Algebraic cycles on a product of two Hilbert modular surfaces

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1 Introduction

Let X be a smooth projective variety of dimension n defined over a number field F and let

$$\bar{X} = X \times_F \overline{\mathbb{Q}}.$$

For a prime number l, let $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ be the étale cohomology of $\bar{X}$. If K is a number field, we denote $\Gamma_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. The Galois group Γ_F acts on $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ by a representation $\phi_{i,l}$. For any $j \in \mathbb{Z}$, let $H^i_{et}(X, \overline{\mathbb{Q}}_l)(j)$ denote the representation of Γ_F on $H^i_{et}(X, \overline{\mathbb{Q}}_l)$ defined by $\phi_{i,l} \otimes \xi_l^j$, where ξ_l is the l-adic cyclotomic character. The elements of $V^i(X,E) := (H^{2i}_{et}(X, \overline{\mathbb{Q}}_l)(i))^{\Gamma_E}$ are called *Tate cycles* on X defined over E. The union

$$V^i(X) := \cup_E V^i(X,E)$$

is the space of all *Tate cycles* on X.

To each algebraic subvariety Y of X of codimension i, one can associate a cohomology class

$$[Y] \in H_{2n-2i}(X(\mathbb{C}), \mathbb{Q}) \cong H^{2i}_B(X(\mathbb{C}), \mathbb{Q})(i),$$

where $H^{2i}_B(X(\mathbb{C}), \mathbb{Q})$ is the Betti cohomology. Then using the isomorphism

$$H^{2q}_B(X(\mathbb{C}), \mathbb{Q})(i) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H^{2i}_{et}(X, \mathbb{Q}_l)(i),$$

we obtain a class $[Y] \in H^{2i}_{et}(X, \mathbb{Q}_l)(i)$. A cohomology class $[Y]$ obtained in this way is called *algebraic*. If Y is defined over an extension E of F, then we obtain a class $[Y] \in H^{2i}_{et}(X, \mathbb{Q}_l)(i)^{\Gamma_E}$. The first part of the Tate conjecture states that every Tate cycle is algebraic.

The L-function $L^{2i}(s, X_{/F})$ (more exactly the Euler product) attached to the representation $\phi_{2i,l}$ converges for $\mathrm{Re}(s) > i+1$. The second part of the Tate conjecture [TA] states that the L-function $L^{2i}(s, X_{/E})$ has a meromorphic continuation to the entire complex plane and has a pole at $s = i+1$ of order equal to

$$\dim_{\overline{\mathbb{Q}}_l} V^i(X, E).$$

In their work [HLR], Harder, Langlands and Rapoport had proved the first part of the Tate conjecture for Hilbert modular surfaces for non-CM sub-motives. In [K] and [MR] it was proved the first part of the Tate conjecture for Hilbert modular surfaces for CM sub-motives and thus using the two results, one gets the full first part of the Tate conjecture asserting the algebraicity of all the Tate cycles of Hilbert modular surfaces over an arbitrary number field. The second part of the Tate conjecture for Hilbert modular surfaces was proved also in [HLR], [K] and [MR], but only for solvable number fields.

In this paper we consider a quadratic real field F and let $G = \mathrm{Res}_{F/\mathbb{Q}} GL_{2/F}$. Let $S_{K_i} := S_{G,K_i}$ be the Hilbert modular surface associated to an open compact subgroup K_i of $G(\mathbb{A}_f)$, for $i = 1, 2$, where $\mathbb{A}_f$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. In this paper we prove the first part of the Tate conjecture for $S_{K_1} \times S_{K_2}$ for the tensor product of non-CM sub-motives of the individual factors S_{K_1} and S_{K_2} (see theorem 7.1 for details). We prove also the second part of the Tate conjecture for $S_{K_1} \times S_{K_2}$, but only for solvable number fields (see proposition 8.1).

We remark that in [MP], it was computed the space of Tate cycles on the product of two arbitrary Hilbert modular surfaces in terms of automorphic representations including the exact determination of their fields of definition, but in [MP] it was not proved the algebraicity of all Tate cycles, but only of those spanned by tensor product of algebraic cycles on individual factors, which from [HLR], [K] and [MR] we know that are algebraic.

We remark the first part of the Tate conjecture is also known for the non-CM submotives of the Shimura surfaces treated in [LA] and [FH], corresponding to a quadratic real field F and to a quaternion algebra $D = B \otimes_{\mathbb{Q}} F$, where B is a quaternion algebra over $\mathbb{Q}$, such that D splits at the real places. Let H be the algebraic group over F associated to $D^\times$ and let $G' = \mathrm{Res}_{F/\mathbb{Q}} H$. Let $\mathbf{S}_{K_i} := S_{G',K_i}$ be the quaternionic Shimura surface corresponding to an open compact subgroup K_i of $G'(\mathbb{A}_f)$, for $i = 1, 2$. Then the surfaces $\mathbf{S}_{K_i}$ for $i = 1, 2$ are compact. In the same way as we do in this paper one can prove the first part of the Tate conjecture for $\mathbf{S}_{K_1} \times \mathbf{S}_{K_2}$ for the tensor product of non-CM sub-motives of the individual factors $\mathbf{S}_{K_1}$ and $\mathbf{S}_{K_2}$. We remark that because the quaternionic Shimura surfaces are compact, one has to replace the intersection cohomology in this paper by the étale cohomology and everything else remains unchanged. The second part of the Tate conjecture for $\mathbf{S}_{K_1} \times \mathbf{S}_{K_2}$, is also true for solvable number fields.

2 Hilbert modular surfaces

Let F be a real quadratic field and let $G = \mathrm{Res}_{F/\mathbb{Q}} GL_{2/F}$. For K sufficiently small open compact subgroup of $G(\mathbb{A}_f)$, let S_K be the smooth toroidal compactification of an open surface S_K^0 which satisfies

$$S_K^0(\mathbb{C}) = G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_\infty K,$$

where $K_\infty = SO_2(\mathbb{R})\mathbb{R}^\times \times SO_2(\mathbb{R})\mathbb{R}^\times \subset G(\mathbb{R})$. Then S_K is a surface defined over $\mathbb{Q}$ and it is called a Hilbert modular surface. Let $\mathbf{S} = S_1 \times S_2$, where $S_i = S_{K_i}$ is the Hilbert modular surface corresponding to a sufficiently small open compact subgroup K_i of $G(\mathbb{A}_f)$, for $i = 1, 2$.

3 Cohomology for Hilbert modular surfaces

If π is a cuspidal automorphic representation of weight 2 of $GL(2)/F$, then from Taylor [T], we know that there exists a λ-adic representation (for λ a prime of the field of coefficients $\mathbf{O}$ of π, such that $\lambda | l$ for some rational prime l)

$$\rho_{\pi,\lambda} : \Gamma_F \to GL_2(\mathbf{O}_\lambda) \hookrightarrow GL_2(\bar{\mathbb{Q}}_l),$$

which satisfies $L_v(s, \pi) = L_v(s, \rho_{\pi,\lambda})$ for almost all finite places v of F and is unramified outside the primes dividing $\mathbf{n}l$, where $\mathbf{n}$ is the level of π. Here if we fix an isomorphism $i : \bar{\mathbb{Q}}_l \to \mathbb{C}$ and if $\rho_{\pi,\lambda}$ is unramified at v, then

$$L_v(s, \rho_{\pi,\lambda}) = \det(1 - i(\rho_{\pi,\lambda}(\mathrm{Frob}_v))Nv^{-s})^{-1},$$

where Frob_v is a geometric Frobenius. In order to simplify the notations we denote by ρ_π the representation $\rho_{\pi,\lambda}$.

Let K be a sufficiently small open compact subgroup of $G(\mathbb{A}_f)$. Then we have a decomposition

$$H^2_{et}(S_K, \bar{\mathbb{Q}}_l) = IH^2_{et}(S_K, \bar{\mathbb{Q}}_l) \oplus H^2(S_K^\infty, \bar{\mathbb{Q}}_l)$$

where $IH^2_{et}(S_K, \bar{\mathbb{Q}}_l)$ is the intersection cohomology of the Baily-Borel compactification $\bar{S}_K$ of S_K^0, and S_K^∞ is the divisor at infinity (a finite set of cusps) such that $\bar{S}_K = S_K^0 \cup S_K^\infty$, and is defined by

$$IH^2_{\mathrm{et}}(S_K, \mathbb{Q}_l) := \mathrm{Im}(H^2_{\mathrm{et}}(\bar{S}_K, \mathbb{Q}_l) \to H^2_{\mathrm{et}}(S_K^0, \mathbb{Q}_l)).$$

If l is a prime number, let $\mathbb{H}_K$ be the Hecke algebra generated by the bi-K-invariant $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $G(\mathbb{A}_f)$ under the convolution. If $\pi = \pi_f \otimes \pi_\infty$ is an automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$, we denote by π_f^K the space of K-invariants in π_f. The Hecke algebra $\mathbb{H}_K$ acts on π_f^K.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group $\Gamma_{\mathbb{Q}}$ on the intersection cohomology $IH^2_{et}(S_K, \bar{\mathbb{Q}}_l)$ and these two actions commute. Then we know (see for example proposition 1.8 of [RT]):

Proposition 3.1. *The representation of $\Gamma_{\mathbb{Q}} \times \mathbb{H}_K$ on the intersection cohomology $IH^2_{et}(S_K, \bar{\mathbb{Q}}_l)(1)$ is isomorphic to*

$$\oplus_\pi \rho(\pi) \otimes \pi_f^K,$$

where $\rho(\pi)$ is a representation of the Galois group $\Gamma_{\mathbb{Q}}$. The above sum is over weight 2 cuspidal automorphic representations π of $G(\mathbb{A}_{\mathbb{Q}})$, such that $\pi_f^K \neq 0$ and the $\mathbb{H}_K$-representations π_f^K are irreducible and mutually inequivalent.

If K' and K are open compact subgroups of $G(\mathbb{A}_f)$ such that $K' \subseteq K$, then we obtain a finite morphism $S_{K'} \to S_K$, and if $g \in G(\mathbb{A}_f)$, we obtain a morphism $S_K \to S_{gKg^{-1}}$. We consider the inverse limit

$$S(\mathbb{C}) := \varprojlim_K S_K(\mathbb{C}) = G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_\infty.$$

The scheme S has a $G(\mathbb{A}_f)$-action that verifies:

$$S/K = S_K.$$

In this paper it is convenient to consider the direct limit of $IH^2_{et}(S_K, \bar{\mathbb{Q}}_l)$ as K shrinks to the identity. We have

$$IH^2_{et}(S, \bar{\mathbb{Q}}_l) = \varinjlim_K IH^2_{et}(S_K, \bar{\mathbb{Q}}_l),$$

and

$$IH^2_{et}(S, \bar{\mathbb{Q}}_l)^K = IH^2_{et}(S_K, \bar{\mathbb{Q}}_l).$$

Then, from proposition 3.1, we get

$$IH^2_{et}(\bar{S}, \bar{\mathbb{Q}}_l)(1) = \oplus_\pi \rho(\pi) \otimes \pi_f.$$

The representation $\rho(\pi)$ which is semisimple (see §4 of HLR] or corollary 3.8 of [G]) and from [MP] for example, we know that $\rho(\pi)$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(\rho_\pi \otimes \rho_\pi^\tau),$$

which verifies

$$\rho(\pi)|_{\Gamma_F} = \rho_\pi \otimes \rho_\pi^\tau,$$

where τ is the non-trivial automorphism of F over $\mathbb{Q}$, and ρ_π^τ is defined by

$$\rho_\pi^\tau(\gamma) = \rho_\pi(\tau\gamma\tau^{-1}).$$

3.1 Betti cohomology

We define the cuspidal part of the Betti cohomology:

$$H^2_{B,\mathrm{cusp}}(S_K) := \mathrm{Im}(H^2(\bar{S}_K(\mathbb{C}), \mathbb{Q}) \to H^2(S^0_K(\mathbb{C}), \mathbb{Q})),$$

where $\bar{S}_K$ is the Baily-Borel compactification of S^0_K.

In this paper it is convenient to consider the direct limit of $H^2_{B,\mathrm{cusp}}(S_K)$ as K shrinks to the identity. We have

$$H^2_{B,\mathrm{cusp}}(S) = \varinjlim_K H^2_{B,\mathrm{cusp}}(S_K).$$

Then

$$H^2_{B,\mathrm{cusp}}(S)^K = H^2_{B,\mathrm{cusp}}(S_K).$$

There exists the Hodge decomposition:

$$H^2_{B,\mathrm{cusp}}(S_K) \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=2} H^{p,q}(S_K),$$

where $H^{p,q}(S_K) = H^p(S_K(\mathbb{C}), \Omega^q)$, with Ω^q the sheaf of holomorphic q-forms.

We have the canonical isomorphisms:

$$\Phi_{\mathrm{et},K} : H^2_{B,\mathrm{cusp}}(S_K) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to IH^2_{\mathrm{et}}(S_K, \mathbb{Q}_l),$$

and

$$\Phi_{\mathrm{et}} : H^2_{B,\mathrm{cusp}}(S) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to IH^2_{\mathrm{et}}(S, \mathbb{Q}_l).$$

We denote by $V_l(\pi)$ the π-component of $IH^2_{\mathrm{et}}(S, \mathbb{Q}_l)(1)$ in the decomposition of proposition 3.1 and by $V_B(\pi)$ the corresponding π-component of $H^2_{B,\mathrm{cusp}}(S)$. Thus

$$V_B(\pi) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong V_l(\pi).$$

For $s = (s_1, s_2)$, with each s_1 and s_2 being $+$ or $-$, we can define a *real analytic automorphism* τ_s of $F \otimes \mathbb{C} - F \otimes \mathbb{R} \cong \mathbb{C}^2 - \mathbb{R}^2$ by

$$\tau_s(z_1, z_2) = (\tau_{s_1}(z_1), \tau_{s_2}(z_2)),$$

where τ_{s_j} is the identity, respectively complex conjugation, if s_j is $+$, respectively $-$. Each such involution τ_s acts on the Hilbert modular surface S and its Betti cohomology. It also commutes with the Hecke action of $G(\mathbb{A}_f)$ and we obtain a decomposition

$$V_B(\pi) \cong \oplus_{s \in \Sigma} V_B(\pi)^s,$$

where $\Sigma = \{s = (\pm, \pm)\}$ and $V_B(\pi)^s$ denotes the s-eigenspace of $V_B(\pi)$. Obviously Σ forms a group under componentwise multiplication with identity $(+,+)$ and each eigenspace $V_B(\pi)^s$ is one-dimensional.

4 Twisted correspondence

In this section we defined some twisted correspondence on Hilbert modular surfaces (see §2 of [MR] for details). Let π be a cuspidal automorphic representation of weight 2 with conductor $\mathfrak{a} = \mathfrak{a}(\pi)$, which is an integral ideal in F. Write $\mathfrak{a}_v = \mathfrak{u}_v^{n_v(\mathfrak{a})} \mathfrak{o}_v$, $n_v(\mathfrak{a}) = \mathrm{ord}_v(\mathfrak{a}) \geq 0$, and set:

$$K(\pi) = K_1(\mathfrak{a}) = \prod_v K_1(\mathfrak{a}_v),$$

where

$$K_1(\mathfrak{a}_v) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{a}_v) | c, (d-1) \in \mathfrak{u}_v^{n_v(\mathfrak{a})} \mathfrak{o}_v \},$$

(when $n_v(\mathfrak{a})$ is zero, $K_1(\mathfrak{a}_v)$ is equal to $GL(2, \mathfrak{o}_v)$).

One knows that $\mathfrak{a}$ is the smallest of all the non-zero integral ideals $\mathfrak{i}$ in O_F such that $\dim \pi_f^{K_1(\mathfrak{i})} \neq 0$. Furthermore we have $\dim \pi_f^{K_1(\mathfrak{i})} = 1$.

Let μ be any finite order character of the ideles group I_F of F of conductor $\mathfrak{c}$. Let $K = K_1(\mathfrak{a})$. Set

$$K[\mu] := K_1(\mathrm{lcm}(\mathfrak{a}, \mathfrak{c}^2))$$

Let $\mathfrak{D}_c$ be the ring of integers of $F_\mathfrak{c} := \prod_{v|\mathfrak{c}} F_v$, and let X be the subset of $F_\mathfrak{c}$ defined by

$$X := \{x = (x_v) \in F_\mathfrak{c} | v(x_v) \geq -v(\mathfrak{c}), \forall v\}.$$

Let $\tilde{X}$ be a set of representatives in X for $X \bmod \mathfrak{D}_\mathfrak{c}$, which is a group isomorphic to $\mathfrak{D}_\mathfrak{c}/\mathfrak{c}\mathfrak{D}_\mathfrak{c}$. For each t in $\tilde{X}$, we define $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. We have that

$$S_K^0(\mathbb{C}) = \cup_{j=1}^{h(K)} \Gamma_j \setminus (\mathbb{C} - \mathbb{R})^2,$$

with

$$\Gamma_j = G(\mathbb{Q}) \cap x_j G(\mathbb{R})^+ K x_j^{-1},$$

where

$$x_j = \begin{pmatrix} b_j & 0 \\ 0 & 1 \end{pmatrix},$$

where $\mathbb{A}_F^\times = \cup_{j=1}^{h(K)} F^\times b_j F_\infty^+ \det K$, with $b_j \in \mathbb{A}_{F,f}$, and $F_\infty^\times$ is the subset of $F \otimes \mathbb{R} \cong \mathbb{R}^2$ consisting of totally positive elements, and $G(\mathbb{R})^+$ is the subgroup of $G(\mathbb{R})$ consisting of totally positive elements.

For every x in $G(\mathbb{A}_f)$ we have the usual *Hecke correspondence* $T(x)$ on S_K^0, which depends only on the double coset KgK, and is given by $T(x)(g) = R(x)(R(1)^{-1}(g))$, where if $K_x := K \cap xKx^{-1}$, then $R(1) : S_{K_x}^0 \to S_K^0$ and $R(x) : S_{K_x}^0 \to S_K^0$, are obtained from the two homomorphisms $K_x \to K$ given by the inclusion and by conjugation by x^{-1}. Then $T(x)$ does not in general preserve the connected components ${S_K^{(j)}}^0$ of S_K^0. Let $T_j(x)$ denote the restriction of $T(x)$ to ${S_K^{(j)}}^0$ when det(x)=1.

We define the *twisting correspondence* $R(\mu) \subset S_K^0 \times S_{K[\mu]}^0$ by

$$R(\mu) = \sum_{j=1}^{h(K)} \mu_f(\det(x_j)) R_j(\mu),$$

with

$$R_j(\mu) = \sum_{t \in \tilde{X}} T_j(u_t).$$

Then for all $x \in G(\mathbb{A}_f)$, we have

$$T(x) \circ R(\mu) = \mu_f(\det(x)) R(\mu) \circ T(x).$$

We remark that the correspondences $R_j(\mu)$ and $R(\mu)$ could be extended to the Baily-Borel compactifications S^j_K and S_K (obtained by adding cusps to $S^{(j)}_K{}^0$ and S^0_K). The twisting correspondence is algebraic and acts on any cohomology group, Betti or étale, of the surface $S = \lim S_K$ and of the variety $S \times S$, and is compatible under the isomorphisms $\Phi_{\mathrm{et},K}$ and Φ_{et}, and $\Phi_{\mathrm{et},K} \times \Phi_{\mathrm{et},K[\mu]}$ and $\Phi_{\mathrm{et}} \times \Phi_{\mathrm{et}}$ defined in §3. The induced operator sends the π-component to the $\pi \otimes \mu$-component. The twisted correspondence $R(\mu)$ is rational over the abelian extension $\mathbb{Q}^{\mu|_{I_\mathbb{Q}}}$ of $\mathbb{Q}$, determined by $\mu|_{I_\mathbb{Q}}$, where $I_\mathbb{Q}$ is the ideles group of $\mathbb{Q}$. We denote by $Z_{g_1,g_2}(\mu)$ the *algebraic cycle of codimension* 2 of $S_K \times S_{K[\mu]}$ induced by $T(g_2) \circ R(\mu) \circ T(g_1)$, where $g_1, g_2 \in G(\mathbb{A}_f)$.

If μ is a finite order character of F, then its component at any archimedian place is 1 or the *sign character*, and we define the *signature* $s(\mu)$ of μ to be $(s(\mu)_1, s(\mu)_2)$, where $s(\mu)_j$ is the sign of μ_1, resp. μ_τ (here τ is the non-trivial automorphism of F over $\mathbb{Q}$) for $j = 1$ or 2 (at any archimedian place u, the sign of μ_u is $+$, resp. $-$, if μ_u is trivial, resp. non-trivial).

It is easy to see that:

Lemma 4.1. *The twisting correspondence $R(\mu)$ sends a vector in $V_B(\pi)^s$ into $V_B(\pi \otimes \mu)^{ss(\mu)}$.*

5 Known results

It is known that (see for example proposition 4.5.4 of [HLR]):

Proposition 5.1. *If π is a cuspidal automorphic representation of weight 2 of $GL(2)/F$, where F is a totally real field. Then one of the following two statements holds:*

(i) $\rho_\pi|_{\Gamma_L}$ is irreducible for each finite extension L/F.

(ii) There exists a quadratic extension L/F and an algebraic Hecke character ψ of L such that $\rho_\pi \cong \mathrm{Ind}^F_L(\psi)$.

We say that a representation ρ of a group G is *dihedral* if there exists a normal subgroup N of index 2 in G and a character $\psi : N \to \mathbb{C}^\times$ such that $\rho = \mathrm{Ind}^G_N \psi$.

We say that an automorphic representation π of $GL(2)/L$ for some number field L is of CM *type* if there exists some quadratic Galois character $\eta : I_L/L^\times \to \mathbb{C}^\times$, where I_L is the ideles group of L, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. If π is an automorphic representation of weight 2 of $GL(2)/L$, then π is of CM type if and only if ρ_π is a dihedral representation.

We know the following result (theorem 2.1 of [MP]):

Proposition 5.2. *The tensor product of two 2 dimensional irreducible complex representations of a group is reducible only if either both representations are dihedral or they are the twist of each other by a character.*

We know (lemma 4.2 of [MP]):

Proposition 5.3. *Let π_1 and π_2 be two cuspidal non-*CM *representations of $GL(2)/F$, where F is a totally real field. Suppose that π_1 and π_2 are twist of each other over an extension of F, then π_1 and π_2 are twist of each other over F.*

We know (proposition 4.1 of [MP]):

Proposition 5.4. *Suppose that π is a cuspidal, non-*CM *automorphic representation of $GL(2)/K$ for some finite extension $K/\mathbb{Q}$. Suppose that K is a quadratic extension of k and τ is the automorphism of K over k. If $\pi^\tau \cong \pi \otimes \chi$ for a Hecke character χ of K, then χ is trivial when restricted to the ideles of k.*

We know (corrolary 2.6 of [MP]):

Proposition 5.5. *Let ρ be a 2-dimensional irreducible representation of a group G. Then $Sym^2(\rho)$ is reducible if and only if ρ is dihedral.*

We know (see the main theorem of [JG]):

Proposition 5.6. *Let π be a cuspidal, non-*CM *automorphic representations of weight* 2 *of $GL(2)/K$ for some finite extension $K/\mathbb{Q}$. Then $Sym^2\pi$ is a cuspidal automorphic representation of $GL(3)/K$.*

We know (lemma 2.9 of [MP]):

Proposition 5.7. *For 2 dimensional irreducible non-dihedral representations σ_1 and σ_2 of a group G, $Sym^2\sigma_1 \cong Sym^2\sigma_2$ if and only if $\sigma_1 \cong \sigma_2 \otimes \chi$ for a quadratic character χ of G.*

We know ([JPSS]):

Proposition 5.8. *If π_1 and π_2 are two cuspidal unitary automorphic representations of $GL(n)/L$ and $GL(m)/L$, where L is a number field, then the function $L(s, \pi_1 \times \pi_2)$ verifies a functional equation and is meromorphic with possible poles only at $s = 0$ and 1, and does not vanish at $s = 1$. The function $L(s, \pi_1 \times \pi_2)$ is holomorphic iff $\pi_1 \ncong \pi_2^*$ and if $\pi_1 \cong \pi_2^*$, then it has a pole of order 1 at $s = 1$.*

6 Tate cycles

Let $S_1 := S_{K_1}$ be the Hilbert modular surface associated to some sufficiently small open compact subgroup K_1 of $G(\mathbb{A}_f)$ and $S_2 := S_{K_2}$ be the Hilbert modular surface associated to some sufficiently small open compact subgroup K_2 of $G(\mathbb{A}_f)$. By the Künneth formula we have

$$IH^4_{et}(S_1 \times S_2, \overline{\mathbb{Q}}_l)(2) = \oplus_{i+j=4} IH^i_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes IH^j_{et}(S_2, \overline{\mathbb{Q}}_l)(1).$$

Also we have a decomposition:

$$H^4_{B,\mathrm{cusp}}(S_1 \times S_2) = \oplus_{i+j=4} H^i_{B,\mathrm{cusp}}(S_1) \otimes H^j_{B,\mathrm{cusp}}(S_2).$$

We have the canonical isomorphism:

$$\Phi_{\mathrm{et}} : H^4_{B,\mathrm{cusp}}(S_1 \times S_2) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to IH^4_{\mathrm{et}}(S_1 \times S_2, \mathbb{Q}_l),$$

which is compatible with the above decompositions.

The essential part of the above intersection cohomology decomposition is

$$IH^2_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes IH^2_{et}(S_2, \overline{\mathbb{Q}}_l)(1).$$

From the proposition 3.1, we obtain

$$IH^2_{et}(S_1, \overline{\mathbb{Q}}_l)(1) \otimes IH^2_{et}(S_2, \overline{\mathbb{Q}}_l)(1)$$
$$= (\oplus_{\pi_1} V(\pi_1) \otimes \pi_{1f}^{K_1}) \otimes (\oplus_{\pi_2} V(\pi_2) \otimes \pi_{2f}^{K_2})$$
$$= \oplus_{\pi_1,\pi_2} (V(\pi_1) \otimes V(\pi_2)) \otimes (\pi_{1f}^{K_1} \otimes \pi_{2f}^{K_2}),$$

where π_1 and π_2 run over the set of cuspidal automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$. The group $\Gamma_{\mathbb{Q}}$ acts on each summand above by $\rho(\pi_1) \otimes \rho(\pi_2) \otimes 1$.

For an extension k of $\mathbb{Q}$ we must compute the Γ_k-invariant subspace of $V(\pi_1) \otimes V(\pi_2)$ which is isomorphic to

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_1), V(\pi_2)^*)$$

which is isomorphic (see the computation of $\rho(\pi_2)$ from §3) to

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\Gamma_k]}(V(\pi_1), V(\pi_2^*)).$$

For k a finite extension of $\mathbb{Q}$, define:

$$\mathbf{V}(\pi_1, \pi_2, k) := \{x \in V(\pi_1) \otimes V(\pi_2) | \rho(\pi_1) \otimes \rho(\pi_2)(a)x = x, \text{ for all } a \in \Gamma_k\}.$$

The elements of $\mathbf{V}(\pi_1, \pi_2, k)$ are called *Tate cycles* defined over k. We denote by $\mathbf{U}(\pi_1, \pi_2, k) \subseteq \mathbf{V}(\pi_1, \pi_2, k)$ the subspace of *algebraic cycles* defined over k.

For ν a finite order character of $\Gamma_{\mathbb{Q}}$, define:

$$\mathbf{V}(\pi_1, \pi_2; \nu) := \{x \in V(\pi_1) \otimes V(\pi_2) | \rho(\pi_1) \otimes \rho(\pi_2)(a)x = \nu^{-1}(a)x, \text{ for all } a \in \Gamma_{\mathbb{Q}}\},$$

and

$$\mathbf{V}(\pi_1, \pi_2, \mathbb{Q}^{ab}) := \cup_\nu \mathbf{V}(\pi_1, \pi_2; \nu).$$

Let $\mathbf{U}(\pi_1, \pi_2; \nu) \subseteq \mathbf{V}(\pi_1, \pi_2; \nu)$ and $\mathbf{U}(\pi_1, \pi_2, \mathbb{Q}^{ab}) \subseteq \mathbf{V}(\pi_1, \pi_2, \mathbb{Q}^{ab})$ be the subspaces of *algebraic cycles*. We remark that in the cases treated in this paper, i.e when either π_1 or π_2 is non-CM, for k sufficiently large we have $\mathbf{V}(\pi_1, \pi_2, k) = \mathbf{V}(\pi_1, \pi_2; \nu)$ for some finite order character ν of $\Gamma_{\mathbb{Q}}$, i.e. for k sufficiently large we have $\mathbf{V}(\pi_1, \pi_2, k) = \mathbf{V}(\pi_1, \pi_2, \mathbb{Q}^{ab})$, i.e. all the Tate cycles are defined over abelian extensions of $\mathbb{Q}$. When both π_1 and π_2 are CM, it is possible to have for all k that $\mathbf{V}(\pi_1, \pi_2, k) \neq \mathbf{V}(\pi_1, \pi_2, \mathbb{Q}^{ab})$.

7 Tate conjecture

In this section we prove the first part of the Tate conjecture for $\mathbf{S} = S_1 \times S_2$:

Theorem 7.1. *Let k be a finite extension of $\mathbb{Q}$. Then we have*

$$\mathbf{U}(\pi_1, \pi_2, k) = \mathbf{V}(\pi_1, \pi_2, k),$$

if either π_1 or π_2 is non-CM.

We remark that by descent, it is sufficient to prove theorem 7.1 for k large. We assume that the representation π_1 is non-CM (the case when π_2 is non-CM is similar), and we prove theorem 7.1 in this case.

We distinguish two cases:

A) The representation π_2 is CM. Thus $\rho_{\pi_2} = \mathrm{Ind}_{\Gamma_M}^{\Gamma_F} \chi$, where M is a quadratic CM-extension of F and χ is a character of Γ_M. From §3 we know that $\rho(\pi_2)$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(\rho_{\pi_2} \otimes \rho_{\pi_2}^{\tau}),$$

which verifies

$$\rho(\pi_2)|_{\Gamma_F} = \rho_{\pi_2} \otimes \rho_{\pi_2}^{\tau}.$$

Thus, we deduce that

$$\begin{aligned}\rho(\pi_2)|_{\Gamma_{kMM^{\tau}}} &\cong (\chi \oplus \bar{\chi}) \otimes (\chi^{\tau} \oplus \bar{\chi}^{\tau}) \\ &\cong \chi\chi^{\tau} \oplus \chi\bar{\chi}^{\tau} \oplus \bar{\chi}\chi^{\tau} \oplus \bar{\chi}\bar{\chi}^{\tau},\end{aligned}$$

where $\bar{\chi}$ is the complex conjugate of χ.

From §3, we know that $\rho(\pi_1)$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(\rho_{\pi_1} \otimes \rho_{\pi_1}^{\tau}),$$

which verifies

$$\rho(\pi_1)|_{\Gamma_F} \cong \rho_{\pi_1} \otimes \rho_{\pi_1}^{\tau}.$$

We distinguish two cases:

a) The representation $\rho(\pi_1)|_{\Gamma_F}$ is irreducible. Then because $\rho(\pi_1)|_{\Gamma_F} \cong \rho_{\pi_1} \otimes \rho_{\pi_1}^{\tau}$, and the representation π_1 is non-CM, from propositions 5.1, 5.2, and 5.3, we deduce that the representation $\rho(\pi_1)|_{\Gamma_{kMM^{\tau}}}$ is irreducible and because $\rho(\pi_2)|_{\Gamma_{kMM^{\tau}}}$ is a sum of 4 one-dimensional characters, we get that $\mathbf{V}(\pi_1, \pi_2, k) = \{0\}$, and thus $\mathbf{U}(\pi_1, \pi_2, k) = \mathbf{V}(\pi_1, \pi_2, k) = \{0\}$, and theorem 7.1 is proved in this case.

b) The representation $\rho(\pi_1)|_{\Gamma_F}$ is reducible. Then because $\rho(\pi_1)|_{\Gamma_F} \cong \rho_{\pi_1} \otimes \rho_{\pi_1}^{\tau}$, and the representation π_1 is non-CM, applying proposition 5.2, we get that $\rho_{\pi_1}^{\tau} \cong \rho_{\pi_1} \otimes \alpha$ for some Hecke character α of F. Hence, from proposition 5.4, we know that α is a Hecke character of I_F which is trivial on $I_{\mathbb{Q}}$. Therefore α can be written as $\alpha = \chi^\tau/\chi$ for some Hecke character χ of I_F. Hence

$$(\pi_1 \otimes \chi^{-1})^\tau \cong \pi_1 \otimes \chi^{-1}.$$

So $\pi_1 \cong \pi_{0/F} \otimes \chi$, where $\pi_{0/F}$ is the base change to F of some automorphic representation π_0 of $GL(2)/\mathbb{Q}$.

Then from the properties of $\rho(\pi_1)$ (see for example [MP]), we have:

$$\rho(\pi_1) \cong (\mathrm{Sym}^2\rho_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}}) \otimes \chi|_{I_{\mathbb{Q}}},$$

where ω_{π_0} is the central character of π_0 and $\omega_{F/\mathbb{Q}}$ is the quadratic character that corresponds to $F/\mathbb{Q}$.

Thus we get

$$\rho(\pi_1)|_{\Gamma_{kNN^\tau}} \cong (\mathrm{Sym}^2\rho_{\pi_0}|_{\Gamma_{kNN^\tau}} \otimes \chi|_{I_{\mathbb{Q}}}|_{\Gamma_{kNN^\tau}})$$
$$\oplus(\omega_{\pi_0}|_{\Gamma_{kNN^\tau}} \cdot \omega_{FkNN^\tau/kNN^\tau} \cdot \chi|_{I_{\mathbb{Q}}}|_{\Gamma_{kNN^\tau}}).$$

Since π_1 is non-CM, the representation π_0 is non-CM, from proposition 5.1, we know that the representation $\rho_{\pi_0}|_{\Gamma_{kNN^\tau}}$ is irreducible and non-dihedral and from proposition 5.5, we deduce that $\mathrm{Sym}^2(\rho_{\pi_0}|_{\Gamma_{kNN^\tau}})$ is irreducible. Because $\rho(\pi_2)|_{\Gamma_{kNN^\tau}}$ is a sum of one-dimensional characters, we obtain that the dimension of $\mathbf{V}(\pi_1, \pi_2, k)$ is 0 or 1, and it is equal to 1 precisely when the Tate cyles obtained are the product of Tate cyles of the individual factors S_1 and S_2 and thus algebraic, because from [HLR], [MR] and [K], we know that all the Tate cycles of S_1 and S_2 are algebraic. Hence in this case theorem 7.1 is proved.

B) The representation π_2 is non-CM. We distinguish two cases:

a) The representation $\rho(\pi_2)|_{\Gamma_F}$ is irreducible. Then because $\rho(\pi_2)|_{\Gamma_F} \cong \rho_{\pi_2} \otimes \rho^\tau_{\pi_2}$, and the representation π_2 is non-CM, from propositions 5.1, 5.2, and 5.3, we deduce that the representation $\rho(\pi_2)|_{\Gamma_{kF}}$ is irreducible. By Shur's lemma, we obtain that the dimension of $\mathbf{V}(\pi_1, \pi_2, kF)$ is 0 or 1, and it is equal to 1 precisely when $\rho(\pi_1)|_{\Gamma_{kF}} \cong \rho(\pi_2^*)|_{\Gamma_{kF}}$. Hence $(\rho_{\pi_1} \otimes \rho^\tau_{\pi_1})|_{\Gamma_{kF}} \cong (\rho_{\pi_2^*} \otimes \rho^\tau_{\pi_2^*})|_{\Gamma_{kF}}$. Then from lemma 2.3 of [MP], and proposition 5.1, we deduce that $\pi_1 \otimes \mu \cong \pi_2^*$, for some character μ of Γ_F. Then from the proprieties of the representations $\rho(\pi_1)$ and $\rho(\pi_2)$, we get that the space $V(\pi_1) \otimes V(\pi_2)$ has only one subspace of dimension 1, namely $\mu^{-1}|_{I_{\mathbb{Q}}}$. Let $\mathbb{Q}^{\mu|_{I_{\mathbb{Q}}}}$ be the abelian extension of $\mathbb{Q}$ defined by $\mu|_{I_{\mathbb{Q}}}$. Then all the Tate cycles of $V(\pi_1) \otimes V(\pi_2)$ are defined over $\mathbb{Q}^{\mu|_{I_{\mathbb{Q}}}}$.

We have that $\pi_1 \otimes \mu \cong \pi_2^*$, and because for any π as above, $\rho(\pi)$ is automorphic (see [R]), from proposition 5.8, we deduce that $L(s, \rho(\pi_1 \otimes \mu) \otimes \rho(\pi_2))$ has a pole at $s = 1$, which implies by the residue formula, that

$$\int_{GL(2,F)Z(\mathbb{A}_F)\backslash GL(2,\mathbb{A}_F)} \phi_1(g)\phi_2(g)\mu(\det(g))dg \neq 0,$$

for some function ϕ_1 in the space of π_1 and some function ϕ_2 in the space of π_2, where Z denotes the center of $GL(2)$. In other words, the integral of the

$(2,2)$-form $\eta_{\phi_1,\phi_2} = ((2\pi i)^2\phi_1(z_1,z_2)dz_1 \wedge \overline{dz_2}) \wedge ((2\pi i)^2\phi_2(z_3,z_4)dz_3 \wedge \overline{dz_4})$ on $S_{K_1} \times S_{K_2}$ defined by (ϕ_1,ϕ_2) has a non-zero twisted period over S_{K_1}. Then the corresponding twisting correspondence (see §4) of $S_{K_1} \times S_{K_2}$, defines (for suitable g_1 and g_2 in $G(\mathbb{A}_f)$) a μ-twisted cycle $Z(\mu) = Z_{g_1,g_2}(\mu)$ of codimension 2 of $S_{K_1} \times S_{K_2}$, and we get that

$$\int_{Z(\mu)} \eta_{\phi_1,\phi_2} \neq 0,$$

and thus $Z(\mu)$ is *homologically non-trivial* and we get

$$\mathbf{U}(\pi_1,\pi_2;\mu|_{I_{\mathbb{Q}}}) \neq \{0\}.$$

Hence both spaces $\mathbf{U}(\pi_1,\pi_2;\mu|_{I_{\mathbb{Q}}})$ and $\mathbf{V}(\pi_1,\pi_2;\mu|_{I_{\mathbb{Q}}})$ have dimension 1 and are equal. Thus, if k contains $\mathbb{Q}^{\mu|_{I_{\mathbb{Q}}}}$, both spaces $\mathbf{U}(\pi_1,\pi_2,k)$ and $\mathbf{V}(\pi_1,\pi_2,k)$ have dimension 1 and are equal, and theorem 7.1 is proved in this case.

b) The representation $\rho(\pi_2)|_{\Gamma_F}$ is reducible. If $\mathbf{V}(\pi_1,\pi_2,kF) = \{0\}$, we get that $\mathbf{U}(\pi_1,\pi_2,k) = \mathbf{V}(\pi_1,\pi_2,k) = \{0\}$, and theorem 7.1 is proved in this case. We assume from now on that $\mathbf{V}(\pi_1,\pi_2,kF) \neq \{0\}$, which implies that $\rho(\pi_1)|_{\Gamma_{Fk}}$ is reducible. Since π_2 is non-CM, from propositions 5.1 and 5.2, we get that $\rho_{\pi_2}^{\tau} \cong \rho_{\pi_2} \otimes \beta$ for some Hecke character β of F. Hence, from proposition 5.4, we know that β is a Hecke character of I_F which is trivial on $I_{\mathbb{Q}}$. Therefore β can be written as $\beta = \psi^{\tau}/\psi$ for some Hecke character ψ of I_F. Hence

$$(\pi_2 \otimes \psi^{-1})^{\tau} \cong \pi_2 \otimes \psi^{-1}.$$

So $\pi_2 \cong \pi'_{0/F} \otimes \psi$, where $\pi'_{0/F}$ is the base change to F of some automorphic representation π'_0 of $GL(2)/\mathbb{Q}$.

Then from the properties of $\rho(\pi_2)$ (see for example [MP]), we have:

$$\rho(\pi_2) \cong (\mathrm{Sym}^2\rho_{\pi'_0} \oplus \omega_{\pi'_0} \cdot \omega_{F/\mathbb{Q}}) \otimes \psi|_{I_{\mathbb{Q}}},$$

where $\omega_{\pi'_0}$ is the central character of π'_0 and $\omega_{F/\mathbb{Q}}$ is the quadratic character that corresponds to $F/\mathbb{Q}$.

Also since $\rho(\pi_2)|_{\Gamma_F}$ is reducible, with the same notations as in case 1, b), we get that $\pi_1 \cong \pi_{0/F} \otimes \mu$ and that

$$\rho(\pi_1) \cong (\mathrm{Sym}^2\rho_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}}) \otimes \mu|_{I_{\mathbb{Q}}}.$$

Now, since π_1 and π_2 are non-CM, also π_0 and π'_0 are non-CM, and thus, from proposition 5.5, we get that the representations $\mathrm{Sym}^2\rho_{\pi_0}$ and $\mathrm{Sym}^2\rho_{\pi'_0}$ are irreducible. Hence for all k sufficiently large $\mathbf{V}(\pi_1,\pi_2,k)$ has dimension 1 or 2, and it has dimension 2 if and only if $\mathrm{Sym}^2\rho_{\pi'^*_0}|_{\Gamma_k} \cong \mathrm{Sym}^2\rho_{\pi_0}|_{\Gamma_k} \otimes \eta$ for some character η of Γ_k. If for k sufficiently large $\mathbf{V}(\pi_1,\pi_2,k)$ has dimension 1, then the Tate cycles are obtained as a tensor product of Tate cyles of the individual factors S_1 and S_2 and thus are algebraic, because from [HLR], [MR] and [K], we know that all the Tate cycles of S_1 and S_2 are algebraic, and thus theorem 7.1 is proved in this case. We assume from now on that for k sufficiently large $\mathbf{V}(\pi_1, \pi_2, k)$ has dimension 2. Then from propositions 5.1, 5.7 and 5.3, we deduce that $\pi_0 \otimes \gamma \cong {\pi'_0}^*$ for some character γ of $\Gamma_{\mathbb{Q}}$. Hence $\pi_1 \otimes \gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1} \cong \pi_2^*$. Then $\rho(\pi_1) \otimes \rho(\pi_2)$ contains in its decomposition as a direct sum of irreducible representations, only two one-dimensional representations, namely $\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}$ and $\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}$. Let $\mathbb{Q}^{\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}}$ be the abelian extension of $\mathbb{Q}$ defined by $\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}$. Then all the Tate cycles of $V(\pi_1) \otimes V(\pi_2)$ are defined over $\mathbb{Q}^{\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}}$. Only one of these two Tate cycles corresponding to $\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}$ is obtained as a tensor product of Tate cyles of the individual factors S_1 and S_2 and thus is algebraic, because from [HLR], [MR] and [K], we know that all the Tate cycles of S_1 and S_2 are algebraic. Hence we know that $\mathbf{V}(\pi_1, \pi_2; \gamma \cdot (\mu\psi)^{-1}|_{I_{\mathbb{Q}}})$ has dimension 2 and $\mathbf{U}(\pi_1, \pi_2; \gamma \cdot (\mu\psi)^{-1}|_{I_{\mathbb{Q}}})$ has dimension at least 1.

But $\pi_1 \otimes \gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1} \cong \pi_2^*$, and because of the above decompositions of $\rho(\pi_1)$ and $\rho(\pi_2)$, from propositions 5.6 and 5.8, we deduce that $L(s, \rho(\pi_1 \otimes \gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1}) \otimes \rho(\pi_2))$ has a pole of order 2 at $s = 1$, which implies as above in the case a), that $Z(\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$ is *homologically non-trivial* and thus is a non-zero algebraic cycle of $\mathbf{U}(\pi_1, \pi_2; \gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}})$.

Lemma 7.2. *There exists a finite order character ξ of F such that*
(i) $s(\xi) = (-, -)$
(ii) $\xi|_{I_{\mathbb{Q}}} = 1$.

Proof: Let λ be a finite order character of F of signature $(+, -)$. Then λ^τ has signature $(-, +)$. Set $\xi = \lambda/\lambda^\tau$. Then we have that $s(\xi) = (-, -)$ and $\xi|_{I_{\mathbb{Q}}} = 1$, because $\mathbb{Q}$ is the fixed field of τ. □

From the proprieties of $\rho(\pi)$ for π as above, we know that for any character η of Γ_F, we have $\rho(\pi \otimes \eta) = \rho(\pi) \otimes \eta|_{I_{\mathbb{Q}}}$. Hence if we choose ξ as in lemma 7.2, since $\xi|_{I_{\mathbb{Q}}} = 1$, we get that $\rho(\pi \otimes \xi) = \rho(\pi)$ for any π. Thus $L(s, \rho(\pi_1 \otimes \xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1}) \otimes \rho(\pi_2)) = L(s, \rho(\pi_1 \otimes \gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1}) \otimes \rho(\pi_2))$ has a pole of order 2 at $s = 1$, which implies as above that $Z(\xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$ is *homologically non-trivial* and thus is a non-zero algebraic cycle in $\mathbf{U}(\pi_1, \pi_2; \gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}})$. Hence in $\mathbf{U}(\pi_1, \pi_2; \gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}})$ we have two *homologically non-trivial* algebraic cycles $Z(\xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$ and $Z(\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$. But these two algebraic cycles are not proportional, because from lemma 7.2, we know that $s(\xi) = (-, -)$, and thus $s(\xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1}) \neq s(\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$, and from lemma 4.1, we know that the twisted correspondence $R(\xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$ sends a vector in $V_B(\pi_1)^s$ into $V_B(\pi_1 \otimes \xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})^{ss(\xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})} = V_B(\pi_1 \otimes \gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})^{ss(\xi\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})}$, and the twisted correspondence $R(\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})$ sends a vector in $V_B(\pi_1)^s$ into $V_B(\pi_1 \otimes \gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})^{ss(\gamma|_{\Gamma_F} \cdot (\mu\psi)^{-1})}$. Hence $\mathbf{U}(\pi_1, \pi_2; \gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}})$ has dimension 2, and we obtain that both spaces $\mathbf{U}(\pi_1, \pi_2; \gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}})$ and $\mathbf{V}(\pi_1, \pi_2; \gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}})$ have dimension 2 and are equal. Thus, if k contains $\mathbb{Q}^{\gamma^{-1} \cdot (\mu\psi)|_{I_{\mathbb{Q}}}}$, both spaces $\mathbf{U}(\pi_1, \pi_2, k)$ and $\mathbf{V}(\pi_1, \pi_2, k)$ have dimension 2 and are equal, and theorem 7.1 is proved in this case. □

8 Poles of L-functions

Proposition 8.1. *If k is a solvable extension of $\mathbb{Q}$, then the order of the pole at $s = 1$ of $L(s, \rho(\pi_1)|_{\Gamma_k} \otimes \rho(\pi_2)|_{\Gamma_k})$ is equal to $\dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi_1, \pi_2, k)$.*

Proof: From [R], we know that the representations $\rho(\pi_1)$ and $\rho(\pi_2)$ are automorphic, and thus using Langlands base change [L] and the decompositions of the representations $\rho(\pi_1)$ and $\rho(\pi_2)$ from §7, and using propositions 5.6, 5.7 and 5.8, one obtains easily proposition 8.1. □

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