

# Tate classes and poles of $L$ -functions of twisted quaternionic Shimura surfaces

Cristian Virdol  
Department of Mathematics  
Columbia University

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## 1 Introduction

Let  $X$  be a smooth projective variety defined over a number field  $F$  and let

$$\bar{X} = X \times_F \bar{\mathbb{Q}}.$$

For a prime number  $l$ , let  $H_{et}^i(X, \bar{\mathbb{Q}}_l)$  be the  $l$ -adic cohomology of  $\bar{X}$ . For  $K$  a number field, we denote  $\Gamma_K := \text{Gal}(\bar{\mathbb{Q}}/K)$ . The Galois group  $\Gamma_F$  acts on  $H_{et}^i(X, \bar{\mathbb{Q}}_l)$  by a representation  $\rho_{i,l}$ . For any  $j \in \mathbb{Z}$ , let  $H_{et}^i(X, \bar{\mathbb{Q}}_l)(j)$  denote the representation of  $\Gamma_F$  on  $H_{et}^i(X, \bar{\mathbb{Q}}_l)$  defined by  $\rho_{i,l} \otimes \xi_l^j$ , where  $\xi_l$  is the  $l$ -adic cyclotomic character.

Let  $U^i(X)$  denote the  $\mathbb{Q}$ -linear space of the algebraic subvarieties of  $X$  of codimension  $i$ . We have the  $l$ -adic cycle map

$$d_{i,l} : U^i(X) \otimes \bar{\mathbb{Q}}_l \rightarrow H_{et}^{2i}(X, \bar{\mathbb{Q}}_l)(i).$$

The cohomology classes in the image of this map are said to be *algebraic*.

For each finite extension  $E$  of  $F$ , we denote by  $U^i(X, E)$  the subspace of  $d_{i,l}(U^i(X) \otimes \bar{\mathbb{Q}}_l)$  left fixed by  $\Gamma_E$ . The first part of the Tate conjecture [TA] states that

$$U^i(X, E) = (H_{et}^{2i}(X, \bar{\mathbb{Q}}_l)(i))^{\Gamma_E}.$$

The elements of  $V^i(X, E) := (H_{et}^{2i}(X, \bar{\mathbb{Q}}_l)(i))^{\Gamma_E}$  are called *Tate classes*.

The  $L$ -function  $L^i(s, X/F)$  attached to the representation  $\rho_{i,l}$  converges for  $\text{Re}(s) > 1 + i/2$ . The second part of the Tate conjecture [TA] states that the  $L$ -function  $L^{2i}(s, X/E)$  has a pole at  $s = i + 1$  of order equal to

$$\dim_{\bar{\mathbb{Q}}_l} U^i(X, E).$$

In their work [HLR], Harder, Langlands and Rapoport had proved the first part of the Tate conjecture for Hilbert modular surfaces for non-CM submotives. In [K] and [MR] it was proved the first part of the Tate conjecture for

Hilbert modular surfaces for CM sub-motives and thus using the two results, one gets the full first part of the Tate conjecture asserting the algebraicity of all the Tate classes of Hilbert modular surfaces over an arbitrary number field.

The second part of the Tate conjecture for Hilbert modular surfaces was proved in [HLR], [K] and [MR] only for solvable number fields.

In this paper we consider a totally real number field  $F$  and a quaternion algebra  $D$  over  $F$ , which is unramified at exactly 2 infinite places of  $F$ . Let  $G$  be the algebraic group over  $F$  defined by the multiplicative group  $D^\times$  of  $D$  and let  $\tilde{G} = \text{Res}_{F/\mathbb{Q}}(G)$ . We consider a prime ideal  $\wp$  of  $O_F$ , such that  $G(F_\wp)$  is isomorphic to  $GL_2(F_\wp)$ . Let  $S_{\tilde{G}, \hat{\Gamma}(\wp)} = S_{\hat{\Gamma}(\wp)}$  be the canonical model of the quaternionic Shimura surface associated to the adelic principal congruence subgroup  $\hat{\Gamma}(\wp)$  of  $\tilde{G}(\mathbb{A}_f)$  of level  $\wp$ , where  $\mathbb{A}_f$  is the finite adeles ring of  $\mathbb{Q}$ . Then  $S_{\hat{\Gamma}(\wp)}$  is a quasi-projective surface defined over a totally real finite extension  $E/\mathbb{Q}$  called the canonical field of definition.

The surface  $S_{\hat{\Gamma}(\wp)}$  has a natural action of  $GL_2(O/\wp)$ . Consider a continuous Galois representation  $\varphi : \Gamma_E \rightarrow GL_2(O/\wp)$  and let  $S'_{\hat{\Gamma}(\wp)}$  be the surface defined over  $E$  obtained from  $S_{\hat{\Gamma}(\wp)}$  via twisting by  $\varphi$  composed with the natural action of  $GL_2(O/\wp)$  on  $S_{\hat{\Gamma}(\wp)}$  (see §2 for details).

When  $D = M_2(F)$ , with  $F$  real quadratic, the quaternionic Shimura surface  $S_{\hat{\Gamma}(\wp)}$  is not smooth and in this case in this article we use the intersection cohomology of the smooth toroidal compactification of  $S_{\hat{\Gamma}(\wp)}$  and to simplify the notations, we denote this compactification also by  $S_{\hat{\Gamma}(\wp)}$ .

The surfaces  $S_{\hat{\Gamma}(\wp)}$  and  $S'_{\hat{\Gamma}(\wp)}$  become isomorphic over  $\mathbb{Q}$  and by descend we deduce that the first part of the Tate conjecture for the surface  $S_{\hat{\Gamma}(\wp)}$  over a given field  $K$  is true if and only if the first part of the Tate conjecture for the surface  $S'_{\hat{\Gamma}(\wp)}$  over the field  $K$  is true. As we mentioned above the first part of the Tate conjecture is known for Hilbert modular surfaces [HLR], [K], [MR]. Also the first part of the Tate conjecture is known in the non-CM case for the Shimura surfaces considered in [L], corresponding to a quadratic real field  $F$  and to a quaternion algebra  $D = B \otimes_{\mathbb{Q}} F$ , where  $B$  is a quaternion algebra over  $\mathbb{Q}$ , such that  $D$  splits at the real places and  $F$  splits over the places where  $B$  ramifies.

In this article we want to generalize the results in [HLR], [K] and [MR] and prove that if the field  $L := \overline{\mathbb{Q}}^{\ker(\varphi)}$  is a solvable extension of a totally real number field, then the  $L$ -function  $L^2(s, S'_{\hat{\Gamma}(\wp)/k})$  has a pole at  $s = 2$  of order equal to

$$\dim_{\overline{\mathbb{Q}}_l} V^1(S'_{\hat{\Gamma}(\wp)}, k),$$

if  $k$  is a solvable extension of a totally real number field which contains the field of definition  $E$ .

## 2 Twisted quaternionic Shimura surfaces

Let  $F$  be a totally real number field of degree  $d \geq 2$  over  $\mathbb{Q}$  and  $O := O_F$  be its the ring of integers. Let  $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_f$  be the adeles ring of  $\mathbb{Q}$  and  $\mathbb{A}_F$  the adeles ring of  $F$ . We denote by  $I_{\mathbb{Q}}$  and  $I_F$  the ideles groups of  $\mathbb{Q}$  and  $F$ , respectively.

We consider a quaternion algebra  $D$  over  $F$  which is unramified at exactly 2 infinite places of  $F$ . We denote by  $S_{\infty}$  the set of infinite places of  $F$  and we identify  $S_{\infty}$  as a  $\Gamma_{\mathbb{Q}}$ -set with  $\Gamma_F \setminus \Gamma_{\mathbb{Q}}$ . Let  $S'_{\infty}$  be the subset of  $S_{\infty}$  at which  $D$  is ramified. Thus the cardinal of  $S_{\infty} - S'_{\infty}$  is equal to 2.

Let  $G$  be the algebraic group over  $F$  defined by the multiplicative group  $D^{\times}$ . By restricting the scalars, we obtain the algebraic group  $\bar{G} = \text{Res}_{F/\mathbb{Q}}(G)$  over  $\mathbb{Q}$  defined by the propriety:  $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$  for all  $\mathbb{Q}$ -algebras  $A$ . It is easy to see that  $\bar{G}(\mathbb{R})$  is isomorphic to  $GL_2(\mathbb{R})^2 \times \mathbf{H}^{*(d-2)}$ , where  $\mathbf{H}$  is the algebra of quaternions over  $\mathbb{R}$ .

For  $v \in S_{\infty} - S'_{\infty}$ , we fix an isomorphism of  $G(F_v)$  with  $GL_2(\mathbb{R})$ . We have  $\bar{G}(\mathbb{R}) = \prod_{v \in S_{\infty}} G(F_v)$ . Let  $J = (J_v) \in \bar{G}(\mathbb{R})$ , where

$$J_v = \begin{cases} 1 & \text{for } v \in S'_{\infty}; \\ 1/\sqrt{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \text{for } v \in S_{\infty} - S'_{\infty}. \end{cases}$$

Let  $K_{\infty}$  be the centralizer of  $J$  in  $\bar{G}(\mathbb{R})$ . Set

$$X = \bar{G}(\mathbb{R})/K_{\infty}.$$

It is well known that  $X$  is complex analytically isomorphic to  $(\mathfrak{H}_{\pm})^2$  where  $\mathfrak{H}_{\pm} = \mathbb{C} - \mathbb{R}$ . For each open compact subgroup  $K \subseteq \bar{G}(\mathbb{A}_f)$  set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/K.$$

For  $K$  sufficiently small,  $S_K(\mathbb{C})$  is a complex manifold which is the set of complex points of a quasi projective variety. The canonical field of definition of  $S_K(\mathbb{C})$  is by definition the subfield  $E$  of  $\bar{\mathbb{Q}}$  such that  $\Gamma_E$  is the stabilizer of  $S'_{\infty} \subseteq \Gamma_F \setminus \Gamma_{\mathbb{Q}}$ . It is known that  $S_K(\mathbb{C})$  has a canonical model over  $E$  which is denoted by  $S_K$ . The dimension of  $S_K$  is equal to 2.

Let  $\wp$  be a prime ideal of  $O_F$  such that  $G(F_{\wp})$  is isomorphic to  $GL_2(F_{\wp})$ . Consider  $\hat{\Gamma}(\wp) =: 1 + \wp O_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ , where  $O_D$  is a maximal order of  $D$ . The group  $GL_2(O/\wp)$  acts on

$$S_{\hat{\Gamma}(\wp)}(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/\hat{\Gamma}(\wp).$$

This action can be described in the following way :  $GL_2(O_{\wp}) \hookrightarrow \bar{G}(\mathbb{A}_{\mathbb{Q}})$  by  $\alpha \mapsto (1, \dots, \alpha, 1, \dots, 1)$ ,  $\alpha$  at the  $\wp$  component. Using the isomorphism  $GL_2(O/\wp) \cong GL_2(O_{\wp})/(\hat{\Gamma}(\wp))_{\wp}$ , the action of an element  $g \in GL_2(O_{\wp})$  is given by the right multiplication at the  $\wp$  component.

We fix a continuous representation

$$\varphi : \Gamma_E \rightarrow GL_2(O/\wp).$$

Let  $L$  be the finite Galois extension of  $E$  defined by  $L := (\overline{\mathbb{Q}})^{\ker(\varphi)}$ .

Let

$$S' = S_{\hat{\Gamma}(\varphi)} \times_{\text{Spec}(E)} \text{Spec}(L).$$

The group  $GL_2(O/\varphi)$  acts on  $S_{\hat{\Gamma}(\varphi)}$ . Since  $\varphi : \text{Gal}(L/E) \hookrightarrow GL_2(O/\varphi)$ , the group  $\text{Gal}(L/E)$  acts on  $S_{\hat{\Gamma}(\varphi)}$ . We denote this action of  $\text{Gal}(L/E)$  on  $S_{\hat{\Gamma}(\varphi)}$  by  $\varphi'$ . The Galois group  $\text{Gal}(L/E)$  has a natural action on  $\text{Spec}(L)$  and we can descend via the quotient process  $S'$  to  $S'_{\hat{\Gamma}(\varphi)}/\text{Spec}(E)$  using the diagonal action

$$\text{Gal}(L/E) \ni \sigma \rightarrow \varphi'(\sigma) \otimes \sigma$$

on  $S'$ . Thus, we obtain a quasi-projective variety  $S'_{\hat{\Gamma}(\varphi)}/\text{Spec}(E)$ . This is the twisted quaternionic Shimura surface that we mentioned in the title.

### 3 Zeta functions of twisted quaternionic Shimura surfaces

From now on, if  $\pi$  is an automorphic representation of  $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ , we denote the automorphic representation of  $GL_2(\mathbb{A}_F)$ , obtained from  $\pi$  by Jacquet-Langlands correspondence (usually denoted  $JL(\pi)$ ) by the same symbol  $\pi$ .

If  $l$  is a prime number, we fix an isomorphism  $j : \overline{\mathbb{Q}}_l \rightarrow \mathbb{C}$ , and from now on we identify these two fields. If  $\pi$  is a cuspidal automorphic representation of weight 2 of  $GL(2)/F$ , then there exists ([T]) a  $\lambda$ -adic representation for  $\lambda \nmid \mathfrak{n}$  ( $\mathfrak{n}$  is the level of  $\pi$ )

$$\rho_{\pi, \lambda} : \Gamma_F \rightarrow GL_2(O_\lambda) \hookrightarrow GL_2(\overline{\mathbb{Q}}_l) \cong GL_2(\mathbb{C}),$$

which satisfies  $L(s - 1/2, \pi) = L(s, \rho_{\pi, \lambda})$  and is unramified outside the primes dividing  $\mathfrak{n}l$ . Here  $O$  is the coefficients ring of  $\pi$  and  $\lambda$  is a prime ideal of  $O$  above some prime number  $l$ . In order to simplify the notations we denote by  $\rho_\pi$  the representation  $\rho_{\pi, \lambda}$ .

Assume that the local and the global Haar measures have been fixed on  $\bar{G}(\mathbb{A}_f)$ . We assume that  $K = \prod_{v < \infty} K_v$  where  $K_v$  is open compact in  $G(F_v)$  and  $K_v = GL_2(O_v)$  for almost all  $v$ , where  $O_v$  is the ring of integers of  $F_v$ . For  $g \in \bar{G}(\mathbb{A}_f)$ , let

$$f_g = \text{char}(KgK)/\text{meas}(K)$$

where  $\text{char}(KgK)$  is the characteristic function of  $KgK$ . Let  $\mathbb{H}_K$  be the Hecke algebra generated by the  $f_g$  under the convolution. If  $\pi = \pi_\infty \otimes \pi_f$  is an automorphic representation of  $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ , we denote by  $\pi_f^K$  the space of  $K$  invariants of  $K$  in  $\pi_f$ . The Hecke algebra  $\mathbb{H}_K$  acts on  $\pi_f^K$ .

We have a Galois-equivariant decomposition

$$H_{et}^2(S_K, \overline{\mathbb{Q}}_l) = IH_{et}^2(S_K, \overline{\mathbb{Q}}_l) \oplus W,$$

where

$$IH_{et}^2(S_K, \overline{\mathbb{Q}}_l) \cong \text{Im}(H_c^2(S_K, \overline{\mathbb{Q}}_l) \rightarrow H_{et}^2(S_K, \overline{\mathbb{Q}}_l)),$$

is the *intersection cohomology* and  $H_c^2(S_K, \bar{\mathbb{Q}}_l)$  denotes the compactly supported cohomology and

$$W = \{x \in H_{et}^2(S_K, \bar{\mathbb{Q}}_l) \mid x \text{ has support in } \tilde{S}_K^\infty\}.$$

The Galois module  $W$  has a decomposition

$$W \cong \bigoplus_{s \in \tilde{S}_K^\infty} (\bar{\mathbb{Q}}_l(-1))^{\oplus m(s)},$$

where  $m(s)$  denotes for each  $s$  the number of irreducible components of the resolution  $\tilde{S}_K^\infty(s)$  of  $s$ .

We have an action of the Hecke algebra  $\mathbb{H}_K$  and an action of the Galois group  $\Gamma_E$  on the intersection cohomology  $IH_{et}^2(S_K, \bar{\mathbb{Q}}_l)$  and these two actions commute. We say that the representation  $\pi$  is *cohomological* if  $H^2(\mathfrak{g}, K_\infty, \pi_\infty) \neq 0$ , where  $\mathfrak{g}$  is the Lie algebra of  $K_\infty$  (the cohomology is taken with respect to  $(\mathfrak{g}, K_\infty)$ -module associated to  $\pi_\infty$ ).

**Proposition 3.1.** *The double representation of  $\Gamma_E \times \mathbb{H}_K$  on the intersection cohomology  $IH_{et}^2(S_K, \bar{\mathbb{Q}}_l)$  is isomorphic to*

$$\bigoplus_{\pi} \rho(\pi) \otimes \pi_f^K,$$

where  $\rho(\pi)$  is a representation of the Galois group  $\Gamma_E$ . The above sum is over weight 2 cohomological automorphic representations  $\pi$  of  $\bar{G}(\mathbb{A}_{\mathbb{Q}})$  and the  $\mathbb{H}_K$ -representations  $\pi_f^K$  are irreducible and mutually inequivalent, i.e. the decomposition is isotypic with respect to the action of  $\mathbb{H}_K$ .

The irreducible automorphic representations that appear in proposition 3.1 are one-dimensional or cuspidal and infinite-dimensional. If  $\pi$  is one-dimensional then  $\rho(\pi)$  has dimension two and if  $\pi$  is infinite-dimensional, then  $\rho(\pi)$  has dimension four.

We fix an isomorphism  $j : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$  and define the  $L$ -function

$$L^2(s, S_{\hat{\Gamma}(\varphi)}) := \prod_{\pi} \prod_q \det(1 - Nq^{-s} j(\rho(\pi)(\text{Frob}_q)) | H_{et}^2(S_{\hat{\Gamma}(\varphi)}, \bar{\mathbb{Q}}_l)^{I_q})^{-1},$$

where  $\text{Frob}_q$  is a geometric Frobenius element at a finite place  $q$  of  $E$  and  $I_q$  is the inertia group at  $q$ .

We consider the injective limit:

$$V := \varinjlim_K IH_{et}^2(S_K, \bar{\mathbb{Q}}_l) \cong \varinjlim_K \bigoplus_{\pi} V(\pi_\infty) \otimes_{\bar{\mathbb{Q}}_l} \pi_f^K,$$

where  $V(\pi_\infty)$  is the  $\bar{\mathbb{Q}}_l$ -space that corresponds to  $\rho(\pi)$  (see proposition 3.1 for notations).

Using the strong multiplicity one for  $\bar{G}$ , we get that the  $\pi$ -component  $V(\pi)$  of  $V$  is isomorphic to  $\rho(\pi) \otimes \pi_f$  as  $\Gamma_E \times \mathbb{H}$ -module. Taking the  $\hat{\Gamma}(\varphi)$ -fixed vectors

we deduce that  $V(\pi)^{\hat{\Gamma}(\varphi)}$  is isomorphic to  $\rho(\pi) \otimes \pi_f^{\hat{\Gamma}(\varphi)}$  as  $\Gamma_E \times GL_2(O/\varphi O)$ -module. Since the varieties  $S_{\hat{\Gamma}(\varphi)}$  and  $S'_{\hat{\Gamma}(\varphi)}$  become isomorphic over  $\bar{\mathbb{Q}}$ , we have the isomorphism  $IH_{et}^2(S_{\hat{\Gamma}(\varphi)}, \bar{\mathbb{Q}}_l) \cong IH_{et}^2(S'_{\hat{\Gamma}(\varphi)}, \bar{\mathbb{Q}}_l)$ . The actions of  $\Gamma_E$  on these cohomologies which give the expression of the zeta functions of these varieties are different. If we consider the component  $V'(\pi)$  that corresponds to  $\pi$  of  $IH_{et}^2(S'_{\hat{\Gamma}(\varphi)}, \bar{\mathbb{Q}}_l)$  (see the decomposition of proposition 3.1), we get that  $V'(\pi)$  is isomorphic to  $\rho(\pi) \otimes (\pi_f^{\hat{\Gamma}(\varphi)} \circ \varphi)$  as  $\Gamma_E$ -module. We denote also by  $W'$  the Galois module obtained from  $W$ . Hence we conclude the following result (this is a particular case of theorem 1.1 of [V]):

**Theorem 3.2.** *The  $L$ -function  $L'^2(s, S'_{\hat{\Gamma}(\varphi)})$  which comes from the intersection cohomology part of  $H_{et}^2(S_{\hat{\Gamma}(\varphi)}, \bar{\mathbb{Q}}_l)$  is given by the formula:*

$$L'^2(s, S'_{\hat{\Gamma}(\varphi)}) = \prod_{\pi} L(s, \rho(\pi) \otimes (\pi_f^{\hat{\Gamma}(\varphi)} \circ \varphi)),$$

where the product is taken over cohomological automorphic representations  $\pi$  of  $\bar{G}(\mathbb{A}_{\mathbb{Q}})$  of weight 2, such that  $\pi_f^{\hat{\Gamma}(\varphi)} \neq 0$ .

For the part of the  $L$ -function of  $S'_{\hat{\Gamma}(\varphi)}$  which comes from the Galois module  $W'$  it is easy to prove the equality between the pole at  $s = 2$  and the dimension of the space of Tate classes since the Galois module  $W'$  is monomial (see section 6.3 for definition and details).

## 4 Base change

We know the following result (theorem 1.1 of [V]):

**Theorem 4.1.** *If  $F$  is a totally real number field,  $\pi$  is a cuspidal automorphic representation of weight 2 of  $GL(2)/F$  and  $F_1$  is a solvable extension of a totally real number field containing  $F$ , then there exists a Galois extension  $F_2$  of  $\mathbb{Q}$  containing  $F_1$ , such that  $F_2$  is a solvable extension of a totally real field, and there exists a prime  $\lambda$  of the field of coefficients of  $\pi$ , such that  $\rho_{\pi, \lambda}|_{\Gamma_{F_2}}$  is modular i.e. there exists a cuspidal automorphic representation  $\pi_1$  of  $GL(2)/F_2$  and a prime  $\beta$  of the field of coefficients of  $\pi_1$  such that  $\rho_{\pi, \lambda}|_{\Gamma_{F_2}} \cong \rho_{\pi_1, \beta}$ .*

In this section we fix an automorphic representation  $\pi$  as in theorem 4.1 and we denote  $\omega := \pi_f^{\hat{\Gamma}(\varphi)} \circ \varphi$ . We assume in the rest of the paper that the field  $L := \bar{\mathbb{Q}}^{\ker(\varphi)}$  is a solvable extension of a totally real number field. Thus the field  $K := \bar{\mathbb{Q}}^{\ker(\omega)}$  is a solvable extension of a totally real number field.

Let  $k$  be a solvable extension of a totally real number field which contains  $E$ . From theorem 4.1 we deduce that there exists a Galois extension  $F_2$  of  $\mathbb{Q}$  containing  $FKk$ , such that  $F_2$  is a solvable extension of a totally real field, and there exists a prime  $\lambda$  of the field coefficients of  $\pi$  and a cuspidal automorphic

representation  $\pi_1$  of  $GL(2)/F_2$  and a prime  $\beta$  of the field of coefficients of  $\pi_1$  such that  $\rho_{\pi,\lambda}|_{\Gamma_{F_2}} \cong \rho_{\pi_1,\beta}$ .

By Brauer's theorem (see theorems 16 and 19 of [SE]), we can find some subfields  $F_i \subset F_2$  such that  $\text{Gal}(F_2/F_i)$  are solvable, some characters  $\chi_i : \text{Gal}(F_2/F_i) \rightarrow \overline{\mathbb{Q}}^\times$  and some integers  $m_i$ , such that the representation

$$\omega|_{\Gamma_k} : \text{Gal}(F_2/k) \rightarrow \text{Gal}(Kk/k) \rightarrow GL_N(\overline{\mathbb{Q}}_l),$$

can be written as  $\omega|_{\Gamma_k} = \sum_{i=1}^{i=k} m_i \text{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$  (a virtual sum). Then

$$\begin{aligned} L(s, (\rho(\pi) \otimes \omega)|_{\Gamma_k}) &= \prod_{i=1}^{i=k} L(s, \rho(\pi)|_{\Gamma_k} \otimes \text{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i)^{m_i} = \\ &= \prod_{i=1}^{i=k} L(s, \text{Ind}_{\Gamma_{F_i}}^{\Gamma_k} (\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i))^{m_i} = \prod_{i=1}^{i=k} L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}. \end{aligned}$$

If  $F \subset F_i$ , since  $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$  is modular and  $\text{Gal}(F_2/F_i)$  is solvable, from Langlands base change one can deduce that  $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$  is modular. We give a short proof of this fact. If the representation  $\pi$  is of CM type (see §5 below for the definition), then the existence of the base change of  $\pi$  to an arbitrary extension  $k/F$  is well known and thus, in this case we are done. We assume now that  $\pi$  is non-CM. Then from proposition 5.1 below we know that the representation  $\pi_1$  is non-CM. By induction we can assume that  $\text{Gal}(F_2/F_i)$  is cyclic of prime order. We denote by  $\theta$  a generator of  $\text{Gal}(F_2/F_i)$ . Then we know (see for example proposition 2.3.1 of [RA]) that  $\pi_1$  is a base change of an automorphic representation  $\pi'$  of  $GL(2)/F_i$  iff  $\pi_1 \cong \pi_1^\theta$ . By strong multiplicity one for  $GL(2)$ , this is equivalent to  $\rho_{\pi_1} \cong \rho_{\pi_1}^\theta$ , where  $\rho_{\pi_1}^\theta(\gamma) = \rho_{\pi_1}(\theta\gamma\theta^{-1})$ . Since  $\rho_{\pi,\lambda}|_{\Gamma_{F_i}} \cong \rho_{\pi_1}^\theta|_{\Gamma_{F_i}}$  by restriction to  $\Gamma_{F_2}$ , we obtain  $\rho_{\pi_1} \cong \rho_{\pi_1}^\theta$ . Hence  $\pi_1$  is a base change of an automorphic representation  $\pi'$  of  $GL(2)/F_i$ . Since  $\pi_1$  is non-CM, from proposition 5.1 below, we know that the representation  $\rho_{\pi_1}$  is irreducible. The restrictions of  $\rho_{\pi'}$  and  $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$  to  $\Gamma_{F_2}$  are equal to the irreducible representation  $\rho_{\pi_1}$  and since  $\Gamma_{F_2}$  is a normal subgroup of  $\Gamma_{F_i}$  one could prove easily that  $\rho_{\pi,\lambda}|_{\Gamma_{F_i}} \cong \rho_{\pi'} \otimes \chi \cong \rho_{\pi'} \otimes \chi$  where  $\chi$  is a Galois character corresponding to  $F_2/F_i$ . Therefore  $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$  is modular.

If  $F \subset F_i$ , then we denote by  $\pi_i$  the cuspidal automorphic representation of  $GL(2)/F_i$  which satisfies  $\rho_{\pi,\lambda}|_{\Gamma_{F_i}} \cong \rho_{\pi_i}$ .

## 5 Known results

It is known that (see for example proposition 4.5.4 [HLR]):

**Proposition 5.1.** *If  $\pi$  is a cuspidal automorphic representation of  $GL(2)/F$ , where  $F$  is a totally real number field, then one of the following two statements holds:*

- (i)  $\rho_\pi|_{\Gamma_L}$  is irreducible for each finite extension  $L/F$ .
- (ii) There exists a quadratic extension  $L/F$  and an algebraic Hecke character  $\psi$  of  $L$  such that  $\rho_\pi \cong \text{Ind}(\psi)$ .

We say that a representation  $\rho$  of a group  $G$  is *dihedral* if there exists a normal subgroup  $N$  of index 2 in  $G$  and a character  $\chi : N \rightarrow \mathbb{C}^\times$  such that  $\rho = \text{Ind}_N^G \chi$ .

We say that an automorphic representation  $\pi$  of  $GL(2)/L$  for some number field  $L$  is of *CM type* if there exists some quadratic Galois character  $\eta : I_L/L^\times \rightarrow \mathbb{Q}_l^\times$ , with  $\eta \neq 1$  such that  $\pi \cong \pi \otimes \eta$ . If  $\pi$  is an automorphic representation of  $GL(2)/L$ , then  $\pi$  is of CM type if and only if  $\rho_\pi$  is a dihedral representation.

We know the following result (theorem 2.1 of [MP]):

**Proposition 5.2.** *The tensor product of two 2 dimensional irreducible complex representations of a group is reducible only if either both representations are dihedral or they are the twist of each other by a character.*

We know (proposition 4.1 of [MP]):

**Proposition 5.3.** *Suppose that  $\pi$  is a cuspidal, non-CM automorphic representation of  $GL(2)/K$  for some finite extension  $K/\mathbb{Q}$ . Suppose that  $K$  is a quadratic extension of  $k$  and  $\tau$  is the automorphism of  $K$  over  $k$ . If  $\pi^\tau \cong \pi \otimes \chi$  for a Hecke character  $\chi$  of  $K$ , then  $\chi$  is trivial when restricted to the ideles of  $k$ .*

We know (corrolary 2.6 of [MP]):

**Proposition 5.4.** *Let  $\rho$  be a 2-dimensional irreducible representation of a group  $G$ . Then  $\text{Sym}^2(\rho)$  is reducible if and only if  $\rho$  is dihedral.*

We know (see theorem M and theorem 2.2.7 of [RA]):

**Proposition 5.5.** *Let  $\pi_1$  and  $\pi_2$  are cuspidal, non-CM automorphic representations of weight 2 of  $GL(2)/K$  for some finite extension  $K/\mathbb{Q}$ . Then  $\pi_1 \otimes \pi_2$  is a cuspidal automorphic representation of  $GL(4)/K$  iff  $\pi_1$  is not isomorphic to  $\tilde{\pi}_2 \otimes \chi$  for some Hecke character  $\chi$ , where  $\tilde{\pi}_2$  is the contragredient representation. If  $\pi_1 \otimes \pi_2$  is cuspidal, then  $L(s, \pi_1 \otimes \pi_2)$  is analytic and does not vanish at  $s = 1$ . If  $\pi_1 \cong \tilde{\pi}_2$ , then  $L(s, \pi_1 \otimes \pi_2)$  has an unique pole of order 1 at  $s = 1$ .*

We know (see the main theorem of [JG]):

**Proposition 5.6.** *Let  $\pi$  be a cuspidal, non-CM automorphic representations of weight 2 of  $GL(2)/K$  for some finite extension  $K/\mathbb{Q}$ . Then  $\text{Sym}^2 \pi$  is a cuspidal automorphic representation of  $GL(3)/K$  and the L-function  $L(s, \text{Sym}^2 \pi)$  is analytic and does not vanish at  $s = 1$ .*

## 6 Tate conjecture for twisted quaternionic Shimura surfaces

Assume that  $k$  be a solvable extension of a totally real number field which contains  $E$  and  $\pi$  is an automorphic representation of  $GL(2)/F$  that appears in

theorem 3.2. The representation  $\pi$  is one-dimensional or cuspidal and infinite-dimensional. Let  $V(\pi)'$  be the space considered in §3 just before theorem 3.2.

We recall that in §4 we denoted  $\omega =: \pi_f^{\hat{\Gamma}(\varphi)} \circ \varphi$  and we assumed that the field  $L := \bar{\mathbb{Q}}^{\ker(\varphi)}$  is a solvable extension of a totally real number field and thus the field  $K := \bar{\mathbb{Q}}^{\ker(\omega)}$  is a solvable extension of a totally real number field.

Define:

$$\mathbf{V}(\pi, k) := \{x \in V(\pi)' | (\rho(\pi) \otimes \omega)(a)x = \xi_l^{-1}(a)x, \text{ for all } a \in \Gamma_k\},$$

where  $\xi_l$  is the  $l$ -adic cyclotomic character. The elements of  $\mathbf{V}(\pi, k)$  are called *Tate classes*.

We will prove the following result:

**Theorem 6.1.** *Assume that  $K := \bar{\mathbb{Q}}^{\ker(\omega)}$  is a solvable extension of a totally real number field and  $k$  is a solvable extension of a totally real field which contains  $E$ , then the order of the pole of the  $L$ -function  $L(s, (\rho(\pi) \otimes \omega)|_{\Gamma_k})$  at  $s = 2$  is equal to  $\dim_{\mathbb{Q}_l} \mathbf{V}(\pi, k)$ .*

We consider

$$\mathbf{V}(\pi, F_i) := \{x \in V(\pi)' | (\rho(\pi) \otimes \chi_i)(a)x = \xi_l^{-1}(a)x, \text{ for all } a \in \Gamma_{F_i}\}.$$

Since  $\omega|_{\Gamma_k} = \sum_{i=1}^{i=k} m_i \text{Ind}_{\Gamma_{F_i}}^{\Gamma_{F_i}} \chi_i$ , in order to prove theorem 6.1, it is sufficient to prove the following proposition:

**Proposition 6.2.** *For each  $i$ , the order of the pole of  $L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)$  at  $s = 2$  is equal to  $\dim_{\mathbb{Q}_l} \mathbf{V}(\pi, F_i)$ .*

We assume for simplicity that  $S_\infty - S'_\infty = \{1, \tau\}$ , where 1 is the trivial embedding of  $F$  in  $\bar{\mathbb{Q}}$ . We denote by the same symbol  $\tau$  the extension of  $\tau$  to  $\bar{\mathbb{Q}}$ . Consider

$$S = \Gamma_F \cup \Gamma_F \tau.$$

The stabilizer of  $S$  is  $\Gamma_E$ . It is easy to check that the stabilizer of  $S$  is equal to  $(\Gamma_{F\tau} \cap \tau^{-1}\Gamma_F) \cup (\Gamma_F \cap \tau^{-1}\Gamma_{F\tau})$ . Thus we get

$$\Gamma_E = (\Gamma_{F\tau} \cap \tau^{-1}\Gamma_F) \cup (\Gamma_F \cap \tau^{-1}\Gamma_{F\tau}).$$

Now we describe the representation  $\rho(\pi)$  which is semisimple (the proof of the semisimplicity of  $\rho(\pi)$  is the same as in the case of Hilbert modular surfaces, see [HLR], §4 or [G], corollary 3.8).

We distinguish two cases:

i)  $\Gamma_{F\tau} \cap \tau^{-1}\Gamma_F = \emptyset$ . Then,  $\Gamma_E = \Gamma_F \cap \tau^{-1}\Gamma_{F\tau}$ . Thus,

$$F \subset E \subset F^{gal}$$

where  $F^{gal}$  is the Galois closure of  $F$ .

If  $\pi$  is infinite-dimensional cuspidal automorphic representation, we denote for simplicity  $\rho_\pi := \rho_{\pi, \lambda}$ . Then we have (see for example §4.3 of [V]):

$$\rho(\pi) \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E}^\tau,$$

where

$$\rho_\pi|_{\Gamma_E}^\tau(\gamma) = \rho_\pi(\tau\gamma\tau^{-1}).$$

If  $\pi$  is one-dimensional, then  $\pi(g) = \rho_\pi(N(g))|N(g)|^{1/2}$ , where  $N$  is the reduced norm map and  $|\cdot|$  denotes the ideles norm and  $\rho_\pi$  is a Hecke character. We denote also by  $\rho_\pi$  the  $\lambda$ -adic representation associated to  $\rho_\pi$ . Then

$$\rho(\pi) \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E}^\tau.$$

ii)  $\Gamma_{F\tau} \cap \tau^{-1}\Gamma_F \neq \emptyset$ . Let  $\Gamma_{E_1} := \Gamma_F \cap \tau^{-1}\Gamma_{F\tau}$ . Thus

$$F \subset E_1 \subset F^{gal}.$$

Since it is obvious now that  $\Gamma_{E_1} \subset \Gamma_E$ ,  $[\Gamma_E : \Gamma_{E_1}] = 2$  and  $\Gamma_E \not\subset \Gamma_F$ , we get  $[E_1 : E] = 2$  and  $F \not\subset E$ . If  $F_1 := E \cap F$ , then  $[F : F_1] = 2$  and we can easily see that  $\tau$  when restricted to  $F_1$  is the trivial embedding. Hence  $\tau$  is the nontrivial automorphism of  $F$  over  $F_1$  and we get that  $\Gamma_{E_1} = \Gamma_F \cap \tau^{-1}\Gamma_{F\tau} = \Gamma_F$ , which means that  $E_1 = F$  and  $E = F_1$  and therefore we have  $[F : E] = 2$  and  $\tau$  is the nontrivial automorphism of  $F$  over  $E$ .

If  $\pi$  is infinite-dimensional cuspidal automorphic, then we know that (see for example §4.3 of [V])  $\rho(\pi)$  is a subrepresentation of

$$\text{Ind}_{\Gamma_F}^{\Gamma_E}(\rho_\pi \otimes \rho_\pi^\tau),$$

which verifies

$$\rho(\pi)|_{\Gamma_F} = \rho_\pi \otimes \rho_\pi^\tau.$$

If  $\pi$  is one-dimensional, then  $\pi(g) = \rho_\pi(N(g))|N(g)|^{1/2}$  and we have (see for example proposition 2.7 of [G])

$$\rho(\pi) \cong \rho_\pi|_{I_E} \oplus \rho_\pi|_{I_E} \cdot \omega_{F/E},$$

where  $\omega_{F/E}$  is the quadratic character corresponding to  $F/E$ .

## 6.1 Non-CM Tate classes case i)

In this section we consider the case i) described above and assume that our automorphic representation  $\pi$  of  $GL(2)/F$  is cuspidal non-CM. Thus  $F \subset E \subset F^{gal}$  and

$$\rho(\pi) \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E}^\tau.$$

Assume that  $\mathbf{V}(\pi, F_i) \neq 0$ . Let  $x \in \mathbf{V}(\pi, F_i)$ , with  $x \neq 0$ . Thus  $(\rho_{\pi_i} \otimes \rho_{\pi_i}^\tau \otimes \chi_i)(a)x = \xi_i^{-1}(a)x$  for  $a \in \Gamma_{F_i}$ . Applying the propositions 5.1 and 5.2, we get that  $\pi_i^\tau \cong \pi_i \otimes \chi$  for some Hecke character  $\chi$  of  $F_i$ . Therefore:

$$\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i} \otimes \rho_{\pi_i} \otimes \chi\chi_i \cong \text{Sym}^2(\rho_{\pi_i}) \cdot \chi\chi_i \oplus \wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i.$$

Since  $\pi$  is non-CM, from proposition 5.1, we know that the representation  $\rho_{\pi_i}$  is irreducible and from proposition 5.4, we deduce that  $\text{Sym}^2 \rho_{\pi_i}$  is irreducible and thus the first factor of the above sum has no nontrivial vector on which  $\Gamma_{F_i}$  acts by  $\xi_l^{-1}$ . Hence we obtain the following result:

**Proposition 6.3.** *If  $\pi$  is cuspidal non-CM, then in case i) the dimension of  $\mathbf{V}(\pi, F_i)$  is equal 1 or 0 according to  $\xi_l^{-1}|_{\Gamma_{F_i}}$  is equal to  $\wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i$  or not.*

The representation  $\pi$  is non-CM and from proposition 5.1, we get that the representation  $\pi_i$  is non-CM. Then

$$L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i) = L(s, \rho_{\pi_i} \otimes \rho_{\pi_i}^{\tau} \otimes \chi_i) = L(s-1, \pi_i \otimes \pi_i^{\tau} \otimes \chi_i).$$

Since the representations  $\pi_i$  and  $\pi_i^{\tau}$  are cuspidal non-CM, from proposition 5.5, we know that the  $L$ -function  $L(s-1, \pi_i \otimes \pi_i^{\tau} \otimes \chi_i)$  has a pole of order 1 at  $s=2$  if and only if  $\pi_i^{\tau} \otimes \chi_i \cong \tilde{\pi}_i \cong \pi_i \otimes \omega_{\pi_i}^{-1}$ , where  $\omega_i$  is the central character of  $\pi_i$ . Otherwise the  $L$ -function  $L(s-1, \pi_i \otimes \pi_i^{\tau} \otimes \chi_i)$  is analytic and does not vanish at  $s=2$ .

We assume now that  $\pi_i^{\tau} \otimes \chi_i \cong \tilde{\pi}_i \cong \pi_i \otimes \omega_{\pi_i}^{-1}$ . Thus  $\pi_i^{\tau} \cong \pi_i \otimes \chi$  for some Hecke character  $\chi$ .

Hence

$$\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i} \otimes \rho_{\pi_i} \otimes \chi\chi_i \cong \text{Sym}^2(\rho_{\pi_i}) \cdot \chi\chi_i \oplus \wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i,$$

and we get

$$L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i) = L(s, \text{Sym}^2(\rho_{\pi_i}) \cdot \chi\chi_i) \cdot L(s, \wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i).$$

Since the representation  $\pi_i$  is cuspidal non-CM, from proposition 5.6, we obtain that the representation  $\text{Sym}^2 \pi_i$  is cuspidal automorphic and the  $L$ -function  $L(s, \text{Sym}^2(\rho_{\pi_i}) \cdot \chi_i \chi)$  is analytic and does not vanish at  $s=2$ .

We deduce that the order of the pole of  $L$ -function

$$L(s, (\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i))$$

at  $s=2$  is 1 or 0 according to  $\xi_l^{-1}|_{\Gamma_{F_i}}$  is equal to  $\wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i$  or not.

From proposition 6.3, we obtain that proposition 6.2 and theorem 6.1 are true in case i) and if the representation  $\pi$  is cuspidal non-CM.

## 6.2 Non-CM Tate classes case ii)

In this section we consider the case ii) described above and assume that the representation  $\pi$  is cuspidal non-CM. Thus  $[F : E] = 2$  and  $\rho(\pi)$  is a subrepresentation of

$$\text{Ind}_{\Gamma_F}^{\Gamma_E}(\rho_{\pi} \otimes \rho_{\pi}^{\tau}),$$

which verifies

$$\rho(\pi)|_{\Gamma_F} \cong \rho_{\pi} \otimes \rho_{\pi}^{\tau}.$$

We distinguish two cases:

a)  $F \subset F_i$ . Then

$$\rho(\pi)|_{\Gamma_{F_i}} \cong \rho_\pi|_{\Gamma_{F_i}} \otimes \rho_\pi|_{\Gamma_{F_i}}^\tau,$$

and proposition 6.2 can be proven in the same way as in section 6.1.

b)  $F \not\subset F_i$ . Assume that  $\mathbf{V}(\pi, F_i) \neq 0$ . Let  $x \in \mathbf{V}(\pi, F_i)$ , with  $x \neq 0$ . Thus, in particular  $(\rho_\pi|_{\Gamma_{FF_i}} \otimes \rho_\pi^\tau|_{\Gamma_{FF_i}})(a)x = \xi_l^{-1}(a)x$  for  $a \in \Gamma_{FF_i}$ . Since  $\rho_{\pi, \lambda}|_{\Gamma_{F_2}}$  is modular and  $F_2$  is a solvable extension of  $FF_i$ , we obtain that  $\rho_\pi|_{\Gamma_{FF_i}}$  is modular and thus there exists a cuspidal automorphic representation  $\pi'_i$  of  $GL(2)/FF_i$  such that  $\rho_{\pi, \lambda}|_{\Gamma_{FF_i}} \cong \rho_{\pi'_i}$ . Since  $\pi$  is non-CM, from propositions 5.1 and 5.2, we get that  $\pi_i'^\tau \cong \pi'_i \otimes \alpha$  for some Hecke character  $\alpha$  of  $IF_i$ .

The element  $\tau$  can be regarded as the nontrivial automorphism of  $FF_i$  over  $F_i$ . Hence, from proposition 5.3 we know that  $\alpha$  is a Hecke character of  $IF_i$  which is trivial on  $I_{F_i}$ . Therefore  $\alpha$  can be written as  $\alpha = \chi^\tau/\chi$  for some Hecke character  $\chi$  of  $IF_i$ . Hence

$$(\pi'_i \otimes \chi^{-1})^\tau \cong \pi'_i \otimes \chi^{-1}.$$

So  $\pi_i' \cong \pi_i/_{FF_i} \otimes \chi$ , where  $\pi_i/_{FF_i}$  is the base change to  $FF_i$  of some automorphic representation  $\pi_i$  of  $GL(2)/F_i$ .

Then from the properties of  $\rho(\pi)$  (see for example [MP]) we have:

$$\rho(\pi)|_{\Gamma_{F_i}} \cong (\text{Sym}^2 \rho_{\pi_i} \oplus \omega_{\pi_i} \cdot \omega_{FF_i/F_i}) \otimes \chi|_{I_{F_i}},$$

where  $\omega_{\pi_i}$  is the central character of  $\pi_i$  and  $\omega_{FF_i/F_i}$  is the quadratic character that corresponds to  $FF_i/F_i$ .

Thus we get

$$\rho(\pi)|_{\Gamma_i} \otimes \chi_i \cong (\text{Sym}^2 \rho_{\pi_i} \otimes \chi|_{I_{F_i}} \cdot \chi_i) \oplus (\omega_{\pi_i} \cdot \omega_{FF_i/F_i} \cdot \chi|_{I_{F_i}} \cdot \chi_i).$$

Since  $\pi$  is non-CM, the representation  $\pi_i$  is non-CM, from proposition 5.1, we know that the representation  $\rho_{\pi_i}$  is irreducible and non-dihedral and from proposition 5.4, we deduce that  $\text{Sym}^2 \rho_{\pi_i}$  is irreducible and thus the first factor of the above sum has no nontrivial vector on which  $\Gamma_{F_i}$  acts by  $\xi_l^{-1}$ .

Therefore we obtain the following result:

**Proposition 6.4.** *If  $\pi$  is cuspidal non-CM, then in case ii) and b) the dimension of  $\mathbf{V}(\pi, F_i)$  is equal to 1 or 0 according to  $\omega_{\pi_i} \cdot \omega_{FF_i/F_i} \cdot \chi|_{I_{F_i}} \cdot \chi_i$  is equal to  $\xi_l^{-1}|_{F_i}$  or not.*

We have

$$\begin{aligned} L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i) L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i \cdot \omega_{FF_i/F_i}) = \\ L(s-1, \pi_i' \otimes \pi_i'^\tau \otimes \chi_i') = L(s, \rho_{\pi_i'} \otimes \rho_{\pi_i'}^\tau \otimes \chi_i'), \end{aligned}$$

where  $\chi_i' = \chi_i|_{\Gamma_{FF_i}}$ .

Since the representations  $\pi'_i$  and  $\pi'^{\tau}_i$  are cuspidal non-CM the  $L$ -function  $L(s-1, \pi'_i \otimes \pi'^{\tau}_i \otimes \chi'_i)$  has a pole of order 1 at  $s=2$  if and only if  $\pi'^{\tau}_i \otimes \chi'_i \cong \tilde{\pi}'_i \cong \pi'_i \otimes \omega_{\pi'_i}^{-1}$ . Otherwise the  $L$ -function  $L(s-1, \pi'_i \otimes \pi'^{\tau}_i \otimes \chi_i)$  is analytic and does not vanish at  $s=2$ . Also the  $L$ -functions  $L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)$  and  $L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i \cdot \omega_{FF_i/F_i})$  have a pole of order at most 1 at  $s=2$  and do not vanish  $s=2$  (for details see propositions 3.11, 3.12 and 3.13 of [HLR]).

We assume now that the  $L$ -function  $L(s-1, \pi'_i \otimes \pi'^{\tau}_i \otimes \chi'_i)$  has a pole of order 1 at  $s=2$ . Then we have  $\pi'^{\tau}_i \otimes \chi'_i \cong \tilde{\pi}'_i \cong \pi'_i \otimes \omega_{\pi'_i}^{-1}$  and as above, we get that  $\pi'_i \cong \pi_{i/FF_i} \otimes \chi$ , where  $\pi_{i/FF_i}$  is the base change to  $FF_i$  of some automorphic representation  $\pi_i$  of  $GL(2)/F_i$ .

As above we obtain

$$\rho(\pi)|_{\Gamma_i} \otimes \chi_i \cong (\text{Sym}^2 \rho_{\pi_i} \otimes \chi|_{I_{F_i}} \cdot \chi_i) \oplus (\omega_{\pi_i} \cdot \omega_{FF_i/F_i} \cdot \chi|_{I_{F_i}} \cdot \chi_i).$$

and thus

$$L(s, \rho(\pi)|_{\Gamma_i} \otimes \chi_i) = L(s, \text{Sym}^2 \rho_{\pi_i} \otimes \chi|_{I_{F_i}} \cdot \chi_i) \cdot L(s, \omega_{\pi_i} \cdot \omega_{FF_i/F_i} \cdot \chi|_{I_{F_i}} \cdot \chi_i).$$

Since for each  $i$  the representation  $\pi_i$  is cuspidal non-CM, we know that the representation  $\text{Sym}^2 \pi_i$  is cuspidal automorphic and that the  $L$ -function  $L(s, \text{Sym}^2 \rho_{\pi_i} \otimes \chi|_{I_{F_i}} \cdot \chi_i)$  is analytic and does not vanish at  $s=2$ .

We get that the order of the pole of  $L$ -function

$$L(s, \rho(\pi)|_{\Gamma_i} \otimes \chi_i)$$

at  $s=2$  is 1 or 0 according to  $\omega_{\pi_i} \cdot \omega_{FF_i/F_i} \cdot \chi|_{I_{F_i}} \cdot \chi_i$  is equal to  $\xi_i^{-1}|_{F_i}$  or not.

From proposition 6.4, we obtain that proposition 6.2 is true in case ii) and b) and if the representation  $\pi$  is cuspidal non-CM.

From the cases a) and b), we deduce that theorem 6.1 is true in case ii) and if the representation  $\pi$  is cuspidal non-CM.

### 6.3 CM and one dimensional Tate classes

We assume first that our representation  $\pi$  is cuspidal of CM type. Thus there exists a quadratic extension  $M/F$  and an algebraic Hecke character  $\Omega$  of weight one of  $M$  such that  $\rho_{\pi} = \text{Ind}_{\Gamma_M}^{\Gamma_F} \Omega$ .

In the case i), from §6 we know that:

$$\rho(\pi) \cong \rho_{\pi}|_{\Gamma_E} \otimes \rho_{\pi}^{\tau}|_{\Gamma_E}.$$

In the case ii), from the proprieties of  $\rho(\pi)$  we have that (see for example §6.3 of [MR]):

$$\Lambda^2(\text{Ind}_{\Gamma_M}^{\Gamma_E} \Omega) \cong \rho(\pi) \oplus \text{Ind}_{\Gamma_F}^{\Gamma_E}(\omega_{\pi}),$$

where  $\omega_{\pi}$  is the central character of  $\pi$ .

From these identities we get that  $\rho(\pi)|_{\Gamma_{F_i}}$  is a virtual sum of monomial representations of  $\Gamma_{F_i}$ . Here a monomial representation of  $\Gamma_{F_i}$  is a representation which is induced from a one-dimensional representation of an open subgroup.

Since  $\rho(\pi)|_{\Gamma_{F_i}}$  is sum of monomial representations, it is easy to see that the pole of  $L(s, (\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i))$  at  $s = 2$  is equal to the dimension of the space of Tate classes  $\mathbf{V}(\pi, F_i)$ .

The same argument works when  $\pi$  is one-dimensional, since in this case, we know from the beginning of §6 that  $\rho(\pi)|_{\Gamma_{F_i}}$  is a sum of one-dimensional representations.

From this section and sections 6.1 and 6.2 where we treated the non-CM case, we obtain proposition 6.2 and theorem 6.1. Actually theorem 6.1 is true for an arbitrary extension  $k$  of  $E$  and a general Artin representation  $\omega$  if  $\pi$  is one-dimensional or of CM type, because in these cases the base change of  $\pi$  to an arbitrary extension  $k/F$  exists.

**Remark 6.5.** *The results in this article remain true if we replace the prime ideal  $\wp$  of  $O_F$  by an arbitrary ideal  $\mathfrak{n}$  of  $O_F$  such that each prime  $\wp|\mathfrak{n}$  is unramified in  $D$ .*

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