# On the rank of Selmer groups for elliptic curves over $\mathbf{Q}$

Eric Urban

January 28, 2013

**Abstract**

The following are extended notes of a lecture given by the author at the international colloquium on L-functions and Automorphic Representation held at TIFR in january 2012. This lecture reported on some joint work of Chris Skinner and the author on the link between central L-values and Selmer groups of elliptic curves. The detailed proofs of our results will appear in [SU13]. The author presents here the main lines of the arguments.

## 1 Introduction

Let $E$ be an elliptic curve over the rational. By the works initiated by Wiles, it is known that $E$ is modular and therefore that its $L$-function $L(E,s)$ is entire on the whole complex plane. The Birch and Swinnerton-Dyer conjecture predicts that

$$ord\ L(E,s)|_{s=1} = \mathrm{rank}_{\mathbf{Z}}\ E(\mathbf{Q})$$

This conjecture is proved thanks to the works of Kolyvagin and Gross-Zagier when the order of vanishing is 0 or 1. When the order of vanishing is higher, very little is known in general for the Mordell-Weil rank. However studying the co-rank of the Selmer group of $E$ seems more accessible. Let $p$ be a rational prime and let $Sel_p(\mathbf{Q},E)$ be the $p$-Selmer group of $E$ over $\mathbf{Q}$. Recall that it is a subgroup of $H^1(\mathbf{Q},E[p^\infty])$ fitting in the Kümmer exact sequence:

$$0 \to E(\mathbf{Q}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to Sel_p(\mathbf{Q},E) \to Ш_p(\mathbf{Q},E) \to 0$$

where $Ш_p(\mathbf{Q},E) \subset H^1(\mathbf{Q},E)$ stands for the $p$-part of the Tate-Shafarevitch group of $E$ over $\mathbf{Q}$. Birch and Swinnerton-Dyer conjecture also that this later is finite. The corank of the Selmer group should therefore be equal to the Mordel-Weil rank of $E$. A special case of our result is the following:

**Theorem 1.1 ([SU13])** *Let $E$ be a semi-stable elliptic curve over $\mathbf{Q}$ having good reduction at $p$. If $L(E,1)=0$ then $Sel_p(\mathbf{Q},E)$ is infinite. Furthermore if $L(E,s)$ vanishes at $s=1$ with a positive even order then the corank of $Sel_p(\mathbf{Q},E)$ is at least 2.*

This result is valid for a larger class of elliptic curves. More generally, we prove a similar result for the Bloch-Kato Selmer group attached to an elliptic cuspidal eigenform of trivial nebentypus. In this note, we explain the main steps of the strategy to prove such a result. In section 2, one recalls the definition of the Bloch-Kato Selmer groups and states the main result. The basic strategy is to construct and use a certain deformation of a reducible Galois representation like in our previous work [SU06a, SU06b] but in a different way than in [SU10] where we proved the Iwasawa conjecture for $p$-ordinary elliptic curves. In section 3, we explain how the existence of such a deformation leads to the construction of a non trivial extension in the Bloch-Kato Selmer group. There isn't much novelty in that part of the argument, except that the Hodge-Tate weights of the reducible Galois representation that is deformed have multiplicities and a slightly different argument is necessary to prove the first part of Theorem 1.1. From this, the even order situation can be resolved like in [SU06b]. In section 4, we explain how to construct a $p$-adic deformation of a certain *critical* $p$-adic Eisenstein series whose Galois representation is isomorphic to the Galois representation we want to deform. This will give rise to the deformation of the Galois representation studied in section 3. In [SU06b], we treated the case of the Bloch-Kato Selmer group for a cuspidal eigenform $f$ of weight $k \geq 4$, because this condition on the weight is necessary to construct the appropriate holomorphic Eisenstein series. The case of weight 2 we treat here[1] is obtained by constructing a $p$-adic Eisenstein series that we can think of as an overconvergent automorphic form with a non arithmetic $p$-adic weight. By overconvergent here, we mean a $p$-adic modular form which defines a point of the Eigenvariety for the quasi-split unitary group $U(2,2)$. This can be achieved under the condition that $L(f,1) = 0$. To construct this $p$-adic Eisenstein series, we put the modular eigenform $f$ into a Coleman family. For each member of this family of weight $\geq 4$, we can construct an Eisenstein series with the arithmetic appropriate weight but that is holomorphic only if the central $L$-value of this member vanishes. However, this Eisenstein series is always nearly holomorphic in a sense defined originally by Shimura in the seventies for elliptic modular forms and generalized later for symplectic and unitary groups (see [Sh04] for an unified treatment of his theory). We are therefore led to study the arithmetic of nearly holomorphic forms and give an algebraic definition[2] of those. This naturally leads us to define the notion of nearly overconvergent forms which can be seen naturally as special $p$-adic modular forms. We have given a taste of this notion here although it is not really necessary for our goal but it gives a good idea of the kind of objects we are dealing with in this work. This notion of nearly overconvergence have other applications in particular for the construction of $p$-adic L-function. We hope to come back to this in the future.

The author would like to thank Michael Harris, Haruzo Hida, Benoit Stroh, Vincent Pilloni and Jacques Tilouine for their interest in this work. The author wishes also to thank the organizers of the International Colloquium on Automorphic Representations and L-functions that was held at the TIFR for their kind invitation to give a lecture and for their hospitality. The author have been founded by the CNRS and the NSF during periods when this research was undertaken and is grateful to these institutions for their financial support.

[1] We actually treat the general case $k \geq 2$.

[2] We learned after this was achieved that Michael Harris had given an equivalent definition in [Ha85, Ha86].

## Contents



**Notations and conventions.** Throughout this paper $p$ is a fixed prime. We denote by $\mathbf{Z}$ and $\mathbf{Z}_p$ the rings of integers and $p$-adic integers with respective field of fractions $\mathbf{Q}$ and $\mathbf{Q}_p$. We denote respectively by $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_p$ the algebraic closures of $\mathbf{Q}$ and $\mathbf{Q}_p$ and by $\mathbf{C}$ the field of complex numbers. We fix embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and we fix an identification $\overline{\mathbf{Q}}_p \cong \mathbf{C}$ compatible with these embeddings. Throughout we implicitly view $\overline{\mathbf{Q}}$ as a subfield of $\mathbf{C}$ and $\overline{\mathbf{Q}}_p$ via the embeddings $\iota_\infty$ and $\iota_p$. All number fields will be considered as subfield of $\overline{\mathbf{Q}}$ and of $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$ via the above embeddings. We denote respectively by $\mathbf{A}$ and $\mathbf{A}_f$ the rings of adeles and finite adeles of $\mathbf{Q}$. For each place $v$ of a number field $K$, we denote $K_v$ the completion of $K$ with respect to the norm $|\cdot|_v$ associated to $v$. If $\mathfrak{X}$ is a rigid analytic variety over an extension of $\mathbf{Q}_p$, we denote by $A(\mathfrak{X})$ the ring of analytic functions on $\mathfrak{X}$. If $H$ is a reductive group over $\mathbf{Z}$, an algebraic representation of $H$ is seen a functorial pair $(\rho, V)$ in the sense that for any ring $R$, we have a group homomorphism $\rho : H(R) \to GL_R(V_R)$ where $V_R = V_{\mathbf{Z}} \otimes R$ is free over $R$ satisfying the obvious base change property for any ring homomorphism $R \to S$.

## 2 Bloch-Kato Selmer groups

### 2.1 Some definitions

We recall the definition of the Bloch-Kato Selmer group attached to a Galois representation and precise our conventions for $L$-functions. Let $K$ be a number field and $G_K = Gal(\overline{\mathbf{Q}}/K)$ be its absolute Galois group. Let $L$ be a finite extension of $\mathbf{Q}_p$ and let $V$ be a finite dimensional $L$-vector space equipped with a continuous linear action of $G_K$. We assume that this Galois representation is geometric in the sense of Fontaine. For such a representation, we denote by $V(n)$ the n-th Tate twist of $V$.

We denote by $H^1(K, V)$ the continuous cohomology of $G_K$ with coefficients in $V$. This space parametrizes the isomorphisms classes $[E]$ of extensions $E$ of the form

$$0 \to V \to E \to L \to 0.$$

where we understand $L$ as the one dimensional $L$-vector space with trivial $G_K$-action. Then $H^1_g(K, V)$ is the subset of $H^1(K, V)$ classifying classes of extensions $[E]$ such that $E$ is also geometric. We now assume that the action of the decomposition subgroups of $G_K$ at the places above $p$ are crystalline. Then, $H^1_f(K, V)$ denotes the subspace of $H^1(K, V)$ of extensions classes $[E]$ such that $E$ is crystalline at all places above $p$ and such that for all places $v$ not divising $p$, we have

$$0 \to V^{I_v} \to E^{I_v} \to L \to 0.$$

where $I_v$ stands for an inertia subgroup at $v$. In general, we have

$$H^1_f(K, V) \subset H^1_g(K, V) \tag{1}$$

We now recall the definition of the $L$-function attached to a Galois representation. For any finite place $v$ not dividing $p$, we put

$$P_v(V, X) := det(1 - XFrob_v; V^{I_v})$$

where $Frob_v$ stands for a geometric Frobenius at $v$ and

$$P_v(V, s) := det(1 - X\varphi_v; D_{crys,v}(V))$$

if $v$ divides $p$ and where $\varphi_v$ stands for the geometric crystalline Frobenius induced on

$$D_{crys,v}(V) := (B_{crys} \otimes V)^{D_v}.$$

For each finite place $v$, let $q_v$ be the cardinal of the residue field of $K$ at $v$. Then we put $L_v(V, s) := P_v(V, q_v^{-s})^{-1}$ and the $L$-function of $V$ is defined as

$$L(V, s) = \prod_{v<\infty} L_v(V, s)$$

It is conjectured that $L(V, s)$ has a meromorphic continuation to the complex plane. This fact is of course established when we know that $V$ is attached to an automorphic representation.

**Remark 2.1** The inclusion (1) is an equality if $P_v(V, q_v) \neq 0$ for all finite place $v$.

## 2.2 Selmer groups for modular forms

Let $f$ be a new cuspidal elliptic eigenform of even weight $k = 2m$ with trivial nebentypus and conductor $N$. Assume $f$ is normalized and let us write its Fourier expansion

$$f(q) = \sum_{n=1}^{\infty} a(n, f) q^n$$

Let $L$ be a finite extension of $\mathbf{Q}_p$ containing the Hecke eigenvalues of $f$. By Eichler-Shimura and Deligne, there exists a two dimensional $L$-vector space $V_f$ with a continuous $G_{\mathbf{Q}}$-linear action such that

$$L(V_f, s) = L(f, s) := \sum_{n=1}^{\infty} a(n, f) n^{-s}.$$

Recall that $L(f, s)$ satisfies a functional equation of the form:

$$L(f, s) = \varepsilon(f, s) L(f, 2m - s)$$

Our main result is the following theorem.

**Theorem 2.2 ([SU13])** *Assume $N$ is prime to $p$. If $k = 2$ and $\varepsilon(f, 1) = 1$, we further assume that $a_\ell \neq 0$ for some prime $\ell | N$. Then,*

*a) if $L(f, m) = 0$, we have $H^1_f(\mathbf{Q}, V_f(m)) \neq 0$,*

*b) if $L(f, m) = 0$ and $\varepsilon(f, m) = 1$, then $\mathrm{rank}_L\, H^1_f(\mathbf{Q}, V_f(m)) \geq 2$.*

**Remark 2.3** Theorem 1.1 follows from Theorem 2.2 because for $f_E$ the weight 2 square free level cusp form associated to $E$, the rank of $H^1_f(\mathbf{Q}, V_{f_E}(1))$ is equal to the co-rank of $Sel_p(\mathbf{Q}, E)$. Notice that we have $V_{f_E}(1) \cong V_p(E) = T_p(E) \otimes \mathbf{Q}_p$ where $T_p(E)$ stands for the Tate module of $E$.

**Remark 2.4** If $f$ is ordinary at $p$, the part a) of this Theorem follows from [SU10]. It follows also from [SU02] if the order of vanishing is odd. The method used in [SU02] works *mutatis mutandis* if $p$ is a supersingular prime for $E$. The construction of the deformation of the corresponding Saito-Kurokawa lift follows from Example 5.5.3 of [Ur11].

**Remark 2.5** If $k > 2$, this Theorem is proved using the strategy outlined in [SU06b]. It is based on the construction of an holomorphic Eisenstein series on the quasi-split unitary group $U(2, 2)$ attached to an imaginary quadratic field $\mathcal{K}$. To include the case $k = 2$, we use an arithmetic theory of nearly holomorphic forms to construct a specific $p$-adic (overconvergent) Eisenstein series (having a non-arithmetic weight).

**Remark 2.6** The condition on the conductor $N$ when $k = 2$ is a simplified assumption making sure $f$ is in the image of the Jacquet-Langlands correspondence for a definite quaternion algebra. This is necessary to show that the $p$-adic (overconvergent) Eisenstein series that will construct as a $p$-adic limit is non-trivial. See (iv) in Theorem 4.8.

### 2.3 The basic strategy

In this note, we explain the main steps of the basic strategy for proving the part a) of this theorem. The main idea is an extension and generalization of a method introduced and developed by C. Skinner and the author in a series of papers[3] [SU02, SU06a, SU06b]. We introduce an imaginary quadratic field $\mathcal{K}$ and construct a generically irreducible deformation of the $G_{\mathcal{K}}$-representation

$$W_f := L \oplus V_f(m) \oplus L(1)$$

From this deformation, we are able to construct a non-split extension of the following form by using a version of Ribet's lemma

$$0 \to L(1) \to E_f \to V_f(m) \to 0$$

Since $V_f^\vee \cong V_f(k-1)$, we get a non trivial class $[E_f] \in H^1(\mathcal{K}, V_f(m))$. We further show that it is contained in $H^1_f(\mathcal{K}, V_f(m))$. Let denote by $\chi_{\mathcal{K}}$ the quadratic character attached to the extension $\mathcal{K}/\mathbf{Q}$. If $\mathcal{K}$ is chosen so that

$$L(f, \chi_{\mathcal{K}}, m) \neq 0 \tag{2}$$

we know by results of Kato or Kolyvagin that $H^1_f(\mathbf{Q}, V_f(m) \otimes \chi_{\mathcal{K}}) = 0$. We deducet that $[E_f] \in H^1_f(\mathbf{Q}, V_f(m))$ and the part a) of Theorem 1.1 follows. For the part b), the strategy is already explained in [SU06b].

## 3 An analytic family of trianguline Galois representations

Using the theory of $p$-adic families of automorphic forms of finite slopes, we will construct a certain family of trianguline Galois representations. We describe the family in Theorem 3.1 and show how we deduce the part a) of Theorem 1.1

### 3.1 Polarized Galois representations

We fix $\mathcal{K} \subset \overline{\mathbf{Q}}$ an imaginary quadratic field. We denote by $c$ the complex conjugation of $\mathbf{C}$ (and hence of $\mathcal{K}$ induced by the embedding $\iota_\infty$). Sometimes we write $\bar{a}$ instead of $a^c$ for $a \in \mathbf{C}$. Let $\mathcal{O}_{\mathcal{K}}$ the ring of integers of $\mathcal{K}$. We assume $p$ splits in $\mathcal{K}$, i. e. $p.O_{\mathcal{K}} = \wp.\wp^c$ where $\wp$ stands for the prime ideal of $\mathcal{O}_{\mathcal{K}}$ induced by $\iota_p$. We denote by $\mathcal{O}_{(\wp)}$ the localization of $\mathcal{O}_{\mathcal{K}}$ at $\wp$ and by $\mathcal{O}_\wp$ its completion.

We will consider Galois representations $(\rho, W)$ of $G_{\mathcal{K}}$. For such a representation, we denote by $(\rho^c, W^c)$ the representation on the space $W$ with the conjugate action by $c$ (i.e. $\rho^c(g) = \rho(cgc)$, $\forall g \in G_{\mathcal{K}}$) and by $(\rho^\vee, W^\vee)$ the contragredient representation. We will say $W$ is polarized if it satisfies:

$$W^c \cong W^\vee(1).$$

Notice that the representation $W_f$ that we have defined in the previous section satisfies this condition. We will consider families of such representations of dimension 4. For this, it is convenient to

[3] In [SU02, SU06a], one does not need to introduce an imaginary quadratic field.

define the parametrizing space of Hodge-Tate-Sen weights. Let $\mathfrak{W}$ be the rigid analytic space over $\mathbf{Q}_p$ such that $\mathfrak{W}(L) = Hom_{cont}((\mathbf{Z}_p^\times)^4, L^\times)$. An element $\underline{\kappa} = (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathfrak{W}(\overline{\mathbf{Q}}_p)$ is called a weight. If $k_1, \ldots, k_4$ are integers, we write $(k_1, k_2, k_3, k_4)$ for the weight defined by

$$(t_1, t_2, t_3, t_4) \mapsto t_1^{k_1} t_2^{k_2} t_3^{k_3} t_4^{k_4}.$$

Such a weight is called arithmetic regular if $k_1 < k_2 < k_3 < k_4$. Using the theory of $p$-adic families of automorphic forms, one shows the following theorem.

**Theorem 3.1 ([SU13])** *Assume that $L(f,m) = 0$ and that the conditions of Theorem 2.2 are satisfied. Let $\alpha$ be an eigenvalue of $X^2 - a(p,f)X + p^{k-1}$. Then there exist:*

*(i) $\mathfrak{V}$ an irreducible finite cover of a one-dimensional affinoid subdomain $\mathfrak{U}$ of $\mathfrak{W}$ with structural map $\underline{\kappa}$,*

*(ii) a point $x_0 \in \mathfrak{V}(L)$ such that $\underline{\kappa}(x_0) = \underline{\kappa}_0 = (-m, -1, 0, m-1) \in \mathfrak{U}(L)$,*

*(iii) A pseudo-character[4] $T : G_{\mathcal{K}} \to A(\mathfrak{V})$ of dimension 4,*

*(iv) An infinite set $\Sigma \subset \mathfrak{V}(\overline{\mathbf{Q}}_p)$ sitting over arithmetic regular weights,*

*(v) Analytic functions $\underline{\varphi} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in A(\mathfrak{V})^4$*

*For $x \in \mathfrak{V}(\overline{\mathbf{Q}}_p)$, let us write $T_x$ for the evaluation of $T$ at $x$. The data $T$, $\Sigma$, $\underline{\varphi}$ and $x_0$ satisfy the following properties:*

*(a) $T_{x_0}$ is the character associated to $W_f$ and $\underline{\varphi}(x_0) = (\alpha p^{-m}, 1, p^{-1}, \alpha^{-1}p^{m-1})$,*

*(b) For all $x \in \Sigma$, $T_x$ is the trace of a semi-simple polarized representation $\rho_x$ such that*

$$\dim(\rho_x)^{I_v} = \dim V_f^{I_v} + 2$$

*for all finite places $v$ not dividing $p$.*

*(c) For all $x \in \Sigma$, the restriction[5] of $\rho_x$ at $G_\wp$ is crystalline with Hodge-Tate weight $\underline{\kappa}(x)$ and Frobenius eigenvalues $(\varphi_1(x)p^{\kappa_1(x)}, \varphi_2(x)p^{\kappa_2(x)}, \varphi_3(x)p^{\kappa_3(x)}, \varphi_4(x)p^{\kappa_4(x)})$.*

*(d) For all integers $N$, the subset $\Sigma_N \subset \Sigma$ of points $x \in \Sigma$ such that $\kappa_{i+1}(x) - \kappa_i(x) > N$ for $i = 1, 2, 3$ is infinite.*

This theorem is proved by constructing a $p$-adic family of cuspidal representations for $U(2,2)$ that specializes to an Eisenstein representation at the point $x_0$ and by looking at corresponding family of Galois representations. The strategy to construct this cuspidal family is explained in the section 4.

**Lemma 3.2** *The pseudo representation $T$ is (generically) irreducible.*

The proof of this generical irreducibility property goes along the same lines as the one of Theorems 3.3.12 and 4.2.7 of [SU06a]. It is done by contradiction. In loc. cit. one uses the fact that there are finitely many units in $\mathbf{Q}$, here the same fact for imaginary quadratic fields is crucial.

[4] See the papers [Ro96] and [Ta91] for the definitions and properties of pseudo-characters.

[5] A similar property holds for the restriction of $\rho_x$ to $G_{\wp^c}$ thanks to the polarization property satisfied by $\rho_x$.

## 3.2 Construction of the desired extension

We keep the hypothesis and notations of Theorem 3.1. After replacing $\mathfrak{V}$ by a finite cover, we may assume that the representation attached to the pseudo-character $T$ is defined over the fraction field of $A(\mathfrak{V})$. We may also assume that $A(\mathfrak{V})$ is a Dedekind domain. Then, we consider a lattice $\mathcal{L}$ of the representation of dimension 4 with trace given by $T$ such that its localization $\mathcal{L}_{(x_0)}$ at the maximal ideal corresponding to $x_0$ has a unique irreducible quotient, this quotient being isomorphic to $V_f(m)$. To see how to construct such a lattice see [SU06b]. Let $\mathcal{L}_{x_0}$ the reduction of $\mathcal{L}$ modulo the maximal ideal corresponding to $x_0$. Notice that by condition (a) the semi-simplification of $\mathcal{L}_{x_0}$ is isomorphic to $W_f$. By construction, $V_f(m)$ is the unique irreducible quotient of $\mathcal{L}_{x_0}$. An important fact is the following.

**Lemma 3.3** *$\mathcal{L}_{x_0}$ contains the trivial representation as a subrepresentation. The quotient $E_f$ of $\mathcal{L}_{x_0}$ by this trivial subrepresentation is a nontrivial extension of the form*

$$0 \to L(1) \to E_f \to V_f(m) \to 0$$

**Proof** Again the argument to prove this fact is already present in [SU06a] and [SU06b] at least when $k > 2$. If the first assertion were not true, then the representation $\mathcal{L}_{x_0}$ would contain a non trivial extension:

$$0 \to L(1) \to E' \to L \to 0$$

By the condition (b) of the theorem, this extension would be unramified away from $p$. It remains to prove that this representation is crystalline at $\wp$ and $\wp^c$ which would give the contradiction we are seeking since $H^1_f(\mathcal{K}, L(1)) = 0$. By a result of B. Perrin-Riou, this extension is semi-stable at $\wp$ because it is ordinary. Let us call $N'$ the monodromy operator on[6] $D_{st,\wp}(E')$. Let us consider the exterior square $\wedge^2\mathcal{L}_{x_0}$. It contains the representation $E' \otimes V_f(m)$ as a subquotient. This latter representation is also semi-stable since $V_f(m)$ is crystalline[7] and

$$D_{st,\wp}(E' \otimes V_f(m)) = D_{st,\wp}(E') \otimes D_{crys,\wp}(V_f(m))$$

Moreover, its monodromy operator is given by $N = N' \otimes Id_{D_{crys,\wp}(V_f(m))}$. On the other hand using a result of Kisin, we know that $D_{crys,\wp}(\wedge^2\mathcal{L}_{x_0})^{\Phi=\alpha p^{-m}} \neq 0$. This implies clearly that

$$\mathrm{rank}_L\ D_{crys,\wp}(E' \otimes V_f(m))^{\Phi=\alpha p^{-m}} = 1. \quad (3)$$

From the relation, $N\Phi = p\Phi N$ in $D_{st,\wp}(E')$, it is easy to see that if $E'$ is not crystalline there exists $v' \in D_{st,\wp}(E')^{\Phi=1}$ such that $N'v' \neq 0$. If $v_\alpha \in D_{crys,\wp}(V_f(m))$ is an eigenvector for the eigenvalue $\alpha p^{-m}$, therefore $v = v' \otimes v_\alpha$ is an eigenvector for the eigenvalue $\alpha p^{-m}$ and $N.v = Nv' \otimes v_\alpha \neq 0$ which contradicts (3) since $D_{crys,\wp}(E \otimes V_f(m))$ is the kernel of $N$. Therefore $E'$ is crystallline at $\wp$. A similar argument applies for $\wp^c$ using the fact that the polarization property implies that the conditions satisfied for the restriction to the decomposition subgroup $D_\wp$ at $\wp$ are also satisfied for

[6] For a representation $W$ of $G_\mathcal{K}$, we put $D_{st,\wp}(W) := (B_{st} \otimes W)^{Gal(\overline{\mathbf{Q}}_p/\mathcal{K}_\wp)}$ where $B_{st}$ stands for the ring of semi-stable periods of Fontaine. It is equipped with a filtration and an action of Frobenius $\phi$.

[7] This is because the conductor $N$ of $f$ is prime to $p$

the restriction to the decomposition subgroup $D_{\wp^c}$ at $\wp^c$. The second assertion of the lemma is clear from the construction.

□

Assuming the hypothesis and the conclusions of Theorem 3.1, we are now in position to finish the proof of the part a) of Theorem 2.2.

**Lemma 3.4** *If $v_p(\alpha) < k-1$, then $[E_f] \in H^1_f(\mathcal{K}, V_f(m))$; in particular $H^1_f(\mathcal{K}, V_f(m)) \neq 0$.*

**Proof** By remark 2.1, in order to prove that the extension class $[E_f]$ belongs to $H^1_f(\mathcal{K}_\wp, V_f(m))$, we only need to show that the restriction to $D_\wp$ of $E_f$ is de Rham. Indeed, since $P_\wp(V_f(m), X) = (1-\alpha p^{-m}X)(1-\alpha^{-1}p^{m-1}X)$ and $\alpha$ is a Weil number of weight $2m-1$, we see that $P_\wp(V_f(m), p) \neq 0$ and $H^1_f(\mathcal{K}_\wp, V_f(m)) = H^1_g(\mathcal{K}_\wp, V_f(m))$. In order to prove that $E_f$ is de Rham at $\wp$, we use Lemma 4.2.3 of [SU06b]. Let $g$ be the projection map from $E_f$ onto $V_f(m)$. We need to show there exists $D' \subset D_{dR,\wp}(E_f)$ such that

$$g \otimes id_{B_{dR}}(D') \oplus Fil^0 D_{dR,\wp}(V_f(m)) = D_{dR,\wp}(V_f(m)). \quad (4)$$

Let $D'$ be the image in $D_{dR,\wp}(E_f)$ of $D_{crys,\wp}(E_f)^{\Phi=\alpha p^{-m}}$. We know that $D' \neq 0$ by the Corollary 5.3 of [Ki03] (see also Proposition 4.2.2 of [SU06b]) . Moreover its image by $g \otimes id_{B_{dR}}$ is the image $D$ of $D_{crys,\wp}(V_f(m))^{\Phi=\alpha p^{-m}}$ in $D_{dR,\wp}(V_f(m))$. Since $v_p(\alpha.p^{-m}) < m-1$ and the Hodge-Tate numbers of $V_f(m)$ are $-m$ and $m-1$, we deduce that $D \cap Fil^0 D^{dR,\wp}(V_f(m)) = \{0\}$ by weak admissibility of $D_{crys,\wp}(V_f(m))$. This finishes the proof of (4) and that $[E_f] \in H^1_f(\mathcal{K}_\wp, V_f(m))$. Similarly we have $[E_f] \in H^1_f(\mathcal{K}_{\wp^c}, V_f(m))$.

One need to prove a similar fact at places $w$ not dividing $p$. More precisely, one has to show that the following sequence is right exact:

$$0 \to L(1)^{I_w} \to E_f^{I_w} \to V_f(m)^{I_w} \to 0$$

but this follows easily by the property (b) of Theorem 3.1. This finishes the proof of our lemma.

□

# 4 Nearly overconvergent Eisenstein series on $U(2,2)$

In this section, we explain how the vanishing of the L-value $L(f,m)$ implies the existence of a certain overconvergent automorphic forms for the quasi-spplit unitary group $U(2,2)$. We will start by recalling standard facts on unitary automorphic forms mainly due to Shimura and then give their algebraic interpretations which enables us to have an arithmetic theory for nearly holomorphic forms. This will be the cornerstone of the construction of $p$-adic families of nearly holomorphic forms and the notion of nearly overconvergent forms. Strictly speaking, the theory of nearly overconvergent forms could be avoided as it will appear in the last subsection of these notes. However, we have introduce them because when $k=2$, the Eisenstein series which is deformed is obtained as a $p$-adic limit of nearly holomorphic Eisenstein series and cannot be seen as a nearly holomorphic form since its ($p$-adic) weight $(1,2,-2,1)$ is not dominant. So it is a purely $p$-adic object. We have not defined it as a section of some $p$-adic sheaf but this could be done easily by using the technics of Andreatta-Iovita-Pilloni [AIP].

## 4.1 Nearly holomorphic unitary automorphic forms

### 4.1.1 Unitary automorphic forms

We consider the skew-hemitian form on $\mathcal{K}^4$ given by the matrix

$$J = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}.$$

Let $\mathbf{G} = GU(2,2) \subset GL_{4/\mathcal{O}_\mathcal{K}}$ be the group scheme of unitary similitudes preserving the skew-hermitian form on $\mathcal{K}^4$ given by $J$ . That is for any ring $R$, we put

$$\mathbf{G}(R) := \{g \in \mathrm{GL}_d(\mathcal{O}_\mathcal{K} \otimes_\mathbf{Z} R) \ : gJ{}^t\bar{g} = \nu(g)J\}.$$

with $\nu(g) \in \mathbf{G}_m(R) = R^\times$. We denote by $G = U(2,2)$ the unitary group defined as the kernel of $\nu : \mathbf{G} \to \mathbf{G}_m$. We similarly define $GU(1,1)$ and $U(1,1)$. A matrix $\gamma \in G(R)$ will be written by blocs of size $2 \times 2$ in the following way:

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

The hermitian tube domain associated to $G$ is the four dimensional complex analytic manifold $\mathcal{D}$ defined by

$$\mathcal{D} := \{z \in M_{2\times 2}(\mathbf{C}) \ : i.(z^* - z) > 0\}$$

where we write $z^* = {}^t\bar{z}$. The identity component $G^+(\mathbf{R})$ of $G(\mathbf{R})$ acts transitively on $\mathcal{D}$ by the usual Möbius transformation

$$\gamma.z = (a_\gamma z + b_\gamma) \cdot (c_\gamma z + d_\gamma)^{-1} \text{ for } \gamma \in G^+(\mathbf{R}).$$

We consider the automorphic factor

$$j(\gamma, z) := (c_\gamma z + d_\gamma, (\bar{c}_\gamma {}^t z + \bar{d}_\gamma))$$

taking values in $H(\mathbf{C})$ with $H := GL_2 \times GL_2$. We also define

$$\Xi(z) := (i(\bar{z} - {}^t z), i(z^* - z)) \text{ and } r(z) := i(\bar{z} - {}^t z)^{-1}$$

Let $(\rho, V)$ be an algebraic representation of $H$ and let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup. We denote by $\mathcal{A}_\rho(K, \mathbf{C})$ the space of $V_\mathbf{C}$-valued real analytic functions on $G(\mathbf{A}_f) \times \mathcal{D}$ such that

$$f(\gamma.g_f.k, \gamma.z) = \rho(j(\gamma, z)).f(g_f, z)$$

for all $\gamma \in G(\mathbf{Q})$ and $k \in K$. Following Shimura [Sh04], we now define some differential operators on the space of automorphic forms. We first introduce a few more notations. Let $(S, M_2(-))$ be the representation of $H = GL_2 \times GL_2$ on the space of $2\times 2$ matrices $M_2(-)$ given $S(g_1, g_2).M = g_1 M\, {}^t g_2$ for $M \in M_2(R)$ and $(g_1, g_2) \in H(R)$. Let $St^+$ (resp. $St^-$) be the standard representation of the first (resp. second) copy of $GL_2$ in $H$ then we have $S \cong St^+ \otimes St^-$.

Let $(\frac{\partial}{\partial r_{ij}})_{i,j}$ be the differential operators on the differentiable functions on $\mathcal{D}$ defined by the relation

$$\frac{\partial}{\partial \bar{z}_{kl}} = \sum_{i,j} \frac{\partial r_{ij}}{\partial \bar{z}_{kl}} \frac{\partial}{\partial r_{ij}}$$

where for $1 \leq i, j \leq 2$, the function $r_{ij}(z)$ stands for the $(i,j)$ entry of the function $r(z)$. We consider the differential operator $\epsilon_\rho$ from $\mathcal{A}_\rho(K, \mathbf{C})$ taking value in the space of real analytic $Hom_{\mathbf{C}}(M_2(\mathbf{C}), V_{\mathbf{C}})$-valued functions on $G(\mathbf{A}_f) \times \mathcal{D}$ defined by:

$$(\epsilon_\rho f)(g_f, z)(u_{ij}) := \sum_{i,j} u_{ij}. \frac{\partial}{\partial \bar{r}_{ij}} f(g_f, z)$$

for $(u_{ij}) \in M_2(\mathbf{C})$. If there is no possible confusion, we sometimes just write $\epsilon$. The image of $\epsilon_\rho$ is contained in $\mathcal{A}_{S^\vee \otimes \rho}(K, \mathbf{C})$. The space of nearly holomorphic forms $\mathcal{N}^r_\rho(K, \mathbf{C})$ of order $\leq r$ is by definition the kernel of the differential operator of degree $r+1$ defined by

$$\epsilon^{r+1}_\rho = \epsilon_{(S^\vee)^{\otimes r} \otimes \rho} \circ \cdots \circ \epsilon_\rho.$$

For $r = 0$, one obtains the usual space of holomorphic unitary automorphic forms of weight $\rho$.

We now recall the generalized Maass-Shimura differential operators. For $f \in \mathcal{N}^r_\rho(K, \mathbf{C})$, one defines $\delta_\rho.f \in \mathcal{N}^{r+1}_{\rho \otimes S}(K, \mathbf{C})$ as the function taking values in $V_{\mathbf{C}} \otimes M_2(\mathbf{C})$ defined by the formula

$$(\delta_\rho.f)(g_f, z) := \sum_{ij} \left( \rho(\Xi(z))^{-1} \frac{\partial}{\partial z_{ij}} [\rho(\Xi(z)).f(g_f, z)] \right) \otimes E_{ij}$$

where $E_{ij}$ stands for the elementary matrix of $M_2(\mathbf{C})$ having the $(i,j)$-entry equal to 1 and zero elsewhere. If $\rho = \det^k \otimes 1$, we denote by $\delta_k$ the Maass-Shimura operator. We record the following lemma which follows from a simple direct computation.

**Lemma 4.1** *Let $f$ be an holomorphic form of scalar weight $k$. Then we have:*

$$\delta_k(f) = \sum_{i,j} \frac{\partial f}{\partial z_{ij}} E_{ij} + k f(z) tr({}^t r(z) E_{**})$$

*where $E_{**}$ stands for the formal matrix with entries $E_{ij}$. Moreover*

$$\epsilon_{det^k \otimes S}(\delta_k f)(z) = k f(z) \sum_{ij} E^*_{ij} \otimes E_{ij}$$

*where $E^*_{ij}$ stands for the dual basis of $E_{ij}$. In particular $\epsilon_{det^k \otimes S}(\delta_k f)$ takes values in the canonical invariant line of $S^\vee \otimes S$.*

It is possible to define algebraic and arithmetic versions of these spaces and differential operators. We will do that in the next sections, we first need to introduce the unitary Shimura variety attached to $\mathbf{G}$.

### 4.1.2 Unitary Shimura variety

We fix a neat open compact subgroup $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$ and we denote by $X = X_K$ the Shimura variety of level $K = K^p.\mathbf{G}(\mathbf{Z}_p)$ given by the usual Shimura data so that its complex points are given by

$$X_K(\mathbf{C}) = \mathbf{G}(\mathbf{Q})\backslash(\mathcal{D} \times \mathbf{G}(\mathbf{A}_f)/K)$$

The Shimura variety $X_K$ is a smooth quasi-projective scheme defined over $\mathcal{K}$ which is not geometrically connected in general. By the work of Kottwitz, we know it has a canonical model over $O_{(\wp)}$ that we denote by $\mathcal{X}_K$. The scheme $\mathcal{X}_K$ represents the functor which sends a $\mathcal{O}_{(\wp)}$-scheme $S$ to the set of the equivalent classes of certain quadruplets $(f : A \to S, \iota, \lambda, \alpha)$ where

- $f : A \to S$ is an abelian scheme over $S$ of relative dimension 4,
- $\iota$ is a a ring homomorphism $\iota : \mathcal{O}_{\mathcal{K}} \to End_S(A)$,
- $\lambda : A \to A^t$ is a polarization of degree prime to $p$,
- $\alpha$ is a $K^p$-level structure, that is to say an isomorphism modulo $K^p$ :

$$\alpha : H_1(A/S, \mathbf{A}_f^p) \cong (\mathbf{A}_f \otimes K)^4_{/S}$$

Let $\omega_{A/S} = f_*\Omega_{A/S}$. It is a locally free sheaf over $S$. Let $\omega^+_{A/S}$ (resp. $\omega^-_{A/S}$) the sub-sheaf of sections on which the complex multiplication by $O_{(\wp)}$ coincides with the one (resp. with the conjugate of the one) induced by the $O_{(\wp)}$-scheme structure of $S$. We denote by $\alpha^t$ the $K$-level structure of $A^t$ induced by $\lambda$ and $\alpha$. These quadruplets are required to satisfy the following conditions.

- $\omega^+_{A/S}$ and $\omega^-_{A/S}$ are locally free of rank 2 over $S$
- The pairing on $(\mathbf{A}_f^p \otimes \mathcal{K})^4$ induced by $\alpha$, $\alpha^t$ and the Weil pairing has matrix $J$

We also consider the relative de Rham cohomology $\mathcal{H}^1_{dR}(A/S) := R^1 f_*\Omega^\bullet_{A/S}$. It is a locally free sheaf of rank 8 and it fits in the canonical exact sequence

$$0 \to \omega_{A/S} \to \mathcal{H}^1_{dR}(A/S) \to \omega^\vee_{A/S} \to 0$$

after we have identified $R^1 f_*\mathcal{O}_A$ with $\omega^\vee_{A/S}$ using Poincaré duality and the polarization $\lambda$. One defines its + part $\mathcal{H}^1_{dR}(A/S)^+$ as we did for $\omega_{A/S}$ and for which we have

$$0 \to \omega^+_{A/s} \to \mathcal{H}^1_{dR}(A/S)^+ \to (\omega^-_{A/S})^\vee \to 0$$

We define $\mathcal{J}(A/S)$ the sheaf obtained by making the following diagramm commutative and the bottom short sequence exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^+_{A/S} \otimes \omega^-_{A/S} & \longrightarrow & \mathcal{H}^1_{dR}(A/S)^+ \otimes \omega^-_{A/S} & \longrightarrow & (\omega^-_{A/S})^\vee \otimes \omega^-_{A/S} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \omega^+_{A/S} \otimes \omega^-_{A/S} & \longrightarrow & \mathcal{J}(A/S) & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}$$

We denote by $\pi : \mathcal{A} \to X$ the universal abelian scheme over $X = X_K$. We consider the sheaves $\omega^\pm := \omega^\pm_{\mathcal{A}/X}$ and $\mathcal{H}^\pm := \mathcal{H}^1_{dR}(\mathcal{A}/X)^\pm$. Let $\bar{X}$ be a smooth toroidal compactification of $X$ over $\mathcal{O}_{(\wp)}$ constructed by K.W. Lan in his thesis [La08]. The boundary $\partial\bar{X} = \bar{X}\backslash X$ is a normal crossing divisor of $\bar{X}$. We denote by $\Omega_{X/O_{(\wp)}}(\log\partial\bar{X})$ the sheaf of Khäler differential having logarithmic poles along $\partial\bar{X}$. Recall we have the Gauss-Manin connexion

$$\nabla : \mathcal{H}^1_{dR}(\mathcal{A}/X) \to \mathcal{H}^1_{dR}(\mathcal{A}/X) \otimes \Omega_{X/O_{(\wp)}}(\log\partial\bar{X})$$

It induces on the + part a map

$$\omega^+ \hookrightarrow \mathcal{H}^1_{dR}(\mathcal{A}/X)^+ \to \mathcal{H}^1_{dR}(\mathcal{A}/X)^+ \otimes \Omega_{X/O_{(\wp)}}(\log\partial\bar{X}) \to (\omega^-)^\vee \otimes \Omega_{X/O_{(\wp)}}(\log\partial\bar{X})$$

which yields the Kodaira-Spencer isomorphism ([La08]):

$$\omega^+ \otimes \omega^- \xrightarrow{\sim} \Omega_{X/O_{(\wp)}}(\log\partial\bar{X})$$

We will identify these sheaves. Notice that we therefore have

$$0 \longrightarrow \Omega_{X/\mathcal{O}_{(\wp)}}(\log\partial\bar{X}) \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

with $\mathcal{J} = \mathcal{J}(\mathcal{A}/X)$ and it could be seen that $\mathcal{J}^\vee$ is isomorphic to the sheaf of 1-jets on $X$.

#### 4.1.3 Automorphic sheaves

We define now locally free coherent sheaves on our unitary Shimura varieties in order to get rational and integral structures on the spaces of automorphic forms we have introduced above. Since $\omega^+$ and $\omega^-$ are locally free of rank 2, we may consider the following $H = GL_2 \times GL_2$-torsor over $X$:

$$\mathcal{T} := Isom(\omega^+ \oplus \omega^-, (O_X)^2 \oplus (O_X)^2)$$

For any algebraic representation $(\rho, V)$ of $H = GL_2 \times GL_2$, we denote by $\omega_\rho$ the coherent sheaf on $X$ defined as the contracted product

$$\omega_\rho := V \times^H \mathcal{T}.$$

In particular we have $\omega_{St^\pm} = \omega^\pm$ and

$$\omega_S = \omega^+ \otimes \omega^- \cong \Omega_{X/\mathcal{O}_{(\wp)}}(\log\partial\bar{X})$$

Let $\omega^1_\rho := \omega_\rho \otimes \mathcal{J}^\vee$. By the isomorphism above we have the short exact sequence of locally free sheaves:

$$0 \to \omega_\rho \to \omega^1_\rho \to \omega_{\rho\otimes S^\vee} \to 0$$

The Hodge decomposition provides the isomorphisms:

$$H^0(X_K, \omega_{\rho/\mathbf{C}}) \cong \mathcal{N}^0_\rho(K, \mathbf{C}) \text{ and } H^0(X_K, \omega^1_{\rho/\mathbf{C}}) \cong \mathcal{N}^1_\rho(K, \mathbf{C})$$

Moreover the map $\omega^1_\rho \to \omega_{\rho\otimes S^\vee}$ induces the differential operator $\epsilon_\rho$ we have defined in Section 4.1.1. Using Leibnitz rule, there is a canonical way to define a connexion on $\mathcal{H}^1_{dR}(\mathcal{A}/X)^{\otimes^r}$, from which we deduce a connexion[8] :

$$\nabla_\rho : \omega_\rho \to \omega^1_{\rho\otimes S}$$

in the sense that it satisties the relation[9] :

$$\nabla_\rho(f.\omega) = df \otimes \omega + f\nabla\omega$$

for $f \in \mathcal{O}_X(U)$ and $\omega \in \omega_\rho(U)$ for any Zariski open set $U \subset X$

**Remark 4.2** We can more generally define $\omega^r_\rho := Hom_{\mathcal{O}_X}(Sym^r(\mathcal{J}), \omega_\rho)$ and verify that

$$H^0(X_K, \omega^r_{\rho/\mathbf{C}}) = \mathcal{N}^r_\rho(K, \mathbf{C})$$

But we will not use this generalization in this note.

We now give a more concrete definition of the sections of $\omega^1_\rho$ à la Katz. Let $Q$ be the standard parabolic of $GL_4$ stabilizing a plane. Then we identify $H$ with the Levi subgroup of $Q$ by the map $(g_1, g_2) \mapsto \mathrm{diag}(g_1, {}^tg_2^{-1})$. Let $X_{ij}$ be formal variables with $i, j \in \{1, 2\}$ and let $R[X_{ij}]_r$ be the polynomials in $X_{ij}$ of total degree at most $r$ with coefficients in a ring $R$. We consider the representation $\rho^r_V$ of $Q$ on $V_R[X_{ij}]_r := V_R \otimes R[X_{ij}]_r$ given by

$$(\rho^r_V(g).P)(\underline{X}) := \rho_V(\mathrm{diag}(a_g, {}^td_g^{-1}).P((a_g^{-1}.\underline{X}d_g - a_g^{-1}b_g)$$

where

$$\underline{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ and } g = \begin{pmatrix} a_g & b_g \\ 0 & d_g \end{pmatrix} \in Q(R)$$

A global section $\varphi$ of $\omega^r_\rho$ can be seen as a functorial rule defined as follows. We consider quintuplet $(f : A \to Spec(R), \iota, \lambda, \alpha, \psi)$ where $(f : A \to Spec(R), \iota, \lambda, \alpha)$ is an in section 4.1.2 and $\psi$ is an isomorphism $\mathcal{H}^1_{dR}(A/Spec(R)^+ \cong R^4$ inducing $\omega^+_{A/Spec(R)} \cong R^2 \oplus \{0\} \subset R^4$. Then $\varphi$ can be seen as a functor on such quintuplets taking values in $V_R[X_{ij}]_r = V \otimes R[X_{ij}]_r$ and such that

$$\varphi(f : A \to Spec(R), \iota, \lambda, \alpha, g \circ \psi) = \rho^r_V(g)\varphi(f : A \to Spec(R), \iota, \lambda, \alpha, \psi)$$

for $g \in Q(R)$.

### 4.1.4 Polynomial $q$-expansions

We define the polynomial $q$-expansion of a nearly holomorphic form by evaluating it on a Mumford-Tate object. The general theory of the $q$-expansion of holomorphic forms is written in great details by K.W. Lan in [La08, La12]. We extend here the definition in the case of nearly holomorphic

[8] The fact that the image of $\omega_\rho$ is contained in $\omega^1_{\rho\otimes S}$ follows from Griffith transversality.

[9] Notice that $df \otimes \omega \in \Omega_{X/\mathcal{O}_{(\wp)}}(\log \partial\bar{X}) \otimes \omega_\rho = \omega_{\rho\otimes S} \subset \omega^1_{\rho\otimes S}$.

forms. For simplicity, we now assume that $K = K_f \subset \mathbf{G}(\hat{\mathbf{Z}})$. We consider the lattice $\mathcal{H} = \mathcal{H}_{K_f}$ of hermitian matrices inside $\mathcal{H}er_2(\mathcal{K}) \subset M_2(\mathcal{K})$ such that

$$\mathcal{H} := \{h \in M_2(\mathcal{K}) | \begin{pmatrix} 1_2 & h \\ 0_2 & 1_2 \end{pmatrix} \in K_f\}$$

We denote by $\mathcal{H}^\vee$ the dual lattice for the pairing on $\mathcal{H}er_2(\mathcal{K})$ defined by $(h, h') = \mathrm{tr}(hh') \in \mathbf{Q}$. We will denote $\mathcal{H}^\vee_{\geq 0}$ the submonoid of $\mathcal{H}^\vee$ of positive hermitian matrices. For any ring $A$ and a monoid $M$ with a neutral element 0, we denote by $A[[q^M]]$ the formal power series ring with coefficient in $A$ over the monoid $M$. We denote element $q^h$ of $H$ inside $A[[q^M]]$ multiplicatively as $q$ and $q^0$ is just denoted 1. We will consider the case $A$ is a $\mathcal{O}_\mathcal{K}$-algebra and $M = \mathcal{H}^\vee_{\geq 0}$.

We consider the decomposition of $\mathcal{K}^4 = W \oplus W'$ where $W$ (respectively $W'$) is the standard totally isotropic subspace of vectors whose last (respectively first) two coordinate entries with respect to the standard basis of $\mathcal{K}^4$ are zero. Let $L$ and $L'$ be respectively the free $\mathcal{O}_\mathcal{K}$-lattices of $W$ and $W'$ such that $L \oplus L' = \mathcal{O}_\mathcal{K}^4$. For any $h \in \mathcal{H}$, $h$ induces a canonical $\mathcal{O}_\mathcal{K}$-linear map $h : L \to L'$. Let $\underline{q}$ the map

$$\underline{q} : L \to L' \otimes \mathbf{G}_{m/\mathcal{O}_{(\wp)}[[q^{\mathcal{H}_{\geq 0}}]]}$$

defined by the composition $L \to Hom(H, L') = H^\vee \otimes L' \to L' \otimes \mathbf{G}_{m_{/\mathcal{O}_{(\wp)}[[q^{\mathcal{H}^\vee_{\geq 0}}]]}}$ where the first map is the obvious map and the last one is defined by $h \otimes l' \mapsto l' \otimes q^h$. By the work of Mumford, there exists a abelian variety $Mum(q)$ over $\mathcal{O}_{(\wp)}((q^\mathcal{H}))$ endowed with a canonical complex multiplication noted $\iota_{can}$ by $\mathcal{O}_\mathcal{K}$ which can be described as the quotient $L' \otimes \mathbf{G}_m / \underline{q}(L)$. In particular, the formal completion along the origin gives a canonical isomorphism:

$$\widehat{Mum(q)} \cong L' \otimes \widehat{\mathbf{G}_m}_{/\mathcal{O}_{(\wp)}((q^\mathcal{H}))}$$

which induces a canonical isomorphism

$$\omega_{Mum(q)/\mathcal{O}_{(\wp)}((q^\mathcal{H}))} \cong L \otimes_\mathbf{Z} \mathcal{O}_{(\wp)}((q^\mathcal{H}))$$

From the decomposition $\mathcal{O}_{(\wp)} \otimes \mathcal{O}_{(\wp)} \cong \mathcal{O}_{(\wp)} \oplus \mathcal{O}_{(\wp)}$ given by $z \otimes a \mapsto (za, \bar{z}a)$, we deduce an isomorphism

$$\omega^+_{Mum(q)/\mathcal{O}_{(\wp)}((q^\mathcal{H}))} \cong L \otimes_{\mathcal{O}_{(\wp)}} \mathcal{O}_{(\wp)}((q^\mathcal{H}))$$

We define $\omega^+_{can} = (\omega_{1,can}, \omega_{2,can})$ the basis of the left hand side of this isomorphism induced by the canonical basis of $L$. We complete it into a basis of $\mathcal{H}^1_{dR}(Mum(q)/\mathcal{O}_{(\wp)}((q^\mathcal{H})))^+ \cong L \otimes_{\mathbf{Z}_p} \mathcal{O}_{(\wp)}((q^\mathcal{H}))$ using the Gauss-Manin connection. Let $D_{ij}$ be the derivation of $\mathcal{O}_{(\wp)}((q^\mathcal{H}))$ such that $D_{ij}(q^h) = h_{ij} q^h$. Then we define[10].

$$\delta_{i,can} = \nabla(D_{ii})(\omega_{i,can}).$$

Then $(\omega_{1,can}, \omega_{2,can}, \delta_{1,can}, \delta_{2,can})$ is a basis of $\mathcal{H}^1_{dR}(Mum(q)/\mathcal{O}_{(\wp)}((q^\mathcal{H}))^+$ which defines an isomorphim

$$\psi_{can} : \mathcal{H}^1_{dR}(Mum(q)/\mathcal{O}_{(\wp)}((q^\mathcal{H}))^+ \cong \mathcal{O}_{(\wp)}((q^\mathcal{H}))^4$$

[10] It can be checked easily using the complex uniformization that it defines an horizontal section.

as in the end of the previous section. Finally, it is not difficult to define a canonical polarization $\lambda_{can}$ and a canonical level structure $\alpha_{can}$. We are now ready to define the polynomial $q$-expansion[11] of a global section of $\omega^1_\rho$. For any $\mathcal{O}_{(\wp)}$-algebra $R$, we consider the map

$$H^0(X_K, \omega^1_{\rho/R}) \to R((q^{\mathcal{H}})) \otimes_R V_R[X_{ij}]_1$$

defined by

$$f \mapsto f(q, X_{ij}) := f(Mum(q)/\mathcal{O}_{(\wp)}((q^{\mathcal{H}})), \lambda_{can}, \iota_{can}, \alpha_{can}, \psi_{can})$$

Moreover, it can be shown as usual that

$$f(q, X_{ij}) \in V_R[[q^{\mathcal{H}^\vee_{\geq 0}}]][X_{ij}]_1$$

On can see the action of $\epsilon_\rho$ on the polynomial $q$-expansion is given by the following formula:

$$(\epsilon_\rho f)(q, X_{ij}) = \sum_{ij} E^*_{ij} \otimes \frac{\partial}{\partial X_{ij}} f(q, X_{ij})$$

## 4.2 $p$-adic unitary automorphic forms

### 4.2.1 $p$-adic unitary automorphic forms

Let $X_{\rm rig}$ be the rigid space obtained as the generic fiber of the formal scheme obtained by taking the formal completion of $\mathcal{X}_K$ along its special fiber at $p$. We consider $X_{\rm ord} \subset X_{\rm rig}$ the ordinary locus (i.e. the open rigid analytic subvariety of points $(f : A \to \text{Spec } \overline{\mathbf{Q}}_p, \iota, \lambda, \alpha)$ for abelian varieties $A$ having good ordinary reduction). The space of $p$-adic forms of weight $\rho$ is defined as

$$M^{p-adic}_\rho(K, \mathbf{Q}_p) := H^0(X_{\rm ord}, \omega_{\rho/\mathbf{Q}_p})$$

We defined the spaces of overconvergent and nearly overconvergent forms of degree at most 1 by:

$$\mathcal{N}^{r,\dagger}_\rho(K, \mathbf{Q}_p) := \lim_{V \supset X_{\rm ord}} H^0(V, \omega^r_{\rho/\mathbf{Q}_p})$$

Here the $V$'s in the injective limit run in the set of strict neighborhood of $X_{\rm ord}$ inside $X_{\rm rig}$. Recall that Dwork defines a canonical splitting $\mathcal{H}_{dR}(\mathcal{A}/X_{\rm ord}) \cong \omega \oplus \mathcal{U}$ where $\mathcal{U}$ is the unit root crystal of $\mathcal{H}_{dR}(\mathcal{A}/X_{\rm ord})$. For any representation $\rho$ of $H$, it induces a splitting

$$\omega^r_{\rho/X_{\rm ord}} \stackrel{split_p}{\longrightarrow} \omega_{\rho/X_{\rm ord}} \to 0.$$

The following proposition follows from the fact that the canonical Dwork splitting does not extend to any strict neighborhood of $X_{\rm ord}$. We will write a proof in a subsequent paper (See [Ur12] in the case $GL(2)$). It shows nearly overconvergent forms or just nearly holomorphic forms can be seen as special $p$-adic forms. It will not be used in this paper.

[11] In fact, we can (and need to) define a polynomial expansion for each connected component of $X_K$ to be able to formulate a polynomial $q$-espansion principle.

**Proposition 4.3** *For any strict neighborhood $V$ of $X_{\text{ord}}$, the compositum of the two following maps*

$$H^0(V, \omega^r_{\rho/\mathbf{Q}_p}) \to H^0(X_{\text{ord}}, \omega^r_{\rho/\mathbf{Q}_p}) \overset{split_p}{\longrightarrow} H^0(X_{\text{ord}}, \omega_{\rho/\mathbf{Q}_p})$$

*induces a canonical injection*

$$\mathcal{N}^{r,\dagger}_\rho(K, \mathbf{Q}_p) \hookrightarrow M^{p-adic}_\rho(K, \mathbf{Q}_p).$$

Let $I \subset \mathbf{G}(\mathbf{Z}_p)$ be the Iwahori subgroup associated to the standard Borel subgroup of $GL_4$. We can define the unitary Shimura variety of Iwahori level above $X$ we denote by $X^I_{\text{rig}}$ the corresponding rigid analytic space and $X^I_{\text{ord}}$ its ordinary part. We can define similarly the space of $p$-adic , overconvergent and nearly overconvergent forms of Iwahori level in a similar way as in [PS11]. We denote them respectively $\mathcal{M}^{p-adic}_\rho(K^pI, \mathbf{Q}_p)$, $\mathcal{M}^\dagger_\rho(K^pI, \mathbf{Q}_p)$ and $\mathcal{N}^{r,\dagger}_\rho(K^pI, \mathbf{Q}_p)$. The proposition above extends to these spaces without difficulty.

#### 4.2.2 Families of finite slope nearly overconvergent forms

*Weights.* For any decreasing quadruplet of integers $\underline{k} := (k_1, k_2; k_3, k_4)$, we write $\omega_{\underline{k}}$ and $\omega^1_{\underline{k}}$ for the sheaves attached to the representation $\rho_{\underline{k}}$ of $H$ given by $V^+_{k_1,k_2} \otimes V^-_{-k_4,-k_3}$ where we denote by $V^+_{a,b}$ (resp. $V^-_{a,b}$) the representation of the first (resp. second) copy of $GL_2$ in $H$ of highest weight $(a, b)$ for any pair of integers $(a, b)$ with $a \geq b$. A unitary automorphic form of one of the types we have defined before will be said of weight $\underline{k}$ if the corresponding representation of $H$ is $\rho_{\underline{k}}$. It is well-known that an holomorphic eigenform of weight $\underline{k}$ is attached to an automorphic representation which archimedean component is a discrete series when $k_2 - 2 \geq k_3 + 2$. It will be convenient to consider the coordinate function functor on $V_{\underline{k}/\mathbf{Z}}$ along the highest weight vector. We denote this function by

$$\pi_{\underline{k}} : \; V_{\underline{k}}(R) \to R.$$

it satisfies $\pi_{\underline{k}}(tn.v) = t_1^{k_1} t_2^{k_2} t_3^{-k_4} t_4^{-k_3} \pi_{\underline{k}}(v)$ for any diagonal $t = \text{diag}(t_1, t_2, t_3, t_4) \in H$ and $n$ in the standard unipotent subgroup of $H$. It is defined up to sign.

*Weight space.* Let $\mathfrak{X}$ be the rigid analytic space such that $\mathfrak{X}(L) := Hom_{cont}((\mathbf{Z}_p^\times)^4, L^\times)$. The points of $\mathfrak{X}(\overline{\mathbf{Q}}_p)$ are called $p$-adic weights. If $\underline{k} = (k_1, k_2, k_3, k_4) \in \mathbf{Z}^4$, we write $[\underline{k}]$ the point of $\mathfrak{X}(\mathbf{Q}_p)$ corresponding to the continuous character $(x_1, x_2, x_3, x_4) \mapsto \prod_{i=1}^4 x_i^{k_i}$. Those points are called algebraic weights if $k_1 \geq k_2 \geq k_3 \geq k_4$. We denote by $\mathfrak{X}(\mathbf{Q}_p)^{alg}$ the subset of algebraic weights of $\mathfrak{X}$.

*Slopes.* For each $t = \text{diag}(t_1, t_2, t_3, t_4) \in T(\mathbf{Q}_p)$ such that

$$v_p(t_1) \leq v_p(t_2) \leq v_p(t_3) \leq v_p(t_4), \tag{5}$$

we consider the Hecke operators $u_t$ attached to the double class $ItI$ and acting on the various spaces automorphic forms of level $K^p.I$ we have defined. It is important to remember that there are two way to normalize the action of these operators on the spaces $\mathcal{N}^r_{\underline{k}}$. We call them the algebraic and $p$-adic normalizations. The difference between the two normalization is given by the following formula:

$$(u_t)^{p-adic} = |\lambda_{\underline{k}}(t)|_p (u_t)^{alg} \tag{6}$$

where $\lambda_{\underline{k}}$ is the algebraic weight[12] $(k_1 - 2, k_2 - 2, k_3 + 2, k_4 + 2)$. Here the algebraic normalization is defined as usual by

$$(f|u_t)(g_f, z) := \sum_i f(g_f \xi_i^{-1}, z)$$

where $I\xi_i$ are the left coset representatives such that $ItI = \sqcup_i I\xi_i$. The $p$-adic normalization extends to an action on the space of overconvergent and $p$-adic forms and this action preserves the integrality of those. We refer to [Hi04] for this fact and to [PS11] for the definition of the action on the overconvergent forms using the theory of the canonical subgroup. This definition extends easily to nearly overconvergent forms. Again we don't really need this fact here but it is an important feature of the general theory which is good to keep in mind.

Let $\mathcal{U}_p$ the Hecke algebra generated by the $u_t$'s for $t$ satisfying (5). It is isomorphic to a polynomial algebra in 4 variables. More precisely if $\theta$ is a character of $\mathcal{U}_p$, one can find a quadruple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \overline{\mathbf{Q}}_p^4$ such that

$$\theta(t) = \prod_{i=1}^{4} \alpha_{5-i}^{v_p(t_i)}$$

We say that $\theta$ is of finite slope if the $\alpha_i$'s are all non zero and we define the slope of $\theta$ as the quadruplet $\underline{s} = (s_1, s_2, s_3, s_4)$ with $s_i = v_p(\alpha_i)$ for $i = 1, 2, 3, 4$. A $p$-adic form is said of finite slope if it belongs to the sum of generalized $\mathcal{U}_p$-eigenspaces of finite slope characters.

**Remark 4.4** If $\theta$ is a character of $\mathcal{U}_p$ acting on the $I$-invariants of an unramified principal series, then the corresponding Hecke polynomial is

$$\prod_{ii=1}^{4} (X - p^{i-1}\alpha_i)$$

**Definition 4.5** Let $\mathfrak{V}$ be an affinoid over a finite extension of $\mathbf{Q}_p$ and $w : \mathfrak{V} \to \mathfrak{X}$ be a finite morphism. We assume that the subset $\Sigma_{\mathfrak{V}} = \mathfrak{V}(\overline{\mathbf{Q}}_p) \cap w^{-1}(\mathfrak{X}^{alg}(\mathbf{Q}_p))$ is Zariski dense. A $\mathfrak{V}$-family[13] of nearly overconvergent forms $F$ of degree at most $r$ is a polynomial $q$-expansion[14] of degree at most $r$ with coefficient in $A(\mathfrak{V})$ (i.e. $F \in A(\mathfrak{V})[X_{ij}]_r[[q^{\mathcal{H}^\vee_{\geq 0}}]]$) such that there exists a Zariski dense set $\Sigma_F \subset \Sigma_{\mathfrak{V}}$ satisfying:

$\forall x \in \Sigma_F$, there exist a *finite slope* nearly holomorphic form $F_x$ of weight $\underline{k}_x$ with $w(x) = [\underline{k}_x]$ and level $K^p.I$ such that $\iota_p(\pi_{\underline{k_x}}(F_x(q, X_{ij}))) \in \overline{\mathbf{Q}}_p[X_{ij}][[q^{\mathcal{H}^\vee_{\geq 0}}]]$ is equal to the evaluation of the formal polynomial $q$-expansion $F$ at the point $x$.

[12] This character is dominant exactly when the weight correspond to holomorphic discrete series and is the cohomological weight of this holomorphic discrete series.

[13] It is possible to give a definition using a theory similar to [AIP] for unitary groups. Instead we use a shortcut here which is sufficient for our application.

[14] In fact, the correct definition is to take a formal $q$-expansion for each connected component of $X_K$. We will not do it for the sake of the notations.

### 4.3 Eisenstein series

#### 4.3.1 Klingen-type Eisenstein series

We recall some results of [SU06b] on certain Eisenstein series for $G$. Let $P$ be the stabilizer in $G$ of the line $\{(0,*,0,0)\in\mathcal{K}^4\ :*\in\mathcal{K}\}$. Then $P$ is a standard, maximal $\mathbf{Q}$-parabolic subgroup of $G$ with standard Levi subgroup $L$ isomorphic to $U(1,1)\times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$. A pair $(g,t)\in U(1,1)\times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$ is identified with

$$m(g,t)=\begin{pmatrix} a & 0 & b & 0\\ 0 & \bar{t}^{-1} & 0 & 0\\ c & 0 & d & 0\\ 0 & 0 & 0 & t\end{pmatrix}\in G.$$

with $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in U(1,1)$ and $t\in\mathbf{G}_{m/\mathcal{K}}$. We write $N$ for the unipotent radical of $P$.The modulus function giving the determinant of the action of $L$ on the Lie $N$ is given by

$$\delta(m(g,t))=|t|_{\mathbf{A}_{\mathcal{K}}}^{-3}$$

Let $f$ be an elliptic cusp form of weight $k=2m>2$ for $\Gamma_0(N)$. We denote by $\phi_f$ be the automorphic form on $U(1,1)$ such that:

$$\varphi_f(\gamma g_\infty kz)=(c_\infty i+d_\infty)^{-m}(\bar{c}_\infty i+\bar{d}_\infty)^{-m}f(g_\infty.i)$$

for $\gamma\in U(1,1)(\mathbf{Q})$, $g_\infty\in U(1,1)(\mathbf{R})$, $z$ in the center of $U(1,1)(\mathbf{A})$ and $k\in U(1,1)(\hat{\mathbf{Z}})$ such that $c_k$ is divisible by $N$. We denote by $(\pi,V_f)$ the irreducible cuspidal representation of $U(1,1)$ generated by $\varphi_f$. For any $s\in\mathbf{C}$, we then consider the induced representation $I(s)$ of smooth functions $\phi_s:G(\mathbf{A})\to V_f$ satisfying

$$\phi_s(h.m(g,t).n)=\delta(m(g,t))^{1/2}|t|_{\mathbf{A}}^{-s}\pi(g).\phi_s(h)$$

for all $h\in G(\mathbf{A}),g\in U(1,1)(\mathbf{A})$ and $t\in\mathbf{A}_{\mathcal{K}}^\times$. Let us decompose $\pi$ into the restricted tensor product of its local component $\pi_f=\bigotimes_v'\pi_v$. For each place $v$, we consider the local induction $I_v(s)=Ind_{P(\mathbf{Q}_c)}^{G(\mathbf{Q}_v)}\pi_v\otimes\delta^{s/3}$. Then we have

$$I(s)=\bigotimes_v{}' I_v(s)$$

For each finite place $v$, we denote $L(\pi_v)$ the Langlands quotient of $I_v(\pi_v):=I_v(s_0)$ for $s_0=1/2$.

Let $w$ be a finite place of $\mathcal{K}$ above $v$. Let us write $W_{K_w}$ for the Weil group of $\mathcal{K}_w$ and denote $rec_w$ the isomorphism of the local class field theory $W_{\mathcal{K}_w}^{ab}\overset{\sim}{\to}\mathcal{K}_w^\times$ sending arithmetic Frobenius onto uniformizor and by $N_w=|\cdot|_w\circ rec_w$. By the local Langlands correspondence, the parameter of the base change $\Pi_w$ of $L(\pi_v)$ to $GL_4(\mathcal{K}_w)$ with $w$ a finite place of $\mathcal{K}$ above $v$ is given by:

$$rec(\Pi_w)(x)=\begin{pmatrix} N_w(x)^{1/2} & & \\ & rec(\pi_w)(x) & \\ & & N_w(x)^{-1/2}\end{pmatrix}\quad\forall x\in W_{\mathcal{K}_w}\tag{7}$$

where $\pi_w$ stands for the base change of $\pi_v$ to $GL_2(K_w)$ and $rec(\pi_w)$ is the image of $\pi_w$ by the local Langlands correspondence for $GL_2(\mathcal{K}_w)$. In particular, the local $L$-function of $\Pi_w$ is given by:

$$L(\Pi_w,s)=(1-q_w^{s+1/2})^{-1}L(\pi_w,s)(1-q_w^{s-1/2})^{-1}$$

Let $\Phi \in I(\pi)$ such that $\Phi(1) \in \Pi \otimes \mathbf{C}(m,-m)]^{U(1)\times U(1)}$. Here we denote $\mathbf{C}(a,b)$ is the one dimensional representaion of $U(1) \times U(1)$ defined by the character $(u_1, u_2) \mapsto u_1^a u_2^{-b}$. Then, we consider the form $f_\Phi$ on $\mathfrak{H}$ defined by:

$$f_\Phi(z) = (ci+d)^m(\bar{c}i+\bar{d})^m\Phi(1)(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$$

with $z = \frac{ai+b}{ci+d}$ belongs to the Poincaré upper half plane $\mathfrak{H}$. It is an holomorphic form of weight $k$. Let $N$ an integer divisible by the conductor of $\pi$ and $\phi \in \prod_{v|N} I(\pi(f)_v)$, we consider

$$\Phi = \phi \otimes (\bigotimes_{\substack{v \text{ such that} \\ v(N)=0}} \phi_v^0) \otimes \phi_\infty$$

where $\phi_v^0$ is a canonical spherical section of $I(\pi_v)$ for $v \not| N$ and $\phi_\infty(1)$ is a basis of $[\pi_\infty \otimes \mathbf{C}(m,-m)]^{U(1)\times U(1)}$. We now define the Eisenstein series attached to $\phi$. For $z \in \mathcal{D}$, $g_f \in G(\mathbf{A}_f)$ and $s \in \mathbf{C}$ such that $Re(s)$ is sufficiently large, we put

$$E(\phi,s)(g_f,z) := \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \rho_{\kappa_m}(j(\gamma,z))|det(c_\gamma z + d_\gamma)|^{-s}.f_{\gamma_f g_f.\Phi}([\gamma.z])v_{\kappa_m}$$

where the map $[\cdot] : \mathcal{D} \to \mathfrak{H}$ is defined by $[z] = z_{11}$ for $z = \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right) \in \mathcal{D}$, the weight $\kappa_m$ is defined by

$$\kappa_m := (m, 2; -2, -m),$$

$v_{\kappa_m}$ is an highest weight vector of $\rho_{\kappa_m}$, $\gamma_f$ is the image of $\gamma$ in $G(\mathbf{A}_f)$ and $\gamma_f g_f.\Phi$ is defined by the action of $G(\mathbf{A}_f)$ on $I(\pi)$.

**Proposition 4.6** *Let $f$ be an eigenform of weight $k = 2m > 2$ for $\Gamma_0(N)$ and $\phi \in \prod_{v|N} I(\pi(f)_v)$, then $E_{\kappa_m}(\phi,s)$ has no pole at $s = 0$ and its evaluation at $s = 0$ is a nearly holomorphic form $E_{\kappa_m}(f,\phi)$ of weight $\kappa_m$ of order at most 1. It is an holomorphic form if $L(f,m) = 0$. This latter condition is necessary if $\phi$ projects non trivially in the Langlands quotient $\otimes_{v|N} L(\pi_v)$. Moreover, after an adequate normalization, $E_{\kappa_m}(f,\phi)$ is defined over $\overline{\mathbf{Q}}$.*

**Proof** This was proved in [SU06b] except the fact that it is nearly holomorphic in general. However this point follows easily from the computation of the constant term realized in loc. cit. This will be explained in greater details in [SU13]. The algebraicity follows from a general result due to M. Harris [Ha81]. His argument is written only for holomorphic Eisenstein series but it extends easily to nearly holomorphic ones.

□

**Remark 4.7** Let $E_{\kappa_m}(f)$ be the image of the $G(\mathbf{A}_f)$-representation generated by the $E_{\kappa_m}(f,\phi)$'s in the product of the local Langlands quotients at the finite places. It is an irreducible representation and we have

$$L(BC(E_{\kappa_m}(f)), s) = L(f, s+m-1/2)\zeta_{\mathcal{K}}(s+1/2)\zeta_{\mathcal{K}}(s-1/2)$$

with $\zeta_{\mathcal{K}}$ the Dedekind zeta function of $\mathcal{K}$. According to the conventions of [SU06b, §4], its associated Galois representation is

$$\rho_{E_{\kappa_m}(f)} = L(-1) \oplus L(-2) \oplus V_f(m-2)$$

Theorem 3.1 establishes the existence of a deformation of $\rho_{E_{\kappa_m}(f)}(2)$ when $L(f,m)=0$. It will follow from the existence of a p-adic family of cusp forms degenerating into a $p$-adic version of $E_{\kappa_m}(f)$. The construction is outlined in the next paragraphs. Notice that when $k=2$, the previous proposition does not hold and we will need to replace $E_{\kappa_m}(f)$ by a finite slope p-adic automorphic representation having the Galois representation $W_f(-2)$.

#### 4.3.2 Families of Eisenstein series

Let $\mathfrak{X}_1$ be the rigid rigid variety over $\mathbf{Q}_p$ such that $\mathfrak{X}_1(\overline{\mathbf{Q}}_p)=Hom_{cont}(\mathbf{Z}_p^\times,\overline{\mathbf{Q}}_p^\times)$. Let $\mathfrak{U}$ be an affinoid, $k:\ \mathfrak{U}\to\mathfrak{X}_1$ be a finite morphism and

$$F=\sum_{n=1}^{\infty}a(n,F)q^n\in A(\mathfrak{U})[[q]]$$

be a Coleman family[15] of normalized new cuspidal eigenforms for $\Gamma_0(N)$ of slope $s_0\in\mathbf{Q}_{\geq 0}$. This means that for each $x\in\mathfrak{V}(\mathbf{Q}_p)$ such that $k(x)=[k_x]$ with $k_x\in\mathbf{Z}_{\geq 2}$ and $k_x>s_0+1$, $F_x=\sum_{n=1}^{\infty}a(n,F)(x)q^n=\iota_p(f_x(q))$ where $f_x(q)$ is the $q$-expansion of a normalized $N$-new eigenform of even weight $k_x=2m_x$ and level $\Gamma_0(Np)$.

**Theorem 4.8 ([SU13])** *We keep the hypothesis and notations as above. Let $\kappa=i\circ k$ with $i$ the closed immersion of $\mathfrak{X}_1$ into $\mathfrak{X}$ given by $i(\xi)=(\xi^{1/2},[2],[-2],\xi^{-1/2})$. Then there exists a $\mathfrak{U}$-family of nearly holomorphic automorphic forms $E(F)$ such that for all $x\in\mathfrak{V}(\mathbf{Q}_p)$ such that $k(x)=[k_x]$ and $f_x$ is of trivial nebentypus and weight $k_x=2m_x\in\mathbf{Z}$, we have:*

*(i) If $k_x\in\mathbf{Z}_{\geq 4}$ and $k_x>s_0+1$, then $E(F)_x$ is the polynomial $q$-expansion of a nearly holomorphic Klingen-Eisenstein series $E_{\kappa_{m_x}}(f_x,\phi_x)$ of weight $\kappa_x=\kappa_{m_x}$ for some section $\phi_x\in\otimes_{v|Np}I_v(\pi(f_x)_v)$ projecting non trivially on the Langlands quotient $\otimes_{v|Np}L(\pi(f_x)_v)$.*

*(ii) $E(F)_x$ is an eigenform of finite slope for a character of $\mathcal{U}_p$ with normalized eigenvalues given by the quadruplet $(a(p,F)(x),p,p^{-1},a(p,F)(x)^{-1}$. In particular, its slope is $\underline{s}_0=(s_0,1,-1,-s_0)$.*

*(iii) If $k_x\geq 2$, $\epsilon(E(F)_x)=0$ when $L(f_x,k_x/2)=0$.*

*(iv) If $k_x\geq 2$, then $E(F)_x$ is non trivial.*

The general construction of this family is done using the doubling method and the use of differential operators on the Siegel Eisenstein series on $U(3,3)$. It uses an explicit description of certain harmonic polynomials and the effect of the Maass-Shimura differential operators on the polynomial $q$-expansion. The fact that we can make an analytic interpolation is easy from these facts by using the standard technic of the $p$-adic Petersson inner product due to Hida.

The point (ii) is done by making a good choice of families of sections at $p$, that is, Iwahori invariant sections which are proper for the Hecke operators $u_t$ with eigenvalues given by the same quadruplet after renormalization. Notice that the slope of the Eisenstein series $E(F)_x$ is critical

[15] This corresponds to an affinoid of the Eigencurve of Coleman-Mazur on which the slope is constant.

with respect to the weight $\lambda_{\kappa_x} = (m_x - 2, 0, 0, 2 - m_x)$ in the sense of [Ur11]. Therefore any larger deformation of this family will no longer be Eisenstein. The point (iii) follows from Proposition 4.6 when $k_x > 2$. For the case $k_x = 2$, it follows by showing that $\epsilon(E(F)_x)$ is divisible by the $p$-adic L-function interpolating the central values $L(f_x, m_x)$. This fact is another consequence of the doubling method. The point (iv) is easy when $k_x > 2$, it follows from the computation of the Fourier coefficients of the Eisenstein series. When $k_x = 2$, one needs[16] to show that some $p$-sdic limit is non trivial and we use crucially that $f$ is in the image of the Jacquet-Langlands correspondence for a definite quaternion algebra.

When $k_x = 2$, $E(F)_x$ is the $q$-expansion of a $p$-adic form as it is obtained as a limit of $p$-adic (since nearly holomorphic) forms. Its weight is $(1, 2; -2, -1)$, so it is not arithmetic since not dominant. Therefore $E(F)_x$ is even not nearly holomorphic in general. However it can be proved it is nearly overconvergent for a suitable $p$-adic sheaf as those constructed in [AIP]. In the next section, we show that when $L(f, 1) = 0$, it defines a point of the Eigenvariety so it can be considered as overconvergent in that sense. To do this we will show that it is a $p$-adic limit of holomorphic forms of arbitrary regular weight and fixed slope $\underline{s}_0$.

**Remark 4.9** The corresponding family of Galois representations $\rho_{E_{\kappa_{m_x}}(F_x)}$ is a finite slope deformation in the sense of the section 3. In that case the function $\underline{\varphi}$ is given by

$$\underline{\varphi}(x) = (a(p, F)(x), p, p^{-1}, a(p, F)(x)^{-1}).$$

However, it does not satisfy the property (d) of Theorem 3.1 and this is why this family is reducible.

**Remark 4.10** In fact there are two ways to do the construction of this family of nearly holomorphic Eisenstein series. The first one is by making the construction directly from the pull-back to obtain a nearly holomorphic form of weight $\kappa_m$. The second one is by constructing a family of Eisenstein series of weights $(a, b; -b, -a)$ with slope $(s_0, 0, 0, -s_0)$ and apply one time a differential operator $\delta^*_{(a,b;-b,-a)}$ to obtain a family of slope $(s_0, 1, -1, -s_0)$ and weights $(a, b+1; -b-1, -a)$ and then evaluate at $a = m$ and $b = 1$. Here $\delta^*_{\underline{k}}$ is the differential operator for $\underline{k} = (k_1, k_2, k_3, k_4)$ obtained as the composition of the generalized Maasss-Shimura operator $\delta_{V_{\underline{k}}}$ and the projection $H^0(X_K, \omega^1_{V_{\underline{k}} \otimes S}) \to H^0(X_K, \omega^1_{k_1, k_2+1; k_3-1, k_4})$ coming from the decomposition when $\underline{k}$ is regular:

$$V_{\underline{k}} \otimes S \cong V_{k_1+1, k_2; k_3-1, k_4} \oplus V_{k_1, k_2+1; k_3-1, k_4} \oplus V_{k_1+1, k_2; k_3, k_4-1} \oplus V_{k_1, k_2+1; k_3, k_4-1}$$

For $GL_2$, the similar construction gives the critical Eisenstein series $E_2^{crit}$ from the ordinary family of Eisenstein series $E_{k-2}^{ord}$ by applying one time the Maass-Shimura operator $\delta_{k-2}$ and then evaluate the result at $k = 2$ (see [Ur12]).

### 4.3.3 Cuspidal deformation of critical Eisenstein series

We sketch the construction of a generically cuspidal deformation of the $p$-adic Eisenstein series[17] $E_{\kappa_m}(\phi, f)$. More precisely, we show that the Eisenstein $E_{\kappa_m}(\phi, f)$ when $f$ is as in Theorem 3.1

[16] This fact is more delicate but much less difficult than proving that the ordinary Eisenstein series appearing in our previous work [SU10] is non zero modulo $p$.

[17] This Eisenstein series is purely $p$-adic only when $m = 1$.

can be seen as a $p$-adic limit of finite slope holomorphic forms of very regular weights. From the theory of the Eigenvariety as developed in [Ur11], this implies our Theorem 3.1 and therefore would conclude the first part of the main theorem stated in these notes. The details of this argument will appear in [SU13].

A common feature to construct $p$-adic families of modular forms is to use high $p$-powers of a lifting of the Hasse invariant. This is insufficient to construct families of very regular weight in general but combining this with our previous family of Eisenstein series it will be sufficient for our goal. Before sketching our argument, we therefore start by explaining how this technic can be extended in the context of nearly holomorphic forms. Let $A$ be a lifting of some power of the Hasse invariant. It is an holomorphic form of weight $k_0 \in (p-1)\mathbf{Z}_{>0}$. For any integer $s > 0$, we write

$$B_s := \frac{1}{4k_0 s}\delta_{sk_0}(A^s)$$

By lemma 4.1, $B_s$ is a global section of $det(\omega^+)^{\otimes^{k_0 s}} \otimes \mathcal{J} \subset \omega^1_{\rho\otimes S}$ for $\rho = det(St_+)^{\otimes^{k_0 s}}$ , because its image $\epsilon(B_s)$ by $\epsilon$ in $H^0(X_K, det(\omega^+)^{\otimes^{k_0 s}} \otimes \omega_{S\otimes S^\vee})$ is $A^s$. In other words, we have:

$$\epsilon(B_s) = A^s \in H^0(X_K, det(\omega^+)^{\otimes^{k_0 s}}) \subset H^0(X_K, det(\omega^+)^{\otimes^{k_0 s}} \otimes \omega_{S\otimes S^\vee})$$

By computing the polynomial $q$-expansion of $B_s$, it is also very easy to verify that $B_s$ is a $p$-adic analytic family in the variable $s$.

We now return to our goal. We treat the general case $k \geq 2$ but the main reason we have to work with nearly holomorphic forms is to treat the special case $k = 2$. Let $f$ be as in Theorem 3.1 and let us choose a $p$-stabilization $f_\alpha$ such that its $U_p$-eigenvalue $\alpha$ satisfies $s_0 := v_p(\alpha) < k-1$. Let $F$ be a Coleman family as in Theorem 4.8 passing through $f_\alpha$ at a point $x_0 \in \mathfrak{U}(\mathbf{Q}_p)$. For a point $x$ in $\mathfrak{U}$ such that $k_x \in \mathbf{Z}_{>2}$ with $[k_x] = k(x)$, we consider

$$G'_{x,s} := A^s E(F)_x - B_s \epsilon(E(F)_x)$$

Since $E(F)_x \in H^0(X_K, \omega_{\kappa_{m_x}} \otimes \mathcal{J}^\vee)$, this is a well defined section of $\omega_{\kappa_{k_x}} \otimes \omega_{S^\vee} \otimes \mathcal{J} \otimes det(\omega^+)^{\otimes^{k_0 s}}$ because $\mathcal{J}^\vee \subset \omega_{S^\vee} \otimes \mathcal{J}$. Moreover since $\epsilon(E(F)_x)$ is an holomorphic[18] form, we have

$$\epsilon(G'_{s,x}) = A^s \epsilon(E(F)_x) - \epsilon(B_s)\epsilon(E(F)_x) = 0$$

Therefore $G'_{x,s}$ is a global section of $\omega_{S^\vee} \otimes \omega_S \otimes det(\omega^+)^{\otimes^{k_0 s}} \otimes \omega_{\kappa_{m_x}}$. We define $G_{x,s}$ as the projection of $G'_{x,s}$ onto $H^0(X, det(\omega^+)^{\otimes^{k_0 s}} \otimes \omega_{\kappa_{m_x}})$. Therefore $G_{s,x}$ is an holomorphc form of weight $\kappa_{x,s} = (sk_0 + m_x, sk_0 + 2; -2, -m_x)$. When $s >> 0$ and $x$ varies, we can make the weight of $G_{s,x}$ arbitrary regular. Since $L(f, m) = 0$, by the point (iii) of Theorem 4.8, we have

$$G'_{x_0,0} = E(F)_{x_0} \tag{8}$$

This implies that $G_{x_0,0} = E(F)_{x_0}$ and in particular that $G_{x,s}$ is non-trivial thanks to the point (iv) of Theorem 4.8. We now make use of the construction of the Eigenvariety in [Ur11] to prove the existence of a deformation that will lead us to the proof of Theorem 3.1. Let $\kappa_0 = (m, 2; -2, -m) \in$

[18] This is because $E(F)_x$ is nearly holomorphic of degree $\leq 1$.

$\mathfrak{X}(\overline{\mathbf{Q}}_p)$ and $\underline{s} = (s_0, 1, -1, -s_0) \in \mathbf{Q}^4$. For each $x$ as in Theorem 4.8, $E(F)_x$ is of slope $\underline{s}$. From the results in [Ur11], one can deduce there exists a neighborhood $\mathfrak{V}$ of $\kappa_0$ and a polynomial $Q(\kappa, X) \in A(\mathfrak{V})[X]$ such that $Q(\kappa, u_0)$ projects the space of nearly holomorphic forms of weight $\kappa$, order $\leq 1$ and level $K^p.I$ onto its subspace of slope $\underline{s}_0$ for any sufficiently regular weight inside $\mathfrak{V}$. Here $u_0$ is the Hecke operator $u_{diag(1,p,p^2,p^3)}$. In particular, that implies that

$$Q(\kappa_{m_x}, u_0).E(F)_x = E(F)_x \tag{9}$$

when $k_x = 2m_x > 2$ and therefore for all $x \in \mathfrak{U}(\overline{\mathbf{Q}}_p) \cap i^{-1}(\mathfrak{V}(\overline{\mathbf{Q}}_p))$ by analytic continuation and in particular for $x = x_0$. Now, we consider the family of forms given by:

$$K_{x,s} := Q(\kappa_{x,s}, u_0).G_{x,s}$$

for $(x, s)$ such that $\kappa_{x,s} \in \mathfrak{V}$. Then $K_{x,s}$ is a well defined holomorphic form for all but finitely many pairs $(x, s)$ for which $\kappa_{x,s}$ is algebraic dominant. Moreover by (8) and (9), we have

$$\lim_{(x,s)\to(x_0,0)} K_{x,s} = E(F)_{x_0} \tag{10}$$

From this, it is easy to see that $E(F)_{x_0}$ gives a point of the cuspidal Eigenvariety for $U(2,2)$ with slope $\underline{s}_0$. Then, by using the theory developed in [Ur11], one shows that there is a 4-dimensional $p$-adic family of automorphic representations of finite slope specializing to the representation $E_{\kappa_0}(f)$ in the sense of [SU06b, §2, Thm 2.3.2] and [Ur11, Thm 5.4.4.].

From this, we construct the corresponding families of Galois representations using the theory of pseudo-characters and the existence of Galois representations for cuspidal representations of $U(2,2)$ and their properties[19] with respect to the local-global compatibility with the Langlands correspondence. We can deduce easily Theorem 3.1 by taking a suitable 1-dimensional subfamily of it. The fact that the sections used for constructing the Eisenstein series project subjectively on the Langlands quotient implies the crucial local property (b) in Theorem 3.1 by (7) and local-global compatibility of the Galois representation with the Langlands correspondence.

# References


[AIP] F. Andreatta, A. Iovita and V. Pilloni, *Families of Siegel modular forms*, preprint 2011.

[Ha81] M. Harris, *Eisenstein series on Shimura varieties*, Ann. of Math., 119, 59-94 (1984).

[Ha85] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. I*, inventiones math. 82 (1985), pp 151-189

[Ha86] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*, compositio math., tome 60 n3 . 3 (1986), pp 323–378.

[Hi04] H. Hida, *p-Adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics, 2004,


[19] This uses essentially the works of Sophie Morel, Chris Skinner and Shrenik Shah.

[Ki03] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*. Invent. Math. 153 (2003), no. 2, 373–454.

[La08] K.W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, Ph.D. thesis, Harvard University, May 20, 2008.

[La12] K. W. Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, J. Reine Angew. Math. 664 (2012), pp. 163–228.

[PS11] B. Stroh and V. Pilloni, *Surconvergence et classicité: le cas déployé.*, preprint 2011.

[Sh04] G. Shimura, *Arithmeticity in the Theory of Automorphic Forms*, American Mathematical Society, Mathematical Surveys and Monographs, Vol. 82, (2004).

[SU02] C. M. Skinner and E. Urban, *Sur les déformations p-adiques des formes de Saito-Kurokawa*, C. R. Math. Acad. Sci. Paris 335 (2002), no. 7, pp 581–586.

[SU06a] C. M. Skinner and E. Urban, *Sur les déformations p-adiques de certaines représentations automorphes*, Journal de l'Institut Mathématique de Jussieu, Volume 5, Issue 04, October 2006, pp 629-698

[SU06b] C. M. Skinner and E. Urban, *Vanishing of L-functions and ranks of Selmer groups*, 28 pages, Proceedings of the International Congress of Mathematician held in Madrid 2006, vol. II, pp 473–500.

[SU10] C. M. Skinner and E. Urban, *The Main conjecture for* $GL_2$, to appear in Inventiones Mathematicae.

[SU13] C. M. Skinner and E. Urban, *Nearly overconvergent automorphic forms and applications*, in preparation.

[Ro96] R. Rouquier, *Caractérisation des caractères et Pseudo-caractres* , Journal of Algebra 180, 571-586 (1996).

[Ta91] R. Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. 63, 281-332, (1991).

[Ur11] E. Urban, *Eigenvarieties for reductive groups*, Annals of math. 174 (2011), pp 1685–1784.

[Ur12] E. Urban, *Nearly overconvergent modular forms*, preprint 2012.



Current Address: Eric Urban, Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

*E-mail:* `urban@math.columbia.edu, urban@math.jussieu.fr`