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On some ideas in Mathematics and Physics

Senior Thesis

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AN ELEMENTARY PROOF OF THE UNCOUNTABILITY OF REAL NUMBERS

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of natural numbers. The following fundamental property of real numbers is key in many analytic arguments involving the real line:

Given a non-decreasing sequence $s_1, s_2, ...$ with $s_i < M$ and $s_i,\ M \in \mathbb{R}$ there exists a **smallest** $s_\infty \in \mathbb{R}$ with $s_i \leq s_\infty$.

Our proof of the uncountability of real numbers has the following characteristics:

i) It is rigorous and yet avoids more elaborate constructions such as compactness and measure.

ii) It follows most directly from this fundamental property of real numbers which can also be used to prove connectedness and compactness of the unit interval. In this manner, the proof avoids decimal expansions which are commonly used in an elementary proof.

THE PROOF

Consider any $f : \mathbb{N} \to \mathbb{R}$. We construct a strictly increasing sequence $s_1, s_2, ...$ of real numbers such that:

a) $s_1 < f(1)$

b) For $k \in \mathbb{N}$, if $s_k < f(k)$ then $s_i < \frac{s_k + f(k)}{2}$ for all $i \in \mathbb{N}$.

Assuming such a construction is possible we conclude that the sequence is bounded above by $f(1)$. Let s_∞ be as above. If $s_\infty = f(k)$ then we have $s_k < s_\infty = f(k)$ since the sequence is strictly increasing. By construction $s_i < \frac{s_k + f(k)}{2}$ therefore $s_\infty \leq \frac{s_k + f(k)}{2} < f(k)$. This is a contradiction.

We show how to construct such a sequence. Let $s_1 = f(1) - 1$. Assume the first n terms of the sequence have been chosen such that:

$$\text{for } k \leq n, \text{ if } s_k < f(k) \text{ then } s_i < \tfrac{s_k + f(k)}{2} \text{ for } i \leq n$$

It follows that if $\epsilon > 0$ is sufficiently small $s_n + \epsilon < \frac{s_k + f(k)}{2}$ for each of the finitely many k's we must consider. Let $s_{n+1} = s_n + \epsilon$. For $k \leq n$, if $s_k < f(k)$ then by construction:

$$s_i < \tfrac{s_k + f(k)}{2} \text{ for } i \leq n + 1$$

If $s_{n+1} < f(n + 1)$ we have:

$$s_i \leq s_{n+1} < \tfrac{s_{n+1} + f(n+1)}{2} \text{ for } i \leq n + 1$$

The s_i form the desired sequence.

THE CAYLEY HAMILTON-THEOREM

Let k denote any algebraically closed field, $M(n,k)$ the set of $n \times n$ matrices with coefficients in k and $GL(n,k)$ the set of $n \times n$ invertible matrices with coefficients in k. We have the following elementary lemma:

Lemma. Given $A \in M(n,k)$ there exists $T \in GL(n,k)$ such that $T^{-1} \cdot A \cdot T$ is upper triangular.

Proof. Assume the lemma holds for $A' \in M(n-1,k)$. A has at least one eigenvector e_λ since $det(A - \lambda I) = 0$ has at least one solution. Complete $\{e_\lambda\}$ to a basis for k^n. If B is the matrix with this basis as columns, $B^{-1}AB$ has the form $\left(\begin{smallmatrix} \lambda & * \\ 0 & A' \end{smallmatrix}\right)$ where $A' \in M(n-1,k)$. By induction, there exists T' such that $T'^{-1}A'T'$ is upper-triangular. Let T be: $B \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & T' \end{smallmatrix}\right)$■

Let $p_A(x)$ be the characteristic polynomial of A ($p_A(x) = det(A - xI)$).

The Cayley-Hamilton Theorem. (Ver. 1) For $A \in M(n,k)$, $p_A(A) = 0$.

Proof. Since $T^{-1}p_A(A)T = p_{T^{-1}AT}(T^{-1}AT)$ it suffices to prove the theorem for an upper-triangular matrix.

Let $\{e_i\}$ be the standard basis for k^n, $V_0 = 0$ and $V_i = Span\{e_1, \cdots, e_i\}$. If A is upper-triangular $(A - \lambda_i) \cdot e_i = 0 \bmod(V_{i-1})$ where $\lambda_i = A_{ii}$. Since $(A - \lambda_i)V_{i-1} \subset V_{i-1}$ it follows that:

$$(A - \lambda_i)V_i = 0 \bmod(V_{i-1}).$$

For an upper-triangular A, $p_A(A) = \prod_{1 \le k \le n}(A - \lambda_k)$. Thus,

$$p_A(A)V_n = (A - \lambda_1)\cdots(A - \lambda_n)V_n = 0. \blacksquare$$

Let R be any commutative ring with an identity.

The Cayley-Hamilton Theorem. (Ver. 2) For $A \in M(n,R)$, $p_A(A) = 0$.

Proof. Let us first consider the special case of a domain. There exists an injection $i : M(n,R) \to M(n,k_R)$ where k_R is the algebraic closure of the quotient field of R. In this case the statement follows from the previous version.

In general, there exists a surjection $j : M(n,R') \to M(n,R)$ where R' is an integral domain. Assuming such a j always exists consider a $A' \in M(n,R')$ such that $j(A') = A$. Since p_A has as coefficients polynomials in the entries of A it follows that $j(p_{A'}) = p_A$. Then, since $p_{A'}(A') = 0$, we have $j(p_{A'}(A')) = p_A(A) = 0$.

It remains to construct j. Consider $1 \cdot \mathbb{Z} \subset R$. There exists a surjection $j'' : \mathbb{Z} \to 1 \cdot \mathbb{Z}$. Consider the polynomial ring $\mathbb{Z}[\{x_r\}]$ having one indeterminate for each $r \in R$. Extend j'' to $j' : \mathbb{Z}[\{x_r\}] \to R$ by mapping x_r to r. This allows us to define $j : M(n, \mathbb{Z}[\{x_r\}]) \to M(n,R)$. ■

GEOMETRIZATION OF CLASSICAL MECHANICS FOR MOTION IN ONE DIMENSION

1. Preliminaries from differential geometry

Let U be an open subset of $\mathbb{R}^n$. The **tangent space** at a point $x \in U$ (denoted by $T_x(U)$) is a copy of $\mathbb{R}^n$. We assume that $T_x(U) \cap T_y(U) = \varnothing$ for distinct x and y. Let e_i denote the standard basis for $\mathbb{R}^n$.

Definition. A **pseudo-riemannian metric** on U is a non-degenerate inner product $\langle , \rangle_x$ on $T_x(U)$ for each $x \in U$ such that $g_{ij}(x) = \langle e_i, e_j \rangle_x$ are smooth functions of x. A pseudo-riemannian metric is **riemannian** when $\langle v, v \rangle_x > 0$ for $v \neq 0 \in T_x(U)$.

Given a path $\gamma : [t_1, t_2] \to U$ one may define the **length** of gamma as:

$$L(\gamma) = \int_{t_1}^{t_2} \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} dt.$$

In this section we will only consider riemannian metrics. In addition, we assume that $\gamma'(t)$ never vanishes. Given x, $y \in U$, in general it is not true that there exists γ such that $L(\gamma)$ is minimal. For example, consider the $\mathbb{R}^2 - pt$ with the standard inner product. However, we have the following (see [MT]):

Theorem. Given $x \in U$ there exists a neighborhood V of x such that any two points in V may be joined by a unique (up to parametrization) path γ such that $L(\gamma)$ is minimal.

Definition. A path $\gamma : (a, b) \to U$ is a **geodesic** if γ' has unit lenght and for any $t \in (a, b)$ there exists a neighborhood V of t such for t_1, $t_2 \in V$, γ restricted to $[t_1, t_2]$ is the shortest path between $\gamma(t_1)$ and $\gamma(t_2)$.

There is an alternate characterization of geodesics as paths with no acceleration. This involves the concept of a connection to which we turn to next.

Definition. Given a path $\gamma : (a, b) \to U$ a **vector field along** γ is a smooth map $v : (a, b) \to \mathbb{R}^n$.

For example, given a smooth $\gamma : (a, b) \to U$ we may define v as $v(t) = \gamma'(t)$. In general, $v(t)$ may be written as $\sum_i v_i(t) \cdot e_i$ where e_i is the constant map to the standard basis vector.

Definition. An **affine connection** is a choice of n^3 smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1,..,n}$. The Γ_{ij}^k's are called the **Christoffel symbols**.

Given an affine connection one can define the derivative $\frac{Dv}{dt}$ of a vector field v along γ as follows:

$$\frac{Dv}{dt} = \sum_k \left(\frac{dv_k}{dt} + \sum_{i,j} \gamma_i' \cdot \Gamma_{ij}^k \cdot v_j\right) e_k$$

A vector field v is **parallel along** γ if $\frac{Dv}{dt} = 0$. Using theorems on existence and uniqueness of solutions to linear differential equations (see for example [H]) we have the following theorem:

Theorem. Given $v_0 \in T_{\gamma(0)}(U)$ there exists a unique vector field v parallel along γ with $v(0) = v_0$.

v is said to be obtained from v_0 by parallel translation. A connection is said to be compatible with a metric if given $v_0,\ w_0 \in T_{\gamma(0)}(U)$, $\langle v_0, w_0\rangle = \langle v(t), w(t)\rangle$. In particular, lenght of tangent vectors does not change under parallel translation. One of the fundamental results of riemannian geometry is that there is a unique connection which is compatible with a given metric (see for example [MT]):

Theorem. Given a metric on U there is a unique connection compatible with this metric. If g^{ij} is the inverse of g_{ij} then:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_r(\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij}) \cdot g^{rk}$$

It is natural to ask when $\frac{D\gamma'}{dt} = 0$ along γ. Using the definition of derivative along γ we see that γ_k satisfies the differential equation:

$$\left(\frac{d^2\gamma_k}{dt^2} + \sum_{i,j} \frac{d\gamma_i}{dt} \cdot \Gamma_{ij}^k \cdot \frac{d\gamma_j}{dt}\right) = 0$$

Given $\gamma(0)$ and $\gamma'(0)$ a unique solution $\gamma(t)$ exists for sufficiently small t. We also have the following theorem (see [MT]):

Theorem. A path γ is a geodesic iff $\frac{D\gamma'}{dt} = 0$ along γ.

2. An application to newtonian mechanics

Newtonian mechanics in one dimension, as interpreted in this text, is as follows. A physical configuration specifies a smooth function $V(x)$ defined for all relevant x. $V(x)$ is known as the potential of a system. Newton's equations of motion specify that the motion of a particle with mass m is governed by:

$$\frac{d^2x}{dt^2} = -\frac{1}{m} \cdot \frac{dV(x)}{dx}$$

We propose to investigate whether such a differential equation arises in a geometric context. More precisely, given an open subset of $\mathbb{R}^2$ with co-ordinates (x, t) and a function $V(x)$ can one define a metric g_{ij} such that geodesics (when parametrized in terms of t) satisfy the equation of motion stated above? In other words, can one encode the physical configuration in the geometry of space-time? Of course, there are several immediate restriction of the possible geodesics. Recall that to specify a geodesic one must specify both the position in space-time as well as a tangent vector. The tangent vector may be interpreted as giving the initial velocity to the particle. Therefore, we must exclude the tangent vector with just an x component as this vector corresponds to motion with infinite velocity. We prove the following result:

Theorem. Given an open set $U \subset \mathbb{R}^2$ with co-ordinates (x, t) and a smooth nonvanishing function $V(x)$ with $V(x) > 0$ there exists a metric g_{ij} on U with the following property: Given $(x_0, t_0) \in U$ and $(a, b) \in T_{(x_0,t_0)}(U)$ with $b > 0$ there exists a neighborhood U' of (x_0, t_0) such that if γ is a geodesic with $\gamma(0) = (x_0, t_0)$ and $\gamma(0)' = (a, b)$ it may be reparametrized as $(x(t), t)$ where:

$$\frac{d^2x}{dt^2} = -\frac{1}{m} \cdot \frac{dV(x)}{dx}$$

Furthermore, g_{ij} has the form:

$$\begin{pmatrix} \frac{m^2}{4V^2(x)} & 0 \\ 0 & \frac{m}{2V(x)} \end{pmatrix}$$

Proof. Let us first compute the Christoffel symbols associated to the metric. A direct computation reveals that:

$$\Gamma^x_{xx} = -\frac{dV(x)}{dx} \cdot \frac{1}{V(x)}; \; \Gamma^x_{xt} = \Gamma^x_{tx} = 0; \; \Gamma^x_{tt} = \frac{1}{m} \cdot \frac{dV(x)}{dx};$$

$$\Gamma^t_{xx} = 0; \; \Gamma^x_{xt} = \Gamma^x_{tx} = -\frac{dV(x)}{2dx} \cdot \frac{1}{V(x)}; \; \Gamma^x_{tt} = 0;$$

The equations for geodesics are:

$$\frac{d^2x}{ds^2} + \left(-\frac{dV(x)}{dx} \cdot \frac{1}{V(x)}\right)\left(\frac{dx}{ds}\right)^2 + \left(\frac{1}{m} \cdot \frac{dV(x)}{dx}\right)\left(\frac{dt}{ds}\right)^2 = 0$$

$$\frac{d^2t}{ds^2} + 2\left(-\frac{dV(x)}{2dx} \cdot \frac{1}{V(x)}\right)\left(\frac{dx}{ds}\right)\left(\frac{dt}{ds}\right) = 0$$

Choose a geodesic $\gamma(s)$ satisfying the initial conditions (we use s instead of t as the parametrizing variable since t is now one of the co-ordinates). Since $\frac{dt}{ds} \neq 0$ at (x_0, t_0) we can express s in terms of t in some neighborhood U' of (x_0, t_0). Using the chain rule we can rewrite the geodesic equations:

$$\left(\frac{d^2x}{dt^2}\right)\left(\frac{dt}{ds}\right)^2 + \left(\frac{dx}{dt}\right)\left(\frac{d^2t}{ds^2}\right) + \left(-\frac{dV(x)}{dx} \cdot \frac{1}{V(x)}\right)\left(\frac{dx}{dt} \cdot \frac{dt}{ds}\right)^2 + \left(\frac{1}{m} \cdot \frac{dV(x)}{dx}\right)\left(\frac{dt}{ds}\right)^2 = 0$$

$$\frac{d^2t}{ds^2} + 2\left(-\frac{dV(x)}{2dx} \cdot \frac{1}{V(x)}\right)\left(\frac{dx}{dt}\right)\left(\frac{dt}{ds}\right)^2 = 0$$

Replacing $\frac{d^2t}{ds^2}$ in the first equation by the expression given in the second equation we have the following differential equation for x in terms of t:

$$\left(\frac{dt}{ds}\right)^2 \cdot \left(\frac{d^2x}{dt^2} + \frac{1}{m} \cdot \frac{dV(x)}{dx}\right) = 0$$

Since $\frac{dt}{ds} \neq 0$ in U' the conclusion follows. ■

Now, assume $\gamma(t) = (x(t), t)$ is a smooth path in U such that: $\frac{d^2x}{dt^2} = -\frac{1}{m} \cdot \frac{dV(x)}{dx}$. At any point (x_0, t_0) if we choose a tangent vector $(x'(t), 1)$ the preceding theorem allows us to conclude that up to parametrization γ coincides with a geodesic. Therefore we have the following corollary:

Corollary. Given $\gamma(t) = (x(t), t)$ where $x(t)$ is a satisfies Newton's equation γ is up to a parametrization a geodesic. ■

We now turn to the Lagrangian formulation of classical mechanics. For a particle with mass m moving in one dimension with a potential $V(x)$ define the **Lagrangian** as:

$$L(a, b) = \frac{m}{2} \cdot b^2 + V(a)$$

Given any smooth $x(t) : [t_1, t_2] \to \mathbb{R}$ we look at the following integral (called the action):

$$\int_{t_0}^{t_1} L\left(x(t), \frac{dx}{dt}\right)dt = \int_{t_0}^{t_1} \left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right]dt$$

The **principle of least action** asserts that the physical path minimizes this action. More precisely, a physical path satisfies the Euler-Lagrange differential equation (see, for example [CV]):

$$\frac{d}{dt}\frac{\partial L}{\partial b} = \frac{\partial L}{\partial a}$$

We will say that such a path satisfies the necessary condition for an extremum. We propose an alternate action principle based on our geometric formulation of one-dimensional motion. As stated above, a path $\gamma(s)$ on a Riemannian manifold is a geodesic (up to parametrization) if it locally minimizes length. By a direct argument (or using the methods developed in [MT]) it follows that such a path satisfies the necessary condition for an extremum of the integral:

$$\int_{s_0}^{s_1} \sqrt{g_{xx} \cdot (\frac{dx}{ds})^2 + g_{tt} \cdot (\frac{dt}{ds})^2} ds = \int_{s_0}^{s_1} \sqrt{\frac{m^2}{4V^2(x)} \cdot (\frac{dx}{ds})^2 + \frac{m}{2V(x)} \cdot (\frac{dt}{ds})^2} ds$$

Now, if we retrict to paths which can be reparametrized as functions of t we obtain the following theorem:

Theorem. Given (x_0, t_0) and (x_1, t_1) the solution to Newton's equation is a path which satisfies the necessary condition for the extremum of:

$$\int_{t_0}^{t_1} \sqrt{\frac{m^2}{4V^2(x)} \cdot (\frac{dx}{dt})^2 + \frac{m}{2V(x)}} dt$$

Furthermore, if (x_0, t_0) and (x_1, t_1) are sufficiently close there exists a unique $\gamma(t)$ with $\gamma(t_0) = x_0$, $\gamma(t_1) = x_1$ which minimizes the value of the above integral. ■

Remark: The author does not know whether the theorems of this section generalize fully to two or three dimensional motion. Certainly, the form of the metric is very special to the case of one-dimensional motion.

Consider $\mathbb{R}^4$ viewed with co-ordinates (x, y, z, t). A motion of a physical body can be described by specifying for each moment in time the space co-ordinates of this body. In general, consider some subset S_{phys} of curves $\gamma : \mathbb{R} \to \mathbb{R}^4$ for which there is a reparametrization $\tilde{\gamma}$ such that $\tilde{\gamma}(t) = (f(t), t)$ where $f(t)$ is a smooth function of t. We say that γ represents a **physical path** if $\gamma \in S_{phys}$

Definition. A choice of a **reference frame** is a pair $(\mathbb{R}^4, S_{phys})$. A **change of a reference frame** is a diffeomorphism $\phi : (\mathbb{R}^4, S_{phys}) \to (\mathbb{R}^4, S'_{phys})$ with the property: $\phi(S_{phys}) = S'_{phys}$

Consider any $\phi : \mathbb{R}^4 \to \mathbb{R}^4$. Given a curve $\gamma : \mathbb{R} \to \mathbb{R}^4$, $\phi \circ \gamma$ defines another curve on $\mathbb{R}^4$ with $(\phi \circ \gamma)'_i = \sum_{k=1}^4 (\frac{\partial \phi_i}{\partial x_k}) \cdot \dot{\gamma}_k(t)$. Therefore, for each $r \in \mathbb{R}^4$, we have a mapping $\phi^* : T_r(\mathbb{R}^4) \to T_{\phi(r)}(\mathbb{R}^4)$.

Definition. Let $\mathbb{R}^4$ have an inner product $\langle , \rangle$. An **isometry** is a diffeomorphism $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ such that if $v, w \in T_r(\mathbb{R}^4)$ then $\langle w, v \rangle = \langle \phi^* w, \phi^* v \rangle$.

In the previous section we provided a mathematical model for the description of solutions to Newton's equations as geodesics on $\mathbb{R}^4$. One can show (see for example [KN]) that the image of a geodesic under an isometry is still a geodesic. Thus, if a change of reference frame is assumed to be an isometry, a solution to Newton's law in one frame may be interpreted as a solution in another. In the present section we specialize to the case where no forces are present. We would like to use the metric structure of space-time to completely characterize a physical situation. We make the following assumption:

Given a reference frame, the underlying $\mathbb{R}^4$ is equipped with an inner product such that the paths of physical bodies are geodesics. If a map $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ is an isometry it is change of reference frame.

We impose several additional constraints on the metric which arise from intuition. First we assume that physical laws do not change with time. Thus, the following transformation is an isometry:

a) Displacements in time: $(x, y, z, t) \mapsto (x, y, z, t + t_0)$
In addition, we assume space is Euclidean in the sense that the following are isometries:

b) Rotations and reflections and translations in the space co-ordinates:
$(x, y, z, t) \mapsto (A \cdot x + x_0, A \cdot y + y_0, A \cdot z + z_0, t)$ where $A \cdot A^T = I$

Finally, we suppose that:
c) The path $\gamma(t) = (0, 0, 0, t)$ is in S_{phys}

For each $pt \in \mathbb{R}^4$ let $g(pt)$ be a 4 by 4 matrix with $g_{ij}(pt) = \langle e_i, e_j \rangle_{pt}$ where e_i is the standard basis of $\mathbb{R}^4$. A diffeomorphism $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ is an isometry exactly when $(\phi^*)^T \cdot g(\phi(pt)) \cdot \phi^* = g(pt)$. Here, by abuse of notation, we let ϕ^* represent the matrix of the linear map in terms of the standard basis.

Theorem. Up to a constant, the only metric on $\mathbb{R}^4$ such that all isometries are changes of reference frame that satisfy the additional conditions a), b) and c) is the constant diagonal metric: $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -c^2$ where $c \neq 0$.

Proof. First, we show that the metric must be constant. Let pt_1, pt_2 be any two points in $\mathbb{R}^4$. If ϕ is a displacement in space-time with $\phi(pt_1) = pt_2$ we have $(\phi^*) = I$. Since ϕ is a composition of translations in space and time it is an isometry so we must have $g(pt_1) = g(pt_2)$.

Next, according to b), the linear map ϕ with $\phi(e_1) = -e_1$, $\phi(e_2) = e_2$, $\phi(e_3) = e_3$ and $\phi(e_4) = e_4$ is an isometry. We have $\phi^* = \phi$, therefore:

$$\langle e_1 + e_4, e_1 + e_4 \rangle = \langle \phi^*(e_1 + e_4), \phi^*(e_1 + e_4) \rangle = \langle -e_1 + e_4, -e_1 + e_4 \rangle$$

This implies $\langle e_1, e_4 \rangle = -\langle e_1, e_4 \rangle$. We conclude that $g_{14} = 0 = g_{41}$. Similarly, $g_{24} = g_{42} = g_{34} = g_{43} = 0$. The matrix g must have the form:

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 \\ g_{21} & g_{22} & g_{23} & 0 \\ g_{31} & g_{32} & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix}$$

Let ϕ be a rotation taking e_1 to e_2. We have:

$$g_{11} = \langle e_1, e_1 \rangle = \langle \phi^* e_1, \phi^* e_1 \rangle = \langle e_2, e_2 \rangle = g_{22}$$

Similarly, $g_{11} = g_{33}$. Now, let ϕ be a rotation taking e_1 to $\frac{e_1+e_2}{\sqrt{2}}$. Then:

$$g_{11} = \langle e_1, e_1 \rangle = \langle \phi^* e_1, \phi^* e_1 \rangle = \langle \tfrac{e_1+e_2}{\sqrt{2}}, \tfrac{e_1+e_2}{\sqrt{2}} \rangle = g_{11} + 2g_{12}$$

We conclude that $g_{12} = g_{21} = 0$. Also, $g_{31} = g_{13} = g_{23} = g_{32} = 0$. Thus, g is a diagonal matrix. After multiplying g by a constant, we can assume that $g_{11} = 1$.

We claim that $g_{44} < 0$. Suppose $g_{44} = c^2 > 0$. Let ϕ be the linear map such that $\phi(e_1) = \frac{e_4}{c}$, $\phi(e_2) = e_2$, $\phi(e_3) = e_3$ and $\phi(e_4) = ce_1$. ϕ is an isometry which maps the path $\gamma(t) = (0, 0, 0, t)$ to the path $(ct, 0, 0, 0)$. However, $\phi \circ \gamma \notin S_{phys}$ since it cannot be reparametrized as a function of time . We conclude that $g_{44} = -c^2$ for some $c \neq 0$. ■

With respect to this inner product we have three types of velocity vectors: $\langle v, v \rangle > 0$, $\langle v, v \rangle < 0$ and $\langle v, v \rangle = 0$. A vector $ae_1 + be_2 + de_3 + he_4$ has norm negative, zero, or positive respectively when:

$$\frac{\sqrt{a^2+b^2+d^2}}{|h|} < c, \ \frac{\sqrt{a^2+b^2+d^2}}{|h|} = c \text{ or } \frac{\sqrt{a^2+b^2+d^2}}{|h|} > c.$$

Thus, it represents motion in space with speed c. The motion of the origin in one reference frame may be represented by a path on the t-axis. Since the metric is constant this path in a different reference frame (x', y', z', t') may be viewed as a linear function of t'. Thus, with respect to each other, the two reference frames are moving with constant velocities of magnitude $< c$. One can show that given an object moving with speed $< c$ in one frame there always exists another frame where the object is at rest. Motion with speed $> c$ is unphysical since there always exist frames where the path is no longer a function of time. We have the following:

Corollary 1. Objects moving with speed c move with the same speed in all reference frames. Furthermore, they are the only objects which have this property.

The isometries which preserve this metric are known as Lorentz transformations. Usually, these transformations are derived from the assumption that the speed of light is constant in all frames. Our formulation derives these transformations from a different set of assumptions. From this viewpoint if Maxwell's equations have the same form in all reference frames then the speed of electromagnetic waves must be invariant. In such case we have:

Corollary 2. The constant c equals the speed of light in a vacuum.

Remark. Before the special theory relativity was accepted most physicists believed that transformation laws between two reference frames moving at constant velocity v with respect to each other had the form: $x \mapsto x + vt$, $t \mapsto t$. However, such transformations are not isometries with respect to any constant metric since for $v \neq 0$:

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = \begin{pmatrix} a + vc + vb + v^2 d & b + vd \\ c + vd & d \end{pmatrix} \neq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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