0 of $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ into $\bar{G}_{0\mathbb{R}}$. The natural diagonal embedding of $\bar{G}_0$ into $\bar{G}$ defines a morphism

$$\delta_{L,K} : S_{0L} \hookrightarrow S_K$$

over $\mathbb{Q}$, if L is contained in K.

For any $g \in \bar{G}(\mathbb{A}_f)$, and any open compact subgroup K of $\bar{G}(\mathbb{A}_f)$, define the corresponding *Hirzebruch-Zagier cycle* (or *H-Z cycle*) (relative to $\bar{G}_0$) to be the algebraic cycle of codimension 2 of S_K given by

$$^{D_0}Z_{g,K} = R(g)(\delta_{\bar{G}_0(\mathbb{A}_f)\cap gKg^{-1}, gKg^{-1}}(S_{0\bar{G}_0(\mathbb{A}_f)\cap gKg^{-1}})),$$

where $R(g) : S_{gKg^{-1}} \to S_K$ is the right translation action on Shimura varieties.

Now for each character of finite order μ of F, we have the usual *twisted correspondence* $R(\mu) \subset S_K \times S_{K[\mu]}$, where $K[\mu]$ is some level which depends on K and μ (see for example §5 of [R] for details). This twisting correspondence is algebraic and acts on any cohomology group, Betti or étale, of the fourfold $S = \varprojlim S_K$. The induced operator sends the π'-component to the $\pi' \otimes \mu$-component. The twisting correspondence $R(\mu)$ is rational over $\mathbb{Q}(\mu_1)$, where $\mu_1 = \mu|_{I_\mathbb{Q}}$, and $I_\mathbb{Q}$ is the idele group of $\mathbb{Q}$.

For each character of finite order μ of F and each H-Z cycle Z on S, let $Z(\mu)$ be the *μ-twisted H-Z cycle* obtained by pushing-forward Z under $R(\mu)$. Then $Z(\mu)$ is algebraic and rational over $\mathbb{Q}(\mu_1)$.

8 Matching algebraic cycles and poles

We prove:

Proposition 8.1. *Let F be a quartic, Galois, totally real number field, and π' be a non-CM cuspidal automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$ of weight 2 that appears in proposition 3.1. Then for any character of finite order ν of $\Gamma_\mathbb{Q}$, we have*

$$r_{alg}(\pi'; \nu) = r_{an}(\pi'; \nu).$$

Proof: From proposition 6.3 we know that $r_{alg}(\pi'; \nu) \leq r_{an}(\pi'; \nu) \leq 2$.

We distinguish 3 cases:

A) $r_{an}(\pi'; \nu) = 0$. Then $r_{alg}(\pi'; \nu) = 0$ and we are done.

B) $r_{an}(\pi'; \nu) = 1$. Then as in the proof of theorem 9.1 of [R], one can find a quadratic subfield F_1 of F and a finite order character μ of F such that $r_{alg}(\pi'; \nu) = r_{alg}(\pi' \otimes \mu; 1)$ and such that $L(s, As_{F/F_1}(\pi' \otimes \mu)) = L(s, As_{F/F_1}(\pi \otimes \mu))$ has a simple pole at $s = 1$, which by the residue formula of [HLR] implies that there exists some function ϕ in the space of π such that

$$\int_{GL(2,F_1)Z(\mathbb{A}_{F_1})\backslash GL(2,\mathbb{A}_{F_1})} \phi(g)\mu(det(g))dg \neq 0,$$

where Z denotes the center of $GL(2)$. From [JL] (the main theorem) and [FH] (appendix), we deduce that that there exists some function ϕ' in the space of π' such that

$$\int_{\bar{G}_1(\mathbb{Q})\bar{Z}_1(\mathbb{A}_{\mathbb{Q}})\backslash\bar{G}_1(\mathbb{A}_{\mathbb{Q}})} \phi'(g)\mu(det(g))dg \neq 0,$$

where $\bar{Z}_1$ denotes the center of $\bar{G}_1 = \mathrm{Res}_{F_1/\mathbb{Q}}(G_1)$, and G_1 is the algebraic group over F_1 defined by the multiplicative group $D_1^\times$ of a suitable quaternion division algebra D_1 over F_1 which satisfies that $D = D_1 \otimes_{F_1} F$ (more exactly let $\mathcal{S}$ be the set of places v of F_1 which split into two different places w and $\bar{w}$ of F such that D_w and $D_{\bar{w}}$ are ramified (we remark that because $D = B \otimes_{\mathbb{Q}} F$, we get that $B \otimes_{\mathbb{Q}} F_1$ is a quaternion division algebra over F_1, and thus we have that for each two different places w and $\bar{w}$ of F dividing a place v of F_1, D_w and $D_{\bar{w}}$ have the same invariant). If $|\mathcal{S}|$ is even then there exists a quaternion division algebra D_1 over F_1 which ramifies at exactly the places v in $\mathcal{S}$ such that $D = D_1 \otimes_{F_1} F$. Then by the main theorem of [JL], D_1 satisfies the above propriety. If $|\mathcal{S}|$ is odd, then from [FH], appendix, we know that there exists a place (actually infinitely many) v_1 of F_1 outside S which does not split in F and a quaternion division algebra D_1 over F_1 which is ramified at exactly the places v in $\mathcal{S} \cup v_1$, such that $D = D_1 \otimes_{F_1} F$ and has the above propriety). Hence, the integral of a $(2,2)$-form $\eta_{\phi'}$ on the compact quaternionic Shimura fourfold S_K defined by ϕ' has a non-zero μ-twisted period over a Hecke translate of the embedded compact quaternionic Shimura surface attached to D_1. Thus the corresponding twisting self-correspondence of the fourfold, defines for some $g \in \bar{G}(\mathbb{A}_f)$ a μ-twisted H-Z cycle $Z(\mu) =^{D_1} Z_{g,K}(\mu)$ of codimension 2 such that

$$\int_{Z(\mu)} \eta_{\phi'} \neq 0,$$

and hence $Z(\mu)$ is *homologically non-trivial*. Thus $r_{alg}(\pi';\nu) \geq 1$, and we obtain that $r_{alg}(\pi';\nu) = 1$, and we are done.

C) $r_{an}(\pi';\nu) = 2$. From part (c) of lemma 6.2 we deduce that $F/\mathbb{Q}$ is biquadratic, and then as in [R], one can find a finite order character μ of F such that $r_{alg}(\pi';\nu) = r_{alg}(\pi' \otimes \mu; 1)$ and such that $\pi \otimes \mu$ is a base change from two quadratic subfields F_1 and F_2 of F. Then as in B) because the functions $L(s, As_{F/F_1}(\pi' \otimes \mu))$ and $L(s, As_{F/F_2}(\pi' \otimes \mu))$ have simple poles at $s = 1$, we get, for some quaternion algebras D_1 and D_2 over F_1 and F_2 and some $g_1, g_2 \in \bar{G}(\mathbb{A}_f)$, two twisted codimension 2 algebraic cycles $Z_1 :=^{D_1} Z_{g_1,K}(\mu)$ and $Z_2 :=^{D_2} Z_{g_2,K}(\mu)$ on S which are *homologically non-trivial* because the period integrals of some $(2,2)$-forms over these cycles are non-zero. But these two cycles could be proportional in the π'-component of the cohomology, and thus one may have to replace one of them by a twisted version. Then in [R], lemma 9.3, a finite order character ξ of F is defined such that it has some special signature at the infinite places, and such that $\xi|_{I_{F_1}} = 1$, and thus $\xi|_{I_{\mathbb{Q}}} = 1$ and hence

$$r_{an}(\pi' \otimes \xi; \nu) = r_{an}(\pi'; \nu).$$

Also $L(s, As_{F/F_1}(\pi' \otimes \xi\mu))$ has a simple pole at $s = 1$, and thus, if we define

$$Z_3 :=^{D_1} Z^*_{g_3,K}(\mu\xi),$$

we have for some $g_3 \in \bar{G}(\mathbb{A}_f)$, and some ϕ' in the space of π' that

$$\int_{Z_3} \eta_{\phi'} \neq 0.$$

Then, from lemma 9.4 of [R], we know by looking at the signatures at infinite places of the classes of Z_1, Z_2 and Z_3 in $V_B(\pi')$ that the space spanned by the classes of Z_1, Z_2 and Z_3 in $V_B(\pi')$ has dimension 2. Thus $r_{alg}(\pi'; \nu) \geq 2$, and we obtain that $r_{alg}(\pi'; \nu) = 2$, and we are done.

We can deduce now the following result:

Theorem 8.2. *Let F be a quartic totally real number field containing a quadratic subfield. Let π' be an automorphic representation as in proposition 3.1. Then*

(a) For any solvable number field k, the function $L(s, \rho(\pi')|_{\Gamma_k})$ has a meromorphic continuation to the complex plane and satisfies a functional equation $s \leftrightarrow 1 - s$.

(b) If $F/\mathbb{Q}$ is Galois, then for any solvable number field k we have that $\dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', k)$ is equal to the order of the pole of the function $L(s, \rho(\pi')|_{\Gamma_k})$ at $s = 1$. If π' is CM this result is true for any number field k.

(c) If $F/\mathbb{Q}$ is Galois and π' is non-CM, then for any number field k we have

$$\dim_{\bar{\mathbb{Q}}_l} \mathbf{U}(\pi', k) = \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', k).$$

Proof: Part (a) is the statement of proposition 4.1. Now assume that $F/\mathbb{Q}$ is Galois and π' is non-CM. Then from propositions 6.3 and 8.1 we get that $r_{alg}(\pi'; \nu) = r_l(\pi'; \nu) = r_{an}(\pi'; \nu) \leq 2$, and from lemma 6.1 we deduce part (c). Now if π' is CM, part (b) is the statement of proposition 6.4. For π' non-CM and k solvable we know from (a) that $L(s, \rho(\pi')|_{\Gamma_k})$ has a meromorphic continuation to the complex plane, and we have to match the order of the pole of $L(s, \rho(\pi')|_{\Gamma_k})$ at $s = 1$ with the dimension of the space of the Tate cycles defined over k. But because k is solvable and π' is non-CM we get that $\rho_{\pi'}|_{\Gamma_{Fk}}$ is cuspidal irreducible. Now because K contains a quadratic subextension F_0 we get that $\rho(\pi')|_{\Gamma_k}$ is a tensor product of Asai representations of degree 4, 2 or 1 associated to cuspidal representations of $GL(2)$. When we have degree 4, i.e $\rho(\pi')|_{\Gamma_k} = As_{Fk/k}\rho_{\pi''}$ for some cuspidal non-CM automorphic representation π'' of $GL(2)/Fk$ (associated to $\rho_{\pi'}|_{\Gamma_{Fk}}$) and Fk has a quadratic subextension k'/k, we obtain part (b) exactly as in [R] §8 (it is proved in the same way as proposition 6.3 above). The rest of the cases are similar. □

References

[B] D. Blasius, *Hilbert modular forms and the Ramanujan conjecture*, Noncommutative geometry and number theory, 35–56, Aspects Math., E37, Vieweg, Wiesbaden, 2006.

[C] H. Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Compositio Mathematica, 59, nr.2 (1986), 151-230.

[FH] Y. Z. Flicker, J. L. Hakim, *Quaternionic Distinguished Representations*, American J. of Math., Vol. 116, No. 3. (Jun., 1994), 683-736.

[HLR] G. Harder, R .P. Langlands, M.Rapoport, *Algebraische Zycklen auf Hilbert-Blumenthal-Flächen*, J. Reine Angew. Math. 366, 53-120 (1986).

[JG] H. Jacquet, S. Gelbart, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, Ann. Sci. École Norm. Sup. 11(1979), 471-542.

[JL] H. Jacquet and K. Lai, *A relative trace formula*, Compositio Math. 54 (1985), 243-310.

[JS] H. Jacquet, J. A. Shalika, *Euler products and the classification of automorphic forms I and II*, Amer. J. of Math. 103(1981), 499-558 and 777-815.

[JPSS] H. Jacquet, I. I. Piatetski-Shapiro, J.A.Shalika, *Rankin-Selberg convolutions*, American J. of Math., 105, nr. 2 (1983), 367-464.

[K] C. Klingenberg, *Die Tate-Vermutungen für Hilbert-Blumenthal-Flächen*, Invent. Math. 89, 291-317(1987).

[KI] H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth power of GL_2*, Journal of the AMS 16, No. 1(2002), 139-183.

[L] K. F. Lai, *Algebraic cycles on compact Shimura surface*, Math. Z. 189, 593-602 (1985).

[LA] R. P. Langlands, *Base change for GL(2)*, Ann. of Math. Studies 96, Princeton University Press, 1980.

[MR] V. K. Murty, D. Ramakrishnan, *Period relations and the Tate conjecture for Hilbert modular surfaces*, Invent. Math. 89, 319-345(1987).

[R] D. Ramakrishnan, *Algebraic cycles on Hilbert modular fourfolds and poles of L-functions*, Proceedings of the International Conference on Algebraic Groups and Arithmetic, 271-274 (2005).

[R1] D. Ramakrishnan, *Modular curves, modular surfaces and modular fourfolds* in Algebraic Cycles and Motives, dedicated to Jacob Murre, Cambridge University Press.

[R2] D. Ramakrishnan, *Modularity of solvable Artin representations of GO(4)-type*, IMRN 2002, No. 1, 1-54.

[T] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Invent, Math., 98, 1989, 265-280.

[TA] J. Tate, *Algebraic cycles and poles of zeta functions*, in: Schilling, O.D.G. (ed.), Arithmetical algebraic geometry, New York: Harper and Row, 1966.