EXCEPTIONAL REGIONS AND ASSOCIATED EXCEPTIONAL HYPERBOLIC 3-MANIFOLDS

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Abstract. We investigate the seven exceptional families as defined in [GMT]. Experimental as well as rigorous evidence suggests that to each family corresponds exactly one manifold. A certain two generator subgroup in PSL$(2, \mathbb{C})$ is specified for each of the seven families in [GMT]. Using Newton's method for finding roots of polynomials in several variables we solve the relation equations specifying the generators to high precision. Then, using the LLL algorithm [Neum] we find exact entries of the generating matrices and in all cases verify with exact arithmetic that they satisfy the relations. This procedure allows us to compute the invariant trace fields [Neum] associated with the conjectured manifolds. In part, our results provide a verification of earlier results of K. Jones and A. Reid [JR] which were obtained by arithmetic methods. We carry out a search of the census of hyperbolic manifolds given in *SnapPea* and find hyperbolic manifolds with fundamental groups isomorphic to some of subgroups mentioned above. In addition, we obtain results on X_3 and X_4 which are not discussed in the K. Jones and A. Reid paper.

1. INTRODUCTION

The following important theorem is proved in [GMT]:

Theorem 1.1: Let N be a closed hyperbolic 3-manifold. Then
i) If $f : M \to N$ is a homotopy equivalence, where M is a closed irreducible 3-manifold, then f is homotopic to a homeomorphism.
ii) If $f, g : M \to N$ are homotopic homeomorphisms, then f is isotopic to g.
iii) The space of hyperbolic metrics on N is path connected.

The proof of this theorem is based on the following theorem of D. Gabai [7] which states that Theorem 1.1 is true if some closed geodesic in N has a *noncoalescable insulator family*. Thus, the main technical result of [1] is:

Theorem 1.2: If δ is a shortest geodesic in a closed hyperbolic 3-manifold, then δ has a non-coalescable insulator family.

A lemma of Gabai [Gabai] ensures that if δ is a core of an embedded hyperbolic tube of radius $ln(2)/3$ then δ has a noncoalescable insulator family. Theorem

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1.1 is proved by showing that all closed hyperbolic 3-manifolds, with seven exceptional cases, have embedded hyperbolic tubes of radius $ln(2)/3$ about their shortest geodesics. Then the authors of [GMT] show that any shortest geodesic in the six of the seven families satisfies a different condition that ensures that they have a non-coalescable insulator family. Finally, the seventh family is shown to correspond to the manifold known as Vol3, which by direct analysis satisfies the insulator condition.

We sketch some of the relevant definitions and the steps of the procedure used to prove theorem 1.2. A detailed description can be found in [GMT]. If a shortest geodesic in a hyperbolic manifold does not have the tube of the desired size there is a subgroup of its fundamental group generated by two elements and does not have this property. Thus, it is necessary to understand specific two-generator subgroups of PSL$(2,\mathbb{C})$. This set is parameterized by a subset of $\mathbb{C}^3$ as described in [GMT]. Each parameter corresponds to a 2-generator group with two specified generators called a *marked group*. Roughly speaking, the parameter space is subdivided into a billion small regions and it is shown that all but 7 cannot contain such a subgroup. Having eliminated all but these seven regions the remaining are eliminated by showing that if in six regions δ is a shortest geodesic with Corona(δ)$> 2\pi$ then the fundamental group of the manifold contains a marked subgroup which lies in the seventh region. As mentioned above the seventh region corresponds to Vol3 which can be handled by geometric techniques. The focus of this paper is the question of existence of manifolds associated to these seven regions.

In [GMT] it is conjectured that each family X_i (i from 1 to 6) contains a unique hyperbolic manifold N_i. K. Jones and A. Reid use arithmetic techniques to prove existence and uniqueness of the manifold associated to region X_0. Their methods allow them to find manifolds for all the regions with the exception of X_3 although questions of uniqueness remain open. In this paper we take an alternate approach by utilizing the invariants for the conjectured hyperbolic manifolds and searching the census of known manifolds to find a match. In addition we provide the calculations needed to identify X_3 in [Lipyan] A more detailed discussion of the invariants is carried out in Section 3.

2. THE TWO-GENERATOR SUBGROUPS

Here we discuss our methods for finding the marked subgroups of PSL$(2,\mathbb{C})$ mentioned above. In [GMT] parameter ranges for the seven regions are specified. *Quasirelators* - words in the two generators what are close to the identity throughout each region and experimentally converge to the identity at some point in the region are specified as well. In [GMT] the allowed values for the two generators are specified by three complex parameters: L', D', R'. Then the generators are defined as:

$$f = \begin{pmatrix} \sqrt{L'} & 0 \\ 0 & 1/\sqrt{L'} \end{pmatrix} \tag{1}$$

(2) $$w = \begin{pmatrix} \frac{\sqrt{R'}*(\sqrt{D'}+1/\sqrt{D'})}{2} & \frac{\sqrt{R'}*(\sqrt{D'}-1/\sqrt{D'})}{2} \\ \frac{(\sqrt{D'}-1/\sqrt{D'})}{2\sqrt{R'}} & \frac{(\sqrt{D'}+1/\sqrt{D'})}{2\sqrt{R'}} \end{pmatrix}$$

Letting $a = \sqrt{L'}$, $b = \sqrt{R'}$ and $c = \sqrt{D'}$ we get that:

(3) $$f = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$$

(4) $$w = \begin{pmatrix} \frac{b*(c+1/c)}{2} & \frac{b*(c-1/c)}{2} \\ \frac{(c-1/c)}{2b} & \frac{(c+1/c)}{2b} \end{pmatrix}$$

Approximate solutions to the relations for the generators are given in [GMT]. In particular, each of the seven regions corresponds to a range of values of L', D', R'. For example, for region X_0 the range is:

Table 1

Parameter	Range $Re(Parameter)$	Range $Im(Parameter)$
L'	-0.84065 to -0.84060	-2.13726 to -2.13722
D'	-0.84064 to -0.84059	-2.13729 to -2.13722
R'	0.999979 to 1.000022	-0.00006103 to 0.00006103

The corresponding quasi-relations for X_0 are:

$$r_1 = fwFwwFwfww$$
$$r_2 = FwfwfWfwfw$$

We solve for a, b and c such that the quasirelators are actually relations in the group. We obtain eight equations in 3 complex variables. From these three are independent and we use Newton's method to find high precision solutions for the parameters satisfying three of the equations. We compute an explicit example to make the procedure more clear. To obtain high precision numerical solutions the following version of Newton's Method was used. Given a smooth $f : \mathbb{C}^n \to \mathbb{C}^n$ we wish to recursively approach a zero of a polynomial by using linear approximations. Let x_i be the i^{th} approximation where $x_i \in \mathbb{C}^n$. Then x_{i+1} is given by:

(5) $$x_{i+1} = x_i - [Df(x_i)]^{-1} f(x_i)$$

Using this relation we calculate the solution to the group relations to say 100 digits of precision using the number theory package PARI-GP [PG]. This allows us to compute $tr(f^2)$, $tr(w^2)$ and $tr(f^2w^2)$ to high precision as well. The parameters for the seven regions are given in the Appendix.

3. THE INVARIANT TRACE FIELD

The results of this section can be found in D. Coulson, O.A. Goodman, C.D. Hodgson, and W.D. Neumann [Neum]. We state them here for convenience. One arithmetic invariant we can compute using high precision is the invariant trace field. Two finite volume, orientable, hyperbolic 3-manifolds are said to be commensurable if they have a common finite-sheeted cover. Subgroups G, $G' \subset \mathrm{PSL}(2,\mathbb{C})$ are commensurable if there exists $g \in$PSL$(2,\mathbb{C})$ such that g^{-1}Gg∩G' is a finite index subgroup of both $g^{-1}Gg$ and G'. It follows by Mostow rigidity (see W.D. Neumann [Neum2]) that finite volume, orientable, hyperbolic 3-manifolds are commensurable if and only if their fundamental groups are commensurable as subgroups of PSL$(2,\mathbb{C})$. Let G be a group of covering transformations and let $\tilde{G}$ be its preimage in SL(2,C). The traces of elements of $\tilde{G}$ generate a number field Q(trG). The *invariant trace field* $k(G)$ of G is defined as the intersection of all the fields $\mathbb{Q}(G_i)$ where G_i ranges over all finite index subgroups of G. The definition already makes clear that the invariant trace field is a commensurability invariant. It can be shown [Neum] that the invariant trace field in case of a subgroup $< f, w | r_1, r_2 >$ can be generated by $tr(f^2), tr(w^2), tr(f^2w^2)$. The three generators expressed in terms of the parameters L', D', R' are:

$$tr(f^2) = L' + \frac{1}{L'} \tag{6}$$

$$tr(w^2) = \frac{(R' + \frac{1}{R'} + 2)(D' + \frac{1}{D'} + 2) - 8}{4} \tag{7}$$

$$tr(f^2w^2) = \frac{(D' + \frac{1}{D'} + 2)(R'L' + \frac{1}{R'L'}) + (D' + \frac{1}{D'} - 2)(L' + \frac{1}{L'})}{4} \tag{8}$$

4. GUESSING THE ALGEBRAIC NUMBERS

Once the traces are obtained to high precision we use the algdep() function of the PARI-GP package [PG] to guess a polynomial over the integers that has the desired number as root. Although the algdep() function cannot prove that the guess is in fact correct, one can prove this by using the guess to perform exact arithmetic and verify the relations (Section 7). PARI-GP computes the guess using the LLL algorithm [Neum]. From experience, the correct polynomial should have relatively small coefficients. Thus, although much of the procedure is trial and error one tends to develop an intuition regarding when the "correct" polynomial is produced by the program. For example, let us consider guessing the polynomial that generates $tr(f^2)$ for the region X_3. Using the numeric approximation given in the first table we look for exact solution for $tr(f^2)$. We use the algdep() function to guess polynomials of degree 5, 11, 12, 14. Here are the results:

Table 3

Degree	Polynomial
5	$1390220588258799676501433704321276274721836221623712457831501851668 * x^5 - 7904560777029848010364803932513432200145307036607458496760395373 00 * x^4 + 111257693769337407863679156957984408268748181520419008363166881672 23 * x^3 + 953747715027490369074782595117814754950772243142419569742230476440 1 * x^2 - 7342400155031870283060569077451151327355162144846111309168811375553 * x - 75436586321908361310577647058029208261838927713975347006978431575 02$
11	$114891486296852866065411950154312 5 * x^{11} - 11449559918048081473732020014772 21 * x^{10} + 105617539840944338171209181702695 65 * x^9 + 273807635094684619032393879326358 9 * x^8 + 749582057079332969044885922591226 * x^7 - 214773941930681392948690446928575 5 * x^6 + 292367078600879721384987986754087 1 * x^5 - 503514865570315070709111448742338 * x^4 - 278433033282773366875185049952494 8 * x^3 - 389551914474206056940850178798410 6 * x^2 - 643442108082801659380678524982333 2 * x + 504943774128591915153457290453996 6$
12	$x^{12} + 6 * x^{11} + 23 * x^{10} + 91 * x^9 + 257 * x^8 + 489 * x^7 + 823 * x^6 + 1054 * x^5 - 13 * x^4 - 2445 * x^3 - 3405 * x^2 - 1847 * x - 337$
14	$8 * x^{14} + 38 * x^{13} + 201 * x^{12} + 960 * x^{11} + 2917 * x^{10} + 8349 * x^9 + 21483 * x^8 + 37855 * x^7 + 52727 * x^6 + 61728 * x^5 - 3791 * x^4 - 168991 * x^3 - 246411 * x^2 - 138849 * x - 25949$

As evident from the table the coefficients of the guessed polynomial become significantly smaller when degree twelve is reached. In fact, raising the degree beyond twelve does not help in this case since the polynomial of degree fourteen, for example, contains the one of degree twelve as a factor. Our initial guesses for the algebraic numbers which represent the roots were verified to be still correct when Newton's Method was allowed to compute the roots to 500 digit precision without recomputing the algebraic number. In this manner one gains confidence regarding the validity of the guess. Further discussion of the LLL algorithm can be found in [Neum]. Once we obtain $tr(f^2)$, $tr(w^2)$ and $tr(f^2w^2)$ we find a primitive element which generates the field that contains all three traces by trying several linear combinations of the three numbers. We summarize our findings in a table:

Table 4

Region	Trace Triple	Primitive Element
	trf^2	Minimal Polynomial
	trw^2	Numerical Value
	trf^2w^2	
X_0	x^2+2x+4	τ^2+3
	x^2+2x+4	$1.7320508075688772935i$
	x^2+2x+4	
X_1	$x^4+6x^3+19x^2+30x+17$	$\tau^4-2\tau^3-\tau^2+2\tau-1$
	$x^4+6x^3+19x^2+30x+17$	$0.5000000000000000000+$ $0.4052327261871812949i$
	$x^4+2x^3+25x^2-114x+103$	
X_2	x^2+4x+8	τ^2+1
	x^2+4x+8	i
	x^2+36	
X_3	$x^{12}+6x^{11}+23x^{10}+91x^9+257x^8$ $+489x^7+823x^6+1054x^5-13x^4$ $-2445x^3-3405x^2-1847x-337$	$\tau^{12}+6\tau^{11}+23\tau^{10}+91\tau^9$ $+257\tau^8+489\tau^7+823\tau^6$ $+1054\tau^5-13\tau^4-2445\tau^3$ $-3405\tau^2-1847\tau-337$
	$x^{12}+6x^{11}+23x^{10}+91x^9+257x^8$ $+489x^7+823x^6+1054x^5-13x^4$ $-2445x^3-3405x^2-1847x-337$	0.6327780003091672724 $-3.0191703764228631 5i$
	$x^{12}+15x^{11}+30x^{10}-408x^9$ $-793x^8+6070x^7+2155x^6-$ $50038x^5+90738x^4-45883x^3$ $-27526x^2+32149x-6227$	
X_4	$x^3+x^2+8x+16$	$\tau^3-\tau-2$
	$x^3+x^2+8x+16$	$-0.7606898534022837848+$ $0.8578736265951786364i$
	$x^3+14x^2+36x-104$	
X_5	x^3+2x^2+4x-8	$\tau^3-\tau^2-\tau-1$
	x^3+2x^2+4x-8	-0.4196433776070805663 $+0.6062907292071993693i$
	$x^3+2x^2+12x-8$	
X_6	x^3-2x^2+4x+8	$\tau^3-\tau^2-\tau-1$
	x^3-2x^2+4x+8	-0.4196433776070805663 $-0.6062907292071993693i$
	$x^3+2x^2+12x-8$	

5. Searching the Census

In [JR] Jones and Reid give the invariant trace field, approximate volume and first homology of the exceptional manifolds associated to the regions X_1, X_2, X_4 and X_5. We find manifolds from the *SnapPea* closed census which have the same invariants as given in [JR] and find isomorphisms from their fundamental groups to the groups $G_i =< f, w | r_1(X_i), r_2(X_i) >$, where $r_1(X_i)$ and $r_2(X_i)$ are quasirelators for the region X_i. The Snap package [Snap] includes a text file, called closed.fields, which lists the invariant trace field and several other fields for each of the manifolds in the closed census. Using this file, we were able to find the manifolds that have the same invariant trace fields as the four regions X_1, X_2, X_4, and X_5. Table 5 shows how many manifolds there are in each of these categories.

Table 5

Region	Invariant Trace Field	Number of Manifolds
X_1	$t^4 - 2t^3 + t^2 - 2t + 1$	13
X_2	$t^2 + 1$	37
X_4	$t^3 - t - 2$	16
X_5 or X_6	$t^3 - t^2 + t + 1$	36

Remark: The polynomials for the invariant trace fields for the regions X_1 and X_5 in Table 5 are different from the ones given in [JR]. It can be checked using PARI-GP that they give isomorphic fields. We give Snap description of fields as it is canonical.

Once we have the manifolds with the correct invariant trace fields listed, we check them against a list of manifolds having the volumes (V), first homologies (H_1), and shortest geodesic lengths (l_{min}) suggested for each region in [GMT] and [JR]. Table 6 below gives the approximate volume, first homology, approximate length of shortest geodesic and the *SnapPea* descriptions of the manifolds with those invariants.

Table 6

Region	V	H_1	l_{min}	Manifolds
X_1	4.11696874	$\mathbf{Z}_7 \oplus \mathbf{Z}_7$	1.0930	2678(2,1), v2796(1,2)
X_2	3.66386238	$\mathbf{Z}_4 \oplus \mathbf{Z}_{12}$	1.061	778(-3,1), v2018(2,1)
X_4	7.517689	$\mathbf{Z}_4 \oplus \mathbf{Z}_{12}$	1.2046	NA
X_5 or X_6	3.17729328	$\mathbf{Z}_4 \oplus \mathbf{Z}_4$	1.0595	479(-3,1), s480(-3,1), s645(1,2), s781(-1,2), v2018(-2,1)

SnapPea rigorously shows that in each region, the manifolds we found are all mutually isometric. The above manifolds include the manifolds mentioned in a remark in [GMT] for the regions X_1,X_2,X_5. It is shown in [JR] that the manifold associated to X_5 is isometric to the manifold associated with X_6. Unfortunately, manifolds with volumes as large as the volume for X_4 (or X_3) are not listed in the Weeks' Census. X_4 is the subject of Section 7. X_3 is discussed in [Lipyan].

6. The Isomorphisms

The manifold associated with each region is expected to have a fundamental group generated by two generators and satisfying two relations given by the quasirelators for that region, which are given in [GMT]. These groups have the form G_i as given above. The fundamental groups of the manifolds from the *SnapPea* census have a presentation with two generators and two relations. We let the generators for the manifolds be a and b. Table 7 shows the quasi-relators for each region and the relations for the fundamental group of each manifold (given by *SnapPea*).

With the help of the program testisom [Rees] we found isomorphisms between the fundamental groups of the above manifolds and the groups G_i. Table 7 shows the groups and Table 8 shows the isomorphisms for the three regions.

Table 7

Region	Quasi-Relators	Manifold	π_1 Relators
X_1	r_1=FFwFWFWfWFWFwFFww r_2=FFwwFwfwfWfwfwFww	$v2678$(2,1)	q_1=aabbabAbABAbAbabb q_2=aBaBABaBabaabbaab
X_2	r_1=FwfwfWffWfwfwFww r_2=FFwFFwwFwfwfwFww	$s778$(-3,1)	q_1=aBabaabbabbaabaB q_2=abbaabaabbabAbAb
X_5	r_1=FwFWFwFwfwfWfwfw r_2=FwfwfWfWFWfWfwfw	$s479$(-3,1)	q_1=abaabbaabaBBAABB q_2=aabbabbaabaBaBab

Table 8

Region	Isomorphism	Inverse
X_1	f⟶A, w⟶b	a⟶F, b⟶w
X_2	f⟶a, w⟶B	a⟶f, b⟶W
X_5	f⟶ab, w⟶b	a⟶fW, b⟶w

This shows that the above manifolds are the exceptional manifolds associated to the regions X_1, X_2 and X_5.

7. THE MANIFOLD ASSOCIATED TO THE REGION X_4

In this section we give a description of the manifold associated to the region X_4 as a double cover of an orbifold commensurable to a manifold in *SnapPea*'s census of closed manifolds[1] .

In Section 5 and 6 using the approximate volumes and other data given by Jones and Reid in [JR] we found manifolds from the *SnapPea*'s census of closed manifolds with fundamental groups isomorphic to the groups for the regions X_1, X_2 and X_5. The regions X_3 and X_4 could not be handled because of their large volumes. However for the region X_4 a list of manifolds was found in the closed census having half the volume of X_4 and the same arithmetic invariants. These manifolds are: $s297(1,3)$, $s298(5,1)$, $s594(1,2)$, $m307(-5,1)$, $m305(-5,1)$, $m369(-1,3)$, $m371(1,3)$, $m290(-1,4)$, $m390(3,1)$, $m293(-2,3)$, $m303(1,3)$, $s594(2,1)$, $s480(3,1)$, $s595(1,2)$, $s235(-4,3)$, and $s287(-3,1)$.

In a hope for obtaining the manifold for X_4 as a double cover of one of these manifolds we compared index two subgroups of the fundamental groups of each of these manifolds to G_4, the group for X_4. The index two subgroups were obtained using *GAP* [GAP]. Most of the subgroups were eliminated as they had different homology from G_4. One index two subgroup of the census manifold $m369(-1,3)$

[1]The first author would like to thank Walter Neumann for many helpful conversations and for showing him the geometry in the following proof.

had the same homology, the same orthodistances and lengths for its elements as for X_4.

Using the program *testisom* [Rees] it was checked that this subgroup was not isomorphic to the G_4. However using the similarity of the geometric information for this subgroup and G_4 we obtained an orbifold quotient of this subgroup which had the G_4 as an index two subgroup. The computations and presentations were obtained using *GAP* and *magnus* and the isomorphisms were checked using *testisom*.

Theorem 7.1 The manifold asociated to the region X_4 is commensurable to the manifold $m369(-1,3)$ in *SnapPea*'s census of closed manifolds. This manifold is obtained as a double cover of a orbifold which is double covered by a double cover of $m369(-1,3)$.

Proof. Let $M = m369(-1,3)$. Using *snap* we get a presentation of $\pi_1(M)$.

$$\pi_1(M) =< \text{a, b, c/aBAccbc, abcbbbAC, acbCBcbacb} >$$

Let $\phi : \pi_1(M) \to \mathbb{Z}_2$ be defined by $\phi(a) = 1, \phi(b) = \phi(c) = 0$. Then ϕ is a homomorphism and $\ker(\phi)$ is an index two subgroup of $\pi_1(M)$ generated by b and c. Let N denote the double cover of M corresponding to this subgroup so that $\pi_1(N) = \ker(\phi)$. Using *GAP* we obtain a presentation of $\pi_1(N)$.

$$\pi_1(N) =< \text{b, c}/\text{r}_1, \text{r}_2 >$$

where r_1 = bcbbbcbCBcbCBcbccbcccbcccbccbcBCbcBC and
r_2 = cbCBcbCBcbccbcccbcccbccbcccbcccbcccbccbcccbcccbcccbcBCbcBCb.

Let $\psi : \pi_1(N) \to \pi_1(N)$ be defined by $\psi(c) = \text{C}$ and $\psi(b) = \text{cccb}$. Then ψ is an isomorphism of $\pi_1(N)$ of order two. Extending the group $\pi_1(N)$ by this isomorphism we obtain a group H whose presentation is

$$H =< \text{b, c, t}/\text{r}_1, \text{r}_2, \text{tcTc}, \text{tbTBCCC}, \text{tt} >$$

.

$\pi_1(N)$ is a subgroup of H of index two and the quotient of $\mathbb{H}^3$ by H is an orbifold O (due to the torsion element t) which is double covered by N. Let $\mu : H \to \mathbb{Z}_2$ be defined by $\mu(c) = 0, \mu(b) = \mu(t) = 1$. Then μ is a homomorphism and $\ker(\mu)$ is an index 2 subgroup of H generated by elements c and bt. Again using *GAP* we obtain the presentation $\ker(\mu)$= G.

$$G = \ker(\mu) =< \text{x, y}/s_1, s_2, s_3 >$$

where s_1 = yXYXyXYXyxxyyxxxyxyxYxyxYxyxyxxxyyxx,
s_2 = yxYxyxYxyxyxxxyyxxyyxxxyyxxyyxxxyxyxYxyxYxyx and
s_3 = YXXXYXYXyXYXyxxyyxxxyxyxxxyyxxyXYXyXYX.

Then G is a torsion free subgroup of H of index 2. Hence it gives a manifold M_4 which double covers the orbifold O. The presentation for G_4 is

$$G_4 =< \text{f, w}/r_1(X_4), r_2(X_4) >$$

where $r_1(X_4) = $ FFwfwFwfWfwfWfwFwfwFFwwFWFwFWFww and
$r_2(X_4) = rmFFwfwFwfwFFwwFWFwFWfWFWfWFwFWFww$.

Using *testisom* we find an isomorphism $\nu : G_4 \to G$ given by $\nu(x)$ =f and $\nu(y)$ =FW. The inverse of ν is given by $\nu^{-1}(f)$ =y and $\nu^{-1}(w)$ =YX. We have the following diagram where every arrow denotes a $2:1$ covering map.

$$\begin{array}{ccc} N & & M_4 \\ \downarrow & \searrow & \downarrow \\ M & & O \end{array}$$

□

Remark 1. *The double cover N of $M = m369(-1,3)$ had the same L, D, R coordinates as for the the region X_4. Using snap one can see that*

```
snap 1.9
1.  :  read census 5 369 surgery -1 3
solution type:  geometric
1.  m369(-1,3):  print geodesics
cutoff radius ?  1.3
[0] 0.68306-2.05524*i bAC
[1] 0.95330-1.36689*i ACe
[2] 0.95330+1.77471*i B
[3] 0.95330+1.77471*i ACebb
[4] 1.20475-1.47049*i ACeb
```

The words are in the unsimplified fundamental group of M. The simplified and unsimflified groups can be obtained as:

```
1.  m369(-1,3):  print group
Fund.  group:  < a b c | aBAccbc abcbbbAC acbCBcbacb >
M0= CCab L0= cba
Unsimplified:  < a b c d e | cadE bceA dbC EaDDb ddEBDBdbbeDDbbeDDbbeDD
>
M0= DbdbeDD L0= bbeDD
```

The map from the unsimplified to the simplified group is a $\to$ ca, b$\to$ BC, c$\to$ C, d$\to$ b, e$\to$ ab.

The geodesic [4] has the same length as parameter L for X_4. In order to find a cover that has geodesic [4] as the shortest we need the first three words to go to 1 *in $\mathbb{Z}_2$. This gives us the map ϕ (with the simplified presentation). snap also finds ortholines.*

```
1.  m369(-1,3):  print ortholines
cutoff radius ?  1.1
geodesics to display ?  4
1.09508+1.90390*i 4:-0.13927-2.84396*i 4:  0.46311-0.43761*i BBEca
```

The first number is the distance between the geodesic [4] and itself. This number agrees with parameter D for X_4 up to πi. The slide can be computed by subtracting the 2nd number from the 3rd. This also agrees with parameter R for X_4 up to πi. The word in the generators of unsimplified group is the word which maps the geodesic onto its conjugate.

Even though the above numbers for N were similar to that of X_4, $\pi_1(N)$ is not isomorpic to G_4. Both groups G_4 and $\pi_1(N)$ are contained in the group of symmetries of the standard position for L, D and R for region X_4 as given in [GMT]. So we hoped that some extension of $\pi_1(N)$ should contain G_4 and so we started out by finding isomorphisms of $\pi_1(N)$. From the geometry we figured that a map of the form $c \to C$ and $b \to c^n b$ should work. This way we arrived at the isomorphism ψ above.

8. EXACT ARITHMETIC COMPUTATIONS

Once $a = \sqrt{L'}$, $b = \sqrt{R'}$, and $c = \sqrt{D'}$ have been found to high precision, Pari-GP can be used to guess minimal polynomials for a, b, and c. Then, if a field extension can be found containing a, b, and c, they can be represented exactly in PARI-GP. For the regions X_0, X_5, and X_6, a, b, and c are all contained in $\mathbb{Q}(a)$. By expressing the matrix entries as algebraic numbers one can verify the relations directly. For example, for X_0, the minimal polynomial for a and c is $x^8 + 2x^6 + 6x^4 + 2x^2 + 1$, and $b = 1$, so we can express a, b, and c as:

```
a = Mod(x, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1),

b = Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1), and

c = Mod(x, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1).
```

Then, using the formulae of Section 2, GP calculates the quasirelators exactly as:

```
[Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1) 0]

[0 Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1)].
```

Thus, exact arithmetic verifies rigorously that the L', D', and R' which were calculated for X_0 using Newton's method are correct – the quasirelators are in fact relators for these points.

In general, to perform exact arithmetic it is best to proceed indirectly. We follow the method described in [Lipyan]: Given that f, w are generic ($fw - wf$ is nonsingular), if f_2, w_2 are any matrices in $\mathrm{SL}(2, \mathbb{C})$ such that $tr(f_2) = tr(f), tr(w_2) = tr(w)$ and $tr(f_2^{-1} w_2) = tr(f^{-1} w)$ then the two pairs are conjugate. Thus, it suffices to solve the word problem for the group generated by f_2, and w_2.

Let $tr_1 = trace(f)$, $tr_2 = trace(w)$, $tr_3 = trace(f^{-1} w)$. Furthermore let:

$$f_2 = \begin{pmatrix} 0 & 1 \\ -1 & tr_1 \end{pmatrix} \tag{9}$$

$$w_2 = \begin{pmatrix} z & 0 \\ tr_1 * z - tr_3 & tr_2 - z \end{pmatrix} \tag{10}$$

Where $(tr_2 - z) * z = 1$. Then, according to [Lipyan], the pair (f_2, w_2) are conjugate to (f, w). Observe that in this form the entries of the matrices are in an at most degree two extension of the trace field. The coefficients of the original f, and w , may have arbitrary index over the trace field. As f_2, w_2 give an efficient way to solve the word problem Table 9 displays the computation of z, tr_1, tr_2 and tr_3 for all regions. In all cases z is the primitive element and $tr_i \in (z : \mathbb{Q})$. One easily verifies the relations using this table.

Table 9

Region	Trace Triple in terms of z	Primitive Element
	$tr(f) = tr_1$	Minimal Polynomial
	$tr(w) = tr_2$	Numerical Value
	$tr(f^{-1}w) = tr_3$	
X_0	$-z - 6z^3 - 2z^5 - z^7$	$\tau^8 + 2\tau^6 + 6\tau^4 + 2\tau^2 + 1$
	$tr(f)$	0.8532306966963658
	$(-5z^2 - 2z^4 - z^6)/2$	$-1.2524486580700i$
X_1	$2 - 4z + 4z^2 - 7z^3 + 4z^4 - 5z^5 + 2z^6 - z^7$	$\tau^8 - 2\tau^7 + 5\tau^6 - 4\tau^5$ $+7\tau^4 - 4\tau^3 + 5\tau^2 - 2\tau + 1$
	$tr(f)$	0.90404719571342
	$tr(f)$	$-1.471654223592238i$
X_2	$2 - 3z + 2z^2 - z^3$	$\tau^4 - 2\tau^3 + 4\tau^2 - 2\tau + 1$
	$tr(f)$	0.7429341358783
	$tr(f)$	$-1.5290855136357i$
X_3	$8 - 34z + 107z^2 - 261z^3 + 538z^4 - 972z^5$ $+1565z^6 - 2282z^7 + 3034z^8 - 3706z^9$ $+4171z^{10} - 4339z^{11} + 4171z^{12} - 3706z^{13}$ $+3034z^{14} - 2282z^{15} + 1565z^{16} - 972z^{17}$ $538z^{18} - 261z^{19} + 107z^{20} - 35z^{21}$ $+8z^{22} - z^{23}$	$\tau^{24} - 8\tau^{23} + 35\tau^{22} - 107\tau^{21} + 261\tau^{20}$ $-538\tau^{19} + 972\tau^{18} - 1565\tau^{17} + 2282\tau^{16}$ $-3034\tau^{15} + 3706\tau^{14} - 4171\tau^{13} + 4339\tau^{12}$ $-4171\tau^{11} + 3706\tau^{10} - 3034\tau^9 + 2282\tau^8$ $-1565\tau^7 + 972\tau^6 - 538\tau^5 + 261\tau^4$ $-107\tau^3 + 35\tau^2 - 8\tau + 1$
	$tr(f)$	1.4042922123248861
	$tr(f)$	$-1.1792672976569768i$
X_4	$3 - 4z + 4z^2 - 5z^3 + 3z^4 - z^5$	$\tau^6 - 3\tau^5 + 5\tau^4 - 4\tau^3 + 5\tau^2 - 3\tau + 1$
	$tr(f)$	1.3546199014688919
	$tr(f)$	$-1.22512545396285i$
X_5	$-z - 7z^3 + 4z^5 - 7z^7 - 2z^9 - z^{11}$	$\tau^{12} + 2\tau^{10} + 7\tau^8 - 4\tau^6 + 7\tau^4 + 2\tau^2 + 1$
	$tr(f)$	0.868063287033412
	$(-6z^2 + 4z^4 - 7z^6 - 2z^8 - z^{10})/2$	$-1.4600236661946i$
X_6	$3z - 7z^3 - 4z^5 - 7z^7 + 2z^9 - z^{11}$	$\tau^{12} - 2\tau^{10} + 7\tau^8 + 4\tau^6 + 7\tau^4 - 2\tau^2 + 1$
	$tr(f)$	1.4600236661946
	$(4 - 6z^2 - 4z^4 - 7z^6 + 2z^8 - z^{10})/2$	$-0.8680632870334i$

9. RESULTS AND CONJECTURES

Here we compare our results with the results and conjectures mentioned in [GMT] as well as [JR]. The invariant trace fields for all the regions with the exception of X_3 were computed by arithmetic methods in [JR]. Our approach, which utilizes a different method, confirms the results of [JR] where there is overlap. In addition, we were able to determine the invariant trace field corresponding to the region X_3 which is not discussed in [JR] as well prove that in fact there is a two generator subgroup in this region which satisfies the relations. A search in the Week's census of the *SnapPea* software failed to find a match for a manifold with the same invariant trace field as X_3. Our results imply progress in resolving the following conjectures stated in [GMT]:

i) Each family X_i (i from 1 to 6) contains a unique hyperbolic manifold N_i such that N_i has the fundamental group $< f, w | r_1(X_i), r_2(X_i) >$.

ii) If (L_i, D_i, R_i) is the parameter corresponding to the solution in X_i, then L_i, D_i, R_i are related as follows:
For $X_0, X_5, X_6 : L = D, R = 0$.
For $X_1, X_2, X_3, X_4 : R = \frac{L}{2}$.

Using exact arithmetic we are able to prove ii) for all regions. In the case i=1,2,5 we are able to find manifolds in the census which have fundamental groups isomorphic with groups found in the respective regions given by the quasirelators. For i=3 see [Lipyan].

10. Appendix

Table 2

Region	Coordinates	Trace Triple
	L'	trf^2
	D'	trw^2
	R'	trf^2w^2
X_0	−0.8406250193166066401943942440378308889772101025491253031197026326145173773487952343630988949291399753 − 2.137255282203144885899336161774001160199816955131967992241582633419447299799144215455249809406887481i	−1.000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 − 1.732050807568877293527446341505872366942805253810380628055806979451933016908800037081146186757248575i
	-0.8406250193166066401943942440378308889772101025491253031197026326145173773487952343630988949291399753 - 2.137255282203144885899336161774001160199816955131967992241582633419447299799144215455249809406887481i	-1.000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 - 1.732050807568877293527446341505872366942805253810380628055806979451933016908800037081146186757248575i
	1.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000	-1.000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 + 1.732050807568877293527446341505872366942805253810380628055806979451933016908800037081146186757248575i
X_1	-1.348464821739557342623304565646963657839403601108753512943730121656262789773970096214449447994424360 - 2.660889747796774236266373287312566203984323743551064633777827256703019216010510102045677418402183586i	-1.500000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 - 2.361869413144213207731349709285394504064256348896521995390218121406170501721821567653756557623043218i
	-0.5432102092500992323450198607193919842545307352034976295087122748535293384191072175962126195437040147 - 2.858605618394846642850684840755730329832775005792551120750888205699947951254745447296370582768503896i	-1.500000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 - 2.361869413144213207731349709285394504064256348896521995390218121406170501721821567653756557623043218i
	0.904047195713429435001656033	-2.621320343559642573202533

	8335212499034985703506483027338189671036750646119111331022021004331503957 - 1.4716542235922380986735713917552828885558358467087670082529826651043388386133922195276519372740917 46i	0863145471178545078130654221097650196069860987176931605582755813014914623 58 + 5.70205716976694237188480385495581122338172481284295040748884015307733905781421645867503193598690 7802i
X_2	-1.78615137775742328606955858584295892952312205783772323766490197010118204762231091371191288915850 8135 - 2.27201964951406896425242246173749149171560804184009624861664038253929757553606801183038421498846025i	-2.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000 - 2.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000i
	-1.0742478930552583201423985407438885608329176562587365880533536660745013054419373966086737767918 22732 - 2.718193355296798726177887906086991972659939689725054287545080082760069768202191472214042429229697 416i	-2.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000 - 2.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000i
	0.742934135878322839091431937947266281096242992001186505475869206219057763956878549059235662914976 0614 - 1.5290855136357461251609905237902252106193650498389097431407711763202398115791894627711485520734 84197i	6.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000i
X_3	0.581365258273134239521088707202614800804661456987585799019521149143221048636033558945740809235521 5268 - 3.312071764698212217535340448036617234993259374164797436334456541990130546085543693426468347795 847218i	0.632778000309167272400959683765424193330644885751627599142251726786421680405424723498396464369712 5073 - 3.019170376422863150395183428370855124001736365941486355888930402021470500988028720113360583124 679246i
	1.156593119241280584014135691114106342675299299162049846325342406262381483133303995213344699889178 062 - 2.75596636829692547441599327396825786748142331438223002020325212430010680063151751807241325556625 7354i	0.632778000309167272400959683765424193330644885751627599142251726786421680405424723498396464369712 5073 - 3.019170376422863150395183428370855124001736365941486355888930402021470500988028720113360583124 679246i
	1.404292212324886160678175784	-7.7444666387271393024411 45

	4383262349143661568038807452790713593996343711400085202368 55861067502040 - 1.1792672976 56976882827409344556527525394099586312704258612966633945307226446080991937630432943882616i	845729635438350823251168612255703877185268038145781611070254234333302947576 - 2.1 771834948638801393892861287791433161509037924969759242845027994368062375541841432 85954707902236518i
X_4	0.33406269950790129529005270210478819027385318980158977416771355111402011103849042612854 84567149980015 - 3.3191586434 683857760527194259848441448207387899293435838152645353063187490501611012312883462613011 47i	0.3640816006661915787790253122496368027833421422386404473184644666306169461158760790810794419078859150 - 3.0 208987796951012265043323487446451656101452645135847792892792135836265287076542644 42984272813688548i
	0.9774769899525187852593235852507483910417317707558491330408777380073255649226554316625 418370800908351 - 2.825096750 991532552472337360344249754464846881349870204719905502952428190606881686071780378167407 444i	0.3640816006661915787790253122496368027833421422386404473184644666306169461158760790810794419078859150 - 3.0 208987796951012265043323487446451656101452645135847792892792135836265287076542644 42984272813688548i
	1.354619901468891950804861034191318695574068815796968126003555790645168332011121744345977655161636381 - 1.2251254539 628540596326482771163730439652973718201679951859602832280197637707295246145014263412222 07i	-7.835975919081319551254191872264041937718485200806430832634565113023235318060012 489985332494274429801 - 0.8 945557998191306345004721956296707165752935592018853366921157631089667134604281289 326595393805086758i
X_5	-1.378135235553237550044770182928896996212172317211913679835301705234452612245132595273906887667431903 - 2.5347858856 470179260216909186416573684008674635167507270168473531831425203356187297264749104016300 30i	-1.543689012692076361570855971801747986525203297650983935240804037831168673927973 866485157914576059125 - 2.2 302850160798747194915292726300281363775561809359092072549360669620818185892710337 14514401374953045i
	-1.378135235553237550044770182928896996212172317211913679835301705234452612245132595273906887667431903 - 2.534785885 647017926021690918641657368400867463516750727016847353183142520335618729726474910401630 030i	-1.543689012692076361570855971801747986525203297650983935240804037831168673927973 866485157914576059125 - 2.2 302850160798747194915292726300281363775561809359092072549360669620818185892710337 14514401374953045i
	1.000000000000000000000000000	-1.295597742522084770980996

	00000000000000000000000000000000000000000000000000000000000000000000000	592851538613898975448446608311537954601573034548153992559253277504852247888 + 3.4428664744942734580102136666860741422766975177087526316847354395998183691515623577771702621337577727i
X_6	1.378135235553237550044770182928896996212172317211913679835301705234452612245132595273906887667431903 - 2.534785885647017926021690918641657368400867463516750727016847353183142520335618729726474910401630030i	1.543689012692076361570855971801747986525203297650983935240804037831168673927973866485157914576059125 - 2.2302850160798747194915292726300281363775561809359092072549360669620818185892710337145144013749530045i
	1.378135235553237550044770182928896996212172317211913679835301705234452612245132595273906887667431903 - 2.534785885647017926021690918641657368400867463516750727016847353183142520335618729726474910401630030i	1.543689012692076361570855971801747986525203297650983935240804037831168673927973866485157914576059125 - 2.2302850160798747194915292726300281363775561809359092072549360669620818185892710337145144013749530045i
	1.000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000	-1.295597742522084770980996592851538613898975448446608311537954601573034548153992559253277504852247888 - 3.4428664744942734580102136666860741422766975177087526316847354395998183691515623577771702621337577727i

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