Appendix A. Application to the three variable Rankin-Selberg p-adic L-functions. A corrigendum to [Ur14].

A.1. **Introduction.** In [Ur14], the author introduced nearly overconvergent modular forms of finite order and their spectral theory. The theory has be refined in [AI17] including intgral structure that allows to define families of nearly overconvergent modular forms of unbounded degree that was missing in [Ur14]. The purpose of this appendix is to fill a gap in [Ur14] about the construction of the three variable Rankin-Selberg p-adic L-functions which we can now solve thanks to the work of F. Andreatta and A. Iovita [AI17]. The gap lies in the construction made in section §4.4.1 a few lines before Proposition 11 where the existence of a finite slope projector denoted $e_{R,\mathfrak{V}}$ is claimed. Here $\mathfrak{V}$ is an affinoid of weight space and R is a polynomail in $A(\mathfrak{V})[X]$ dividing the Fredholm determinant of U_p acting on the space of $\mathfrak{V}$-families of nearly overconvergent modular forms. It was falsely claimed on top of page 434 that $e_{R,\mathfrak{V}}$ can be defined as $S(U_p)$ for some $S \in X.A(\mathfrak{V})[[X]]$ when it would actually be a limit of polynomial in the Hecke operator U_p with coefficient in the fractions ring of $A(\mathfrak{V})$ that may have unbounded denominators making the convergence a difficult question. In the following pages, we will explain how the existence of this projector in the theory of [AI17] can actually be used to define the missing ingredient of the construction in [Ur14, §4.4.1]. For the sake of brevity, we will use freely the notations of [Ur14] and [AI17] without recalling all of them.

I would like to thank Zheng Liu for pointing out the gap to me when she was working in her thesis on a generalization of my work to the Siegel modular case. I would like also to thank F. Andreatta and A. Iovita for telling me about their work and for including this corrigendum as an appendix of their paper.

A.2. **Families of nearly overconvergent modular forms.** Let p be an odd prime. The purpose of this paragraph is to collect some results of [Ur14]and [AI17] and harmonize the notations. Recall that for any rigid analytic variety X over a non archimedean field, we denote respectively by $A(X)$ and $A^0(X)$ the ring of rigid analytic function on X and its subring of functions bounded by 1 on X. Recall also that we denote weight space by $\mathfrak{X}$. It is the rigid analytic space over $\mathbf{Q}_p$ such that $\mathfrak{X}(\overline{\mathbf{Q}}_p) = Hom_{cont}(\mathbf{Z}_p^\times, \overline{\mathbf{Q}}_p^\times)$. For any integer k, we denote by $[k] \in \mathfrak{X}(\mathbf{Q}_p)$ the weight given by $x \mapsto x^k, \forall x \in \mathbf{Z}_p^\times$. For any p-power root of unity ζ, we denote χ_ζ the finite order character of $\mathbf{Z}_p^\times$ trivial on μ_{p-1} and such that $\chi_\zeta(1+p) = \zeta$.

Let $\mathfrak{U} \subset \mathfrak{X}$ be an affinoid subdomain of weight space and choose $I = [0, p^c]$ such that $A^0(\mathfrak{X}) = \Lambda \subset \Lambda_I \subset A^0(\mathfrak{U})$ with Λ and Λ_I as defined in [AI17, §3.1]. We also fix integers r and n compatible with I as in loc. cit. We consider the Frechet space over the Banach algebra $A(\mathfrak{U})$

$$\mathcal{N}_{\mathfrak{U}}^\dagger := \varinjlim_r (H^0(\mathfrak{X}_{r,I}, \mathbb{W}_{k_I}) \otimes_{\Lambda_I} A(\mathfrak{U}))$$

Here $\mathfrak{X}_{r,I}$ is the formal scheme defined in [AI17, §3.1] attached to a strict neighborhood (in rigid geometry) of the ordinary locus of the modular curve.

It is easily seen that the filtration on $\mathbb{W}_{k_I}$ of [AI17, Thm 3.11] induces the filtration

$$\mathcal{M}_{\mathfrak{U}}^{\dagger} = \mathcal{N}_{\mathfrak{U}}^{0,\dagger} \subset \mathcal{N}_{\mathfrak{U}}^{1,\dagger} \subset \cdots \subset \mathcal{N}_{\mathfrak{U}}^{s,\dagger} \subset \cdots \subset \mathcal{N}_{\mathfrak{U}}^{\dagger}$$

where for each integer s, $\mathcal{N}_{\mathfrak{U}}^{s,\dagger}$ denotes the space of $\mathfrak{U}$-families of nearly overconvergent modular forms as defined in [Ur14, §3.3]. The work done in [AI17, §3.1] that we use here is the rigorous construction using the correct integral structure of what was alluded to in [Ur14, Remark 10]. Moreover, it follows from [AI17, §3.6] that there is a completely continuous action of the U_p operator on $\mathcal{N}_{\mathfrak{U}}^{\dagger}$ that respect the above filtration and that is compatible with the one defined in [Ur14]. Moreover, we easily see for example using [Ur14, Prop. 7 (ii)] that the Fredholm determinant $P_{\mathfrak{U}}^{\infty}(\kappa, X)$ of U_p acting on $\mathcal{N}_{\mathfrak{U}}^{\dagger}$ satisfied the relation

$$P_{\mathfrak{U}}^{\infty}(\kappa, X) = \prod_{i=0}^{\infty} P_{\mathfrak{U}[-2i]}(\kappa.[-2i], p^i X)$$

where $P_{\mathfrak{U}[-2i]}$ stands for the Fredholm determinant of U_p acting on the space of families of overconvergent modular forms of weights varying in the translated affinoid $\mathfrak{U}$ by the weight $[-2i]$.

Recall finally that an admissible pair for nearly overconvergent forms is a data $(R, \mathfrak{V})$ where $R \in A(\mathfrak{U})[X]$ is a monic polynomial such that there is a factorization $P_{\mathfrak{U}}^{\infty}(\kappa, X) = R^*(X)Q(X)$ where $R^*(X) = R(1/X)X^{deg\ R}$ and $Q(X)$ are relatively prime in $A(\mathfrak{U})\{\{X\}\}$. To such a pair, one can associate a decomposition

$$\mathcal{N}_{\mathfrak{U}}^{\dagger} = \mathcal{N}_{R,\mathfrak{U}} \oplus \mathcal{S}_{R,\mathfrak{U}}$$

which is stable under the action of U_p and such that $det(1 - X.U_p|\mathcal{N}_{R,\mathfrak{U}}) = R(X)$. We will call $e_{R,\mathfrak{U}}$ the projection of $\mathcal{N}_{\mathfrak{U}}^{\dagger}$ onto $\mathcal{N}_{R,\mathfrak{U}}$. This later subspace consist in familes of nearly overconvergent modular forms of bounded order. This is well-known and follows from the generalization by Coleman and others of the spectral theory of completely continuous operators originally due to J.P. Serre.

A.3. **The nearly overconvergent Eisenstein family.** Recall that we have defined in [Ur14, §4.3] the nearly overconvergent Eisenstein family q-expansion $\Theta.E \in A^0(\mathfrak{X}\times\mathfrak{X})[[q]]$ by

$$\Theta.E(\kappa, \kappa') := \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \langle n \rangle_{\kappa} a(n, E, \kappa') q^n$$

It satisfied the following interpolation property [Ur14, Lemma 6]. If $\kappa = [r]$ and $\kappa' = [k]\psi$ with ψ a finite order character and k and r positive integers, the evaluation at (κ, κ') of $\Theta.E$ is $\Theta.E(\kappa, \kappa') = \Theta^r.E_k^{(p)}(\psi)(q)$ and is the p-adic q-expansion of the nearly holomorphic Eisenstein series $\delta_k^r E_k^{(p)}(\psi)$.

A generalization of this statement is the crucial lemma below which will follow from [AI17, Thm 4.6]. Because of the hypothesis 4.1 of loc. cit. , we need to introduce the following notation. We denote by $\mathfrak{X}' \subset \mathfrak{X}$ the affinoid subdomain of $\mathfrak{X}$ of the weights κ

such that $|\kappa(1+p)-\zeta(1+p)^n|_p \leq 1/p^2$ for some integer n and some p-power root of unity ζ. Notice that $\mathfrak{X}'(\overline{\mathbf{Q}}_p)$ contains all the classical weights.

Lemma A.3.1. *There exists $\Theta E_{\mathfrak{X}',\mathfrak{X}'} \in A(\mathfrak{X}')\hat{\otimes}\mathcal{N}^{\dagger}_{\mathfrak{X}'}$ such that its q-expansion is given by the canonical image of $\Theta.E$ into $A(\mathfrak{X}')\hat{\otimes}_{\mathbf{Q}_p}A(\mathfrak{X}')[[q]]$ induced by the canonical map $\Lambda\otimes_{\mathbf{Z}_p}\Lambda \to A(\mathfrak{X}')\hat{\otimes}_{\mathbf{Q}_p}A(\mathfrak{X}')$.*

Proof. For a given integer n and p-power root of unity ζ, we denote by $\mathfrak{X}'_{n,\zeta}\subset\mathfrak{X}'$ the affinoid subdomain of the weights κ such that $|\kappa(1+p)-\zeta(1+p)^n|_p\leq 1/p^2$. When $\zeta=1$, we just write $\mathfrak{X}'_n$ for $\mathfrak{X}'_{n,1}$. Since $\mathfrak{X}'$ is the disjoint union

$$\mathfrak{X}' = \bigsqcup_{n=0}^{p-1}\bigsqcup_{\zeta}\mathfrak{X}'_{n,\zeta}$$

it is sufficient to construct $E_{\mathfrak{X}'_{n,\zeta},\mathfrak{X}'_{m,\eta}} \in A(\mathfrak{X}'_{n,\zeta})\hat{\otimes}\mathcal{N}^{\dagger}_{\mathfrak{X}'_{m,\eta}}$ satisfying the corresponding condition on the q-expansion. Notice also that $\mathfrak{X}'_{n,\zeta}=[n]\chi_\zeta.\mathfrak{X}'_0$ and that, with the notations of [AI17], we have $A^0(\mathfrak{X}'_0)=\Lambda_{I'}$ with $I'=[0,p^2]$.

It clearly exists $E^{(p)}_{\mathfrak{X}'_{m,\eta}} \in \mathcal{M}^{\dagger}_{\mathfrak{X}'_{m,\eta}} \subset \mathcal{N}^{\dagger}_{\mathfrak{X}'_{m,\eta}}$ such that its q-expansion in $A(\mathfrak{X}'_{m,\eta})[[q]]$ is given by $\Theta E([0],\kappa)$. Indeed it is defined by $E^{(p)}_{\mathfrak{X}'_{m,\eta}} = E^{ord}_{\mathfrak{X}'_{m,\eta}} - E^{ord}_{\mathfrak{X}'_{m,\eta}}|V_p$ where $E^{ord}_{\mathfrak{X}'_{m,\eta}} \in e_{ord}.\mathcal{M}^{\dagger}_{\mathfrak{X}'_{m,\eta}}$ denotes the $\mathfrak{X}'_{m,\eta}$-family of ordinary Eisenstein series and V_p denotes the Frobenius operator inducing raising q to its p-power on the q-expansion.

We have the isomorphism $\Lambda\cong\mathbf{Z}_p[(\mathbf{Z}/p\mathbf{Z})^\times][[T]]$ done by choosing the topological generator $1+p\in 1+p\mathbf{Z}_p$,. Let $\kappa_{\mathfrak{X}'_0}$ be the universal weight $\mathbf{Z}_p^\times\to A(\mathfrak{X}'_0)^\times$. We can easily see that $Log(\kappa_{\mathfrak{X}'_0})=\frac{log(1+T)}{log(1+p)}=u_\kappa$ where u_κ is the notation defined in [AI17] while $Log(\kappa_{\mathfrak{X}'_0})$ is the notation defined in [Ur14]. The assumption 4.1 of [AI17], now reads easily as $I\subset[0,p^2]$ and is therefore satisfied since $A^0(\mathfrak{X}'_0)=\Lambda_{[0,p^2]}$.

Before pursuing, we note that we will use the notation ∇^χ following the definition 4.11 of [AI17] for the twist of nearly overconvergent forms by a finite order character χ of $\mathbf{Z}_p^\times$. We refer the reader to loc. cit. for its properties.

Let κ_s the generic weight $\mathbf{Z}_p^\times\to A(\mathfrak{X}'_{n,\zeta})$. Since $\mathfrak{X}'_{n,\zeta}=[n]\chi_\zeta.\mathfrak{X}'_0$, the weight $\kappa_s.[-n]\chi_\zeta^{-1}$ satisfies the assumption 4.1 of [AI17]. Let m' be a natural integer such that $m+2m'$ is divisible by p and let η' be a p-power root of unity so that $\eta'^2=\eta^{-1}$. Then $([m'].\chi_{\eta'})^2\mathfrak{X}'_{m,\eta}=\mathfrak{X}'_0$ and therefore the weight of $\nabla^{\chi_{\eta'}}\nabla^{m'}E^{(p)}_{\mathfrak{X}'_{m,\eta}}$ satisfies also the assumption 4.1. of [AI17]. According to [AI17, Thm 4.6], one can therefore define $\nabla^{s-n\chi_\zeta}(\nabla^{\chi_{\eta'}}\nabla^{m'}E^{(p)}_{\mathfrak{X}'_{m,\eta}})$ where $\nabla^{s-n\chi_\zeta}$ stands for $\nabla^{s'}$ with s' the weight corresponding to $\kappa_{s'}=\kappa_s[-n]\chi_\zeta^{-1}$ taking values in $A^0(\mathfrak{X}'_0)$. Since $\mathfrak{X}'_{n,\zeta}$ depends only on n modulo p, we may and do assume that $n>m'$, and we can therefore set

$$\Theta E_{\mathfrak{X}'_{n,\zeta},\mathfrak{X}'_{m,\eta}} := \nabla^{\chi_{\zeta\eta}}\nabla^{n-m'}(\nabla^{s-n\chi_\zeta}(\nabla^{\chi_{\eta'}}\nabla^{m'}E^{(p)}_{\mathfrak{X}'_{m,\eta}}))$$

From the effect of ∇ on the q-expansion, it is now easy to verify that $\Theta E_{\mathfrak{X}'_{n,\zeta},\mathfrak{X}'_{m,\eta}}$ satisfies the condition on the q-expansion claimed in the Lemma. ■

A.4. **Final construction of $G^E_{Q,\mathfrak{U},R,\mathfrak{V}}$.** In this paragraph, we explain how to replace the bottom of page 433 of [Ur14]. We now assume that $\mathfrak{U}$ and $\mathfrak{V}$ are affinoid subdomains of $\mathfrak{X}'$. Let $(Q,\mathfrak{U})$ be an admissible pair for overconvergent forms of tame level 1 and let $T_{Q,\mathfrak{U}}$ be the corresponding Hecke algebra over $A(\mathfrak{U})$. By definition it is the ring of analytic function on the affinoid subdomain $\mathcal{E}_{Q,\mathfrak{U}}$ sitting over the affinoid subdomain $Z_{Q,\mathfrak{U}}$ associated to $(Q,\mathfrak{U})$ of the spectral curve of the U_p-operator. Recall that

$$Z_{Q,\mathfrak{U}} = Max(A(\mathfrak{U})[X]/Q^*(X)) \subset Z_{U_p} \subset \mathbf{A}^1_{\mathbf{r}ig} \times \mathfrak{U}$$

where Z_{U_p} is the spectral curve attache to U_p and

$$T_{Q,\mathfrak{U}} = A(\mathcal{E}_{Q,\mathfrak{U}}) \text{ with } \mathcal{E}_{Q,\mathfrak{U}} = \mathcal{E} \otimes_{Z_{U_p}} Z_{Q,\mathfrak{U}}$$

where $\mathcal{E}$ stands for the Eigencurve. The universal family of overconvergent modular eigenforms of type $(Q,\mathfrak{U})$ is an element of $\mathcal{M}_{Q,\mathfrak{U}} \otimes_{A(\mathfrak{U})} T_{Q,\mathfrak{U}}$ whose q-expansion is given by

$$G_{Q,\mathfrak{U}}(q) := \sum_{n=1}^{\infty} T(n) q^n \in T_{Q,\mathfrak{U}}[[q]]$$

Tautologically, for any point $y \in \mathcal{E}_{Q,\mathfrak{U}}$ of weight $\kappa_y \in \mathfrak{U}$, the evaluation $G_{Q,\mathfrak{U}}(y)$ at y of $G_{Q,\mathfrak{U}}$ is the overconvergent normalized eigenform g_y of weight κ_y associated to y.

We set

$$G^E_{Q,\mathfrak{U}} := G_{Q,\mathfrak{U}}.\Theta.E_{\mathfrak{X}'\mathfrak{X}'} \in T_{Q,\mathfrak{U}} \otimes A(\mathfrak{X}')\hat{\otimes}\mathcal{N}^\dagger_{\mathfrak{X}'} = A(\mathcal{E}_{Q,\mathfrak{U}}) \otimes A(\mathfrak{X}')\hat{\otimes}\mathcal{N}^\dagger_{\mathfrak{X}'}$$

Let now $(R,\mathfrak{V})$ be an admissible pair for nearly overconvergent forms as in [Ur14, §4.1]. We consider

$$G^E_{Q,\mathfrak{U},R,\mathfrak{V}} \in A(\mathfrak{V} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}')\hat{\otimes}\mathcal{N}_{R,\mathfrak{V}}$$

defined by

$$G^E_{Q,\mathfrak{U},R,\mathfrak{V}}(\kappa, y, \nu) := e_{R,\mathfrak{V}}.G^E_{Q,\mathfrak{U}}(y,\nu,\kappa\kappa_y^{-1}\nu^{-2}) \in \mathcal{N}^\dagger_\kappa$$

for any $(\kappa, y, \nu) \in \mathfrak{V} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}'(\overline{\mathbf{Q}}_p)$. Notice that since $\mathfrak{U}$ and $\mathfrak{V}$ are contained in $\mathfrak{X}'$, so is $\kappa\kappa_y^{-1}\nu^2$ which allows to evaluate $G^E_{Q,\mathfrak{U}}$ at $(y,\nu,\kappa\kappa_y^{-1}\nu^{-2})$. Its gives a nearly overconvergent modular form of weight κ which is the running variable in $\mathfrak{V}$. We can therefore apply the finite slope projector $e_{R,\mathfrak{V}}$ from $\mathcal{N}^\dagger_{\mathfrak{V}}$ onto $\mathcal{N}_{R,\mathfrak{V}}$ specialized at κ.

A.5. **Final Remarks.** We denote $G^E_{Q,\mathfrak{U},R,\mathfrak{V}}(q) \in A(\mathfrak{V} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}')[[q]]$ the q-expansion of the family of nearly overconvergent forms we have defined above. This is the family of q-expansion that we wanted to define in [Ur14, §4.4.1]. The rest of the statements and results of [Ur14, §4] are now valid under the condition that we replace $\mathfrak{X}$ by $\mathfrak{X}'$ and $\mathcal{E}$ by $\mathcal{E}' = \mathcal{E} \times_{\mathfrak{X}} \mathfrak{X}'$ in all of them. To obtain a more general result, we would need to extend the work of [AI17] to relax their assumption 4.1. This seems possible by noticing that the condition $u_\kappa \in p.\Lambda_I$ can be replaced by u_κ topologically nilpotent in Λ_I and by using a congruence for $\nabla^{(p-1)p^n} - id$ for n sufficiently large. This would allow to replace

$\mathfrak{X}'$ by $\mathfrak{X}$ in Lemma A.3.1 above which is the only reason we needed to restrict ourself to $\mathfrak{X}'$.

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