Moduli problem and points on some twisted Shimura varieties of PEL type

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1 Introduction

In this article we describe the moduli problem of a "twist" of some simple Shimura varieties of PEL-type that appear in Kottwitz's papers [K1], [K2] and [K5] and then, using the moduli problem, we compute the cardinality of the set of points over finite fields of the twisted Shimura varieties. Using this result, we compute the zeta function of the twisted varieties. The twist of the Shimura varieties is done by a mod q representation of the absolute Galois group of the reflex field of the Shimura varieties. We remark that in the modular curves case, these are the Galois twists that appear in Wiles's paper [W].

More exactly we consider a simple $\mathbb{Q}$-algebra B and $*$ a positive involution on B. Let V be a non-degenerate skew-Hermitian B-module and let G be the algebraic group over $\mathbb{Q}$ of similitudes of the skew-Hermitian B-module V (other conditions are imposed on B and V to insure that G is unramified at some given rational prime p; see §2 for details). Let $h : \mathbb{C} \to \mathrm{End}_B(V_{\mathbb{R}})$ be an $\mathbb{R}$-algebra homomorphism such that $h(\bar{z}) = h(z)^*$ and the symmentric real-valued bilinear form $(v, h(i)w)$ on $V_{\mathbb{R}}$ is positive definite, where we have written $*$ for the involution on $\mathrm{End}_B(V)$ obtained from the alternating form on V. By restricting h to $\mathbb{C}^{\times}$ we obtain a homomorphism $h : \mathbb{C}^{\times} \to G_{\mathbb{R}}$ of algebraic group over $\mathbb{R}$. For q a rational prime distinct from p, such that G is unramified at q, let $\mathbf{S_K}$ be the Shimura variety associated to (G, h^{-1}) and to an open compact subgroup $\mathbf{K} := K_q \times H$ of $G(\mathbb{A}_f)$, where K_q is a normal subgroup of a hyperspecial maximal compact subgroup H_q of $G(\mathbb{Q}_q)$, H is a sufficiently small open compact subgroup of $G(\mathbb{A}_f^q)$, and $\mathbb{A}_f$ and $\mathbb{A}_f^q$ are the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$ and the finite part of the ring of adeles of $\mathbb{Q}$ away from q, respectively. Then $\mathbf{S_K}$ is a smooth quasi-projective variety defined over a number field E called the reflex field.

Consider the following moduli problem over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. If S is a locally noetherian scheme over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, then S-valued points on the moduli problem are the isomorphism classes of quadruples $(A, \lambda, i, \bar{\eta})$ where: $A \to S$ is a projective abelian scheme over S up to prime-to-p isogeny, $\lambda : A \to \hat{A}$ is a polarization of A, $i : O_B \to \mathrm{End}(A)$ is a $*$-homomorphism for $*$ on O_B and the Rosati involution on $\mathrm{End}(A)$ obtained from λ, and $\bar{\eta}$ is a level structure of type K^p on A and we require that $(A, \lambda, i, \bar{\eta})$ satisfy some "determinant condition" (see §3 below), which depends on h.

For K^p sufficiently small our moduli problem is representable by a quasi-projective scheme S_{K^p} over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. The variety over E obtained from S_{K^p} is a disjoint union of $|\ker^1(\mathbb{Q}, G)|$ copies of $\mathbf{S_K}$, where our $\mathbf{K}$ defined above has a decomposition of the form $\mathbf{K} = K^p K_p$, where $K_p \subset G(\mathbb{Q}_p)$ is some hyperspecial maximal compact (see §2 below for details).

The varieties S_{K^p} and $\mathbf{S_K}$ have natural actions of H_q/K_q. For H sufficiently small these actions are free. We fix such a small group H and consider a continuous Galois representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to H_q/K_q$ unramified at p and let S'_{K^p} be the variety defined over E obtained from S_{K^p} via twisting by ρ composed with the natural action of H_q/K_q on S_{K^p} (see §2 for details). Let v be a prime ideal of the ring of integers O_E lying over p, let k the residue field at v and let k' be a finite extension of k. In this article we compute, using the moduli problem for S'_{K^p}, the cardinality of the set of k'-rational points of the twisted Shimura variety S'_{K^p} as a sum of products of twisted orbital integrals and orbital integrals (see formula (8.1) below for details). From this result, using the trace formula and following the ideas in [K5], one can describe also easily the zeta function of S'_{K^p} in terms of automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ (see formula (10.12) below).

2 Shimura varieties of PEL type

Let p be a prime number. We are interested in certain Shimura varieties of PEL type that appear in [K2] having good reduction at all places of the reflex field lying above the prime p (in the next few sections we are following very closely [K2]).

We consider the PEL data $(B, *, V, (\ ,\ ))$, where:

1) B is a finite-dimensional simple algebra over $\mathbb{Q}$,

2) $*$ is a positive involution on B over $\mathbb{Q}$,

3) V is a nonzero finitely generated left B-module,

4) $(\ ,\ )$ is a nondegenerate $\mathbb{Q}$-valued $*$-Hermitian alternating pairing on V, that satisfy the following conditions:

a) there exists a $\mathbb{Z}_{(p)}$-order O_B on B whose p-adic completion is a maximal order in $B_{\mathbb{Q}_p}$ and which is preserved by $*$;

b) there exists a lattice Λ_0 in $V_{\mathbb{Q}_p}$ which is self-dual for $(\ ,\ )$ and is preserved by O_B;

c) $B_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.

Let F be the be the center of B. Then c) implies that F is a number field unramified at p. Let D be the $\mathbb{Q}$-algebra $\mathrm{End}_B(V)$; it is a simple $\mathbb{Q}$-algebra with center F and has an involution $*$ (the adjoint map for $(\cdot,\cdot)$). Let G be the algebraic group over $\mathbb{Q}$ of the B-linear automorphisms of V, which preserve the pairing $(\; , \;)$ up to a scalar multiple; then for any $\mathbb{Q}$-algebra R we have $G(R) = \{x \in D \otimes_{\mathbb{Q}} R | xx^* \in R^\times\}$. Let $h : \mathbb{C} \to D_{\mathbb{R}}$ be an $\mathbb{R}$-algebra homomorphism such that $h(\bar{z}) = h(z)^*$ and the symmetric real-valued bilinear form $(v, h(i)w)$ on $V_{\mathbb{R}}$ is positive definite, i.e $(v, h(i)v) > 0$ for all nonzero $v \in V_{\mathbb{R}}$.

The homomorphism h determines a decomposition of the $B_{\mathbb{C}}$-module $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$, where V_1 (respectively, V_2) is the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts by z (respectively, by $\bar{z}$). The field of definition of the isomorphism class of the complex representation V_1 of B is a number field E called the *reflex field*, whose ring of integers we denote by O_E.

We denote also by h the restriction of h to $\mathbb{C}^\times$, regarded as a homomorphism of real algebraic groups $\mathbb{C}^\times \to G$. Then from Lemma 4.1 of [K2], we know that the pair (G, h^{-1}) satisfies the conditions (1.5.1), (1.5.2), (1.5.3) of [D]. Therefore the set X_∞ of conjugates of h^{-1} under $G(\mathbb{R})$ is a finite union of copies of the symmetric space for the identity component of $G(\mathbb{R})$ and this symmetric space is of Hermitian type. Let K^p be a sufficiently small open compact subgroup of $G(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p$ denotes the ring of finite adeles of $\mathbb{Q}$ with trivial p-adic component. The stabilizer K_p of Λ_0 in $G(\mathbb{Q}_p)$ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We define $\mathbf{K} := K^p K_p$. We denote by $\mathbf{S_K}$ the Shimura variety associated to (G, h^{-1}) and $\mathbf{K}$. Then $\mathbf{S_K}$ is a smooth quasi-projective variety and has a canonical model defined over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

For any compact open subgroup K_1^p of K^p there is an etale covering $\mathbf{S}_{\mathbf{K}_1} \to \mathbf{S_K}$, where $\mathbf{K}_1 := K_1^p K_p$ and this covering map is Galois with Galois group $\mathbf{K}/\mathbf{K}_1 = K^p/K_1^p$ if K_1^p is normal in K^p. For $g \in G(\mathbb{A}_f)$ there is a natural isomorphism $\mathbf{S_K} \to \mathbf{S}_{g^{-1}\mathbf{K}g}$.

3 Moduli problem

In this section we formulate the PEL moduli problem having good reduction at all places of the reflex field lying over a given rational prime p (in the case D of [K2], page 375, we assume that $p > 2$). For an open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, we consider the set-valued contravariant functor from the category of locally noetherian schemes S over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ that associates to a connected scheme S and to a geometric point s on S the set of isomorphism classes of quadruples $(A, \lambda, i, \bar{\eta})$, where

1) A is a projective abelian scheme over S;

2) $\lambda : A \to \hat{A}$ is a prime-to-p polarization;

3) $i : O_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a morphism of $\mathbb{Z}_{(p)}$-algebras such that $\lambda \circ i(b^*) = i(b)\hat{} \circ \lambda$ and $\det(b, \mathrm{Lie}(A)) = \det(b, V_1)$ for all $b \in O_B$;

4) $\bar{\eta}$ is a $\pi_1(S, s)$-invariant K^p-orbit of isomorphisms of skew-Hermitian B-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to H_1(A_s, \mathbb{A}_f^p)$, where $H_1(A_s, \mathbb{A}_f^p)$ is the Tate $\mathbb{A}_f^p$-module of the abelian variety A_s.

We remark that the level structure $\bar{\eta}$ of type K^p is independent of the choice of the geometric point s of S and thus our functor is independent of s. For K^p

sufficiently small this functor is representable by a quasi-projective scheme S_{K^p} over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Two quadruples $(A, \lambda, i, \bar{\eta})$ and $(A', \lambda', i', \bar{\eta}')$ are said to be isomorphic if there is a prime-to-p isogeny $\varphi : A \to A'$, commuting with the action of O_B, carrying $\bar{\eta}$ into $\bar{\eta}'$, and carrying λ into a scalar multiple of λ' (a scalar in $\mathbb{Z}_{(p)}^\times$) i.e. $\varphi^*\lambda' = c\lambda$ for some $c \in \mathbb{Z}_{(p)}^\times$, where $\varphi^*\lambda' = \hat{\varphi} \circ \lambda' \circ \varphi$ and $\hat{\varphi} : \hat{A}' \to \hat{A}$ is the dual of φ.

For any compact open subgroup K_1^p of K^p there is an etale covering $S_{K_1^p} \to S_{K^p}$, sending $(A, \lambda, i, (\bar{\eta})_1)$ to $(A, \lambda, i, \bar{\eta})$, where $(\bar{\eta})_1$ denotes the K_1^p-orbit of η, and this covering map is Galois with Galois group K^p/K_1^p if K_1^p is normal in K^p. For $g \in G(\mathbb{A}_f^p)$ there is an isomorphism $S_{K^p} \to S_{g^{-1}K^p g}$ sending $(A, \lambda, i, \bar{\eta})$ $(A, \lambda, i, \overline{\eta g})$.

4 Complex points

We want to describe the set of complex points of $S_{K^p}(\mathbb{C})$ relative to the map $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathbb{C}$ induced by the inclusion $E \subset \mathbb{C}$. Hence we are interested in quadruples $(A, \lambda, i, \bar{\eta})$, where this time A is an abelian variety over $\mathbb{C}$ up to prime-to-p isogeny. Then the homology group $H := H_1(A, \mathbb{Q})$ is a skew-Hermitian B-module, with the B-action coming from i and the alternating form coming from λ.

From §8 of [K2], we know that for every place v of $\mathbb{Q}$ the skew-Hermitian $B_{\mathbb{Q}_v}$-modules $H_{\mathbb{Q}_v}$ and $V_{\mathbb{Q}_v}$ are isomorphic.

The isomorphism classes of skew-Hermitian B-modules of the same dimension as V are classified by $H^1(\mathbb{Q}, G)$, and thus we get there are finitely many isomorphic classes of skew-Hermitian B-modules $(V', (.,.)')$ such that $V'_{\mathbb{Q}_p}$ and $V_{\mathbb{Q}_p}$ are isomorphic for all places v of $\mathbb{Q}$. Let $V^{(1)}, \cdots, V^{(m)}$ be some representatives in these isomorphism classes. We fix local isomorphisms $V^{(i)}_{\mathbb{Q}_p} \cong V_{\mathbb{Q}_p}$ for all places v of $\mathbb{Q}$ and let $G^{(i)}$ denote the group of automorphisms of $V^{(i)}$ and thus we have $G^{(i)}_{\mathbb{Q}_p} \cong G_{\mathbb{Q}_p}$ for all p. We obtain that the set $S_{K^p}(\mathbb{C})$ is a disjoint union $\coprod_{i=1}^m \mathbf{S_K}(\mathbb{C})^{(i)}$, where $\mathbf{S_K}(\mathbb{C})^{(i)}$ consists of quadruples $(A, \lambda, i, \bar{\eta})$ such that H is isomorphic to $V^{(i)}$, and hence it is enough to study each set $\mathbf{S_K}(\mathbb{C})^{(i)}$.

Assume that $V^{(1)}$ is V and we start studying $\mathbf{S_K}(\mathbb{C})^{(1)}$. Let $(A, \lambda, i, \bar{\eta})$ be a point of $\mathbf{S_K}(\mathbb{C})^{(1)}$ and choose an isomorphism between H and V. Choosing a level stucture $\bar{\eta}$ is equivalent to choosing an element of $G(\mathbb{A}_f^p)/K^p$. The complex structure on $H_{\mathbb{R}} = \mathrm{Lie}(A)$ determines a complex structure on $V_{\mathbb{R}}$ and yields a homomorphism $h' : \mathbb{C} \to D_{\mathbb{R}}$. From Lemma 4.2 of [K2], we get that h' belongs to the set X_∞ of $G(\mathbb{R})$-conjugates of h. The self-dual O_B-lattice $\Lambda' := H_1(A, \mathbb{Z}_p)$ in $H_{\mathbb{Q}_p}$ becomes an O_B-lattice in $V_{\mathbb{Q}_p}$ that it is self-dual up to a scalar. The group K_p is the stabilizer of the self-dual O_B-lattice Λ_0 in $G(\mathbb{Q}_p)$. From Corollary 7.3 of [K2], we know that there exists $g \in G(\mathbb{Q}_p)$ such that $\Lambda' = g\Lambda_0$, and g determines a well-defined element of $G(\mathbb{Q}_p)/K_p$. The choice of isomorphism between H and V gives us a well-defined element of $(G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty$. Two isomorphisms between H and V differ by an automorphism of V, thus by an element of $G(\mathbb{Q})$ and the corresponding elements of $(G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty$ are in the same $G(\mathbb{Q})$-orbit. Hence $(A, \lambda, i, \overline{\eta})$ gives rise to a well-defined element of the quotient $G(\mathbb{Q}) \setminus (G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$. We get a bijection from $\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(1)}$ to $G(\mathbb{Q}) \setminus ((G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$. Replacing V by $V^{(i)}$, we obtain a bijection from $\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)}$ to $G(\mathbb{Q})^{(i)} \setminus ((G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$. Then $\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)}$ is the set of complex points of a variety $\mathbf{S}_{\mathbf{K}}^{(i)}$ over E, which is isomorphic to $\mathbf{S}_{\mathbf{K}}$.

Hence, we get (see §8 of [K2]) that variety S_{K^p} over E is a disjoint union of $|\ker^1(\mathbb{Q}, G)|$-copies of $\mathbf{S}_{\mathbf{K}}$.

5 Twisted Shimura varieties

Let $q \neq p$ be a rational prime such that G is unramified at q, i.e. G is quasi-split over $\mathbb{Q}_q$ and split over an unramified extension of $\mathbb{Q}_q$. Consider now our $\mathbf{K}$ of the form $\mathbf{K} := K_q \times H$, where K_q is a normal subgroup of a hyperspecial maximal compact subgroup H_q of $G(\mathbb{Q}_q)$. Then it is well known (see for example Corollary 1.4.1.3 of [CA]) that for H sufficiently small, the group H_q/K_q acts freely on S_{K^p}. The action of H_q/K_q on $S_{K^p}(\mathbb{C})$ can be described in the following way: the set $S_{K^p}(\mathbb{C})$ is a disjoint union $\coprod_{i=1}^m \mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)}$ (a disjoint union of $|\ker^1(\mathbb{Q}, G)|$ copies of $\mathbf{S}_{\mathbf{K}}(\mathbb{C})$) and H_q/K_q acts on each

$$\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)} = G(\mathbb{Q})^{(i)} \setminus ((G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$$

where $H_q \hookrightarrow G(\mathbb{A}_{\mathbb{Q}})$ by $\alpha \mapsto (1, ..., \alpha, 1, ..., 1)$, α at the q-component, and the action of an element $\alpha \in H_q$ is given by the right multiplication at the q-component.

We fix a continuous representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to H_q/K_q,$$

unramified at p. Let L be the finite Galois extension of E defined by $L := (\overline{\mathbb{Q}})^{\mathrm{Ker}(\rho)}$.

Let

$$S' = S_{K^p} \times_{\mathrm{Spec}(E)} \mathrm{Spec}(L).$$

The group H_q/K_q acts on S_{K^p}. Since $\rho : \mathrm{Gal}(L/E) \hookrightarrow H_q/K_q$, the group $\mathrm{Gal}(L/E)$ acts on S_{K^p}. We denote this action of $\mathrm{Gal}(L/E)$ on S_{K^p} by ρ'. The Galois group $\mathrm{Gal}(L/E)$ has a natural action on $\mathrm{Spec}(L)$ and we can descend via the quotient process S' to $S'_{K^p}/\mathrm{Spec}(E)$ using the diagonal action

$$\mathrm{Gal}(L/E) \ni \sigma \to \rho'(\sigma) \otimes \sigma$$

on S'. Thus, we obtain a quasi-projective variety $S'_{K^p}/\mathrm{Spec}(E)$ having the same dimension as S_{K^p}. This is the twisted Shimura variety that we mentioned in the title.

Now we describe the PEL moduli problem for S'_{K^p}. We consider again quadruples as in §3 above, only η will be replaced by η'. Thus, with the same notations as above, for the twisted Shimura variety S'_{K^p} a level structure of type K^p on A is a K^p-orbit $\bar{\eta}'$ of isomorphisms $\eta' : V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to H_1(A_s, \mathbb{A}_f^p)$ of skew-Hermitian B-modules such that $\bar{\eta}'$ is fixed by the action $\rho(\tau) \times \tau$ for all $\tau \in \pi_1(S, s)$, where $\rho(\tau)$ acts on the left using the natural action of H_q/K_q and τ acts on the right using the natural action of $\tau \in \pi_1(S, s)$ on A_s.

6 Fixed points of Φ_v^j

Let k be the residue field of E at some place v above p, let $\bar{k}$ be the algebraic closure of $\mathbb{F}_p$, let σ be the Frobenius automorphism $x \mapsto x^p$ of $\bar{k}$, let $\Phi_v := \sigma^{[k:\mathbb{F}_p]}$ and for any integer r denote by k_r the fixed field of σ^r on $\bar{k}$, so k_r is a field with p^r elements and Φ_v fixes the field k. Consider a fixed point of $S'_{K^p}(\bar{k})$ under the action of Φ_v^j for some positive integer j. Hence a fixed point is a $\bar{k}$-point $(\bar{A}, \lambda, i, \bar{\eta}')$ of the moduli problem for S'_{K^p} having the propriety that $(\bar{A}, \lambda, i, \bar{\eta}')$ be isomorphic to $\sigma^r(\bar{A}, \lambda, i, \bar{\eta}')$, where $r := j[k : \mathbb{F}_p]$. Here $\sigma^r(\bar{A}, \lambda, i, \bar{\eta}')$ is the object $(\sigma^r(\bar{A}), \sigma^r(\lambda), \sigma^r(i), \sigma^r(\bar{\eta}'))$ over $\bar{k}$ obtained from $(\bar{A}, \lambda, i, \bar{\eta}')$ by extension of scalars for $\sigma^r : \bar{k} \to \bar{k}$. Thus $(\bar{A}, \lambda, i, \bar{\eta}')$ is a fixed point of S'_{K^p} iff there is a prime-to-p isogeny $u : \sigma^r(\bar{A}) \to \bar{A}$, that commutes with the action of O_B, sending $\sigma^r(\bar{\eta}')$ into $\bar{\eta}'$, and sending $\sigma^r(\lambda)$ into a scalar multiple of λ (a scalar in $\mathbb{Z}_{(p)}^\times$). Since K^p is assumed small enough such that our objects have no automorphisms, the isogeny u is unique. We call the pair $A := (\bar{A}, u)$ a *virtual abelian variety* over k_r up to prime-to-p isogeny. Thus we can associate to any fixed point $(\bar{A}, \lambda, i, \bar{\eta}')$ a *polarized virtual B-abelian variety* (A, λ, i) over k_r up to prime-to-p isogeny.

From the definition we get that an endomorphism of $A := (\bar{A}, u)$ is an endomorphism $f : \bar{A} \to \bar{A}$ such that $fu = u\sigma^r(f)$ which is equivalent to the condition $f\pi_A = \pi_A f$, where we define the *Frobenius element* $\pi_A \in \mathrm{End}(\bar{A})$ by $\pi_A := u \circ \Phi_r$, where Φ_r is the Frobenius morphism $\Phi_r : \bar{A} \to \sigma^r(\bar{A})$. Thus $\mathrm{End}(A)$ is the centralizer of π_A in $\mathrm{End}(\bar{A})$, which implies that π_A belongs to the center of $\mathrm{End}(A)$. Choose an isomorphism $H_1(\bar{A}, \mathbb{A}_f^p) \simeq V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ of skew-Hermitian B-modules. The element π_A is an automorphism of the skew-Hermitian B-module $H_1(\bar{A}, \mathbb{A}_f^p)$ (up to a scalar), and by means of the above isomorphism, π_A can be regarded as an element $\pi_A \in G(\mathbb{A}_f^p)$. Define $\gamma := \pi_A^{-1}$. Then $\gamma \in G(\mathbb{A}_f^p)$ and its conjugacy class is independent of the choice of isomorphism between $H_1(\bar{A}, \mathbb{A}_f^p)$ and $V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$.

We denote by L the fraction field of the Witt ring $W(\bar{k})$ and by L_r the fixed field of σ^r on L, which is the fraction field of the Witt ring $W(k_r)$. As in §10 of [K2], one gets that $H := \{x \in \bar{H} | ux = x\}$, (here $\bar{H}$ is the tensor product over $W(\bar{k})$ with L of the dual of the $W(\bar{k})$-module $H^1_{cris}(\bar{A}/W(\bar{k}))$) is a skew-Hermitian B-module over L_r such that $H \cong V \otimes_{\mathbb{Q}} L_r$. The σ-linear operator Φ on $H^1_{cris}(\bar{A}/W(\bar{k}))$ is carried to a a σ-linear bijection $\Phi : H \to H$ commuting with B and satisfying $\Phi^r = \pi_A^{-1}$ (the element $\pi_A \in \mathrm{End}(\bar{A})$ is carried to $\mathrm{End}(H)$) and $(\Phi v, \Phi w) = c\sigma(v,w)$ for all $v, w \in H$, where c is some element of $L_r^\times$. Choose an isomorphism $H \cong V \otimes_{\mathbb{Q}} L_r$ of skew-Hermitian B-modules and use it to carry Φ over to a σ-linear bijection $\Phi : V \otimes_{\mathbb{Q}} L_r \to V \otimes_{\mathbb{Q}} L_r$. Then we get that $\Phi = \delta\sigma$ for a linear automorphism δ of $V \otimes_{\mathbb{Q}} L_r$, where σ is the σ-linear bijection $\mathrm{id}_V \otimes \sigma$ from $V \otimes_{\mathbb{Q}} L_r$ to itself. We get that $\delta \in G(L_r)$ and changing our choice of isomorphism replaces δ by a σ-conjugacy $x\delta\sigma(x)^{-1}$ $(x \in G(L_r))$. Hence, the σ-conjugacy class of $\delta \in G(L_r)$ is well defined.

We want to define now an element $a \in G(L_r)$ that will be used in the next section. By restricting $h_{\mathbb{C}} : (R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} \to G_{\mathbb{C}}$ to the factor of $(R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$ indexed by the identity map from $\mathbb{C}$ to $\mathbb{C}$, we obtain a map $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$. Then the $G(\mathbb{C})$-conjugacy class of μ_h gives a $G(\bar{\mathbb{Q}}_p)$-conjugacy class of homomorphisms $\mu : \mathbb{G}_m \to G_{\bar{\mathbb{Q}}_p}$, fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_v)$ and hence by its subgroup $\mathrm{Gal}(\bar{\mathbb{Q}}_p/L_r)$. Let S be a maximal O_{L_r}-split torus in G over O_{L_r}. Then S is also a maximal L_r-split torus in G over L_r, and μ can be choosen such that it factors through S. Define $a := \mu(\pi_{L_r}^{-1}) \in S(L_r)$, where π_{L_r} is a uniformizing element in L_r.

7 Counting fixed points of Φ_v^j within a single isogeny class

Now we fix such a polarized virtual B-abelian variety (A_0, λ_0, i_0) over k_r. In this section we want to count the number of fixed points $(\bar{A}, \lambda, i, \bar{\eta}')$ of our correspondence for which the associated polarized virtual B-abelian variety (A, λ, i) is isogenous to (A_0, λ_0, i_0), in the sense that there exists a $\mathbb{Q}$-isogeny $\varphi : A \to A_0$ compatible with the B-actions and the polarizations (up to $\mathbb{Q}^\times$).

Let $M := \mathrm{End}_B(A_0)$ and I be the $\mathbb{Q}$-group $\{x \in M | xx^* \in \mathbb{G}_m\}$ (the group of isogenies from (A_0, λ_0, i_0) to itself). Then the set of fixed points for which (A, λ, i) isogenous to (A_0, λ_0, i_0) equal to the quotient $I(\mathbb{Q}) \setminus Y$, where Y is the set of such fixed points together with an isogeny $\varphi : A \to A_0$ as above.

Fix two isomorphisms $H_1(\bar{A}_0, \mathbb{A}_f^p) \simeq V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and $H \simeq V \otimes_{\mathbb{Q}} L_r$ of skew-Hermitian B-modules and use them as above to get $\gamma \in G(\mathbb{A}_f^p)$ and $\delta \in G(L_r)$. We have a self-dual O_B-lattice Λ_0 in $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and in $V \otimes_{\mathbb{Q}} L_r$. The stabilizer K_r of Λ_0 in $G(L_r)$ is a hyperspecial maximal compact subgroup $G(L_r)$. Giving (A, φ) is the same as giving an element $x \in G(L_r)/K_r$ such that $x^{-1}\delta\sigma(x) \in K_r\sigma(a)K_r$ (see page 432 of [K2]). Giving a level structure of type K^p on $\bar{A}_0$ is equivalent to giving an element $y \in G(\mathbb{A}_f^p)/K^p$, and it equivalent to giving a level structure $\bar{\eta}'$ on $\bar{A}$ if we have a given $\varphi : A \to A_0$. The condition that u carries $\sigma^r(\bar{\eta}')$ into $\bar{\eta}'$ is equivalent to the condition that $\eta' = \pi_A \eta' \rho(\sigma^r)$(modulo K^p), which becomes the condition $y = \gamma^{-1} y \rho(\sigma^r)$ (modulo K^p), or, equivalently, $y^{-1}\gamma y \in K^p \rho(\sigma^r)$ (because $\rho(\sigma^r)K^p = K^p\rho(\sigma^r)$). Thus we obtain that there exists a bijection from the set of fixed points $(\bar{A}, \lambda, i, \bar{\eta}')$ with (A, λ, i) isogenous to (A_0, λ_0, i_0) to the quotient set $I(\mathbb{Q}) \setminus (Y^p \times Y_p)$, where

$$Y^p = \{y \in G(\mathbb{A}_f^p)/K^p | y^{-1}\gamma y \in K^p \rho(\sigma^r)\},$$

and

$$Y_p = \{x \in G(L_r)/K_r | x^{-1}\delta\sigma(x) \in K_r\sigma(a)K_r\}.$$

Thus the cardinality of this set is the integral

$$\int_{I(\mathbb{Q})\backslash(G(\mathbb{A}_f)\times G(L_r))} f^p(y^{-1}\gamma y)\tilde{\phi}_r(x^{-1}\delta\sigma(x)),$$

where f^p is the characteristic function of $K^p\rho(\sigma^r) = K^p\rho(\sigma^r)K^p$ and $\tilde{\phi}_r$ is the characteristic function of $K_r\sigma(a)K_r$. Here we use the Haar measure on $I(\mathbb{Q})$ giving points measure 1, the Haar measure on $G(\mathbb{A}_f^p)$ giving K^p measure 1, and the Haar measure on $G(L_r)$ giving K_r measure 1. From Lemma 10.7 and Lemma 10.8 of [K2], we know that $I(\mathbb{A}_f^p) = G_\gamma(\mathbb{A}_f^p)$, where $G_\gamma(\mathbb{A}_f^p)$ is the centralizer of γ in $G(\mathbb{A}_f^p)$, and that $I(\mathbb{Q}_p) = G_{\delta\sigma}(\mathbb{Q}_p)$, where $G_{\delta\sigma}(\mathbb{Q}_p)$ is the twisted centralizer of δ in $G(L_r)$. Hence the above integral is equal to

$$\mathrm{vol}(I(\mathbb{Q}) \setminus I(\mathbb{A}_f))O_\gamma(f^p)TO_\delta(\tilde{\phi}_r),$$

where

$$O_\gamma(f^p) = \int_{I(\mathbb{A}_f^p)\backslash G(\mathbb{A}_f^p)} f^p(y^{-1}\gamma y),$$

$$TO_\delta(\tilde{\phi}_r) = \int_{I(\mathbb{Q}_p)\backslash G(L_r)} \tilde{\phi}_r(x^{-1}\delta\sigma(x)).$$

Replacing x by $\sigma(x)$ in the integral defining $TO_\delta(\tilde{\phi}_r)$ and using the fact that $\sigma^{-1}(\delta)$ is σ-conjugate to δ (because $\sigma^{-1}(\delta) = c\delta\sigma(c)^{-1}$ for $c = \sigma^{-1}(\delta)$) shows that $TO_\delta(\tilde{\phi}_r) = TO_{\sigma^{-1}(\delta)}(\phi_r) = TO_\delta(\phi_r)$, where ϕ_r is the characteristic function of K_raK_r. Therefore we proved that the number of fixed points $(\bar{A}, \lambda, i, \bar{\eta}')$ for which (A, λ, i) is isogenous to (A_0, λ_0, i_0) is equal to

$$\mathrm{vol}(I(\mathbb{Q}) \setminus I(\mathbb{A}_f))O_\gamma(f^p)TO_\delta(\phi_r).$$

8 The fixed points of Φ_v^j

It remains only to describe the isogeny classes (A, λ, i) (see the beginning of §7 for details). This is done in §17 and §18 of [K2]. One can define as in §14 of [K2], an element $\gamma_0 \in G(\mathbb{Q})$. The $G(\bar{\mathbb{Q}})$-conjugacy class of γ_0 is determined uniquely by the requirement that γ_0 be conjugate in $G(\bar{\mathbb{Q}}_l)$ to the l-adic component of γ for all primes l different from p. Also we have that $\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta) \in G(L_r)$ is conjugate under $G(\bar{\mathbb{Q}}_p)$ to γ_0. The group I is an inner form of the centralizer I_0 of γ_0 in G. Thus we obtain a map $(A, \lambda, i) \mapsto (\gamma_0; \gamma, \delta)$.

In §17 and §18 of [K2], it was proved that the image of the map $(A, \lambda, i) \mapsto (\gamma_0; \gamma, \delta)$ is the set of triples satisfying the above conditions and such that $\alpha(\gamma_0; \gamma, \delta) = 1$, where $\alpha(\gamma_0; \gamma, \delta)$ was defined in §15 of [K2]. From §17 and §18 of [K2], we know also that having a triple $(\gamma_0; \gamma, \delta)$ satisfying the above conditions and such that $\alpha(\gamma_0;\gamma,\delta)=1$, the cardinality of the set of isogeny classes (A,λ,i) giving rise to $(\gamma_0;\gamma,\delta)$ is equal to

$$|\ker[\ker^1(\mathbb{Q},I_0)\to\ker^1(\mathbb{Q},G)]|.$$

Let $c(\gamma_0;\gamma,\delta)$ be the product of $\mathrm{vol}(I(\mathbb{Q})\setminus I(\mathbb{A}_f))$ and $|\ker[\ker^1(\mathbb{Q},I_0)\to\ker^1(\mathbb{Q},G)]|$.

Combining the above remarks with the result obtained in §7 and with the fact that S_{K^p} is a disjoint union of $|\ker^1(\mathbb{Q},G)|$-copies of the canonical model of the Shimura variety associated to (G,h^{-1}) and to $\mathbf{K}=K^pK_p$, we get that the cardinality of the set of fixed points for the action of $\Phi_v^j=\sigma^r$ on $S'_{K^p}(\bar{k})$ is equal to

$$|S'_{K^p}(k_r)|=|\ker^1(\mathbb{Q},G)|\sum_{\gamma_0}\sum_{(\gamma,\delta)}c(\gamma_0;\gamma,\delta)\cdot O_\gamma(f^p)\cdot TO_\delta(\phi_r), \tag{8.1}$$

with $(\gamma_0;\gamma,\delta)$ running through all equivalence classes of triples satisfying the conditions below and having the propriety that $\alpha(\gamma_0;\gamma,\delta)=1$. The first sum is taken over a set of representatives γ_0 for the stable conjugacy classes in $G(\mathbb{Q})$ (i.e. classes of elements in $G(\mathbb{Q})$ that are conjugates in $G(\bar{\mathbb{Q}})$). The second sum is over a set of representatives for the equivalence classes of pairs $(\gamma,\delta)\in G(\mathbb{A}_f^p)\times G(L_r)$ such that

(a) for each prime $l\neq p$ the l-adic component γ_l of γ is stably conjugate to γ_0,

(b) $\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)\in G(L_r)$ is conjugate under $G(\bar{\mathbb{Q}}_p)$ to γ_0,

Two pairs (γ_1,δ_1) and (γ_2,δ_2) are equivalent if γ_1 and γ_2 are conjugate under $G(\mathbb{A}_f^p)$ and δ_1 and δ_2 are σ-conjugate under $G(L_r)$.

9 Some representations

In this section we define some representations that will be used in §10. To any unramified complex representation π_p of $G(\mathbb{Q}_p)$ one can associate an unramified complex representation $V(\pi_p,E_v)$ of the Weyl group W_{E_v} as follows. To such a π_p one can associate an unramified admissible homomorphism

$$\phi(\pi_p):W_{\mathbb{Q}_p}\to^L(G_{\mathbb{Q}_p}),$$

well-defined up to conjugation by $\hat{G}$, where ${}^L(G_{\mathbb{Q}_p})$ is the L-group formed using $\hat{G}$ and $W_{\mathbb{Q}_p}$. By restricting $\phi(\pi_p)$ to W_{E_v} one gets an unramified admissible homomorphism

$$\phi_{E_v}(\pi_p):W_{E_v}\to^L(G_{E_v}).$$

The homomorphism of $\mathbb{R}$-algebraic groups $h:\mathbb{C}^\times\to G_\mathbb{R}$ determines a representation r of ${}^L(G_{E_v})$ (see [L]). Let $V(\pi_p,E_v)$ be the unramified representation of W_{E_v} defined by

$$(r\circ\phi_{E_v}(\pi_p))\otimes\chi,$$

where χ is the unramified character of W_{E_v} whose value at the Frobenius element Φ_v element of $W_{E_v}^{\mathrm{un}}$ is $\sqrt{p}^{-[E_v:\mathbb{Q}_p]\dim S_{K^p}}$. Let $P(\pi_p, E_v)$ be the characteristic polynomial of Φ_v acting on $V(\pi_p, E_v)$.

We now fix an admissible irreducible representation π_f of $G(\mathbb{A}_f)$ unramified at p and having component π_p at p. Since ρ is unramified at p, by restriction one gets an unramified representation of W_{E_v} and let $V(\pi_p, E_v)'$ be the unramified representation of W_{E_v} defined by

$$((r \circ \phi_{E_v}(\pi_p)) \otimes \chi) \otimes (\pi_f^{\mathbf{K}} \circ \rho),$$

where we have used the natural action of H_q/K_q on $\pi_f^{\mathbf{K}}$ (recall that $\mathbf{K} = K_q \times H$): if $h \in H_q$ is a representative of an element $\bar{h} \in H_q/K_q$, and $v \in \pi_f^{\mathbf{K}}$, then $\bar{h} \cdot v := h \cdot v$, where we regard $h \in H_q$ as an element of $\mathbf{K}$ using the natural inclusion $H_q \hookrightarrow \mathbf{K}$, where $\alpha \mapsto (1, \cdots, \alpha, 1, \cdots, 1)$, α at the q-component. This definition is of course independent of the representative of the class $\bar{h}$ and it is easy to check that $\bar{h} \cdot v \in \pi_f^{\mathbf{K}}$, because for $k \in \mathbf{K}$, we have $k \cdot (\bar{h} \cdot v) = k \cdot h \cdot v = h(h^{-1}kh) \cdot v = h \cdot v$ since $h^{-1}kh \in \mathbf{K}$, the subgroup K_q being normal in H_q.

Let $P(\pi_p, E_v)'$ be the characteristic polynomial of Φ_v acting on $V(\pi_p, E_v)'$.

10 The zeta function of S'_{K^p}

Using the computations from §8, we want to determine the zeta function of S'_{K^p} (in this section we are following closely [K5]). The formula (8.1) takes the form (see formula (5.1) of [K5]):

$$|S'_{K^p}(k_r)| = |\ker^1(\mathbb{Q}, G)| \tau(G) \sum_{\gamma_0} \sum_{(\gamma,\delta)} e(\gamma,\delta) O_\gamma(f^p) TO_\delta(\phi_r) \cdot \mathrm{vol}\left(\frac{A_G(\mathbb{R})^0}{I(\infty)(\mathbb{R})}\right)^{-1}, \tag{10.1}$$

where $\tau(G)$ is the Tamagawa number of G, A_G is the split component of the center of G and the number $e(\gamma, \delta) = \pm 1$ is defined as follows: for each place w of $\mathbb{Q}$ we have a group $I(w)$ over $\mathbb{Q}_w$: for $w \neq p, \infty$ the group $I(w)$ is the centralizer of γ_w in G; for $w = p$ the group $I(w)$ is the twisted centralizer of δ; for $w = \infty$ the group $I(w)$ is the inner form of the centralizer of γ_0 in G that is anisotropic modulo the center of G.

Then the number $e(\gamma, \delta) = \pm 1$ is equal to the product $e(\gamma)e(\delta)e_\infty(\gamma_0)$ where $e(\delta) = e(I(p))$, $e_\infty(\gamma_0) = e(I(\infty))$, and $e(\gamma) = \prod_{w \neq p,\infty} e(I(w))$.

Let f_∞ be $(-1)^{\dim S_{K^p}}$ times a pseudo-coefficient of some irreducible admissible representation π_∞ of $G(\mathbb{R})$ having trivial central and infinitesimal characters. The function f_∞ is compactly supported modulo $A_G(\mathbb{R})^0$ and transforms under $A_G(\mathbb{R})^0$ by the trivial character.

We know (see Lemma 3.1 of [K5]):

Lemma 10.1. *Let γ_∞ be a semisimple element of $G(\mathbb{R})$. Then the stable orbital* *integral SO_{γ_∞} vanishes unless γ_∞ is elliptic, in which case*

$$SO_{\gamma_\infty}(f_\infty) = vol(\frac{A_G(\mathbb{R})^0}{I(\mathbb{R})})^{-1} \cdot e(I),$$

where I denotes the inner form of the centralizer of γ_∞ in G that is anisotropic modulo A_G.

For an automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ with central character χ_π, we define $m(\pi)$ to be the multiplicity of π in the space of automorphic forms on $G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})$ transforming by χ_π under the center of $G(\mathbb{A}_\mathbb{Q})$.

Consider a smooth function f on $G(\mathbb{A}_\mathbb{Q})$, which transforms under $A_G(\mathbb{R})^0$ by the trivial character and whose support is compact modulo $A_G(\mathbb{R})^0$. Because G/A_G is anisotropic over $\mathbb{Q}$, the trace formula for f is

$$\sum_\gamma \tau(G_\gamma) O_\gamma(f) = \sum_\pi m(\pi) \mathrm{tr}\pi(f), \tag{10.2}$$

where γ runs over the representatives for the conjugacy classes in $G(\mathbb{Q})$, G_γ is the centralizer of γ in G, $\tau(G_\gamma)$ is the Tamagawa of G_γ, $O_\gamma(f)$ is the orbital integral of f over the conjugacy class of $G(\mathbb{A}_\mathbb{Q})$, and π runs over the automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ whose central character is the trivial character on $A_G(\mathbb{R})^0$.

We know (see Lemma 4.1 of [K5]):

Lemma 10.2. *The left hand side of (10.2) is equal to*

$$\sum_{\gamma_0} \tau(G) SO_\gamma(f),$$

where γ_0 runs through a set of representatives for the stable conjugacy classes in $G(\mathbb{Q})$ and $SO_\gamma(f)$ denotes the stable orbital integral of f over the stable conjugacy class of γ in $G(\mathbb{A}_\mathbb{Q})$.

Applying the base change homomorphism to ϕ_r, we get a function f_r in the Hecke algebra of $G(\mathbb{Q}_p)$ with respect to K_p. From §2 of [K3], we know that for any irreducible unramified representation π_p of $G(\mathbb{Q}_p)$ we have

$$\mathrm{tr}\pi_p(f_r) = \mathrm{tr}(\sigma^r; V(\pi_p, E_v)) = \mathrm{tr}(\Phi_v^j; V(\pi_p, E_v)), \tag{10.3}$$

where $V(\pi_p, E_v)$ was defined in §9.

From [C], we know that the "fundamental lemma" for ϕ_r and f_r holds: for every semisimple $\gamma_p \in G(\mathbb{Q}_p)$

$$SO_{\gamma_p}(f_r) = \sum_\delta e(\delta) TO_\delta(\phi_r), \tag{10.4}$$

where SO_{γ_p} is the stable orbital integral (see §5 of [K4]) and δ runs over a set of representatives for the σ-conjugacy classes of $\delta \in G(L_r)$ such that $\delta\sigma(\delta) \cdots \sigma^{r-1}(\delta)$ is conjugate to γ_p in $G(\bar{\mathbb{Q}}_p)$.

Thus from (10.4) and Lemma 10.1, we deduce that (10.1) is equal to

$$|\ker^1(\mathbb{Q}, G)|\tau(G)\sum_{\gamma_0} SO_{\gamma_0}(f^p \cdot f_r \cdot f_\infty), \tag{10.5}$$

where γ_0 runs over a set of representatives for the stable conjugacy classes in $G(\mathbb{Q})$. From Lemma 10.2 and the trace formula (10.2) applied to the function $f^p \cdot f_r \cdot f_\infty$, we get that (10.5) is equal to

$$|\ker^1(\mathbb{Q}, G)|\sum_{\pi} m(\pi)\mathrm{tr}\pi(f^p \cdot f_r \cdot f_\infty), \tag{10.6}$$

which is equal to

$$|\ker^1(\mathbb{Q}, G)|\sum_{\pi_f} a(\pi_f)\mathrm{tr}\pi_f(f^p f_r), \tag{10.7}$$

where π_f ranges over the irreducible representations of $G(\mathbb{A}_f)$ and $a(\pi_f) \in \mathbb{Z}$ is given by

$$a(\pi_f) = \sum_{\pi_\infty} m(\pi_f \otimes \pi_\infty)\mathrm{tr}\pi_\infty(f_\infty),$$

where the sum runs over $\pi_\infty \in \Pi_\infty$, and Π_∞ is the set of irreducible admissible representations up to isomorphism of $G(\mathbb{R})$ having trivial central and infinite characters and for an automorphic representation $\pi = \pi_f \otimes \pi_\infty$ of $G(\mathbb{A}_\mathbb{Q})$ with central character χ_π, we define $m(\pi)$ to be the multiplicity of π in the space of automorphic forms on $G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})$ transforming by χ_π under the center of $G(\mathbb{A}_\mathbb{Q})$.

Since f_r is bi-invariant under K_p, the number $\mathrm{tr}\pi_f(f^p f_r)$ is zero unless the p-component π_p of π_f is unramified, in which case we get

$$\mathrm{tr}\pi_f(f^p f_r) = \mathrm{tr}\pi_f(f_1^p)\mathrm{tr}\pi_p(f_r), \tag{10.8}$$

where we recall that $f^p = K^p\rho(\sigma^r) = K^p\rho(\sigma^r)K^p$ and the function f_1^p is defined by $f_1^p := K^pK_p\rho(\sigma^r) = \mathbf{K}\rho(\sigma^r) = \mathbf{K}\rho(\sigma^r)\mathbf{K}$. Now one could see easily that

$$\mathrm{tr}\pi_f(f_1^p) = \mathrm{tr}(\sigma^r; \pi_f^{\mathbf{K}} \circ \rho), \tag{10.9}$$

because if we denote by $f_1'^p$ the characteristic function of $K^pK_p = \mathbf{K}$, we have

$$\pi_f(f_1^p) = \int f_1^p(g)\pi_f(g)dg = \int f_1'^p(g\rho(\sigma^{-r}))\pi_f(g)dg$$

$$= \int f_1'^p(g)\pi_f(g\rho(\sigma^r))dg = \int_{\mathbf{K}} \pi_f(g)\pi_f(\rho(\sigma^r))dg.$$

Thus from (10.3), (10.8) and (10.9) we get

$$\mathrm{tr}\pi_f(f^p f_r) = \mathrm{tr}(\sigma^r; V(\pi_p, E_v))\mathrm{tr}(\sigma^r; \pi_f^{\mathbf{K}} \circ \rho)$$

$$= \mathrm{tr}(\sigma^r; V(\pi_p, E_v) \otimes (\pi_f^{\mathbf{K}} \circ \rho)) = \mathrm{tr}(\sigma^r; V(\pi_p, E_v)'), \tag{10.10}$$

where $V(\pi_p, E_v)')$ was defined in §9 and from (10.1) and (10.7) we get that

$$|S'_{K^p}(k_r)| = |\ker^1(\mathbb{Q}, G)| \sum_{\pi_f} a(\pi_f) \mathrm{tr}(\sigma^r; V(\pi_p, E_v)'), \tag{10.11}$$

where π_f ranges over the irreducible representations of $G(\mathbb{A}_f)$.

Consider now a smooth algebraic variety S over a number field E. Then the zeta function of S is defined by

$$Z(S, s) =: \prod_v Z_v(S, s),$$

where v runs over all places of E, and at the places v where S has good reduction, we have

$$Z_v(S, s) := Z(S(v), q_v^{-s}),$$

where $S(v)$ is the reduced variety at v and q_v is the cardinality of the residue field $k(v)$, and

$$Z(S(v), T) = \exp(\sum_{j>0} |S(v)(\mathbb{F}_{q_v^j})| \cdot \frac{T^j}{j}),$$

where $\mathbb{F}_{q_v^j}$ is the finite field of cardinality q_v^j.

From (10.11) we deduce that

$$Z_v(S'_{K^p}, s) = \exp(\sum_{j>0} |\ker^1(\mathbb{Q}, G)| \sum_{\pi_f} a(\pi_f) \mathrm{tr}(\Phi_v^j; V(\pi_p, E_v)') \cdot \frac{q_v^{-js}}{j})$$

$$= \prod_{\pi_f} \det(1 - q_v^{-s + \frac{\dim S_{Kp}}{2}} \cdot ((r \circ \phi_{E_v}(\pi_p)) \otimes (\pi_f^{\mathbf{K}} \circ \rho))(\Phi_v))^{-a(\pi_f) \cdot |\ker^1(\mathbb{Q}, G)|}, \tag{10.12}$$

where π_f ranges over the irreducible representations of $G(\mathbb{A}_f)$.

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