The Casimir Effect on a 1-Dimensional Manifold of p Periodicity

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Abstract

Using the Abel-Plana formula, we calculate the Casimir energy of a massless scalar field on a one-dimensional twisted manifold of periodicity p.

Introduction

In the years since the first calculation of the Casimir effect [1], much effort has been put into studying this phenomenon [2, 3, 4, 5, 6]. The best-known macroscopic result of the vacuum fluctuations predicted by quantum field theory, the Casimir effect is the response of the vacuum state to external fields or constraints. Typically, one finds a finite difference between the non-renormalized (infinite) energy of the vacuum state and that of the altered configuration, and this observable energy difference is termed the Casimir energy. The constraints may take the form of physical objects imposing boundary conditions, as in the classic case of parallel, uncharged conducting plates; or, as has more recently been studied, a Casimir energy may result from boundary conditions imposed on the vacuum field by the topology of different spaces. Calculations have been done for the topologies of the circle and the Möbius strip, represented with periodic and anti-periodic boundary conditions, respectively [6]. In the present paper, we generalize these results, calculating the Casimir effect on a one-dimensional manifold of periodicity p.

The Twisted Manifold

To represent the massless scalar field φ on S^1, we define the periodic boundary condition

$$\varphi(x+a) = \varphi(x);$$

on the Möbius strip S^1_2, we have the anti-periodic condition

$$\varphi(x+a) = -\varphi(x),$$

where a is the length of the dimension. On these two manifolds the Casimir energies are found to be $-\frac{\pi}{6a}$ and $\frac{\pi}{12a}$, respectively [6].

Generally, then, on a p-periodic manifold S^1_p, we write

$$\varphi(x+a) = e^{i\frac{2\pi}{p}}\varphi(x). \tag{1}$$

The allowed frequencies ω_n on this manifold are given by

$$\omega_n = \frac{2\pi(n+\frac{1}{p})}{a}.$$

Calculation of the Casimir energy $\mathcal{E}_p$ on S^1_p clearly requires modification of the usual renormalization procedure to include a sum over non-integer values ($\hbar = c = 1$):

$$\mathcal{E}_p = \frac{1}{2}\sum_{n=0}^{\infty}\frac{2\pi(n+\frac{1}{p})}{a} - \frac{1}{2}\int_0^{\infty}\frac{2\pi x}{a}\,dx. \tag{2}$$

We can use a modified form of the Abel-Plana formula (A-2) to evaluate the difference on the RHS (see Appendix). Then, putting in $f(x) = \frac{2\pi x}{a}$, we obtain

$$-\frac{1}{a}\left[\int_0^{\infty}\frac{2\pi t}{e^{2\pi i/p}\,e^{2\pi t}\,-\,1}\,dt \,+\, \int_0^{\infty}\frac{2\pi t}{e^{-2\pi i/p}\,e^{2\pi t}\,-\,1}\,dt\right]. \tag{3}$$

Letting $u = 2\pi t$, $y = e^{-2\pi i/p}$, and $z = e^{2\pi i/p}$, (3) becomes

$$-\frac{1}{2\pi a}\left[y\int_0^{\infty}\frac{ue^{-u}}{1\,-\,ye^{-u}}\,du \,+\, z\int_0^{\infty}\frac{ue^{-u}}{1\,-\,ze^{-u}}\,du\right]. \tag{4}$$

The integrals can be transformed into known functions by noting that

$$x\int_0^{\infty}\frac{te^{-t}}{1\,-\,xe^{-t}}\,dt \;=\; \frac{\partial}{\partial\beta}\bigg|_{\beta=1}\int_0^{\infty}\ln(1\,-\,xe^{-\beta t})\,dt. \tag{5}$$

If we let $\alpha = xe^{-\beta t}$, the RHS of (5) becomes

$$\frac{\partial}{\partial\beta}\bigg|_{\beta=1}\frac{-1}{\beta}\int_x^0\frac{\ln(1\,-\,\alpha)}{\alpha}\,d\alpha \;=\; \int_x^0\frac{\ln(1\,-\,\alpha)}{\alpha}\,d\alpha. \tag{6}$$

This is just the integral definition of the dilogarithm function $Li_2(x)$, so (4) becomes

$$-\frac{1}{2\pi a}\left[Li_2(e^{-2\pi i/p}) + Li_2(e^{2\pi i/p})\right]. \tag{7}$$

Now using the sum definition of the dilogarithm, we write

$$\begin{aligned}\mathcal{E}_p &= -\frac{1}{2\pi a}\left[\sum_{n=1}^{\infty}\frac{e^{-2\pi i n/p}}{n^2} + \sum_{n=1}^{\infty}\frac{e^{2\pi i n/p}}{n^2}\right]\\ &= -\frac{1}{\pi a}\sum_{n=1}^{\infty}\frac{\cos(\frac{2\pi n}{p})}{n^2}.\end{aligned} \tag{8}$$

Finally, using $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2} = \frac{\pi^2}{6} - \frac{\pi}{2}x + \frac{1}{4}x^2$ [7], we have

$$\mathcal{E}_p = -\frac{1}{2\pi a}\left[\frac{\pi^2}{6} - \frac{\pi}{2}\left(\frac{2\pi}{p}\right) + \frac{1}{4}\left(\frac{2\pi}{p}\right)^2\right],$$

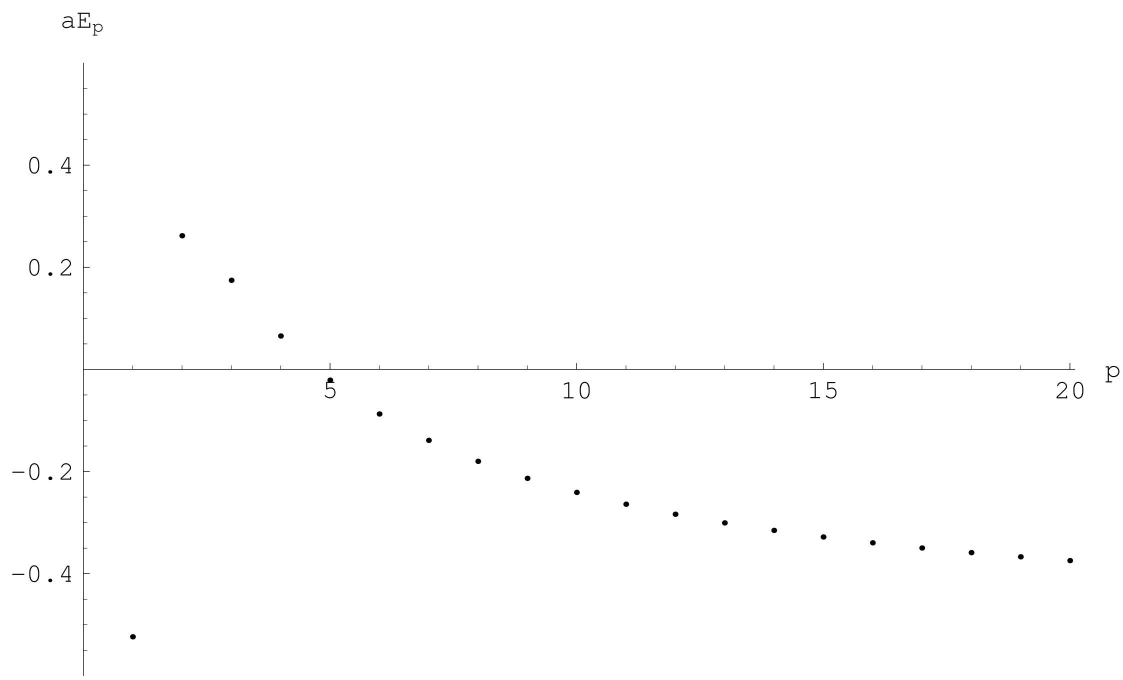

Figure 1: Energy $\times$ length of dimension vs. Periodicity

or more suggestively,

$$\boxed{\mathcal{E}_p = \frac{1}{a}\left[\frac{\pi}{12} - \frac{\pi(p-2)^2}{4p^2}\right].} \tag{9}$$

With the expression for the Casimir energy in this form, we can easily check the limiting cases $p = 1$ and $p \to \infty$:

$$\mathcal{E}_1 = \mathcal{E}_\infty = -\frac{\pi}{6a}.$$

The $p = 1$ case agrees with the result given in [6], and the energy for $p \to \infty$ is as expected, since in this limit the boundary condition (1) approaches

$$\varphi(x + a) = \varphi(x).$$

Conclusions

In this paper we have given the Casimir energy of a massless scalar field on a one-dimensional generalized Möbius strip S^1_p as a fucntion of its periodicity p. The behavior of this function is shown in figure 1. It is interesting to note the sign changes: the Casimir energy is only postive for $2 \leq p \leq 4$.

One may note that, as these topologies occur naturally in string theory, this type of calculation may have applications therein. For example, the closed string is topologically S^1_1, while the closed string with fermionic modes corresponds to the S^1_2 topology [6]. Similarly, one may expect that the closed string with anyonic modes corresponds to the S^1_p topology. In considering the behavior and stability of closed strings with bosonic, fermionic, and anyonic modes, then, one may employ calculations of the type done in the present paper.

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Appendix: The Abel-Plana formula

The usual Abel-Plana formula (APF) reads [3, 6]

$$\sum_{n=0}^{\infty} f(n) - \int_0^\infty f(x)\,dx \;=\; \frac{f(0)}{2} - \frac{1}{2}\int_0^\infty f(it)\,[\cot(\pi it) + i]\,dt - \frac{1}{2}\int_0^\infty f(-it)\,[\cot(-\pi it) - i]\,dt,$$

if $f(z)$ is analytic for $\Re(z) > 0$. The sum on the LHS is the function f evaluated at the poles of the auxillary function $\cot(\pi z)$. We want to look at the function $f(x) = \frac{2\pi x}{a}$ summed over integers plus $\frac{1}{p}$, so we shift the poles of the cotangent:

$$\begin{aligned}\sum_{n=0}^{\infty} f\left(n+\frac{1}{p}\right) - \int_0^\infty f(x)\,dx \;&=\; -\frac{1}{2}\int_0^\infty f(it)\left[\cot\left(\pi it - \frac{\pi}{p}\right) + i\right]dt \\ &\quad -\frac{1}{2}\int_0^\infty f(-it)\left[\cot\left(-\pi it - \frac{\pi}{p}\right) - i\right]dt.\end{aligned} \tag{A-1}$$

Notice that the $\frac{f(0)}{2}$ term disappears, since now there is no pole at the origin. Using the identity

$$\cot\left(\pm\pi it - \frac{\pi}{p}\right) \pm i \;=\; \frac{\mp 2i}{e^{\pm 2\pi i/p}\,e^{2\pi t} - 1},$$

we can write (A-1) as

$$\sum_{n=0}^{\infty} f\left(n+\frac{1}{p}\right) - \int_0^\infty f(x)\,dx \;=\; i\int_0^\infty \frac{f(it)}{e^{2\pi i/p}\,e^{2\pi t} - 1}\,dt - i\int_0^\infty \frac{f(-it)}{e^{-2\pi i/p}\,e^{2\pi t} - 1}\,dt. \tag{A-2}$$

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