A COMPUTER-ASSISTED APPLICATION OF POINCARE'S FUNDAMENTAL POLYHEDRON THEOREM

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Abstract. We describe a rigorous, computer-assisted approach to the solution of the following question: Given a finite set of elements of $\mathrm{PSL}(2,\mathbb{C})$, is the group G generated by these elements discrete? In addition, what can one say about the geometry of the quotient $\mathbb{H}^3/G$? After a review of Poincare's Theorem for compact polyhedra, we derive some relevant formulae for the the intersections of planes in the projective disk model. We discuss some of the practical advantages for working in the projective disk model and compare it to other standard models. Given F, a finite algebraic extension of $\mathbb{Q}$ containing the trace field of a given group G, we sketch a method for conjugating the group to obtain a group with entries in a degree two extension of F. This provides a convenient way of solving the word problem in G. We describe our program which constructs a Dirichlet domain for G and then applies Poincare's Fundamental Polyhedron Theorem to determine if G is discrete. The program also computes hyperbolic volume using the Lobachevsky function. Finally, using our program, we construct manifolds associated to the "exceptional regions" defined in the paper of D. Gabai, R. Meyerhoff and N. Thurston [5] and further analyzed in [2], [7]. In particular, we address parts of the conjectures posed in [5] that are inaccessible by the arithmetic methods discussed in [7].

1. Introduction

First we review Poincare's Polyhedron Theorem. For a proof of Poincare's Theorem and a discussion of fundamental domains see [10] and [13]. Suppose G is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ such that $\mathbb{H}^3/G$ is a compact manifold. Let x be an arbitrary point in $\mathbb{H}^3$. Then the *Dirichlet domain* D at x is defined as:

$$D = \{y \in \mathbb{H}^3 | d(x,y) < d(g(x),y), \forall g \neq 1 \in G\}$$

Here $d(x,y)$ denotes the distance between x and y in the hyperbolic metric. $\overline{D}$, the closure of D, is a convex finite-sided fundamental polyhedron. Here sides are maximal convex subsets of ∂D. Each side S is a convex subset of the plane P_{g_S} which is equidistant from x and $g_S(x)$. More precisely, if $\{S_i\}$ is the set of sides and $H_g = \{y \in \mathbb{H}^3 | d(x,y) \leq d(g(x),y)\}$ then:

$$\overline{D} = \bigcap_{S_j \in \{S_i\}} H_{g_{S_j}} \text{ and } S_i = (\cap_{S_j \in \{S_i\}} H_{g_{S_j}}) \cap P_{g_{S_i}}$$

In addition, the following five conditions are satisfied:

(i) For each side S there is a paired side S' such that $g_S(S') = S$

(ii) $g_{S'} = g_S^{-1}$ (*side-pairing relation*)

I wish to thank W. Neumann for his guidance and insight which have made this project possible.

Let R be an edge contained in S. Define a sequence of sides $\{S_i\}_{i=1}^{\infty}$:
Let $S_1 = S$. Let S_2 be the side of P adjacent to S'_1 such that $g_{S_1}(S'_1 \cap S_2) = R$. For $i > 1$, let S_{i+1} be the side adjacent to S'_i such that $g_{S_i}(S'_i \cap S_{i+1}) = S'_{i-1} \cap S_i$.

(iii) There is a least positive integer k such that $g_{S_1} g_{S_2} ... g_{S_k} = 1$ (*cycle relation*)

Let $\theta(S'_i, S_{i+1})$ denote the dihedral angle between S'_i, S_{i+1}. We have:
(iv) $\sum_{i=1}^{k} \theta(S'_i, S_{i+1}) = 2\pi$

(v) No point of $\overline{D}$ is fixed by a non-trivial element of G

Suppose one is given a finite collection $\{g_i\}$ of elements of $\mathrm{PSL}(2, \mathbb{C})$. Let $\overline{D} = \bigcap_{g_j \in \{g_i\}} H_{g_j}$. Assume $\overline{D}$ is a compact nonempty 3-polyhedron such to each g_i corresponds a non-empty side and that conditions (i)-(iv) are met. Then Poincare's Fundamental Polyhedron Theorem asserts that the group G generated by $\{g_i\}$ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and the images of $\overline{D}$ under this group form an exact tessellation of $\mathbb{H}^3$. Furthermore, if (v) is satisfied then G is torsion-free and therefore, $\mathbb{H}^3/G$ is a compact hyperbolic 3-manifold. Also, if $\tilde{G}$ is a group generated by the set of symbols $\{g_i\}$ with the side-pairing relations as defined in (ii) and the cycle relations as defined in (iii) then $\tilde{G} \cong G$.

Given a set of generators as matrices in $\mathrm{SL}(2, \mathbb{C})$ with the matrix entries expressed both to high precision as well as algebraic numbers we propose a rigorous computer application of the above theorem [1]. First, we construct an approximate Dirichlet domain by computing the bisecting planes defined by the words in the given generators. We obtain a polyhedron defined by floating-point entries for the co-ordinates. One can then check that the conditions (i)-(v) are approximately satisfied. In this manner, one obtains a finite collection of sides and the generating elements which one hopes indeed satisfy (i)-(v). In Section 3 we show that knowing the generators to high precision often allows one to obtain explicit algebraic expressions for the entries. Note that knowing explicit algebraic expressions for the entries solves the word problem for the group. The formulae derived in Section 2 imply that given a field containing the entries of the orbits of a point one need not extend the field any further to express the vertices of the polyhedron defined by these orbits as algebraic numbers. This, with the solution to the word problem in the group, allows one to rigorously verify conditions (i)-(v). As we shall see later, in many cases it is even unnecessary to express vertices as explicit algebraic numbers to verify the conditions.

2. Constructing Dirichlet Domains in the Projective Disk Model

We now consider the construction of the Dirichlet domain in the projective disk model. First we review relevant hyperbolic geometry. The main reference (except issues related to the coefficient field) is [13]. Let us denote the set of quaternions

[1] First work on computer application's of Poincare's Theorem was initiated by R. Riley. However, due to his unfortunate death we were not able to find any information about the existing program. The publicly available paper which describes his work in this direction does not resolve/address many of the issues described here.

$\{x+y\mathbf{i}+z\mathbf{j}|z>0\}$ by $\mathbb{U}^3$. Let the arc lenght element at a point $x+y\mathbf{i}+z\mathbf{j}$ be $\frac{\sqrt{dx^2+dy^2+dz^2}}{z}$. $\mathbb{U}^3$ with this metric is known as the upper-half space model of hyperbolic 3-space. Let D^3 be the open unit ball in $\mathbb{R}^3$. Denote the Euclidean norm of a vector r by $|r|$ and let the arc length element at a point r equal $\frac{[(1-|r|^2)|dr|^2+(r\cdot dr)^2]^{\frac{1}{2}}}{1-|r|^2}$. This is known as the projective disk model. Also, the arc length element defines a metric $d_{D^3}(r,r')$ on D^3 such that:

$$\cosh(d_{D^3}(r,r')) = \frac{1-r\cdot r'}{\sqrt{1-|r|^2}\sqrt{1-|r'|^2}} \tag{1}$$

Let $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}(2,\mathbb{C})$. Then g acts on $\mathbb{U}^3$ as an orientation preserving isometry:

$$g(w) = (a*w+b)*(c*w+d)^{-1} \tag{2}$$

Here w=$x+y\mathbf{i}+z\mathbf{j}$ and $*$ represents quaternion multiplication. Let K be the field generated by the entries of g. By writing out the action on the three co-ordinates it follows that the field generated by the three co-ordinates of $g(\mathbf{j})$ is contained in K', the field generated by $\{Re(y), Im(y)|y\in K\}$. For example, K' is contained in the splitting field of a polynomial which defines K.

For the construction of Dirichlet domains we need to analyze the intersection of planes bisecting the distance from $\mathbf{j}$ to $g(\mathbf{j})$. Since hyperbolic planes are defined by spheres centered at $z=0$, to obtain algebraic expressions for the vertices ones needs to obtain repeated degree two extensions of K'. This appears to be a very inefficient procedure for determining whether the conditions stated in the Introduction are satisfied.

Consider $\phi : D^3 \to \mathbb{U}^3$ given by:

$$(x,y,z) \longmapsto \frac{x+y\mathbf{i}+(\sqrt{1-x^2-y^2-z^2})\mathbf{j}}{1-z} \tag{3}$$

It is easy to check that ϕ is an isometry with inverse:

$$x+y\mathbf{i}+z\mathbf{j} \longmapsto \frac{(2x,2y,x^2+y^2+z^2-1)}{1+x^2+y^2+z^2} \tag{4}$$

Let us now attempt to construct a Dirichlet domain for a group G based at $\vec{0}=(0,0,0)$ in the projective disk model. The bisecting plane between $\vec{0}$ and $g(\vec{0})$ is:

$$P_g = \{r\in D^3 | d_{D^3}(\vec{0},r) = d_{D^3}(r,g(\vec{0}))\}$$

From equation (1) we see that P_g is also defined as:

$$P_g = \{r\in D^3 | \sqrt{1-|g(\vec{0})|^2} = 1 - r\cdot g(\vec{0})\}$$

Clearly, this is an equation of a plane of the type $\mathbf{n}\cdot r = t$ where $\mathbf{n}=g(\vec{0})$ and $t = 1-\sqrt{1-|g(\vec{0})|^2}$. Observe, once K' (the field containing the co-ordinates of images of $\mathbf{j}$ under G) has been obtained, the images of these co-ordinates under ϕ^{-1} are still in K'. Thus, if $g(\mathbf{j}) = x + y\mathbf{i} + z\mathbf{j}$ with $x, y, z \in K'$ and $g(\vec{0}) = (x', y', z')$ then $x', y', z' \in K'$. By construction, $g(\vec{0})$ is the image $\phi^{-1}(x + y\mathbf{i} + z\mathbf{j})$. Using equation (3) we see that:

$$\sqrt{1 - |g(\vec{0})|^2} = z(1 - z')$$

Therefore, if the bisecting plane defined by $g(\vec{0})$ is $\mathbf{n} \cdot r = t$, then $t \in K'$. If $g_1(\vec{0}), g_2(\vec{0}), g_3(\vec{0})$ are three orbits then the vertex defined by the three corresponding planes is the simultaneous solution of $\mathbf{n_1} \cdot r = t_1$, $\mathbf{n_2} \cdot r = t_2$ and $\mathbf{n_3} \cdot r = t_3$.

Let $R = (r_1, r_2, r_3)^T$ and $N = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}$ $(n_{ij} = (\vec{\mathbf{n_i}})_j)$.

If $T = (t_1, t_2, t_3)^T$ we have:

$$N \cdot R = T \Rightarrow R = N^{-1} \cdot T$$

Note, that our argument holds even for orientation-reversing isometries. In addition, if G has torsion and the origin is not a fixed point of G we can still construct a Dirichlet domain based at the origin. Thus, in the projective disk model we have obtained expressions for vertices in terms of the coordinates of the orbits as well as the following theorem:

Theorem 2.1: Let G be a discrete group of isometries of the projective disk model D^3 such that the origin is not fixed by any element of G. Let K' be a field containing the co-ordinates of the images of the origin under the group action. Then the co-ordinates of all the vertices of the Dirichlet domain based at the origin are contained in K'. $\square$

Remark: By inspecting the transformations relating $\mathbb{U}^3$, the conformal ball model and the hyperboloid model one can show that the other two models have the same complication as $\mathbb{U}^3$ since the transformations involve only rational expressions in the coordinates.

3. Constructing The Coefficient Field

We outline a general technique for obtaining explicit algebraic entries for a co-compact discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. Most of the results are based on the methods explained in [8]. Let A, $B \in \mathrm{SL}(2,\mathbb{C})$. We say that the pair (A, B) is *generic* if the matrices I, A, B, AB are linearly independent. By putting A in canonical form it is easy to show that a pair (A, B) is generic iff $AB - BA$ is non-singular. Following [8] we say that $G \subset \mathrm{SL}(2,\mathbb{C})$ is *elementary* if it is either finite or contains no generic pair. For discrete groups this agrees with the usual definition. Furthermore, if G is discrete and co-compact then G is never elementary. Let $\mathbb{Q}(tr(G))$ denote the field generated by the traces of elements of G. It follows, [8, Theorem 4.1] that $\mathbb{Q}(tr(G))$ is a finite algebraic extension of $\mathbb{Q}$. Also, we have the following:

Theorem 3.1: Let $G \in \mathrm{SL}(2,\mathbb{C})$ and let (A, B) be a generic pair. Then the coefficient field K of G is generated by $\mathbb{Q}(tr(G))$ and the coefficients of A, B.

Proof. Let $A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}$ and $AB = \begin{pmatrix} a_1b_1 + a_3b_2 & a_1b_3 + a_3b_4 \\ a_2b_1 + a_4b_2 & a_2b_3 + a_4b_4 \end{pmatrix}$. If $C \in G$, $C = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}$. We have the equations: $tr(C) = t_1, tr(CA) = t_2, tr(CB) = t_3$ and $tr(CAB) = t_4$. Here $t_i \in K$. In matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1b_1 + a_3b_2 & a_2b_1 + a_4b_2 & a_1b_3 + a_3b_4 & a_2b_3 + a_4b_4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \\ c_2 \\ c_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} \tag{5}$$

Since the pair is generic the square matrix is invertible. □

In general, there is no reason that $(K : \mathbb{Q}(tr(G)))$ is small or even finite. However, one is interested in the properties of G which are preserved under conjugation in $\mathrm{SL}(2, \mathbb{C})$. Assume again that (A, B) is a generic pair. The following elementary lemma can be proved by putting A in canonical form:

Lemma 3.1: Let (A, B) be a generic pair. Put $tr(A) = t_1, tr(B) = t_2, tr(A^{-1}B) = t_3$. If (A', B') is another pair such that $tr(A') = t_1, tr(B') = t_2, tr({A'}^{-1}B') = t_3$ then there exist $T \in \mathrm{SL}(2, \mathbb{C})$ such that $A = TA'T^{-1}$ and $B = TB'T^{-1}$. □

We omit the proof of the lemma since the explicit form of T is never used in the program. Given $\mathbb{Q}(tr(G))$ the proof of the following provides an explicit construction of a coefficient field of a group conjugate to G:

Theorem 3.2: Let G be a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ with generic pair (A, B).
G is conjugate to a group $\tilde{G}$ such that if K' is the coefficient field of $\tilde{G}$ then $(K' : \mathbb{Q}(tr(G))) \leq 2$.

Proof. Keeping the notation of Lemma 3.1 consider:

$$A' = \begin{pmatrix} 0 & 1 \\ -1 & t_1 \end{pmatrix}, B' = \begin{pmatrix} z & 0 \\ t_1 z - t_3 & t_2 - z \end{pmatrix} \text{ with } (t_2 - z)z = 1.$$

The pair (A', B') fulfills the hypothesis of Lemma 3.1 and z satisfies a quadratic polynomial with coefficients in $\mathbb{Q}(tr(G))$. Now use Theorem 3.1. □

As remarked, $\mathbb{Q}(tr(G))$ is a finite extension of $\mathbb{Q}$. The following theorem proved in [7] provides a convenient way of obtaining the trace field:

Theorem 3.3: Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be generated by $\{g_1, ..., g_n\}$. The trace field is generated by: $tr(g_i), tr(g_i g_j), i < j$, and (if $n > 2$) the trace of one triple product of generators.

Often knowing the traces to high numeric precision is sufficient to obtain the trace field and hence the coefficient field of a group conjugate to G. One uses the LLL-algorithm as applied in the algdep() and lindep() routines of the PARI-GP package [12]. See [2] for applications and examples.

4. Volume of Hyperbolic Polyhedra

In this section we summarize relevant formulae for volumes of non-ideal hyperbolic tetrahedra and discuss how our program computes volume. Recall, that the Lobachevsky function may be defined as [13]:

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t|dt$$

As discussed in [13], can be shown that:

$$2i\Lambda(\theta) = \psi(e^{2i\theta}) - \psi(1) + \pi\theta - \theta^2 \text{ where } \psi(\theta) = \sum_{n=1}^{\infty} \frac{\theta^n}{n^2}$$

Let $IT(A, B, C, A', B', C')$ represent a tetrahedron in hyperbolic 3-space with one ideal vertex and dihedral angles A, B, C, A', B', and C'. Let edges corresponding to A, B, and C share exactly one vertex. Also, let the edges corresponding to A and A' have no vertex in common (similarly for the others). It is irrelevant which vertex is ideal. In terms of the Lobachevsky function, the volume has the following form [3]:

$$VolIT(A, B, C, A', B', C') =$$

$$\begin{gathered}
\tfrac{1}{2}[\Lambda(\tfrac{A-B-C+\pi}{2}) + \Lambda(\tfrac{-A+B-C+\pi}{2}) + \Lambda(\tfrac{-A-B+C+\pi}{2}) \\
-\Lambda(\tfrac{A+B+C+\pi}{2}) + \Lambda(\tfrac{A-B'-C'+\pi}{2}) + \Lambda(\tfrac{-A+B'-C'+\pi}{2}) + \Lambda(\tfrac{-A-B'+C'+\pi}{2}) \\
-\Lambda(\tfrac{A+B'+C'+\pi}{2}) + \Lambda(\tfrac{A'-B-C'+\pi}{2}) + \Lambda(\tfrac{-A'+B-C'+\pi}{2}) + \Lambda(\tfrac{-A'-B+C'+\pi}{2}) \\
-\Lambda(\tfrac{A'+B+C'+\pi}{2}) + \Lambda(\tfrac{A'-B'-C+\pi}{2}) + \Lambda(\tfrac{-A'+B'-C+\pi}{2}) + \Lambda(\tfrac{-A'-B'+C+\pi}{2}) \\
-\Lambda(\tfrac{A'+B'+C+\pi}{2})] + \Lambda(\tfrac{A+A'+B+B'}{2}) + \Lambda(\tfrac{A+A'+C+C'}{2}) + \Lambda(\tfrac{B+B'+C+C'}{2})
\end{gathered}$$

The paper of Yunhi Cho and Hyuk Kim [3] also derives a volume formula for an arbitrary non-ideal tetrahedron. However, from a computational viewpoint, it is easier to represent the volume of an arbitrary tetrahedron as the difference of the volumes of two tetrahedra each with one ideal point. Suppose we have chosen to work in the upper-half space model. Represent a tetrahedron as the convex hull of its four vertices: $T(v_1, v_2, v_3, v_4)$. Then, by applying a Mobius transformation, we can assume that say v_1 and v_2 are on the z-axis with z co-ordinate of v_1 larger than that of v_2. Then $VolT(v_1, v_2, v_3, v_4) = VolT(\infty, v_2, v_3, v_4) - VolT(v_1, \infty, v_3, v_4)$. Notice, since this model is conformal, positioning an edge on the z-axis makes the computation of the dihedral angle easy - the two planes which meet at the edge are Euclidean planes. Thus, given an arbitrary compact polyhedron in hyperbolic 3-space our program first represents it as a union of tetrahedra and then computes hyperbolic volume by applying the formula above.

5. Program Routines

The purpose of this section is to discuss the basic structure of the functions we have written for constructing Dirichlet domains and checking that the conditions of Poincare's theorem are met. All routines have been written using the PARI-GP package [12] with an option for output to Geomview [6]. The actual routines as well as a user's guide are also available [15]. After discussing how our routines construct the domain we sketch a method to verify the conditions of the theorem.

For the moment, one might ignore that the projective disk model is restricted to the unit ball and simply consider the intersection of planes in $\mathbb{R}^3$. We start out with the initial standard unit box centered at the origin. Then we modify the domain by cutting with planes which bisect the hyperbolic distance from the origin to the orbits. This list of orbits is constructed by computing all words up to a given length. Since for purposes of efficiency we use only the decimal approximation of the words, elements very close to the identity are ignored. Thus, the routine "guesses" that an element close to the identity must be the identity. This is an example of the general philosophy of our approach: the main task of the program is to find manifolds defined by a subgroup of $\mathrm{PSL}(2, \mathbb{C})$, rather then prove that this subgroup is not discrete. In practice, this means that the routines will make choices which may fail to construct a domain, even if one exists, given that the specific case is highly pathological. However, for many interesting and useful examples the procedure will prove to be effective.

The domain is represented as a set of sides where each side is a collection of edges and each edge is a pair of vertices - points in $\mathbb{R}^3$. The intersection of the domain with a plane is computed by considering each edge (pair of vertices) separately. For the pair we compute whether vertices lie below or above the plane. If both lie above - the edge is discarded. If one lies below and the other above the necessary adjustment to the edge is made (one of the vertices is replaced). When a vertex appears sufficiently close to the plane the program "guesses" that the vertex is actually exactly on the plane and adjusts accordingly.

In principle, the length of words can be increased until one obtains the desired domain. However, even with the examples at hand, the length of desired words can be so large that finding the right words by searching all the words up to a given length can take weeks. Instead, we first construct a preliminary domain by trying a few hundred words. Each side of the resulting domain corresponds to a word. Then, we consider words that are two-word combinations of those forming the sides of the existing domain to modify the domain. This procedure is iterated until the domain no longer changes in the process. The resulting domain is then tested as a possible Dirichlet domain for the group. This procedure has proven to be fast and effective in practice.

Before finding exact expressions for the words generating the sides of the possible Dirichlet domain one might check if the domain meets a few preliminary criteria: along with each word corresponding to a side the inverse word corresponds to some side as well, number of vertices of sides corresponding to inverse words is the same, etc. Once the approximate domain is obtained we employ exact arithmetic to rigorously verify the conditions of Poincare's theorem. There are several ways to proceed. At least in principle, one can verify the conditions of Poincare's theorem by simply finding the field containing the entries of orbits and using formulae of Section 2 to obtain expressions for the vertices (as well as their images under isometries) as algebraic numbers. Then the verification comes down to the ability to perform exact arithmetic on the vertices. Since this involves finding the orbit field, the procedure is quite lengthy except for very small extensions of $\mathbb{Q}$. Also, even in this case, one cannot avoid using decimal approximations, since to compare entries in different field extensions one must have the decimal approximations of the roots as well as their expressions as algebraic numbers. We propose an alternate approach that avoids the construction of the orbit field at the cost of additional assumptions on the geometry of the domain.

The procedure outlined above allows one to find a finite collection $\{g_i\}$ of elements of the group in question which experimentally seems to generate a convex domain that meets all the hypothesis of the Introduction. We show that under some additional assumptions our program is capable of verifying the hypothesis of Poincare's Theorem for the group (or at least the subgroup generated by $\{g_i\}$) without using algebraic expressions for the vertices. For clarity, issues related to precision are deferred till the next section. Just by using decimal approximations of the orbits we can assume they are all distinct and do not equal the origin. Thus, we have a nonempty convex domain. Assume:

1. All edge-cycles have length three.

2. Each vertex of the Dirichlet domain is common to exactly three sides.

Below we discuss program routines which test **sufficient** conditions to prove that the group is discrete, co-compact and torsion-free. If the two assumptions on the actual Dirichlet domain are satisfied and we have mananged to find the collection of sides that really generate the domain the conditions of the tests are also **necessary**. Let us define $\overline{D} = \bigcap_{g_i \in \{g_i\}} H_{g_i}$ where $H_{g_i} = \{x \in \mathbb{R}^3 | d(0,x) \leq d(x, g_i(0))\}$. Thus, each side is:

$$S_{g_i} = (\cap_{g_j \in \{g_i\}} H_{g_j}) \cap P_{g_i} \tag{6}$$

where $P_{g_i} = \{x \in D^3 | d(0,x) = d(x, g_i(0))\}$. In the expression for S_{g_i} some H_{g_j} can be ignored. In fact, H_{g_j} can be ignored iff S_{g_i} and S_{g_j} do not share an edge. Given a side S and a plane P contaning this side we wish to find the subcollection Φ ($\Phi = \Phi_S$) of $\{g_i\}$ generating exactly the sides having a edge in common with S using only high precision estimates of the orbits.

Note that the construction of the approximate Dirichlet domain outlined above allows us to guess which elements generate sides belonging to Φ. This collection consists of group elements that generate sides which share an edge with S in the approximate domain. To verify that this Φ is indeed the minimal collection we proceed in several steps. First we verify that $\tilde{S} = (\cap_{g_i \in \Phi} H_{g_i}) \cap P$ is a nonempty subset of the open unit ball. This will force it to be compact. We must also verify that none of the elements of Φ are redundant. Each $g_i \in \Phi$ should correspond to an actual edge so we associate a pair g_{j_1}, g_{j_2} such that:

1) g_i, g_{j_1} and g_{j_2} are distinct.
2) If $r_1 = P \cap P_{g_i} \cap P_{g_{j_1}}$ and $r_2 = P \cap P_{g_i} \cap P_{g_{j_2}}$ then $|r_1| < 1$, $|r_2| < 1$ and $r_1 \neq r_2$. (all the planes are assumed to be in general position)
3) For any other element $g \in \Phi$, $\mathbf{n_g} \cdot r_i < t_g$ $(i = 1, 2)$ where $\mathbf{n_g}$ and t_g, are the parameters that define the bisecting plane corresponding to g (as in Section 2).

Finally, to check that $\tilde{S} = S$ it is sufficient to check that for $g \in \{g_i\}$ whenever $g \notin \Phi$ and r is a vertex of $\tilde{S}$ then $\mathbf{n_g} \cdot r < t_g$. In other words, the other bisecting planes do not modify S. Observe that since the additional assumptions on the domain are met all these verifications involve strict inequalities that may be verified using high precision without explicitly computing the vertices as algebraic numbers.

To verify the side-pairing relations we must show that given $g_S \in \{g_i\}$ there exists $g_{S'} \in \{g_i\}$ such that:

$$\text{(a) } g_{S'} = g_S^{-1} \text{ and (b) } g_S(S') = S$$

(a) can be verified from having the solution to the word problem. From (a) we have $g_S(P_{g_{S'}}) = P_{g_S}$. We need only to check that $g_S(S')$ and S have the same edges. Thus, given $h \in \Phi_S$ we verify that there exists $h' \in \Phi_{S'}$ such that:

$$P_{g_S} \cap P_h = P_{g_S} \cap g_S(P_{h'})$$

The left side of the equality is the line of points equidistant from 0, $g_S(0)$ and $h(0)$. The right side is the line of points equidistant from 0, $g_S(0)$ and $g_S \cdot h'(0)$. However, since by assumption edge-cycles have length three we have $h = g_S \cdot h'$. This equality is verified by once more using the solution of the word problem in the group. Since cycles have length three, condition (iv) is immediately satisfied because the sum of the three dihedral angles is at most 3π. But the sum of the angles along any edge-cycle is a whole multiple of 2π. Finally, to show that condition (v) of the Introduction is satisfied it suffices to compare the vertices of the finite number of images of $\overline{D}$ that still share a vertex with $\overline{D}$.

6. Precision Issues

Since the method outlined in the preceding section attempts to use precision arguments to provide a rigorous approach to proving that a group is discrete it is necessary to address the reliability of these arguments. One must use a computer package capable of working with specified precision that allows one to estimate the precision loss under arithmetic operations. The main issue is that as the program computes using the approximations, the real precision drops below the initially specified precision. Note that the elements of the group are specified as solutions to polynomials and hence can easily be computed to any precision.

Using the formulae of Section 2 we see that once a guess for a minimal collection has been obtained to compute the orbits under the action of desired elements we must compute the group elements starting from the generators and convert from the upper half space model to the projective disk model. Furthermore, to compute intersections of planes we must use these orbits in the matrices as described in Section 2. Note that for any given group it is possible to calculate the precision loss since for a given orbit the operations must be applied one at a time and never consecutively. Once the maximum word length in the collection has been obtained, it is easy to ensure that both the orbits and the intersection of planes are correct to specified precision.

Recall from the last section that precision is used in two types of computation: (a) to show two vectors in are distinct, (b) to show that one scalar is bigger than another. For (a) let x' and y' be approximations of x and y known to a given precision. We wish to prove that $x \neq y$. Let x_i, y_i, etc. be the co-ordinates of the vectors ($i = 1, 2, 3$). Assume $|x_i - x'_i| < \delta$ and $|y_i - y'_i| < \delta$. Then $\sum |x'_i - y'_i| > 6\delta$

implies $x \neq y$. For (b) suppose we want to prove $\mathbf{n} \cdot z > t$ where we have approximations $\mathbf{n}'$, z' and t' within δ. If $\mathbf{n}' \cdot z' - t' > 4\delta$, then $\mathbf{n} \cdot z > t$. It should be stressed that the program always uses sufficient conditions for the inequality. When the compared values are far enough apart to distinguish with approximations these conditions are also necessary.

7. Applications

One of the main results of the paper by D. Gabai, R. Meyerhoff and N. Thurston [5] is that if N is a closed irreducible 3-manifold homotopy-equivalent to a closed hyperbolic 3-manifold M then N and M are homeomorphic (for further details and sharper formulation see [5]). The proof, which makes use of computers to analyze a parameter space, isolates seven exceptional families that must be considered separately to complete the argument. From the initial analysis it is not clear that the exceptional families contain manifolds. However, the analysis does single out two-generator subgroups with supposed relators (quasi-relators) that appear to converge to actual relators somewhere inside each region. It is conjectured in [5] that each family X_i contains a unique manifold M_i with $\pi_1(M_i) = \langle f, w | r_1(X_i), r_2(X_i) \rangle$ where r_i are the quasi-relators for the family. In fact, the existence of a unique manifold associated to one the of the regions is crucial to the completion of the proof. Using arithmetic techniques K. Jones and A. Reid [7] show that there is a unique hyperbolic manifold associated to this region. Their methods extend to construct a hyperbolic manifold associated to all but one of the remaining regions. The main obstruction is that the region (known as X_3) does not appear to contain an arithmetic group. In [2], using the LLL-algorithm, explicit expressions as algebraic numbers are found for the generators of subgroups which satisfy the relations for each parameter family. Therefore, the investigation of the existence of a manifold associated to the region X_3 is possible with the methods of the present work.

We provide examples of how the program was applied to the seven families discussed in [5]. As noted, in [7] a unique manifold associated to region X_0 is found using arithmetic methods. Its volume is computed using arithmetic techniques as well. As an independent verification we used the methods described above to find the Dirichlet domain and calculate the volume. Our calculation of its volume, 1.014941... is in perfect agreement with [7] and the Week's census [14]. In all calculations we use the PARI-GP precision capabilities. Although this package allows one to specify arbitrary high initial precision at the present we cannot rigorously estimate the precision loss that occurs during the computation. Thus, although we are confident of the results, the theorems of this section can strictly be regarded only as experimental.

We used the methods described in this paper to successfully construct a Dirichlet domain for all the seven families. In particular, we have provided independent verifications of trace field and volume calculations for the regions already worked out in [7] as well as obtained a closed hyperbolic manifold associated to X_3 which is inaccessible by arithmetic techniques. We summarize our results as follows:

Theorem 7.1: There is a closed hyperbolic manifold associated to each of the seven families such that $\pi_1(M_i) = \langle f, w | r_1(X_i), r_2(X_i)\rangle$. In particular, $H_1(M_3) = \mathbb{Z}_7 \oplus \mathbb{Z}_7$ and Volume(M_3)= 7.73809...

This theorem is proved by first constructing the approximate domain as above and then using the sufficient tests of Section 5. In all cases the additional geometric assumptions on the domain are met. The presentation of the group available from the cycle and side-pairing relations is used to construct an isomorphism of $\pi_1(M_i)$ with $\langle f, w | r_1(X_i), r_2(X_i)\rangle$. This amounts to verifying that the words specified by cycle and side-pairing relations are trivial in $\langle f, w | r_1(X_i), r_2(X_i)\rangle$. For this we use the MAGNUS package [9] which for the groups we considered is able to show that the words are trivial in a matter of seconds. The presentation for the group derived from the application of Poincare's theorem to the Dirichlet domain as well as input to Geomview for all the regions is available at [15]. Further information about the representation of the fundamental groups in $\mathrm{PSL}(2,\mathbb{C})$ can be found in [2]. Although Theorem 7.1 does not address the uniqueness of the manifolds associated to the regions it provides a partial answer to the conjectures stated in [5].

The construction of a Dirichlet domain depends on the choice of a base point in $\mathbb{H}^3$ and on the particular representation of the group in $\mathrm{PSL}(2,\mathbb{C})$. The methods described above allow one to investigate how the geometry of the domain depends on these choices. For example, the domain for X_0 manifold with base point at the origin and the subgroup with entries in the parameter family specified by [5] has 22 sides. The domain with base point at the origin and the group conjugated to the form described in Section 3 has 28 sides. Figure 2 and 3 illustrate the output to Geomview. Similar phenomena occur for other examples. The geometry of the domain may be of consequence for the application of the program. For example, the method described in Section 5 assumes all edge-cycles are of length three. Can this be achieved for any discrete, torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ provided the basepoint is chosen appropriately? For instance, in the theory of Fuchsian groups one can prove that, except for a set of measure zero, all base points for the construction of a Dirichlet domain result in a domain with edge-cycles with length at most three [1]. One apparent obstruction to such a result for 3-orbifolds is the presence of parabolic elements (for example $z \to z+1, z \to z+i$). In general, with a proper choice of basepoint, is possible for the Dirichlet domain of any co-compact discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ to satisfy the geometric conditions stated in Section 5?

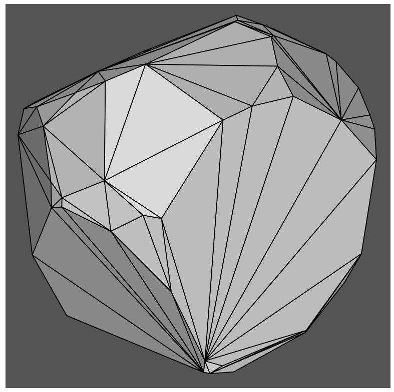

FIGURE 1. M_3 - manifold associated to X_3. Base point at the origin, group conjugated as in Section 3

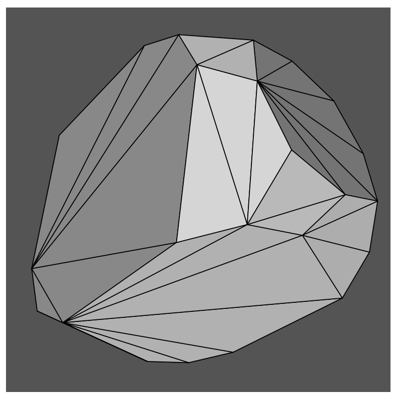

FIGURE 2. Vol3 - manifold associated to X_0. Base point at the origin, group same as in [GMT]

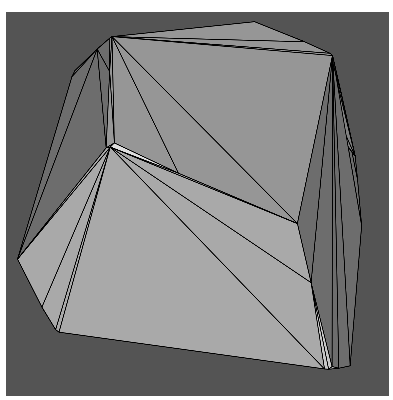

FIGURE 3. Vol3 - manifold associated to X_0. Base point at the origin, group conjugated as in Section 3

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