On Deligne's conjecture for Hilbert motives over totally real number fields

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1 Introduction

Let M be a motive defined over a number field F with coefficients in a number field E. One can associate to M an L-funcion $\mathbb{L}(M,s)$ having values in $E\otimes_{\mathbb{Q}}\mathbb{C}$. From the proprieties of the restriction of scalars, one knows that $\mathbb{L}(M,s)=\mathbb{L}(\mathrm{Res}_{F/\mathbb{Q}}M,s)$. When M is critical one has the "+"-period defined by Deligne $c^+(\mathrm{Res}_{F/\mathbb{Q}}M)\in E\otimes_{\mathbb{Q}}\mathbb{C}$. Then Deligne's conjecture states that:

Conjecture 1.1. *If M is a critical motive defined over F with coefficients in E, then:*

$$\mathbb{L}(M,0)/c^+(Res_{F/\mathbb{Q}}M)\in E\otimes 1\subset E\otimes_{\mathbb{Q}}\mathbb{C}.$$

This conjecture is known to be true for rank 1 motives if F is either totally real or a CM field (see [B]) and for motives associated to classical modular forms of $\mathrm{GL}(2)/\mathbb{Q}$ (see [D]).

In this paper, we prove the following result (we remark that in the proof of this theorem we assume the Tate conjecture for motives; see §4 for details):

Theorem 1.2. *Let F be a totally real number field, let I_F be the set of infinite places of F, let f be a Hilbert cuspform of weight $k=(k_\tau)_{\tau\in I_F}$ of $GL(2)/F$, where all k_τ have the same parity and all $k_\tau\geq 3$. Let $M(f)(j)$ the j-Tate twist of the motive $M(f)$ associated to f, where j is an integer such that $(k_0+1)/2\leq j<(k_0+k^0)/2$, where $k_0=\max\{k_\tau|\tau\in I_F\}$ and $k^0=\min\{k_\tau|\tau\in I_F\}$. Assume that conjecture 1.1 is true for all the motives of the form $M(g)(j)$ where g is an arbitrary modular form of weight k of $GL(2)/L$ and L is an arbitrary totally real finite extension of F. Then conjecture 1.1 is true for all the motives of the form $M(f)(j)/F'$, where F' is an arbitrary totally real finite extension of F.*

Note the following point: We don't know that the motive $M(f)(j)/F'$ corresponds to a Hilbert modular form, since arbitrary totally real base change is not yet established.

2 Periods for motives

Consider a motive M defined over a number field F with coefficients in a number field E. Denote by Γ_F the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$. We recall now the definition of the L-function $\mathbb{L}(M,s)$ of M. Consider the étale cohomology $H_\lambda(M)$ for each prime ideal λ of E. It is conjectured that the Galois representation $\rho_\lambda : \Gamma_F \to \mathrm{GL}(H_\lambda(M))$ is unramified outside the residual characteristic l of λ and a finite set S of primes of F independent of λ. Denote by $V := H_\lambda(M)$ the representation space of ρ_λ. If $\wp$ is a prime ideal of F prime to l, we choose an inertia group $I_\wp$ at $\wp$ and a geometric Frobenius $\mathrm{Frob}_\wp$. It is conjectured that the characteristic polynomial $Z_\wp(M,X) = \det(1-\rho_\lambda(\mathrm{Frob}_\wp)|_{V^{I_\wp}}X)$ has coefficients in E and is independent of λ. Assume all these conjectures. Denote by I_E the set of infinite places of E. For $\tau \in I_E$, put

$$L_\wp(\tau, M, s) = \tau Z_\wp(M, N(\wp)^{-s})^{-1}$$

and

$$L(\tau, M, s) = \prod_\wp L_\wp(\tau, M, s).$$

One has the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{I_E}$ given by $e \otimes z \to (z \cdot \tau(e))_{\tau \in I_E}$. One can define a function $\mathbb{L}(M,s)$ taking values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ by arranging $L(\tau, M, s)$.

Let $\mathbb{L}_\infty(M,s)$ be the infinite part of the L-function of M which is a product of Γ-functions. If one puts $\Lambda(M,s) = \mathbb{L}(M,s)\mathbb{L}_\infty(M,s)$, then the conjectural functional equation has the following form:

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(M^\vee, 1-s),$$

where $\epsilon(M,s)$ is a multiple of an exponential function of s with values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ and $M^\vee$ is the dual of M. We say that an integer n is critical for M if neither $\mathbb{L}_\infty(M,s)$ nor $\mathbb{L}_\infty(M^\vee, 1-s)$ has a pole at $s = n$. We call M *critical* if M is critical at 0.

Consider now a motive M defined over $\mathbb{Q}$ with coefficients in E. Let $H_B(M)$ denote the Betti realization of M. Then $H_B(M)$ is a finite dimensional vector space over E. The complex conjugation F_∞ acts on $H_B(M)$ and one gets a decomposition

$$H_B(M) = H_B^+(M) \oplus H_B^-(M),$$

where $H_B^\pm(M)$ denote the eigenspaces of $H_B(M)$ with eigenvalues ± 1.

Assume that the motive M is homogeneous of weight w. Then one has the Hodge decomposition

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=w} H^{pq}(M),$$

where $H^{pq}(M)$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module.

Let $H_{DR}(M)$ denote the de Rham realization of M. Then $H_{DR}(M)$ is a finite dimensional vector space over E. One has the comparison isomorphism

$$I : H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR}(M) \otimes_{\mathbb{Q}} \mathbb{C}$$

as $E \otimes_{\mathbb{Q}} \mathbb{C}$ modules.

Define the Hodge filtration $\{F^m\}$ on $H_{DR}(M)$ by

$$I^{-1}(F^m(H_{DR}(M)) \otimes_{\mathbb{Q}} \mathbb{C}) = \oplus_{p \geq m} H^{pq}(M).$$

For M a motive of odd weight $w = 2p+1$, define $F^{\pm}(M) =: F^{p+1}(H_{DR}(M))$ (for a motive M of even weight $w = 2p$, one can define in a similar way $F^{\pm}(M)$, see §2 of [Y]). If one defines $H^{\pm}_{DR}(M) = H_{DR}(M)/F^{\mp}(M)$, then one has the comparison isomorphisms

$$I^{\pm} : H^{\pm}_{B}(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^{\pm}_{DR}(M) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.1}$$

Let $c^{\pm}(M) = \det(I^{\pm})$ be the determinants calculated using E-rational basis. Hence $c^{\pm}(M) \in (E \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$ are determined up to multiplication by elements of E.

If M is a motive defined over F with coefficients in E, let I_F be the set of infinite places of F. Then $H_{DR}(M)$ is a free $E \otimes_{\mathbb{Q}} F$-module of some rank $d(M)$ and for each $\sigma \in I_F$ one has the Betti realization $H_B(M^{\sigma})$ which is a vector space of dimension $d(M)$ over E. The number $d(M)$ is called the rank of M.

We recall now the definition of the restriction of scalars $\mathrm{Res}_{F/F'}(M)$ of M to a subfield F' of F. For the de Rham side one forgets the F-vector space structure and put $H_{DR}(\mathrm{Res}_{F/F'}(M)) = H_{DR}(M)$ as a F'-vector space. For the Betti side, one sets $H_B(\mathrm{Res}_{F/F'}(M)^{\sigma}) = \oplus_{\tau|_{F'}=\sigma} H_B(M^{\tau})$. Hence $\mathrm{Res}_{F/F'}(M)$ is a motive over F' of rank $[F : F']d(M)$ with coefficients in E.

3 L-functions

Let F be a totally real number field and let I_F be the set of infinite places of F. If π is an automorphic representation of weight $k = (k_{\tau})_{\tau \in I_F}$ of $\mathrm{GL}(2)/F$, where all k_{τ} have the same parity and all $k_{\tau} \geq 2$, then there exists ([T]) a λ-adic representation

$$\rho_{\pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(O_{\lambda}) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L(\rho_{\pi,\lambda}, s) = L(f, s)$ and is unramified outside the primes dividing $\mathbf{n}l$. Here O is the coefficients ring of π and λ is a prime ideal of O above some prime number l, $\mathbf{n}$ is the level of π and f is the modular form of $\mathrm{GL}(2)/F$ of weight k corresponding to π. In order to simplify the notations we denote by ρ_{π} the representation $\rho_{\pi,\lambda}$ (by fixing an isomorphism $i : \bar{\mathbb{Q}}_l \to \mathbb{C}$ we can regard always ρ_{Π} as a complex-valued representation). We know the following result (theorem 1.1 of [V]):

Theorem 3.1. *If π is a cuspidal automorphic representation of weight k as above of $GL(2)/F$ for some totally real number field F and F' is a totally real extension of F, then there exists a totally real finite Galois extension F'' of $\mathbb{Q}$ containing F' and a prime λ of the field coefficients of π, such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}}$ is modular i.e. there exists an automorphic representation π'' of weight k of $GL(2)/F''$ and a prime β of the field of coefficients of π'' such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}} \cong \rho_{\pi'',\beta}$.*

Fix a cuspidal automorphic representation π as in the theorem 3.1. Let F'/F be a totally real finite extension. Then one can find a totally real finite Galois extension F'' of $\mathbb{Q}$ containing F', a prime λ of the field coefficients of π and an automorphic representation π'' of $\mathrm{GL}(2)/F''$ and a prime β of the field of coefficients of π'' such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}} \cong \rho_{\pi'',\beta}$.

By Brauer's theorem (see theorems 16 and 19 of [SE]), one can find some subfields $F_i \subset F''$ such that $\mathrm{Gal}(F''/F_i)$ are solvable, some characters $\varphi_i : \mathrm{Gal}(F''/F_i) \to \bar{\mathbb{Q}}^\times$ and some integers m_i, such that the trivial representation

$$1 : \mathrm{Gal}(F''/F') \to \bar{\mathbb{Q}}^\times,$$

can be written as $1 = \sum_{i=1}^{i=u} m_i \mathrm{Ind}_{\mathrm{Gal}(F''/F_i)}^{\mathrm{Gal}(F''/F')} \varphi_i$ (a virtual sum). Then

$$L(\rho_\pi|_{\Gamma_{F'}}, s) = \prod_{i=1}^{i=u} L(\rho_\pi|_{\Gamma_{F'}} \otimes \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_{F'}} \varphi_i, s)^{m_i} =$$

$$\prod_{i=1}^{i=u} L(\mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_{F'}} (\rho_\pi|_{\Gamma_{F_i}} \otimes \varphi_i), s)^{m_i} = \prod_{i=1}^{i=u} L(\rho_\pi|_{\Gamma_{F_i}} \otimes \varphi_i, s)^{m_i}.$$

Since $\rho_\pi|_{\Gamma_{F''}}$ is modular and $\mathrm{Gal}(F''/F_i)$ is solvable, from Langland's base change for solvable extensions, one can deduce easily that the representation $\rho_\pi|_{\Gamma_{F_i}}$ is modular, and thus there exists an automorphic representation π_i of weight k such that $\rho_\pi|_{\Gamma_{F_i}} \cong \rho_{\pi_i}$. Denote by f_i the modular form corresponding to π_i. Then:

$$L(\rho_\pi|_{\Gamma_{F'}}, s) = \prod_{i=1}^{i=u} L(\rho_{\pi_i} \otimes \varphi_i, s)^{m_i} = \prod_{i=1}^{i=u} L(f_i, \varphi_i, s)^{m_i}, \tag{3.1}$$

where $L(f_i, \varphi_i, s)$ are defined below in §4.

4 Deligne's conjecture for $M(f)(j)/_{F'}$

Let F be a totally real number field and f be a modular form of weight k as in §3 of $\mathrm{GL}(2)/F$. Let θ be a Hecke character of F of finite order. For $\mathbf{n}$ an ideal of the ring of integers O_F of F, define $a(\mathbf{n})$ by $T(\mathbf{n})f = a(\mathbf{n})f$, where $T(\mathbf{n})$ is the Hecke operator of level $\mathbf{n}$. The field of coefficients of f is by definition the field $\mathbb{Q}_f$ generated by the values $a(\mathbf{n})$ over $\mathbb{Q}$. It is well known that $\mathbb{Q}_f$ is a finite extension of $\mathbb{Q}$. We consider a number field E which contains $\mathbb{Q}_f$ and the field of coefficients $\mathbb{Q}(\theta)$ of θ. Put

$$L(f, \theta, s) = \sum_{\mathbf{n}} a(T(\mathbf{n}))\theta(\mathbf{n})N(\mathbf{n})^{-s}.$$

For $\tau \in I_E$ we define

$$L(\tau, f, \theta, s) = \sum_{\mathbf{n}} a(T(\mathbf{n}))^\tau \theta(\mathbf{n})^\tau N(\mathbf{n})^{-s}.$$

Using the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{I_E}$, one gets an $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued L-function $\mathbb{L}(f, \theta, s)$, by arranging the factors $L(\tau, f, \theta, s)$. In the same way one can define the L-function $\mathbb{L}(f, s)$.

Let f be a modular form of weight k of $\mathrm{GL}(2)/F$ and $M(f)$ be the motive conjecturally corresponding to f. Then $M(f)$ is a motive of rank 2 over F with coefficients in $\mathbb{Q}_f$. By the definition of $M(f)$ we have $\mathbb{L}(M(f), s) = \mathbb{L}(f, s)$. Since the modular form f has weight k, if we define $k_0 = \max\{k_\tau | \tau \in I_F\}$ and $k^0 = \min\{k_\tau | \tau \in I_F\}$, then any integer $(k_0 - k^0)/2 < j < (k_0 + k^0)/2$ is a critical value for $M(f)$.

Let $m \in \mathbb{Z}$ and $T(m)$ the Tate motive over F. Put $M(f)(m) = M(f) \otimes T(m)$. One has

$$\mathbb{L}(M(f)(m), s) = \mathbb{L}(M(f), m + s).$$

Hence, from the fact that $M(f)$ is critical at j for $(k_0 - k^0)/2 < j < (k_0 + k^0)/2$, one gets that $M(f)(j)$ is critical at 0. If θ is a finite order character of a number field, then we denote by $M(\theta)$ the motive corresponding to θ. Then $M(\theta)$ satisfies $L(\theta, s) = L(M(\theta), s)$.

Now we prove theorem 1.2. Thus, we assume from now on that $k_\tau \geq 3$ for all $\tau \in I_F$. Using the same notations as in §3, we assume that f is the cuspform corresponding to the cuspidal automorphic representation π which appears in theorem 3.1. Since $k^0 \geq 3$, we know from proposition 4.16 of [S], that for each integer j such that $(k_0 + 1)/2 \leq j < (k_0 + k^0)/2$, we have $L(f_i, \varphi_i, j) \neq 0$. Thus for such a j, from formula (3.1) above, we obtain the identity

$$L(M(f)_{/F'}, j) = \prod_{i=1}^{i=u} L(f_i, \varphi_i, j)^{m_i}.$$

Define $E_1 := \mathbb{Q}_f \cup_{i=1}^{i=u} \mathbb{Q}(\varphi_i)$, where $\mathbb{Q}(\varphi_i)$ is the field of coefficients of φ_i. By extending their fields of coefficients, we regard the funtions $\mathbb{L}(M(f)_{/F'}, s)$ and $\mathbb{L}(f_i, \varphi_i, s)$ as having values in $E_1 \otimes_{\mathbb{Q}} \mathbb{C}$. Hence we get

$$\mathbb{L}(M(f)_{/F'}, j) = \prod_{i=1}^{i=u} \mathbb{L}(f_i, \varphi_i, j)^{m_i} \in E_1 \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.1}$$

Since $1 = \sum_{i=1}^{i=u} m_i \mathrm{Ind}_{\mathrm{Gal}(F''/F_i)}^{\mathrm{Gal}(F''/F')} \varphi_i$, we get the equality of motives (by assuming the Tate conjecture for motives)

$$\mathrm{Res}_{F'/\mathbb{Q}}(M(f)_{/F'}(j))) = \bigoplus_{i=1}^{i=u} \mathrm{Res}_{F_i/\mathbb{Q}}(M(f)_{/F_i} \otimes M(\varphi_i))(j))^{m_i},$$

from which, by looking at the E_1 rational basis (see (2.1) above), we obtain trivially

$$c^+(\mathrm{Res}_{F'/\mathbb{Q}}(M(f)_{/F'}(j))) = \prod_{i=1}^{i=u} c^+(\mathrm{Res}_{F_i/\mathbb{Q}}(M(f)_{/F_i} \otimes M(\varphi_i))(j))^{m_i}. \tag{4.2}$$

Under the assumptions of theorem 1.2, we have

$$\frac{\mathbb{L}(M(g)(j),0)}{c^+(\mathrm{Res}_{L/\mathbb{Q}}(M(g)(j)))} \in \mathbb{Q}_g \otimes 1,$$

for any modular form g of weight k of $\mathrm{GL}(2)/L$, for L totally real number field.

Thus we get

$$\frac{\mathbb{L}(M(f_i) \otimes M(\varphi_i)(j),0)}{c^+(\mathrm{Res}_{F_i/\mathbb{Q}}(M(f_i) \otimes M(\varphi_i))(j))} \in E_1 \otimes 1,$$

because $f_i \otimes \varphi_i$ is a modular form. From 4.1 and 4.2, we deduce theorem 1.2:

$$\frac{\mathbb{L}(M(f)(j)_{/F'},0)}{c^+(\mathrm{Res}_{F'/\mathbb{Q}}(M(f)(j)_{/F'}))} \in E_1 \otimes 1.$$

Actually in this last result one can replace E_1 by $\mathbb{Q}_f$ since $M(f)(j)_{/F'}$ has coefficients in $\mathbb{Q}_f$. $\square$

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