On the cohomology of some twisted Shimura varieties of PEL-type

Cristian Virdol
Department of Mathematics
Columbia University

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1 Introduction

In this article we consider the moduli problem of a "twist" of some simple Shimura varieties of PEL-type that appear in Kottwitz's papers [K1] and [K2] (these varieties have a hyperspecial level subgroup at some prime p). The twist of the Shimura varieties is done by a mod q (for some prime $q \neq p$) representation of the absolute Galois group of the reflex field of the Shimura varieties. Then we consider the Newton polygon stratification of the special fiber at p of twisted Shimura varieties and show that each Newton polygon stratum can be described in terms of the products of the reduced fibers of the corresponding PEL-type Rapoport-Zink spaces with certain smooth twisted Igusa varieties and of the action on them of a p-adic group that depends on the stratum. Then we extend our results to characteristic 0, and if $l \neq p$ is a prime number, we describe the l-adic cohomology of proper twisted Shimura varieties of not necessarily hyperspecial level at p, in terms of the l-adic cohomology with compact supports of the twisted Igusa varieties and of the Rapoport-Zink spaces. These results generalize Mantovan's paper [M].

More exactly, we consider a simple $\mathbb{Q}$-algebra B and $*$ a positive involution on B. Let V be a non-degenerate skew-Hermitian B-module and let G be the algebraic group over $\mathbb{Q}$ of similitudes of the skew-Hermitian B-module V (other conditions are imposed on B and V to insure that G is unramified at some given rational prime p; see §2 for details). Let $h : \mathbb{C} \rightarrow \mathrm{End}_B(V_{\mathbb{R}})$ be an $\mathbb{R}$-algebra homomorphism such that $h(\bar{z}) = h(z)^*$ and the symmentric real-valued bilinear form $(v, h(i)w)$ on $V_{\mathbb{R}}$ is positive definite, where we have written $*$ for the involution on $\mathrm{End}_B(V)$ obtained from the alternating form on V. By restricting h to $\mathbb{C}^{\times}$ we obtain a homomorphism $h : \mathbb{C}^{\times} \rightarrow G_{\mathbb{R}}$ of algebraic group over $\mathbb{R}$. For q a rational prime distinct from p, such that G is unramified at q, let $\mathbf{S_K}$ be the Shimura variety associated to (G, h^{-1}) and to an open compact subgroup $\mathbf{K} := K_q \times H$ of $G(\mathbb{A}_f)$, where K_q is a normal subgroup of a hyperspecial maximal compact subgroup H_q of $G(\mathbb{Q}_q)$, H is a sufficiently small open compact subgroup of $G(\mathbb{A}_f^q)$, and $\mathbb{A}_f$ and $\mathbb{A}_f^q$ are the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$ and the finite part of the ring of adeles of $\mathbb{Q}$ away from q, respectively. Then $\mathbf{S_K}$ is a smooth quasi-projective variety defined over a number field E called the reflex field.

Consider the following moduli problem over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. If S is a locally noetherian scheme over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, then S-valued points on the moduli problem are the isomorphism classes of quadruples $(A, \lambda, i, \bar{\eta}^p)$ where: $A \to S$ is a projective abelian scheme over S up to prime-to-p isogeny, $\lambda : A \to \hat{A}$ is a polarization of A, $i : \mathcal{O}_B \to \mathrm{End}(A)$ is a $*$-homomorphism for $*$ on $\mathcal{O}_B$ and the Rosati involution on $\mathrm{End}(A)$ obtained from λ, and $\bar{\eta}^p$ is a level structure of type K^p on A and we require that $(A, \lambda, i, \bar{\eta}^p)$ satisfy some "determinant condition" (see §3 below), which depends on h.

For K^p sufficiently small our moduli problem is representable by a quasi-projective scheme S_{K^p} over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. The variety over E obtained from S_{K^p} is a disjoint union of $|\ker^1(\mathbb{Q}, G)|$ copies of $\mathbf{S_K}$, where our $\mathbf{K}$ defined above has a decomposition of the form $\mathbf{K} = K^p K_p$, where $K_p \subset G(\mathbb{Q}_p)$ is some hyperspecial maximal compact (see §2 below for details).

The varieties S_{K^p} and $\mathbf{S_K}$ have natural actions of H_q/K_q. For H sufficiently small these actions are free. We fix such a small group H and consider a continuous Galois representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to H_q/K_q$ unramified at p and let S'_{K^p} be the variety defined over E obtained from S_{K^p} via twisting by ρ composed with the natural action of H_q/K_q on S_{K^p} (see §2 for details). We show that each Newton polygon stratum at p of S'_{K^p} or $\mathbf{S'_K}$ can be described in terms of the products of the reduced fibers of the corresponding PEL-type Rapoport-Zink spaces with certain smooth twisted Igusa varieties and the action on them of a p-adic group that depends on the stratum (see theorems 9.4 and 9.5 and corollary 9.6 for details). Then we show (see theorem 9.7) that if $l \neq p$ is a prime number, the l-adic cohomology of proper twisted Shimura varieties S'_K or $\mathbf{S'_K}$, of level not necessarily hyperspecial at p, can be described in terms of the l-adic cohomology with compact supports of the twisted Igusa varieties and of the Rapoport-Zink spaces.

2 Shimura varieties of PEL type

Let p be a prime number. We are interested in certain Shimura varieties of PEL type that appear in [K2] having good reduction at all places of the reflex field lying above the prime p (in the next few sections we are following very closely [K2]).

We consider the PEL data $(B, *, V, (\ ,\ ))$, where:

1) B is a finite-dimensional simple algebra over $\mathbb{Q}$,

2) $*$ is a positive involution on B over $\mathbb{Q}$,

3) V is a nonzero finitely generated left B-module,

4) $(\ ,\ )$ is a nondegenerate $\mathbb{Q}$-valued $*$-Hermitian alternating pairing on V, that satisfy the following conditions:

a) there exists a $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ on B whose p-adic completion is a maximal order in $B_{\mathbb{Q}_p}$ and which is preserved by $*$;

b) there exists a lattice Λ in $V_{\mathbb{Q}_p}$ which is self-dual for $(\ ,\ )$ and is preserved by $\mathcal{O}_B$;

c) $B_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.

Let F be the be the center of B. Then c) implies that F is a number field unramified at p. Let D be the $\mathbb{Q}$-algebra $\mathrm{End}_B(V)$; it is a simple $\mathbb{Q}$-algebra with center F and has an involution $*$ (the adjoint map for $(\cdot,\cdot)$). Let G be the algebraic group over $\mathbb{Q}$ of the B-linear automorphisms of V, which preserve the pairing $(\ ,\ )$ up to a scalar multiple; then for any $\mathbb{Q}$-algebra R we have $G(R) = \{x \in D \otimes_{\mathbb{Q}} R | xx^* \in R^\times\}$. Let $h : \mathbb{C} \to D_{\mathbb{R}}$ be an $\mathbb{R}$-algebra homomorphism such that $h(\bar{z}) = h(z)^*$ and the symmetric real-valued bilinear form $(v, h(i)w)$ on $V_{\mathbb{R}}$ is positive definite, i.e $(v, h(i)v) > 0$ for all nonzero $v \in V_{\mathbb{R}}$.

The homomorphism h determines a decomposition of the $B_{\mathbb{C}}$-module $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$, where V_1 (respectively, V_2) is the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts by z (respectively, by $\bar{z}$). The field of definition of the isomorphism class of the complex representation V_1 of B is a number field E called the *reflex field*, whose ring of integers we denote by O_E.

We denote also by h the restriction of h to $\mathbb{C}^\times$, regarded as a homomorphism of real algebraic groups $\mathbb{C}^\times \to G$. Then from Lemma 4.1 of [K2], we know that the pair (G, h^{-1}) satisfies the conditions (1.5.1), (1.5.2), (1.5.3) of [D]. Therefore the set X_∞ of conjugates of h^{-1} under $G(\mathbb{R})$ is a finite union of copies of the symmetric space for the identity component of $G(\mathbb{R})$ and this symmetric space is of Hermitian type. Let K^p be a sufficiently small open compact subgroup of $G(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p$ denotes the ring of finite adeles of $\mathbb{Q}$ with trivial p-adic component. The stabilizer K_p of Λ in $G(\mathbb{Q}_p)$ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We define $\mathbf{K} := K^p K_p$. We denote by $\mathbf{S_K}$ the Shimura variety associated to (G, h^{-1}) and $\mathbf{K}$. Then $\mathbf{S_K}$ is a smooth quasi-projective variety and has a canonical model defined over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

For any compact open subgroup K_1^p of K^p there is an etale covering $\mathbf{S}_{\mathbf{K}_1} \to \mathbf{S_K}$, where $\mathbf{K}_1 := K_1^p K_p$ and this covering map is Galois with Galois group $\mathbf{K}/\mathbf{K}_1 = K^p/K_1^p$ if K_1^p is normal in K^p. For $g \in G(\mathbb{A}_f^p)$ there is a natural isomorphism $\mathbf{S_K} \to \mathbf{S}_{g^{-1}\mathbf{K}g}$.

3 Moduli problem

In this section we formulate the PEL moduli problem having good reduction at all places of the reflex field lying over a given rational prime p (in the case D of [K2], page 375, we assume that $p > 2$). For an open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, we consider the set-valued contravariant functor from the category of locally noetherian schemes S over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ that associates to a connected scheme S and to a geometric point s on S the set of isomorphism classes of quadruples $(A, \lambda, i, \bar{\eta}^p)$, where

1) A is a projective abelian scheme over S;

2) $\lambda : A \to \hat{A}$ is a prime-to-p polarization;

3) $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a morphism of $\mathbb{Z}_{(p)}$-algebras such that $\lambda \circ i(b^*) = i(b)\hat{} \circ \lambda$ and $\det(b, \mathrm{Lie}(A)) = \det(b, V_1)$ for all $b \in \mathcal{O}_B$;

4) $\bar{\eta}^p$ is a $\pi_1(S, s)$-invariant K^p-orbit of isomorphisms of skew-Hermitian B-modules $\eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to H_1(A_s, \mathbb{A}_f^p)$, where $H_1(A_s, \mathbb{A}_f^p)$ is the Tate $\mathbb{A}_f^p$-module of the abelian variety A_s.

We remark that the level structure $\bar{\eta}^p$ of type K^p is independent of the choice of the geometric point s of S and thus our functor is independent of s. For K^p sufficiently small this functor is representable by a quasi-projective scheme S_{K^p} over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Two quadruples $(A, \lambda, i, \bar{\eta}^p)$ and $(A', \lambda', i', \bar{\eta'}^p)$ are said to be isomorphic if there is a prime-to-p isogeny $\varphi : A \to A'$, commuting with the action of $\mathcal{O}_B$, carrying $\bar{\eta}^p$ into $\bar{\eta'}^p$, and carrying λ into a scalar multiple of λ' (a scalar in $\mathbb{Z}_{(p)}^\times$) i.e. $\varphi^*\lambda' = c\lambda$ for some $c \in \mathbb{Z}_{(p)}^\times$, where $\varphi^*\lambda' = \hat{\varphi} \circ \lambda' \circ \varphi$ and $\hat{\varphi} : \hat{A}' \to \hat{A}$ is the dual of φ.

For any compact open subgroup K_1^p of K^p there is an etale covering $S_{K_1^p} \to S_{K^p}$, sending $(A, \lambda, i, (\bar{\eta})_1^p)$ to $(A, \lambda, i, \bar{\eta}^p)$, where $(\bar{\eta})_1^p$ denotes the K_1^p-orbit of η^p, and this covering map is Galois with Galois group K^p/K_1^p if K_1^p is normal in K^p. For $g \in G(\mathbb{A}_f^p)$ there is an isomorphism $S_{K^p} \to S_{g^{-1}K^p g}$ sending $(A, \lambda, i, \bar{\eta}^p)$ $(A, \lambda, i, \overline{\eta^p g})$.

4 Complex points

We want to describe the set of complex points of $S_{K^p}(\mathbb{C})$ relative to the map $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathbb{C}$ induced by the inclusion $E \subset \mathbb{C}$. Hence we are interested in quadruples $(A, \lambda, i, \bar{\eta}^p)$, where this time A is an abelian variety over $\mathbb{C}$ up to prime-to-p isogeny. Then the homology group $H := H_1(A, \mathbb{Q})$ is a skew-Hermitian B-module, with the B-action coming from i and the alternating form coming from λ.

From §8 of [K2], we know that for every place v of $\mathbb{Q}$ the skew-Hermitian $B_{\mathbb{Q}_v}$-modules $H_{\mathbb{Q}_v}$ and $V_{\mathbb{Q}_v}$ are isomorphic.

The isomorphism classes of skew-Hermitian B-modules of the same dimension as V are classified by $H^1(\mathbb{Q}, G)$, and thus we get there are finitely many isomorphic classes of skew-Hermitian B-modules $(V', (.,.)')$ such that $V'_{\mathbb{Q}_p}$ and $V_{\mathbb{Q}_p}$ are isomorphic for all places v of $\mathbb{Q}$. Let $V^{(1)}, \cdots, V^{(m)}$ be some representatives in these isomorphism classes. We fix local isomorphisms $V^{(i)}_{\mathbb{Q}_p} \cong V_{\mathbb{Q}_p}$ for all places v of $\mathbb{Q}$ and let $G^{(i)}$ denote the group of automorphisms of $V^{(i)}$ and thus we have $G^{(i)}_{\mathbb{Q}_p} \cong G_{\mathbb{Q}_p}$ for all p. We obtain that the set $S_{K^p}(\mathbb{C})$ is a disjoint union $\coprod_{i=1}^m \mathbf{S_K}(\mathbb{C})^{(i)}$, where $\mathbf{S_K}(\mathbb{C})^{(i)}$ consists of quadruples $(A, \lambda, i, \bar{\eta}^p)$ such that H is isomorphic to $V^{(i)}$, and hence it is enough to study each set $\mathbf{S_K}(\mathbb{C})^{(i)}$.

Assume that $V^{(1)}$ is V and we start studying $\mathbf{S_K}(\mathbb{C})^{(1)}$. Let $(A, \lambda, i, \bar{\eta}^p)$ be a point of $\mathbf{S_K}(\mathbb{C})^{(1)}$ and choose an isomorphism between H and V. Choosing a level stucture $\bar{\eta}^p$ is equivalent to choosing an element of $G(\mathbb{A}_f^p)/K^p$. The complex structure on $H_{\mathbb{R}} = \mathrm{Lie}(A)$ determines a complex structure on $V_{\mathbb{R}}$ and yields a homomorphism $h' : \mathbb{C} \to D_{\mathbb{R}}$. From Lemma 4.2 of [K2], we get that h' belongs to the set X_∞ of $G(\mathbb{R})$-conjugates of h. The self-dual $\mathcal{O}_B$-lattice $\lambda' := H_1(A, \mathbb{Z}_p)$ in $H_{\mathbb{Q}_p}$ becomes an $\mathcal{O}_B$-lattice in $V_{\mathbb{Q}_p}$ that it is self-dual up to a scalar. The group K_p is the stabilizer of the self-dual $\mathcal{O}_B$-lattice Λ in $G(\mathbb{Q}_p)$. From Corollary 7.3 of [K2], we know that there exists $g \in G(\mathbb{Q}_p)$ such that $\lambda' = g\Lambda$, and g determines a well-defined element of $G(\mathbb{Q}_p)/K_p$. The choice of isomorphism between H and V gives us a well-defined element of $(G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty$. Two isomorphisms between H and V differ by an automorphism of V, thus by an element of $G(\mathbb{Q})$ and the corresponding elements of $(G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty$ are in the same $G(\mathbb{Q})$-orbit. Hence $(A, \lambda, i, \bar{\eta}^p)$ gives rise to a well-defined element of the quotient $G(\mathbb{Q}) \setminus (G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$. We get a bijection from $\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(1)}$ to $G(\mathbb{Q}) \setminus ((G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$. Replacing V by $V^{(i)}$, we obtain a bijection from $\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)}$ to $G(\mathbb{Q})^{(i)} \setminus ((G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$. Then $\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)}$ is the set of complex points of a variety $\mathbf{S}_{\mathbf{K}}^{(i)}$ over E, which is isomorphic to $\mathbf{S}_{\mathbf{K}}$.

Hence, we get (see §8 of [K2]) that variety S_{K^p} over E is a disjoint union of $|\ker^1(\mathbb{Q}, G)|$-copies of $\mathbf{S}_{\mathbf{K}}$.

5 Twisted Shimura varieties

Let $q \neq p$ be a rational prime such that G is unramified at q, i.e. G is quasi-split over $\mathbb{Q}_q$ and split over an unramified extension of $\mathbb{Q}_q$. Consider now our $\mathbf{K}$ of the form $\mathbf{K} := K_q \times H$, where K_q is a normal subgroup of a hyperspecial maximal compact subgroup H_q of $G(\mathbb{Q}_q)$. Then it is well known (see for example Corollary 1.4.1.3 of [CA]) that for H sufficiently small, the group H_q/K_q acts freely on S_{K^p}. The action of H_q/K_q on $S_{K^p}(\mathbb{C})$ can be described in the following way: the set $S_{K^p}(\mathbb{C})$ is a disjoint union $\coprod_{i=1}^m \mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)}$ (a disjoint union of $|\ker^1(\mathbb{Q}, G)|$ copies of $\mathbf{S}_{\mathbf{K}}(\mathbb{C})$) and H_q/K_q acts on each

$$\mathbf{S}_{\mathbf{K}}(\mathbb{C})^{(i)} = G(\mathbb{Q})^{(i)} \setminus ((G(\mathbb{A}_f)/\mathbf{K}) \times X_\infty)$$

where $H_q \hookrightarrow G(\mathbb{A}_{\mathbb{Q}})$ by $\alpha \mapsto (1, ..., \alpha, 1, ..., 1)$, α at the q-component, and the action of an element $\alpha \in H_q$ is given by the right multiplication at the q-component.

We fix a continuous representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to H_q/K_q,$$

unramified at p. Let L be the finite Galois extension of E defined by $L := (\overline{\mathbb{Q}})^{\mathrm{Ker}(\rho)}$.

Let

$$X = S_{K^p} \times_{\mathrm{Spec}(E)} \mathrm{Spec}(L).$$

The group H_q/K_q acts on S_{K^p}. Since $\rho : \mathrm{Gal}(L/E) \hookrightarrow H_q/K_q$, the group $\mathrm{Gal}(L/E)$ acts on S_{K^p}. We denote this action of $\mathrm{Gal}(L/E)$ on S_{K^p} by ρ'. The

Galois group $\mathrm{Gal}(L/E)$ has a natural action on $\mathrm{Spec}(L)$ and we can descend via the quotient process X to $S'_{K^p}/\mathrm{Spec}(E)$ using the diagonal action

$$\mathrm{Gal}(L/E) \ni \sigma \to \rho'(\sigma) \otimes \sigma$$

on X. Thus, we obtain a quasi-projective variety $S'_{K^p}/\mathrm{Spec}(E)$ having the same dimension as S_{K^p}. This is the twisted Shimura variety that we mentioned in the title.

Now we describe the PEL moduli problem for S'_{K^p}. We consider again quadruples as in §3 above, only η^p will be replaced by η'^p. Thus, with the same notations as above, for the twisted Shimura variety S'_{K^p} a level structure of type K^p on A is a K^p-orbit $\bar{\eta}'^p$ of isomorphisms $\eta'^p : V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to H_1(A_s, \mathbb{A}_f^p)$ of skew-Hermitian B-modules such that $\bar{\eta}'^p$ is fixed by the action $\rho(\tau) \times \tau$ for all $\tau \in \pi_1(S, s)$, where $\rho(\tau)$ acts on the left using the natural action of H_q/K_q and τ acts on the right using the natural action of $\tau \in \pi_1(S, s)$ on A_s.

6 The Newton polygon stratification

For a place v of E above p, we denote by E_v the completion of E at v. From the condition c) of §2, we get that $E_v/\mathbb{Q}_p$ is an unramified extension. Let $k/\mathbb{F}_p$ be the residue field of $\mathcal{O}_{E_v}$, $|k| = q = p^f$, and let $\sigma : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ be the Frobenius map $x \mapsto x^p$.

For $K^p \subset G(\mathbb{A}_f^p)$ a sufficiently small open compact open subgroup, we write $S = S_{K^p}$ and $\bar{S} = \bar{S}_{K^p}/k$ for its reduction. If $\mathcal{A}$ is the universal abelian variety over $\bar{S}$, we define $\mathcal{G} = \mathcal{A}[p^\infty]$. Then $\mathcal{G}$ is a Barsotti-Tate group endowed with a quasi-polarization and a compatible action (i.e it satisfies the determinant condition as in [K2]) of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ of a polarized Barsotti-completion of $\mathcal{O}_B$ in $B_{\mathbb{Q}_p}$. We call a Barsotti-Tate group endowed with a quasi-polarization and of a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ a Barsotti-Tate group with *additional structures*.

We denote by $B(G)$ the set of σ-conjugacy classes in $G(L)$, where $L = \mathbb{Q}_p^{nr} \supset E_v$. Hence, $B(G)$ is the set of isogeny classes over $\bar{\mathbb{F}}_p \supset k$ of polarized Barsotti-Tate groups, which are endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$. If $\mathcal{G}$ is a Barsotti-Tate group with additional structures, we denote by $b(\mathcal{G}) \in B(G)$ the corresponding isogeny class.

For any element $b \in B(G)$ define the strata

$$\bar{S}^{(b)} = \{x \in \bar{S} | b(\mathcal{G}_{\bar{x}}) = b\} \subset \bar{S}.$$

Then $\bar{S}^{(b)} \subset \bar{S}$ is a reduced subscheme, and as the level K^p varies, these strata are preserved by the natural projection $\bar{S}_{K_1^p} \to \bar{S}_{K^p}$ (for $K_1^p \subset K^p$) and also by the morphism $g : \bar{S}_{K^p} \to \bar{S}_{gK^pg^{-1}}$ for all $g \in G(\mathbb{A}_f^p)$.

We know (proposition 1 of [M]):

Proposition 6.1. *Let $\Sigma/\bar{\mathbb{F}}_p \supset k$ be a polarized Barsotti-Tate group endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$, and denote by $b \in B(G)$ its isogeny class. We* *define*

$$C_\Sigma = \{x \in \bar{S} | \mathcal{G}_{\bar{x}} \simeq \Sigma_{\overline{k(x)}}\} \subset \bar{S}^{(b)} \times_k \bar{\mathbb{F}}_p.$$

This is a closed subset of the stratum $\bar{S}^{(b)}$; and as a subscheme of $\bar{S}^{(b)} \times_k \bar{\mathbb{F}}_p$ endowed with the induced reduced structure, it is smooth.

Then C_Σ is called a leaf. As the level K^p varies, the leaves are preserved by both the natural projection $\bar{S}_{K_1^p} \to \bar{S}_{K^p}$ (for $K_1^p \subset K^p$) and also by the morphism $g : \bar{S}_{K^p} \to \bar{S}_{gK^pg^{-1}}$ for all $g \in G(\mathbb{A}_f^p)$.

The universal Barsotti-Tate group $\mathcal{G}$ admits a slope filtration when restricted to the leaves (see page 239 of [Z]).

If $\Sigma/\bar{\mathbb{F}}_p$ is a completely slope-divisible Barsotti-Tate group with additional structures, we write $b \in B(G)$ for its isogeny class and $1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$ for the slopes of its Newton polygon. Then $\Sigma = \oplus_i \Sigma^i$, where for each i, $1 \leq i \leq k$, Σ^i is the slope-divisible isoclinic Barsotti-Tate group of the slope λ_i.

For $C = C_\Sigma$ a leaf contained in $\bar{S}^{(b)}$ associated to a completely slope-divisible Barsotti-Tate group Σ with additional structures, we denote by $\mathcal{G}$ the universal Barsotti-Tate group over C. Then $\mathcal{G}$ is a completely slope-divisible Barsotti-Tate group with slope filtration

$$0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G},$$

where the subquotients $\mathcal{G}^i = \mathcal{G}_i/\mathcal{G}_{i-1}$ are isoclinic slope-divisible Barsotti-Tate groups of slope λ_i. Then $\mathcal{G}_i$ and $\mathcal{G}^i$ inherit an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$.

7 Igusa varieties

For a completely slope-divisible Barsotti-Tate group Σ as above, we denote by $b = b(\Sigma) \in B(G)$ the corresponding isogeny class, by $C_b = C_\Sigma \subset \bar{S}^{(b)}$ the corresponding leaf and by $\mathcal{G}$ the universal Barsotti-Tate group over C_b.

Definition 7.1. *For any $m \geq 1$, the Igusa variety of level m, $J_{b,m} \to C_b$ is the universal space for the existence of isomorphisms*

$$j_{m,i} : \Sigma^i[p^m] \to \mathcal{G}^i[p^m]$$

of finite flat group schemes over C_b such that

(1) they extend étale locally to any level $m^{'} \geq m$;

(2) they are $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant;

(3) they commute up to $(\mathbb{Z}/p^m)^\times$-multiple with the isomorphisms induced by the quasi-polarizations on $\mathcal{G}$ and Σ.

The group $\Gamma_b = \mathrm{Aut}(\Sigma) = \prod_i \mathrm{Aut}(\Sigma^i)$ acts on $J_{b,m}$. If $\gamma = \oplus_i \gamma^i \in \Gamma_b = \mathrm{Aut}(\Sigma) = \prod_i \mathrm{Aut}(\Sigma^i)$, then γ acts on $J_{b,m}$ by

$$(A, \lambda, i, \bar{\eta}^p; j_{m,1}, \cdots, j_{m,k}) \mapsto (A, \lambda, i, \bar{\eta}^p; j_{m,1} \circ \gamma_{|[p^m]}, \cdots, j_{m,k} \circ \gamma_{|[p^m]}),$$

where a point of $J_{b,m}$ is represented by the data $(A, \lambda, i, \bar{\eta}^p; j_{m,1}, \cdots, j_{m,k})$, as in the definition 7.1. This action factors through the quotient $\Gamma_{b,m}$ of Γ_b by the subgroup of automorphisms of Σ which restrict to the identity on $\Sigma[p^m]$.

We know (see proposition 4 of [M]):

Proposition 7.2. *For any $m \geq 1$, the Igusa variety $J_{b,m} \to C_b$ is finite étale and Galois with Galois group $\Gamma_{b,m}$. In particular, the Igusa varieties are smooth.*

We have natural projections

$$q_b = q_{b,m',m} : J_{b,m'} \to J_{b,m}, \quad \forall\ m^{'} \geq m,$$

which correspond to restricting the isomorphisms $j_{m',i}$ to the p^m-torsion subgroups for all i. The projections q_b are invariant under the action of Γ_b on the Igusa varieties.

Thus, as the level m varies, the Igusa varieties $J_{b,m}$ form a projective system under the morphisms q_b which is endowed with an action of Γ_b. In §4 of [M] it is shown that this action naturally extends to a submonoid $S_b \supset \Gamma_b$ of the group T_b of the quasi-self-isogenies of Σ, preserving its additional structures.

Now we define S_b. Any $\rho \in T_b$ can be written as $\rho = \oplus_i \rho^i$, where ρ^i is the quasi-self-isogeny of Σ^i obtained by restricting ρ to Σ^i. If ρ^{-1} is an isogeny, then, for each $i = 1, \cdots, k$, define two integers $e_i = e_i(\rho) \geq f_i = f_i(\rho) \geq 0$ to be, respectively, the minimal and maximal integers such that

$$\ker[p^{f_i}] \subset \ker[\rho^{i^{-1}}] \subset \ker[p^{e_i}].$$

Define

$$S_b = \{\rho \in T_b | \rho^{-1} \text{ is an isogeny},\ f_{i-1}(\rho) \geq e_i(\rho),\ \forall i \geq 2\}.$$

Then S_b is a submonoid of T_b, and p^{-1}, $\mathrm{fr}^{-B} = \oplus_i p^{-\lambda_i B} \in S_b$, where the integer $B \geq 1$ is defined such that for all $i = 1, \cdots, k$, $F^B p^{-\lambda_i B} : \Sigma^i \to \Sigma^{i(p^B)}$ are well defined isogenies, where $F : \Sigma \to \Sigma^{(p)}$ is the Frobenius element. Also we know that $T_b = < S_b, p, \mathrm{fr}^B >$, where $\mathrm{fr}^{-B} := \oplus_i p^{-\lambda_i B} \in S_b$ (see lemma 2.11 of [M1]).

The action of Γ_b on the system of Igusa varieties extends to $S_b \subset T_b$ (lemma 5 of [M]):

Lemma 7.3. *Let $(\mathcal{A}, \lambda, i, \bar{\eta}^p; j_{m,1}, \cdots, j_{m,k})$ denote the universal object over the Igusa variety $J_{b,m}$, let $\rho = \oplus_i \rho_i \in S_b$, and assume $m \geq e = e_1(\rho)$.*

There exists a unique finite flat subgroup $\mathcal{K}_\rho \subset \mathcal{G}[p^e]$ such that the corresponding subgroups inside the isoclinic subquotients $\mathcal{G}^i$ of $\mathcal{G}$ are $j_{m,i}(ker(\rho_i^{-1}))$ for all i.

Moreover, the quotient abelian variety $\mathcal{A}/\mathcal{K}_\rho$ inherits the additional structures of $\mathcal{A}/J_{b,m}$ and thus gives rise to a morphism between the Igusa varieties:

$$\rho : J_{b,m} \to J_{b,m-e}.$$

For all $\rho \in S_b$, the morphisms $\rho : J_{b,m} \to J_{b,m-e}$ are compatible with the projections among Igusa varieties of different levels $m \geq e = e_1(\rho)$ and they give rise to an action of the monoid S_b on the system of Igusa varieties which extends the action of Γ_b.

One can to define a σ-semilinear action of Frobenius on the varieties as arising from the Frobenius morphism of the Barsotti-Tate group Σ, $F : \Sigma \to \Sigma^{(p)}$, that is, to define

$$\mathrm{Frob} : J_{b,m} \to J_{b,m-1}$$

to be the σ-semilinear morphism corresponding to the quotient abelian variety $\mathcal{A}^{(p)} = \mathcal{A}/\mathcal{A}[p]$ endowed with the induced additional structures. Then Frob is the absolute Frobenius of $J_{b,m}$ followed by the projection $q_{b,m,m-1}$.

By reintroducing in our notation the level K^p, we write

$$J_{b,K^p,m} \to C_{b,K^p} \subset \bar{S}^{(b)}_{K^p} \subset \bar{S}_{K^p}$$

for the Igusa varieties of level K^p and level m at p. For any $b \in B(G)$ and $m \geq 1$, we the group $G(\mathbb{A}_f^p)$ acts on the Igusa varieties

$$\forall g \in G(\mathbb{A}_f^p) : J_{b,K^p,m} \to J_{b,g^{-1}K^p g,m},$$

$$(A, \lambda, i, \bar{\eta}^p; j_{m,1}, \cdots, j_{m,k}) \mapsto (A, \lambda, i, \overline{\eta^p \circ g}; j_{m,1}, \cdots, j_{m,k}).$$

This action is compatible under the projection map with the action of $G(\mathbb{A}_f^p)$ on the leaves C_{b,K^p}, commutes with the natural projections among the Igusa varieties as the level m at p varies, and also commutes with the action of S_b defined above.

Thus, for $l \neq p$ be a prime number, the direct limit of cohomologies with compact support

$$H^i_c(J_b, \mathbb{Q}_l) = \lim_{\to\, K^p,m} H^i_c(J_{b,K^p,m}, \mathbb{Q}_l), \quad \forall\ i,$$

is endowed with an action of the submonoid $S_b \times G(\mathbb{A}_f^p) \subset T_b \times G(\mathbb{A}_f^p)$. This action cannot be extended to an action of $T_b \times G(\mathbb{A}_f^p)$ on the varieties, but it is possible to extend it on the cohomology groups. Because of the equality $T_b =< S_b, p, \mathrm{fr}^B >$, it suffice to prove that the action of p^{-1}, $\mathrm{fr}^{-B} \in S_b$ on the cohomology groups of Igusa varieties is invertible.

We know (lemma 6 of [M]):

Lemma 7.4. *Maintaining the above notation, we have the following:*

(1) for any $m \geq 1$, $(p^{-1}, 1) \in S_b \times G(\mathbb{A}_f^p)$ acts on the Igusa varieties as $(p, 1) \circ q_{b,m,m-1}$;

(2) for any $m \geq \lambda_1 B$, $(fr^{-B}, 1) \in S_b \times G(\mathbb{A}_f^p)$ acts on the Igusa varieties as $q_{b,m,m-\lambda_1 B} \circ Frob$.

Thus we get the first part of the following proposition:

Proposition 7.5. *The action of the submonoid $S_b \times G(\mathbb{A}_f^p)$ on the cohomology groups of the Igusa varieties extends uniquely to an action of the group $T_b \times G(\mathbb{A}_f^p)$.*

Moreover, as representations of $T_b \times G(\mathbb{A}_f^p)$, the $H_c^i(J_b, \mathbb{Q}_l)$ are admissible for all $i \geq 0$.

The second part regarding the admissibility of the representations follows from

$$H_c^i(J_b, \mathbb{Q}_l)^{\Gamma_{b,m} \times K^p} = H_c^i(J_{b,K^p,m}, \mathbb{Q}_l),$$

where $\Gamma_{b,m}$ is the subgroup of T_b consisting of the isomorphisms of Σ which restrict to the identity on $\Sigma[p^m]$.

8 Twisted leaves and twisted Igusa varieties

Let $q \neq p$ be the rational prime, and let $K_q \subset G(\mathbb{Q}_q)$ be the open compact subgroup considered in §5. Let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup of the form $K^p = K_q \times K^{pq}$, where K^{pq} is an open compact subgroup of $G(\mathbb{A}_f^{pq})$, and we write $S' = S'_{K^p}$ for the twisted variety defined in §5 and $\bar{S}' = \bar{S}'_{K^p}/k$ for its reduction.

Using the same notations as in §6, one can define for any element $b \in B(G)$ the twisted strata

$$\bar{S}'^{(b)} = \{x \in \bar{S}' | b(\mathcal{G}_{\bar{x}}) = b\} \subset \bar{S}'.$$

Then $\bar{S}'^{(b)} \subset \bar{S}'$ is a reduced subscheme, and as the level $K^p = K_q \times K^{pq}$ varies, these twisted strata are preserved by the natural projection $\bar{S}'_{K_1^p} \to \bar{S}'_{K^p}$ (for $K_1^p = K_q \times K_1^{pq} \subset K^p = K_q \times K^{pq}$) and also by the morphism $g : \bar{S}'_{K^p} \to \bar{S}'_{gK^pg^{-1}}$ for all $g \in G(\mathbb{A}_f^{pq})$.

If $\Sigma/\bar{\mathbb{F}}_p \supset k$ is a polarized Barsotti-Tate group endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$, and $b \in B(G)$ is its isogeny class, we define the twisted leaf

$$C'_b = C'_\Sigma = \{x \in \bar{S}' | \mathcal{G}_{\bar{x}} \simeq \Sigma_{\overline{k(x)}}\} \subset \bar{S}'^{(b)} \times_k \bar{\mathbb{F}}_p.$$

This is a closed subset of the stratum $\bar{S}'^{(b)}$, and as a subscheme of $\bar{S}'^{(b)} \times_k \bar{\mathbb{F}}_p$ endowed with the induced reduced structure, it is smooth. As the level $K^p = K_q \times K^{pq}$ varies, these twisted leaves are preserved by the natural projection $\bar{S}'_{K_1^p} \to \bar{S}'_{K^p}$ (for $K_1^p = K_q \times K_1^{pq} \subset K^p = K_q \times K^{pq}$) and also by the morphism $g : \bar{S}'_{K^p} \to \bar{S}'_{gK^pg^{-1}}$ for all $g \in G(\mathbb{A}_f^{pq})$.

In the same way, one can define the twisted Igusa varieties $J'_{b,m} \to C'_b$, where a point of $J'_{b,m}$ is represented by the data $(A, \lambda, i, \bar{\eta}'^p; j_{m,1}, \cdots, j_{m,k})$. As in §7, we have that the group Γ_b acts on $J'_{b,m}$, and this action factors through $\Gamma_{b,m}$. From proposition 7.2, we know that the map $J'_{b,m} \to C'_b$ is finite étale and Galois with Galois group $\Gamma_{b,m}$, and the twisted Igusa varieties $J'_{b,m}$ are smooth.

As in §8, we have natural projections

$$q'_b = q_{b,m',m} : J'_{b,m'} \to J'_{b,m}, \quad \forall\, m' \geq m,$$

which correspond to restricting the isomorphisms $j_{m',i}$ to the p^m-torsion subgroups for all i. The projections q'_b are invariant under the action of Γ_b on the Igusa varieties.

Thus, as the level m varies, the twisted Igusa varieties $J'_{b,m}$ form a projective system under the morphism q_b which is endowed with an action of Γ_b. As in §7, this action naturally extends to the submonoid $S_b \supset \Gamma_b$ of the group T_b of the quasi-self-isogenies of Σ, preserving its additional structures.

By reintroducing in our notation the level $K^p = K_q \times K^{pq}$, we write

$$J'_{b,K^p,m} \to C'_{b,K^p} \subset \bar{S}'^{(b)}_{K^p} \subset \bar{S}'_{K^p}$$

for the twisted Igusa varieties of level $K^p = K_q \times K^{pq}$ and level m at p. For any $b \in B(G)$ and $m \geq 1$, we define an action of the group $G(\mathbb{A}_f^{pq})$ on the twisted Igusa varieties of level m at p (as the level $K^p = K_q \times K^{pq}$ away from p varies) as

$$\forall g \in G(\mathbb{A}_f^{pq}) : J'_{b,K^p,m} \to J'_{b,g^{-1}K^p g,m},$$
$$(A, \lambda, i, \bar{\eta}'^p; j_{m,1}, \cdots, j_{m,k}) \mapsto (A, \lambda, i, \overline{\eta'^p \circ g}; j_{m,1}, \cdots, j_{m,k}).$$

This action is clearly compatible under the projection map with the action of $G(\mathbb{A}_f^{pq})$ on the twisted leaves C'_{b,K^p}, commutes with the natural projections among the Igusa varieties as the level at p varies, and also commutes with the action of S_b.

Thus, for $l \nmid pq$ be a prime number, the direct limit of cohomologies with compact support

$$H^i_c(J'_{b,K_q}, \mathbb{Q}_l) = \lim_{\to\, K^p = K_q \times K^{pq},m} H^i_c(J'_{b,K^p,m}, \mathbb{Q}_l), \quad \forall\ i,$$

is endowed with an action of the submonoid $S_b \times G(\mathbb{A}_f^{pq}) \subset T_b \times G(\mathbb{A}_f^{pq})$. As in §7, this action cannot be extended to an action of $T_b \times G(\mathbb{A}_f^{pq})$ on the varieties, but it is possible to extend it on the cohomology groups.

We have the identities

$$H^i_c(J'_{b,K_q}, \mathbb{Q}_l)^{\Gamma_{b,m} \times K^{pq}} = H^i_c(J'_{b,K^p,m}, \mathbb{Q}_l),$$

where $K^p = K_q \times K^{pq}$.

9 The geometry of the twisted Newton polygon strata

In this section, following closely §5 of [M], we construct a system of finite morphisms

$$\pi'_N : J'_{m,b} \times \bar{\mathcal{M}}'^{n,d}_b \to \bar{S}'^{(b)} \times_k \bar{\mathbb{F}}_p,$$

indexed by positive integers $m,\ n,\ d,\ N$ ($m \geq d$ and $N \geq d/\delta f$, where $\delta \in \mathbb{Q}$ depends only on $b \in B(G)$), which are compatible under the projections among the twisted Igusa varieties and the inclusion among the twisted truncated Rapoport-Zink spaces $\bar{\mathcal{M}}'^{n,d}$, and which are also T_b-invariant and W_{E_v}-invariant, where W_{E_v} is the Weil group. Also for any geometric closed point $x \in \bar{S}'^{(b)}$, the fibers of π'_N at x form a T_b-principal homogeneous space.

Now we recall the definition of the Rapoport-Zink space $\mathcal{M}_{b/W(\bar{\mathbb{F}}_p)}$. An isogeny class $b \in B(G)$ (corresponding to S_{K^p}) is called *descent* if it contains a Barsotti-Tate group defined over a finite field. In [OZ], it was proved that this is equivalent to containing a completely slope-divisible Barsotti-Tate group. We choose a completely slope-divisible Barsotti-Tate group $\Sigma = \Sigma_b$ in the isogeny class b.

To any descent class $b \in B(G)$, Rapoport-Zink associated a formal scheme $\mathcal{M}_b$ over $W(\bar{\mathbb{F}}_p) \supset O_{E_v}$ as follows: for any scheme $S/W(\bar{\mathbb{F}}_p)$, where p is locally nilpotent, $\mathcal{M}_b(S)$ is the set of isomorphisms classes of Barsotti-Tate groups with additional structures H/S together with a quasi-isogeny $\beta : \Sigma \times_{\bar{\mathbb{F}}_p} \bar{S} \to H \times_S \bar{S}$ defined over $\bar{S} = Z(p) \subset S$.

The group T_b acts on the Rapoport-Zink space $\mathcal{M}_b$ by

$$\forall \rho \in T_b : (H, \beta) \mapsto (H, \beta \circ \rho).$$

One can define also a σ-semilinear isomorphism of $\mathcal{M}_b$ arising from the Frobenius of Σ,

$$\mathrm{Frob} : \mathcal{M}_b \to \mathcal{M}_b,$$
$$(H, \beta) \mapsto (H, F^{-1} \circ \beta),$$

which obviously commutes with the action of T_b.

Because the formal scheme $\mathcal{M}_b$ is only locally of finite type, it is convenient to define certain subspaces of $\mathcal{M}_b$, which are called *truncated Rapoport − Zink spaces*, which are of finite type and which form a cover of $\mathcal{M}_b$. If $n,\ d$ are positive integers, define $\mathcal{M}_b^{n,d} \subset \mathcal{M}_b$ as the locus classifying pairs (H, β) such that the quasi-isogenies $p^n\beta$ and $p^{d-n}\beta^{-1}$ are isogenies (or, equivalently, such that $p^n\beta$ is an isogeny whose kernel is killed by p^d). Then $\mathcal{M}_b^{n,d}$ are closed subspaces of $\mathcal{M}_b$ and, as $n,\ d$ vary, they form a direct system under the natural inclusions $i_b = i^{n,d}_{n',d'}$ with $\mathcal{M}_b$ as a limit. The truncated Rapoport-Zink spaces $\mathcal{M}_b^{n,d}$ are not preserved by the actions of T_b and Frob, but one can think of the action of T_b and Frob on $\mathcal{M}$ as arising from the action on the direct system. More exactly, if $\rho \in T_b$, let $n(\rho)$ and $d(\rho)$ be the smallest integer such that $p^{n(\rho)}\rho$ and $p^{d(\rho)-n(\rho)}\rho^{-1}$ are isogenies. Then the morhism $\rho : \mathcal{M}_b \to \mathcal{M}_b$ induces by restriction some morphisms

$$\rho : \mathcal{M}_b^{n,d} \to \mathcal{M}_b^{n+n(\rho),d+d(\rho)}, \quad \forall\ n,\ d,$$

which obviously commute with the natural inclusions. Also by restricting Frob to $\mathcal{M}_b^{n,d} \subset \mathcal{M}_b$, we obtain a σ-linear morphism

$$\mathrm{Frob} : \mathcal{M}_b^{n,d} \to \mathcal{M}_b^{n+1,d+1}, \quad \forall\ n,\ d.$$

We denote by $\bar{\mathcal{M}}_b$ (resp. $\bar{\mathcal{M}}_b^{n,d}$) over $\bar{\mathbb{F}}_b$, the reduced fiber of $\mathcal{M}_b$ (resp., $\mathcal{M}_b^{n,d}$ for any n, d). Then $\bar{\mathcal{M}}_b$ uniformizes the isogeny classes of the abelian varieties classified by $\bar{S}^{(b)}$.

In the same way, starting with an isogeny class $b \in B(G)$ corresponding to S'_{K^p}, one can define the twisted Rapoport-Zink space $\mathcal{M}'_b$ and the truncated Rapoport-Zink spaces ${\mathcal{M}'}_b^{n,d}$, verifying the same proprieties as above.

We write Fr=$\mathrm{Fr}_{\bar{S}'_{K^p}}$ for the absolute Frobenius. Then Fr^f is the morphism $\bar{S}'_{K^p} \to \bar{S}'_{K^p}$ which sends the quadruple $(A, \lambda, i, \bar{\eta'}^p)$ to $(A^{(q)}, \lambda^{(q)}, i^{(q)}, \bar{\eta'}^{p(q)})$.

From proposition 9 of [M], and the description of S' in §5, and of $J'_{m,b}$ in §7, we obtain the following result:

Proposition 9.1. *For any positive integers $m, n, d, N, m \geq d$ and $N \geq d/\delta f$, there exists some morphisms*

$$\pi'_N : J'_{m,b} \times \bar{\mathcal{M}'}_b^{n,d} \to \bar{S'}^{(b)} \times_k \bar{\mathbb{F}}_p,$$

such that

(1) $\pi'_{N+1} = (Fr^f \times 1) \circ \pi'_N$;

(2) $\pi'_N \circ q_b = \pi'_N$;

(3) $\pi'_N \circ i_b = \pi'_N$;

(4) $\pi'_N \circ \rho \times \rho = \pi'_N$ for any $\rho \in S_b$, provided that $m \geq d + d(\rho) + e(\rho)$ and $N \geq (d + d(\rho))/\delta f$;

(5) $\pi'_N \circ (Frob^f \times Frob^f) = (1 \times \sigma^f) \circ \pi'_N$ for all $m \geq d+1$ and $N \geq (d+1)/\delta f$.

We now describe the fibers on the morphisms π'_N. If x is a point of $\bar{S'}^{(b)}(\bar{\mathbb{F}}_p)$, then for any $N \geq N_0 \geq d/\delta f$, we consider the fibers over x of the maps

$$\pi'_N : J'_{m,b} \times \bar{\mathcal{M}'}_b^{n,d} \to \bar{S'}^{(b)} \times_k \bar{\mathbb{F}}_p.$$

Using proposition 9.1 (1), we can identify

$${\pi'}_N^{-1}(\mathrm{Fr}^{fN} x) = {\pi'}_{N_0}^{-1}(\mathrm{Fr}^{fN_0} x).$$

As m varies, the sets ${\pi'}_N^{-1}(\mathrm{Fr}^{fN} x)$ form an inverse system under the projections q_b, and the corresponding limits are a direct system under the inclusions i_b as n, d vary. We define the *fiber above* x to be the set

$${\Pi'}^{-1}(x) = \lim_{\to\, n,d} \lim_{\leftarrow\, m} {\pi'}_N^{-1}(\mathrm{Fr}^{fN} x),$$

which is endowed with the topology of the direct limit of inverse limits of discrete sets. Then from proposition 9.1 (4), we get that ${\Pi'}^{-1}(x)$ is also endowed with continous action of S_b, and because of proposition 9.1 (5) and lemma 7.4, this action extends uniquely to a continous action of the group $T_b =< S_b, p, \mathrm{fr}^B >$.

We obtain (see propositions 10 and 11 of [M]):

Proposition 9.2. *For any $x \in \bar{S'}^{(b)}(\bar{\mathbb{F}}_p)$, its fiber ${\Pi'}^{-1}(x)$ is a principle homogeneous space for the continuous action of T_b.*

Proposition 9.3. *For any positive integers $m,\ n,\ d,\ N,\ m \geq d$ and $N \geq d/\delta f$, the morphism $\pi'_N : J'_{m,b} \times \bar{\mathcal{M}'}^{n,d}_b \to \bar{S'}^{(b)} \times \bar{\mathbb{F}}_p$ is finite.*

Let $U'^{n,d}_b$ be the maximal open subspaces of $\mathcal{M}'_b$ contained in $\mathcal{M}'^{n,d}_b$, and let $\bar{U'}^{n,d}_b$ be their reduced fibers. Then the sets $U'^{n,d}_b$ are an open cover of $\bar{\mathcal{M}'}$, and thus for any sheaf $\mathcal{F}/\bar{\mathcal{M}'}_b$, we have

$$H^t_c(\bar{\mathcal{M}'}_b, \mathcal{F}) = \lim_{\rightarrow\, n,d} H^t_c(\bar{U'}^{n,d}_b, \mathcal{F}), \qquad t \geq 0.$$

Let $\dot{\pi'}_N$ be the restrictions of the morphisms π'_N to the subschemes $J'_{b,m} \times \bar{U'}^{n,d}_b \subset J'_{b,m} \times \bar{\mathcal{M}'}^{n,d}_b$ and let $\bar{\mathrm{pr}} : J'_{b,m} \times \bar{U'}^{n,d}_b \to \bar{\mathcal{M}'}_b$ be the projections to the second factor.

We denote by $\tau \mapsto \bar{\tau} = (\sigma^f)^{r(\tau)}$ the projection $W_{E_v} \to \mathrm{Gal}(\bar{\mathbb{F}}_p/k)$. If $\mathcal{L}$ is a torsion étale over $\bar{S'}^{(b)} \times \bar{\mathbb{F}}_p$, we define an action of the Weil group W_{E_v} on $\mathcal{L}$ to be the data of isomorphisms

$$\tau : (1 \times \bar{\tau})^* \mathcal{L} \simeq \mathcal{L}, \qquad \forall \tau \in W_{E_v},$$

such that $\tau \circ \tau' = \tau' \tau$. Also, we call an *action of the Weil group W_{E_v} on a sheaf $\mathcal{G}$ over $\bar{\mathcal{M}'}$* the data of isomorphisms $\tau : (\mathrm{Frob}^f)^{r(\tau)^*} \mathcal{G} \simeq \mathcal{G}$ for all $\tau \in W_{E_v}$, $r(\tau) \geq 0$, such that $\tau \circ \tau' = \tau' \tau$.

We obtain (see theorems 12 and 13 and corollary 14 of [M]):

Theorem 9.4. *For any $b \in B(G)$ and $K^p = K_q \times K^{pq} \subset G(\mathbb{A}^p_f)$, let $\mathcal{L}$ be an abelian torsion étale sheaf over $\bar{S'}^{(b)}$ with torsion orders prime to p and endowed with an action of the Weil group W_{E_v}. Then there exists a W_{E_v}-equivalent spectral sequence*

$$E^{p,q}_2 = H^q(T_b, \lim_{\rightarrow\, m,n,d} H_c(J'_{b,m} \times \bar{U'}^{n,d}_b, \dot{\pi'}^*_N \mathcal{L})) \Rightarrow H^{p+q}_c(\bar{S'}^{(b)}_{K^p} \times \bar{\mathbb{F}}_p, \mathcal{L}).$$

For $l \neq p$ a prime and an integer $r \geq 1$, we denote by $\mathcal{H}_r(T_b)$ the Hecke algebra of T_b with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$.

Theorem 9.5. *For any $b \in B(G)$ and $K^p = K_q \times K^{pq} \subset G(\mathbb{A}^p_f)$, let $\mathcal{L}$ (resp. $\mathcal{G}$) be an étale sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules over $\bar{S'}^{(b)} \times \bar{\mathbb{F}}_p$ (resp. $\bar{\mathcal{M}}$) which is endowed with an action of the Weil group W_{E_v}.*

Suppose there exists a system of W_{E_v}-equivariant isomorphisms of étale sheaves

$$\{\dot{\pi'}^*_N \mathcal{L} \simeq \bar{pr}^* \mathcal{G}\}_{m,n,d,N,m \geq d, N \geq d/\delta f}$$

which are compatible under the natural transaction maps induced by q_b, i_b and $\rho \in S_b$.

Then there exists a W_{E_v}-equivariant spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-modules:

$$\oplus_{t+s=q} Tor^p_{\mathcal{H}(T_b)}(H^s_c(\bar{\mathcal{M}'}, \mathcal{G}), \lim_{\rightarrow\, m} H^t_c(J'_{b,K^p,m}, \mathbb{Z}/l^r\mathbb{Z})) \Rightarrow H^{p+q}_c(\bar{S'}^{(b)}_{K^p} \times \bar{\mathbb{F}}_p, \mathcal{L}).$$

If $\mathcal{L}$ (resp. $\mathcal{G}$) is the constant étale sheaf over $\bar{S'}^{(b)}$ (resp., over $\bar{\mathcal{M}'}$) associated to $\mathbb{Z}/l^r\mathbb{Z}$, then the hypothesis of theorem 9.5 is satisfies and thus we get:

Corollary 9.6. *For any $b \in B(G)$ and $K^p = K_q \times K^{pq} \subset G(\mathbb{A}_f^p)$, there exists a W_{E_v}-equivariant spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-modules:*

$$\oplus_{t+s=q} Tor_{\mathcal{H}(T_b)}(H_c^s(\bar{\mathcal{M}'}, \mathbb{Z}/l^r\mathbb{Z}), H_c^t(J'_{b,K^p}, \mathbb{Z}/l^r\mathbb{Z})) \Rightarrow H_c^{p+q}(\bar{S'}^{(b)}_{K^p} \times \bar{\mathbb{F}}_p, \mathbb{Z}/l^r\mathbb{Z}),$$

where we write $H_c^t(J'_{b,K^p}, \mathbb{Z}/l^r\mathbb{Z}) = \lim_{\rightarrow m} H_c^t(J'_{b,K^p,m}, \mathbb{Z}/l^r\mathbb{Z})$ for all $t \geq 0$.

We remark that all the above results are proved for the varieties $S_{K^p} = S_{K^p(0)/O_{E_v}}$, corresponding to $K^p(0) = K^p K_p$, where K_p is a hyperspecial maximal compact of $G(\mathbb{Q}_p)$ (more exactly K_p is the stabilizer of the lattice Λ). For any $m \geq 0$, one can define (see §6 of [M]) also the varieties $S_{K^p(m)} \to S_{K^p(0)}$, where $K^p(m) = K^p K_m$, and $K_m \subset K_p$ is a congruence subgroup of level m (i.e. K_m is the stabilizer of the lattice Λ modulo p^m, see [M] for details). Also for any $m \geq 0$, one can define the Rapoport-Zink spaces of level m, $\mathcal{M}_{b,m} \to \mathcal{M}_b = \mathcal{M}_{b,0}$. Moreover, this construction could be extended (see §6 of [M]), and for any

$$g \in G(\mathbb{Q}_p)^+ = \{g \in G(\mathbb{Q}_p) | g^{-1}(\Lambda) \subset \Lambda\} \subset G(\mathbb{Q}_p),$$

and for any $m \geq e(g)$, where $e(g)$ is some number which depends on g, one can define the varieties $S_{K^p(m),g} \to S_{K^p(m)}$ and the Rapoport-Zink spaces $\mathcal{M}_{b,m,g} \to \mathcal{M}_{b,m}$ (for $g \in K_p$, we have $S_{K^p(m),g} = S_{K^p(m)}$ and $\mathcal{M}_{b,m,g} = \mathcal{M}_{b,m}$).

As above for K^p of the form $K^p = K_q \times K^{pq}$, one can define the twisted varieties $S'_{K^p(m),g}$, and the twisted Rapoport-Zink spaces $\mathcal{M}'_{b,m,g}$, and following §6, 7 and 8 of [M], we obtain (see theorem 22 of [M]):

Theorem 9.7. *There is an equality of virtual representations of the group $G(\mathbb{A}_f^q) \times 1 \times W_{E_v}$,*

$$\sum_{t \geq 0} (-1)^t \lim_{\rightarrow K^q \subset G(\mathbb{A}_f^q)} H^t(S'_{K_q \times K^q} \times_{E_v} \bar{E}_v, \mathbb{Q}_l)$$

$$= \sum_{b \in B(G)} \sum_{i,j,k \geq 0} (-1)^{i+j+k} \lim_{\rightarrow U \subset G(\mathbb{Q}_p)} \mathcal{E}_{b,U}^{i,j,k},$$

where for all $b \in B(G)$ and $i, j, k \geq 0$, we write

$$\mathcal{E}_{b,U}^{i,j,k} = Ext^i_{T_b-smooth}(H_c^j({\mathcal{M}'}^{rig}_{b,U} \times_{E_v^{nr}} \bar{E}_v, \mathbb{Q}_l(-D)), H_c^k(J'_{b,K_q}, \mathbb{Q}_l)),$$

where D is the dimension of the Shimura varieties.

We remark that from §4, we know that variety S_{K^p} is a disjoint union of $|\ker^1(\mathbb{Q}, G)|$-copies of $\mathbf{S_K}$. Also we have that the variety $J_{b,K^p,m}$ is a disjoint union of $|\ker^1(\mathbb{Q}, G)|$-copies of the corresponding varieties $\mathbf{J}_{b,\mathbf{K},m}$, and thus all the above results remain true if we replace S'_{K^p} by $\mathbf{S}'_{\mathbf{K}}$ and $J'_{b,K^p,m}$ by $\mathbf{J}'_{b,\mathbf{K},m}$, etc.

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