Alternating sign matrices and tilings of Aztec rectangles

Undergraduate Thesis

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Abstract

The problem of counting numbers of tilings of certain regions has long interested researchers in a variety of disciplines. In recent years, many beautiful results have been obtained related to the enumeration of tilings of particular regions called Aztec diamonds. Problems currently under investigation include counting the tilings of related regions with holes and describing the behavior of random tilings.

Here we derive a recurrence relation for the number of domino tilings of Aztec rectangles with squares removed along one or both of the long edges. Through an interpretation of a sequence of alternating sign matrix rows as a family of nonintersecting lattice paths, we relate this enumeration to that of lozenge tilings of trapezoids, and use the Lindström-Gessel-Viennot theorem to express the number in terms of determinants.

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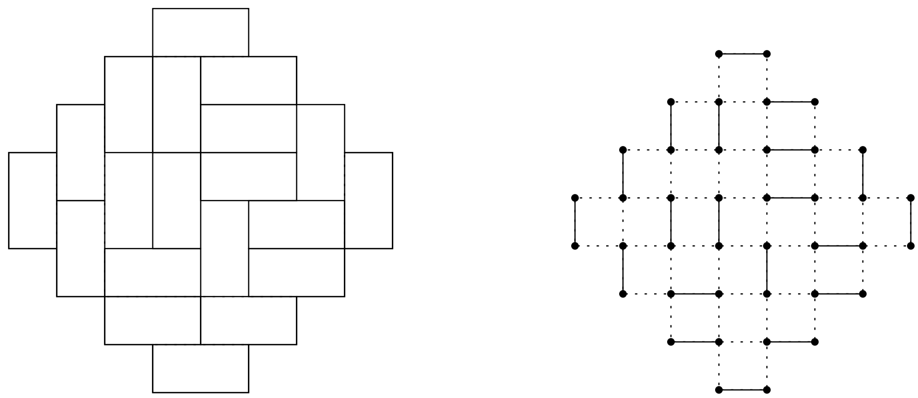

Figure 1: A domino tiling of the order-4 Aztec diamond and the corresponding matching of the Aztec diamond graph

1 Introduction

A **tiling** of a region R is a set of non-overlapping tiles whose union is R, where a **tile** may be any closed connected region in $\mathbb{R}^2$. For certain choices of regions and tiles, a tiling is equivalent to a perfect matching of a related graph. Given a bipartite graph G, a **perfect matching** of G is a set of pairs of adjacent vertices such that each vertex of G is in exactly one pair. (In the following we will frequently say "matching" to mean "perfect matching," the distinction not being relevant for our purposes.) For some time, the problem of enumerating the matchings of a graph has interested physicists and chemists. Chemists are typically concerned with the honeycomb lattice, since graphite and benzene rings bond in hexagonal configurations; physicists consider matchings of subsets of the infinite grid graph in studying dimer models in statistical mechanics.

In 1980, Grensing, Carlsen, and Zapp first considered one such family of subgraphs of the infinite grid, now known as Aztec diamond graphs [GCZ]. The **Aztec diamond** of order n is the union of unit squares lying entirely within the region $\{(x,y) : |x| + |y| \leq n+1\}$, and an Aztec diamond graph is the graph dual to this region, with vertices replacing squares, and an edge connecting two vertices if the squares they replace share an edge. It is not hard to see that matchings of Aztec diamond graphs correspond to tilings of Aztec diamonds by **dominoes** (i.e., 1×2 or 2×1 rectangles), as shown in Figure 1. Similarly, if we define a **lozenge** as the rhombus formed by the union of two equilateral unit triangles sharing an edge, a matching on a subset of the honeycomb graph is equivalent to a lozenge tiling of a hexagon (Figure 2).

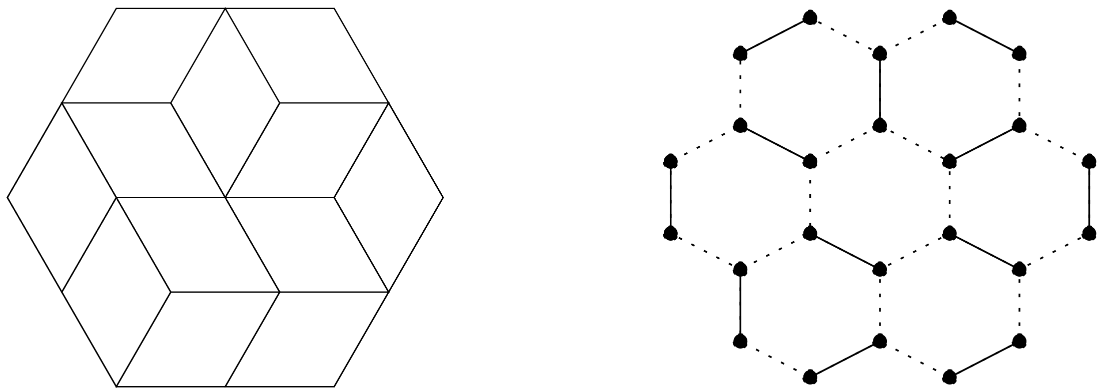

Figure 2: A lozenge tiling of the $(2, 2, 2)$ hexagon and the corresponding matching of the honeycomb graph

More recently, the combinatorial properties of matchings on certain graphs (and tilings of related regions) have attracted the attention of mathematicians. In the late 1980s, Elkies, Kuperberg, Larsen, and Propp examined domino tilings of Aztec diamonds, and proved by four distinct methods that the number of tilings is $2^{n(n+1)/2}$ [EKLP]. Since then, a number of interesting results have been obtained concerning domino tilings of Aztec diamonds, lozenge tilings of hexagons, tilings of regions with holes, and random tilings. See [P] for a more detailed history with references.

In the present paper, we shall examine tilings of subsets of Aztec diamonds called Aztec rectangles, and discuss the relationship between domino tilings of these regions and lozenge tilings of trapezoids (considered as subsets of hexagons). Related to both regions are combinatorial objects called alternating sign matrices, and these will appear in two separate connections.

An **Aztec rectangle** is an $(n-k) \times n$ subset of an order-n Aztec diamond, where the dimensions refer to the number of squares along the diagonal boundaries. (More precisely, it is the union of unit squares lying in $\{(x, y) : |x| + |y| \leq n + 1\}$ and above the line $y = x - (n + 1 - 2k)$.) An (a, b, a) **trapezoid** is a trapezoid with side lengths (counterclockwise from top) $a, b, a, a + b$. (See Figure 3.) An **alternating sign matrix** (ASM) of order n is an $n \times n$ square matrix whose entries are all 0, 1, or -1, nonzero entries alternate in sign along rows and columns, and the row sums and column sums are equal to 1. Finally, an $(n-k) \times n$ **partial ASM** is an $(n-k)$ by n matrix that can be obtained from an ASM by removing the last k rows. Let us denote the set of all order-n ASMs by $\mathcal{A}(n)$, and the set of all $(n-k) \times n$ partial ASMs by $\mathcal{A}(n, k)$.

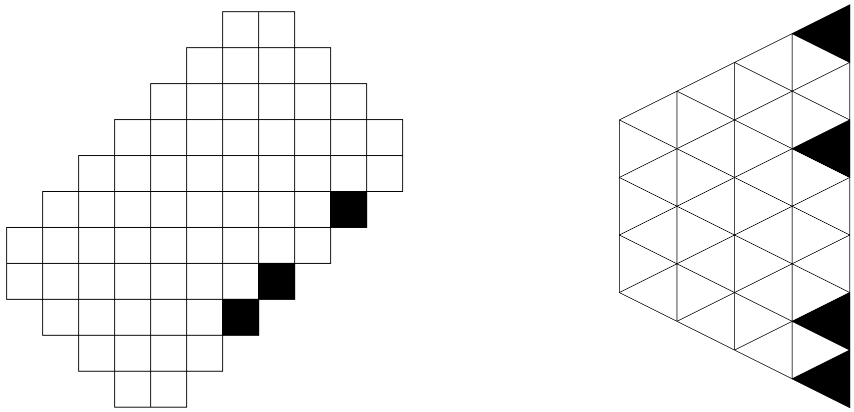

Figure 3: The Aztec rectangle $AR_{4\times 7}(2,0,1,1)$ and the trapezoid $TR_{4,3,4}(2,0,1,1)$

Note that for $k > 0$, an Aztec rectangle is not tileable: if the squares are colored in checkerboard fashion, we find that there are k more squares of one color than the other. Removing k of the dominant-colored squares from the bottom right edge, however, allows the region to be tiled, and these are the regions we shall consider in the following. There are $\binom{n}{k}$ ways of removing squares, so we introduce some notation to describe the boundary of an Aztec rectangle.

The boundary condition can be represented as a $(k+1)$-tuple of nonnegative integers, adding to $n-k$. Reading along the lower right edge from south to east, count the number of squares that remain in the rectangle between each square that is removed, including those before the first removed square and after the last. Thus the 4×7 rectangle in Figure 3 has boundary configuration (2,0,1,1); let us indicate this region by the notation $AR_{4\times 7}(2,0,1,1)$. For convenience we will also use this notation to indicate the number of domino tilings of the region.

Similarly, an (a,b,a) trapezoid is not tileable unless exactly b triangles are *present* on the right edge. Let $TR_{a,b,a}(P)$ be the number of lozenge tilings of the (a,b,a) trapezoid, where the entries of P count the number of triangles removed from the right edge between triangles present (reading from bottom to top).

Our main result is a recurrence relation on the number of domino tilings of Aztec rectangles. We adapt one of the proof methods of [EKLP] to obtain the following:

Theorem 3.1 *For Aztec rectangles, we have the following recurrence relation on the numbers of domino tilings:*

$$AR_{(n-k)\times n}(P) = 2^{n-k} \sum_{r_i} AR_{((n-1)-k)\times(n-1)}(P - r_i),$$

where the sum is taken over all order-$(k+1)$ ASM rows r_i, P is the $(k+1)$-tuple specifying the configuration of the larger rectangle, and we define $AR_{(n-k)\times n}(Q) = 0$ if Q has any negative entries.

For example, consider $AR_{3\times 5}(1,0,2)$. The theorem claims

$$AR_{3\times 5}(1,0,2) = 8 \times [AR_{2\times 4}(0,0,2) + AR_{2\times 4}(1,-1,2) + AR_{2\times 4}(1,0,1) + AR_{2\times 4}(0,1,1)].$$

Taking the second term on the RHS to be zero (as prescribed by the theorem), we can count the tilings of each rectangle to verify this equality. (For small n, there are computational methods that quickly count numbers of tilings.) Also, note that in the diamond case we have $k = 0$, $P = (n)$, and the only ASM row is $r = (1)$, so we recover the recurrence relation of [EKLP].

It follows that we can enumerate the tilings of an Aztec rectangle in terms of lattice paths in $\mathbb{R}^{k+1}$ whose steps are rows of ASMs. An **ASM-path** is a sequence of points $v_1, v_2, \ldots$ in $\mathbb{Z}^{k+1}$ such that $v_{i+1} = v_i + r_i$, where r_i is a row of an order-$(k+1)$ ASM. We have

$$AR_{(n-k)\times n}(P) = 2^{(n-k)[(n-k)+1]/2} \times \#(\Pi(P)),$$

where $\Pi(P)$ is the set of ASM-paths from the origin to P that remain in the nonnegative orthant.

The enumeration of domino tilings of Aztec rectangles with squares missing along one edge is not new; nor is that of lozenge tilings of trapezoids. A product formula for the former can be found in [HG] (Section 4 of [EKLP] also contains a proof using monotone triangles), and a similar formula for the latter appears in [CLP] and [HG]. Here, though, we emphasize the role of ASM-paths and the connection between Aztec rectangles and trapezoids.

In Section 2, we describe the height functions introduced in [EKLP] and their role in relating alternating sign matrices to tilings of Aztec rectangles. We prove Theorem 3.1 in Section 3, first rehearsing the proof of the case $k = 0$ given in Section 3 of [EKLP]. There we also express the number of tilings of Aztec rectangles in terms of ASM-paths. In Section 4, we relate ASM-paths to tilings of trapezoids and use the Lindström-Gessel-Viennot theorem to obtain an expression in terms of determinants. In Section 5, we generalize

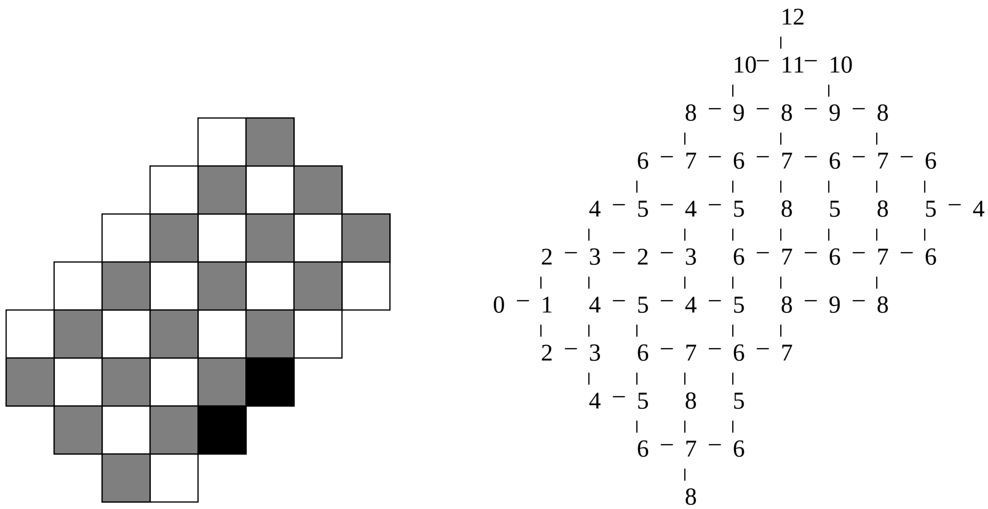

Figure 4: A height function on $AR_{3\times 5}(1,0,2)$

Theorem 3.1 to rectangles with squares removed from both long edges (Theorem 5.1), and introduce the related double-trapezoids (called “butterflies”). Finally, in Section 6 we mention some possibilities for future investigation.

2 Alternating sign matrices and height functions

The main idea of the proof of Theorem 3.1 is to draw a correspondence between partial ASMs and height functions on an Aztec rectangle. A **height function**, as defined in [EKLP], is a map that assigns nonnegative integer values to the vertices of the grid, subject to certain constraints. It can be shown that tilings of an Aztec rectangle are in one-to-one correspondence with height functions on its vertices. Roughly, one colors the squares checkerboard fashion (as in Figure 4) and stipulates that the heights of vertices increase by 1 as one follows the outline of a domino clockwise around a white square (or counterclockwise around a black square), and decrease by 1 going counterclockwise around a white square (or clockwise around a black square); for details see [EKLP]. (The origin of the name is more apparent if one looks at a lozenge tiling of a hexagon as a visual representation of a stack of boxes. In this context a height function describes the relative heights of the boxes. See Figure 2.)

A height function on an Aztec rectangle gives rise to two interlaced rectangular arrays of numbers – one containing the values of the height function on vertices (x, y) with $(x+y)$

odd, and one containing the values on vertices with $(x+y)$ even. Each array contains the values of diagonally adjacent vertices; since the values of horizontally or vertically adjacent vertices always differ by an odd number, the values of diagonally adjacent vertices are always of the same parity. Thus the values in the first array ($(x+y)$ odd) are all even, and the values in the second are all odd. If we form matrices from each of these arrays, a height function on an $(n-k)\times n$ Aztec rectangle yields one $[(n-k)+2]\times(n+2)$ matrix (with even entries) and one $[(n-k)+1]\times(n+1)$ matrix (with odd entries). Call the former an (n,k)**-even matrix** and denote the set of all such matrices by $\mathcal{E}(n,k)$; likewise call the latter (n,k)**-odd** and denote the set by $\mathcal{O}(n,k)$. For example, the $(5,2)$-even and -odd matrices associated with the height function in Figure 4 are

$$E=\begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 & 12 \\ 2 & 4 & 2 & 4 & 6 & 8 & 10 \\ 4 & 6 & 4 & 6 & 8 & 6 & 8 \\ 6 & 8 & 6 & 8 & 6 & 8 & 6 \\ 8 & 6 & & & 8 & 6 & 4 \end{pmatrix}, \qquad D=\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 3 & 5 & 3 & 5 & 7 & 9 \\ 5 & 7 & 5 & 7 & 5 & 7 \\ 7 & 5 & 7 & 9 & 7 & 5 \end{pmatrix}.$$

Notice that the even matrix has entries missing from its bottom row, corresponding to the squares removed from the Aztec rectangle. A few other observations are worth noting at this point:

i. All horizontally or vertically adjacent entries differ by 2.

ii. In particular, the entries increase by 2 along the top row and left column, and decrease by 2 along the right column.

iii. In the bottom row of the odd matrix, the entries increase by 2 where a removed square separates the corresponding vertices of the rectangle, and otherwise decrease by 2.

iv. The bottom two rows of the even matrix exhibit the following general pattern:

$$\begin{pmatrix} \cdots\cdots\cdots & & & b_{i+1} & \cdots & b_{i+l} & \cdots\cdots\cdots & \\ a_0 & \cdots & a_i & \emptyset & \cdots & \emptyset & a_{i+l+1} & \cdots \end{pmatrix},$$

where adjacent a_j decrease by 2, adjacent b_j increase by 2, and for any group of l consecutive missing entries, $a_i=b_{i+1}$ and $a_{i+l+1}=b_{i+l}$. A missing entry at position j of the bottom row of the matrix corresponds to a removed square in position j on the bottom-right edge of the rectangle.

All these facts are consequences of the definition of a height function.

Facts (i) and (ii) allow us to construct a bijection between $\mathcal{A}(n,k)$ and (n,k)-odd matrices, and something like a bijection between $\mathcal{A}(n+1,k)$ and (n,k)-even matrices. To see the connection between height functions and ASMs, we introduce the skew-summation operation described in [EKLP].

Given a partial ASM $A = (a_{ij}) \in \mathcal{A}(n,k)$, define the **skew-summation** of A to be $A^* = (a^*_{ij})$, where

$$a^*_{ij} = i + j - 2\left(\sum_{i'=1}^{i}\sum_{j'=1}^{j} a_{i'j'}\right),$$

for $0 \le i \le n-k$, $0 \le j \le n$. Thus if

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix},$$

we have

$$A^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 & 2 & 3 \\ 3 & 2 & 3 & 4 & 3 & 2 \end{pmatrix}.$$

Conversely, given a matrix B whose entries differ by 1 along rows and columns, increase along the first row and column, and decrease along the last column, we can invert this operation to obtain a partial ASM. Let the inverse skew-summation $\widetilde{B}$ be the matrix obtained by replacing each submatrix of B of the form

$$\begin{pmatrix} b_{i-1,j-1} & b_{i-1,j} \\ b_{i,j-1} & b_{i,j} \end{pmatrix}$$

with the entry

$$\frac{1}{2}(b_{i-1,j} + b_{i,j-1} - b_{i-1,j-1} - b_{i,j}).$$

Facts (i) and (ii) above ensure that if we take an (n,k)-odd matrix, subtract one from each entry and divide by 2, we can perform the inverse skew-summation operation to obtain a partial ASM. If D is an (n,k)-odd matrix, call the $(n-k)\times n$ partial ASM obtained in this way the inverse skew-sum of D, and denote it by $\widetilde{D}$; similarly for $E \in \mathcal{E}(n,k)$.

(Note, however, that the missing entries in the bottom row of E will result in missing entries in the bottom row of $\widetilde{E}$, so $\widetilde{E}$ is not a proper partial ASM.)

Facts (iii) and (iv) will be important in proving Theorem 3.1, since they tell us how to read the boundary configuration of an Aztec rectangle from the height function matrices. The entries of P count the number of decreasing steps between each increasing step in the bottom row of an odd matrix; they also count the number of present entries between each missing entry in the bottom row of an even matrix (excluding the first and last entries, which are always present). If we wish to denote the set of (n, k)-even or -odd matrices that determine a given configuration P, we will write $\mathcal{E}(n, k; P)$ or $\mathcal{O}(n, k; P)$. (Warning: these sets no longer correspond bijectively with $\mathcal{A}(n, k)$ or $\mathcal{A}(n+1, k)$, but only with particular subsets.)

3 A recurrence relation for Aztec rectangles

We restate our main result:

Theorem 3.1 *For Aztec rectangles, we have the following recurrence relation on the numbers of domino tilings:*

$$AR_{(n-k)\times n}(P) = 2^{n-k} \sum_{r_i} AR_{((n-1)-k)\times(n-1)}(P - r_i), \tag{1}$$

where the sum is taken over all order-$(k+1)$ ASM rows r_i, P is the $(k+1)$-tuple specifying the configuration of the larger rectangle, and we define $AR_{(n-k)\times n}(Q) = 0$ if Q has any negative entries.

Before proving Theorem 3.1, we recount the proof of the case $k = 0$, as given in Section 3 of [EKLP]. The argument is clearer in this case, and it will serve to highlight the differences between the diamond and rectangular cases. We shall denote the number of domino tilings of the order-n Aztec diamond by $AD_n = AR_{n\times n}(n)$.

Theorem 3.2 (Diamond case) *The following recurrence relation holds on tilings of Aztec diamonds:*

$$AD_n = 2^n \times AD_{n-1}.$$

Proof. Since a tiling of AD_n corresponds to a height function on its vertices, it yields a pair of matrices: one containing the even heights, and one containing the odd heights.

Conversely, any "compatible" pair of matrices satisfying conditions (i) and (ii) from page 6 determine a tiling of the diamond. The pair must be compatible in the sense that when superimposed on the grid, they give a legal height function on the diamond. In fact, given an odd matrix D, the rules of height functions determine the entries of a compatible even matrix E, except where D has a submatrix of form

$$\begin{pmatrix} a_{i-1,j-1} & a_{i-1,j} \\ a_{i,j-1} & a_{i,j} \end{pmatrix} = \begin{pmatrix} 2m-1 & 2m+1 \\ 2m+1 & 2m-1 \end{pmatrix};$$

there the compatible even entry $b_{i,j}$ may be either $2m+2$ or $2m-2$. This submatrix maps to a 1 in the inverse skew-sum $\widetilde{D}$, so for every 1 in $\widetilde{D}$, we have a choice between two compatible even matrices – and therefore two tilings of the diamond. Thus we obtain the enumeration

$$AD_n = \sum_{\widetilde{D}} 2^{N_+(\widetilde{D})},$$

where $N_+(\widetilde{D})$ is the number of 1's in the ASM $\widetilde{D}$, and the sum is taken over all $\widetilde{D}$ such that D is an n-odd matrix. In the diamond case, D is square and the inverse skew-summation maps the set of all D to the set of all order-n ASMs, so in fact we have

$$AD_n = \sum_{A \in \mathcal{A}(n)} 2^{N_+(A)}. \tag{2}$$

Similar consideration applies to the even matrices. Here we find that for an n-even matrix E, ambiguity in the choice of a compatible odd matrix corresponds to the occurrence of a -1 in $\widetilde{E}$. As before, we have

$$AD_n = \sum_{\widetilde{E}} 2^{N_-(\widetilde{E})},$$

and hence, since in the diamond case E is square and is missing no entries,

$$AD_n = \sum_{B \in \mathcal{A}(n+1)} 2^{N_-(A)}. \tag{3}$$

In any row of an ASM, there must be one more $+1$ than there are -1's, so for $A \in \mathcal{A}(n)$, $N_+(A) = N_-(A) + n$. Thus (2) becomes

$$\begin{aligned} AD_n &= \sum_{A \in \mathcal{A}(n)} 2^{N_-(A)+n} \\ &= 2^n \sum_{A \in \mathcal{A}(n)} 2^{N_-(A)}. \end{aligned} \tag{4}$$

Finally, if we relabel in (3) by $n \to n-1$ and $B \to A$, and substitute into (4), we obtain the desired relation. ∎

We now proceed to prove the recurrence relation (1) for general k.

Proof of Theorem 3.1. As in the diamond case, a tiling of $AR_{(n-k)\times n}(P)$ is equivalent to a compatible pair of matrices, one from $\mathcal{E}(n,k;P)$ and one from $\mathcal{O}(n,k;P)$. The argument proceeds identically as above to obtain

$$AR_{(n-k)\times n}(P) = \sum_{\widetilde{D}} 2^{N_+(\widetilde{D})}, \tag{5}$$

summing over all $\widetilde{D}$ such that $D \in \mathcal{O}(n,k;P)$.

Since entries are missing from the bottom row of E, the bottom row of $\widetilde{E}$ is undetermined. However, in specifying the boundary configuration P, we have already ruled out any ambiguity in the bottom row of the compatible odd matrix, so we ignore the bottom row of $\widetilde{E}$ anyway. Just counting -1's in the rest of the matrix, we get

$$AR_{(n-k)\times n}(P) = \sum_{\widetilde{E}} 2^{N'_-(\widetilde{E})}, \tag{6}$$

for $E \in \mathcal{E}(n,k;P)$, where $N'_-(\widetilde{E})$ is the number of -1's in the partial ASM $\widetilde{E}$, excluding the bottom row.

In analogy with the proof of the diamond case, we want an expression for the number of tilings of $[(n-1)-k] \times (n-1)$ rectangles in terms of elements of $\mathcal{A}(n,k)$. Using $N_+(\widetilde{D}) = N_-(\widetilde{D}) + (n-k)$, we can manipulate Equation (5) to obtain

$$\begin{aligned}
AR_{(n-k)\times n}(P) &= \sum_{\widetilde{D}\,|\,D\in\mathcal{O}(n,k;P)} 2^{N_-(\widetilde{D})+(n-k)} \\
&= \sum_{\widetilde{D}\,|\,D\in\mathcal{O}(n,k;P)} 2^{N'_-(\widetilde{D})+(\text{number of -1's in bottom row of } \widetilde{D})+(n-k)} \\
&= 2^{n-k} \sum_{\widetilde{D}\,|\,D\in\mathcal{O}(n,k;P)} 2^{N'_-(\widetilde{D})} 2^{(\text{number of -1's in bottom row of } \widetilde{D})},
\end{aligned} \tag{7}$$

which already looks good, except for the noisome extra power of 2 inside the sum.

To find an expression for numbers of tilings of $[(n-1)-k] \times (n-1)$ rectangles in the RHS of (7), we must rewrite the sum index in terms of $(n-1,k)$-even matrices, as in (6). Let E_D be the $[(n-k)+1] \times (n+1)$ matrix obtained by subtracting 1 from each entry of D. While all its entries are even, E_D is not a proper $(n-1,k)$-even matrix because no entries are missing from its bottom row. We can remove entries to form a proper even matrix, but there may be more than one way to do this in compliance with the requirement of Fact (iv) on page 6. For example, if

$$D = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 3 & 5 & 3 & 5 & 7 & 9 \\ 5 & 7 & 5 & 7 & 5 & 7 \\ 7 & 5 & 7 & 9 & 7 & 5 \end{pmatrix}, \quad E_D = \begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 2 & 4 & 2 & 4 & 6 & 8 \\ 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 8 & 6 & 4 \end{pmatrix},$$

then we can make two different $(n-1,k)$-even matrices from E_D:

$$\begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 2 & 4 & 2 & 4 & 6 & 8 \\ 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & & 6 & & 6 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 2 & 4 & 2 & 4 & 6 & 8 \\ 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & & & 6 & 4 \end{pmatrix}.$$

This is significant, as these two matrices describe Aztec rectangles with different boundary conditions ($AR_{2\times 4}(0,1,1)$ and $AR_{2\times 4}(1,0,1)$, respectively).

Another way of viewing this ambiguity is to notice that the partial ASM $\widetilde{E_D}$ has a -1 in its bottom row, so there is an ambiguity in the corresponding bottom-row entry of a compatible $(n-1,k)$-odd matrix:

$$\begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 2 & 4 & 2 & 4 & 6 & 8 \\ 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 8 & 6 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & \ldots & \ldots & \ldots & 7 \\ 5 & 7 & 5 & 7 & 5 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & \ldots & \ldots & \ldots & 7 \\ 5 & 3 & 5 & 7 & 5 \end{pmatrix}.$$

Let us say that these different odd bottom rows are **compatible** with E_D, and since each odd bottom row determines the boundary configuration P_i of an $[(n-1)-k]\times(n-1)$ rectangle, we can also say these P_i are compatible with E_D. Finally, D determines a configuration P for an $(n-k)\times n$ rectangle, so we say P_i is compatible with P if P_i is compatible with E_D, for some $D \in \mathcal{O}(n,k;P)$.

Evidently, there are $2^{(\text{number of -1's in bottom row of } \widetilde{E_D})}$ different odd bottom rows compatible with E_D, and hence this many configurations P_i compatible with E_D. This explains the power of 2 in (7), since $\widetilde{E_D} = \widetilde{D}$ (recall the definition of $\widetilde{D}$ on page 7). Now we can write

$$AR_{(n-k)\times n}(P) = 2^{n-k} \sum_{P_i \text{ compatible with } P} \; \sum_{\widetilde{E}\,|\,E\in\mathcal{E}(n-1,k;P_i)} 2^{N'_-(\widetilde{E})}. \tag{8}$$

The power of 2 has been absorbed by double-counting: the same partial ASM may appear in different terms of the outer sum, in the guise of different bottom-row configurations, when multiple P_i's are compatible with the same E_D (with $D \in \mathcal{O}(n,k;P)$). Using the expression from (6), we have

$$AR_{(n-k)\times n}(P) = 2^{n-k} \sum_{P_i \text{ compatible with } P} AR_{((n-1)-k)\times(n-1)}(P_i). \tag{9}$$

All that remains is to show that the P_i which are compatible with a given P are exactly those obtained by subtracting an ASM row from P. Consider the configuration $P = (e_1, \ldots, e_{k+1})$. According to Fact (iii) on page 6, the bottom row of the corresponding (n,k)-odd matrix is

$$(\underbrace{a_0 \underset{-2}{,} a_1 \underset{-2}{,} \ldots \underset{-2}{,} a_{j_1}}_{e_1} \underset{+2}{,} \underbrace{a_{j_1+1} \underset{-2}{,} \ldots \underset{-2}{,} a_{j_2}}_{e_2} \underset{+2}{,} \underbrace{a_{j_2+1} \underset{-2}{,} \ldots \underset{-2}{,} a_{j_3}}_{e_3}, \ldots); \tag{10}$$

i.e., the entries decrease by e_i consecutive steps between each increase.

Before we remove entries, the $(n-1,k)$-even bottom row will have the same form, with $b_i = a_i - 1$. Fact (iv) on page 6 says that in the bottom row of a proper $(n-1,k)$-even matrix, adjacent entries must always decrease by 2. It follows from the condition on the penultimate row that in general, if bottom-row entries b_j and b_{j+l+1} are separated by l consecutive removed entries, then $b_{j+l+1} = b_j + 2(l-1)$. We want to know all ways of removing k entries such that this rule is satisfied. Since there are exactly k places where the entries increase by 2, we must simply remove an entry to the left or right of the increase. That is, wherever $b_j = b_{j-1} + 2$, remove either b_j or b_{j-1}. An even matrix bottom row looks like this:

$$(b_0, \underbrace{b_1, \ldots, b_{i_1-1}}_{f_1}, \emptyset, \underbrace{b_{i_1+1}, \ldots, b_{i_2-1}}_{f_2}, \emptyset, \underbrace{b_{i_2+1}, \ldots,}_{f_3} \emptyset, \ldots), \tag{11}$$

for configuration $(f_1, f_2, \ldots, f_{k+1})$. Comparing (10) and (11), we see that configurations $(f_1, \ldots, f_{k+1})$ obtained by removing entries at jumps correspond to configurations obtained by subtracting a row of an order-$(k+1)$ ASM from $(e_1, \ldots, e_{k+1})$. (Moving from left to right, a 1 in the ASM row tells us to remove from the right side of a jump, a -1 says to remove from the left side, and a 0 says to remove from the same side as the previous removal.)

Finally, because of the pattern of $+2$ and -2 steps shown in (10), removing an entry anywhere other than at the jumps violates the requirement that the entries increase by $2(l-1)$ across a gap of l removed entries. ■

The base of the recurrence relation (1) is the $n = k$ case $AR_{0\times k}(0,0,\ldots,0)$. In this trivial case, there is exactly one tiling: all squares must be removed, and no dominoes

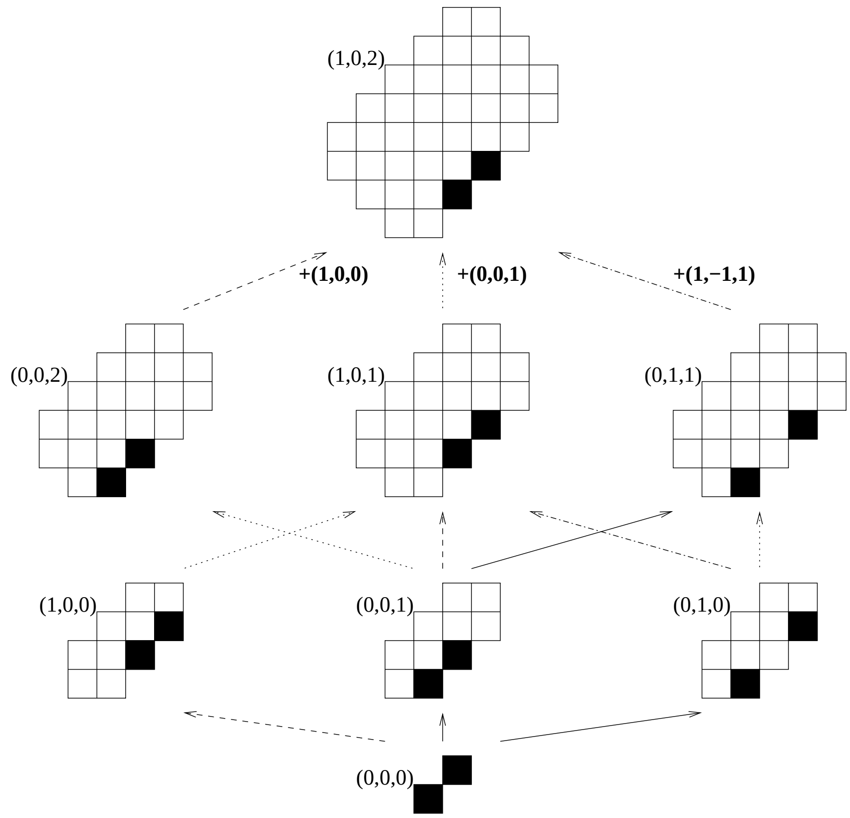

Figure 5: The evolution of $AR_{3\times 5}(1,0,2)$ from the base case $AR_{0\times 2}(0,0,0)$

are used. Theorem 3.1 thus says that every order-$(k+1)$ ASM-path from the origin to P represents a sequence of counts of tilings of smaller rectangles, and each path contributes a factor of 2^i at the ith step to $AR_{(n-k)\times n}(P)$. (Figure 5 retraces a rectangle to the base case. An ASM-path from the origin to $(1,0,2)$ corresponds to a path following the arrows from bottom to top.) If we let $\Pi(P)$ be the set of all ASM-paths in the nonnegative orthant from the origin to P, we have

Corollary 3.3 *The number of domino tilings of the Aztec rectangle $AR_{(n-k)\times n}(P)$ is*

$$\begin{aligned} AR_{(n-k)\times n}(P) &= \left(\prod_{i=1}^{n-k} 2^i\right) \times \#(\Pi(P)) \\ &= 2^{(n-k)[(n-k)+1]/2} \times \#(\Pi(P)). \end{aligned} \tag{12}$$

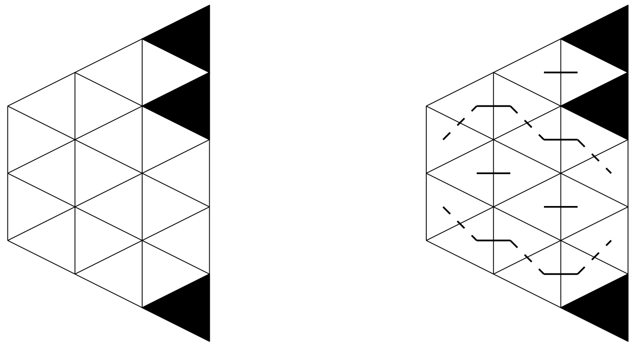

Figure 6: The trapezoid $TR_{3,2,3}(1,0,2)$ and a family of nonintersecting lattice paths corresponding to the ASM-path $(0,1,0)+(0,0,1)+(1,-1,1)$

4 ASM-paths and trapezoids

An ASM-path in the nonnegative orthant, considered as a series of ASM rows, can be interpreted as a family of nonintersecting lattice paths. From bottom to top, let $t_0^{(0)}, t_1^{(0)}, \ldots, t_{k+1}^{(0)}$ be points aligned vertically in the plane, each separated by a distance at least 1. From each $t_j^{(0)}$, take a step by either $(1, \frac{1}{2})$ or $(1, -\frac{1}{2})$ to get $t_j^{(1)}$. For $1 \leq j \leq k+1$, let $d_j^{(i)} = t_j^{(i)} - t_{j-1}^{(i)}$. If we stipulate that $t_0^{(0)}$ steps down and $t_{k+1}^{(0)}$ steps up, the $(k+1)$-tuple of integers

$$(d_1^{(1)} - d_1^{(0)}, d_2^{(1)} - d_2^{(0)}, \ldots, d_{k+1}^{(1)} - d_{k+1}^{(0)})$$

is a row of an ASM.

Such families of nonintersecting lattice paths correspond to lozenge tilings of $(n-k, k, n-k)$ trapezoids, as shown in Figure 6. Consider the dual graph, and remove all but the horizontal edges (solid lines). Now, starting from the k vertices on the left, draw nonintersecting paths connecting the endpoints of the edges (dotted lines). The set of all diagonal edges of the paths and all horizontal edges not included in any path constitutes a matching on the graph, and hence a lozenge tiling of the trapezoid. If we also imagine paths along the upper and lower boundaries of the trapezoid, we see that this construction of nonintersecting paths is equivalent to the above interpretation of an ASM-path. Thus, every order-$(k+1)$ ASM-path in the nonnegative orthant from the origin to the point P corresponds to a lozenge tiling of $TR_{n-k,k,n-k}(P)$, and we have

Corollary 4.1 *Let AD_n be the number of domino tilings of the order-n Aztec diamond. Then*

$$\begin{aligned} AR_{(n-k)\times n}(P) &= 2^{(n-k)[(n-k)+1]/2} \times TR_{n-k,k,n-k}(P) \\ &= AD_{n-k} \times TR_{n-k,k,n-k}(P). \end{aligned}$$

Also, since

$$\#(\Pi(P)) = \sum_{r_i} \#(\Pi(P - r_i)),$$

we have an analogue to Theorem 3.1:

Theorem 4.2 *The following recurrence relation holds on lozenge tilings of trapezoids:*

$$TR_{n-k,k,n-k}(P) = \sum_{r_i} TR_{(n-1)-k,k,(n-1)-k}(P - r_i), \tag{13}$$

where the sum is taken over all order-$(k+1)$ ASM rows r_i.

We can express this enumeration of tilings in terms of a determinant. In the graph dual to the trapezoid $TR_{n-k,k,n-k}(P)$, call the left-most vertices $u_1, \ldots, u_k$ (from bottom to top), and call the right-most vertices $v_1, \ldots, v_n$ – including removed vertices. If $P = (x_0, \ldots, x_k)$, the right-most present vertices will be $v_{l_1}, \ldots, v_{l_k}$, where $l_j = j + \sum_{m=0}^{j-1} x_m$. One version of the Lindström-Gessel-Viennot theorem [L, GV] says that the number of families of nonintersecting lattice paths from $\{u_1, \ldots, u_k\}$ to $\{v_{l_1}, \ldots, v_{l_k}\}$ is $\det(a_{ij})_{i,j=1}^m$, where a_{ij} is the number of lattice paths from u_i to v_{l_j}. The number of paths from u_i to v_{l_j} is a binomial coefficient: a path takes $(n-k)$ steps, and exactly $(l_j - i)$ of them must move upward. Thus,

$$a_{ij} = \binom{n-k}{l_j - i},$$

where we use the convention $\binom{x}{y} = 0$ if $y < 0$ or $y > x$. In terms of entries of P, we can write

$$n - k = \sum_{m=0}^{k} x_m = X$$

and

$$l_j - i = (j - i) + \sum_{m=0}^{j-1} x_m = Y_{ij},$$

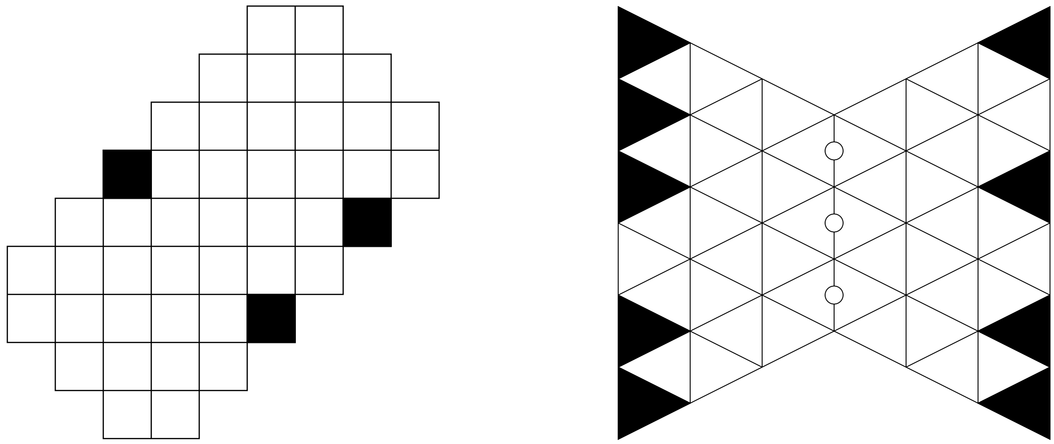

Figure 7: The Aztec rectangle $AR_{3\times 6}(2,3;2,1,1)$ and the butterfly $B_{3,3,3}(2,3;2,1,1)$

so we have

$$TR_{n-k,k,n-k}(P) = \left| \begin{pmatrix} X \\ Y_{ij} \end{pmatrix} \right|_{i,j=1}^{k}, \tag{14}$$

and by Corollary 4.1,

$$AR_{(n-k)\times n}(P) = 2^{(n-k)[(n-k)+1]/2} \left| \begin{pmatrix} X \\ Y_{ij} \end{pmatrix} \right|_{i,j=1}^{k}. \tag{15}$$

5 Butterflies and more Aztec rectangles

The proof of Theorem 3.1 extends naturally to include the case of an $(n-k)\times n$ Aztec rectangle with squares removed from both long edges. If we let P represent the configuration on the South $\rightarrow$ East edge (as before), and let Q represent the configuration on the West $\rightarrow$ North edge, denote the rectangle (and the number of tilings) by $AR_{(n-k)\times n}(Q;P)$. Note that P has k_1+1 entries and Q has k_2+1 entries, with $k=k_1+k_2$; the entries of P sum to $n-k_1$ and the entries of Q sum to $n-k_2$. We can generalize the recurrence relation (1):

Theorem 5.1 *For Aztec rectangles with removed squares in configuration P on the bottom edge and configuration Q on the top edge, we have*

$$AR_{(n-k)\times n}(Q;P) = 2^{n-k}\sum_{r_i,s_j} AR_{((n-1)-k)\times(n-1)}(Q-s_j;P-r_i), \tag{16}$$

taking the sum over all order-(k_1+1) ASM rows r_i and order-(k_2+1) ASM rows s_j.

Proof. Simply apply the argument in the proof of Theorem 3.1 to both sides. We need to expand the definition of an $(n-k)\times n$ partial ASM to include matrices obtained by removing the first k_2 rows, as well as the last k_1 rows, from order-n ASMs. The proof follows as a straightforward generalization. ∎

The base of the recurrence is not trivial in this case. As in the one-sided case, we have a $0\times k$ rectangle in which all k squares are removed – but we consider k_1 of them to be removed from the "bottom" and k_2 removed from the "top." Instead of starting at the origin, then, the recurrence starts at a pair of configurations $(O_2;O_1)$ specifying which squares are removed from the top and which from the bottom. (Obviously, O_1 and O_2 must specify complementary configurations, since a given square can be removed only once. Also, the entries of O_1 sum to k_2, and the entries of O_2 sum to k_1.) The statement analogous to Corollary 3.3 is therefore

Corollary 5.2 *The number of domino tilings of the Aztec rectangle $AR_{(n-k)\times n}(Q;P)$ is*

$$AR_{(n-k)\times n}(Q;P) \;=\; 2^{(n-k)[(n-k)+1]/2}\sum_{\textit{pairs }(O_2;O_1)} \#(\Pi(O_1,P))\times\#(\Pi(O_2,Q)), \tag{17}$$

where $\Pi(X,Y)$ is the set of ASM-paths in the nonnegative orthant from X to Y.

Once again, there is an interpretation of the RHS of (17) in terms of lozenge tilings. We define the **butterfly** $B_{a,b,a}(Q;P)$ as follows: Take the trapezoid $TR_{a,b,a}(P)$ and the mirror image of $TR_{a,b,a}(Q)$, and identify the edges of length b. For purposes of tiling, the b pairs of triangles adjacent along this identified edge should be considered identified, as well. (The circles in Figure 7 are meant to symbolize this identification.) A tiling of $B_{a,b,a}(Q;P)$ corresponds to a pair of families of nonintersecting lattice paths, as shown in Figure 8, so the number of tilings is

$$B_{n-k,k,n-k}(Q;P) = \sum_{\text{pairs }(O_2;O_1)} \#(\Pi(O_1,P))\times\#(\Pi(O_2,Q)). \tag{18}$$

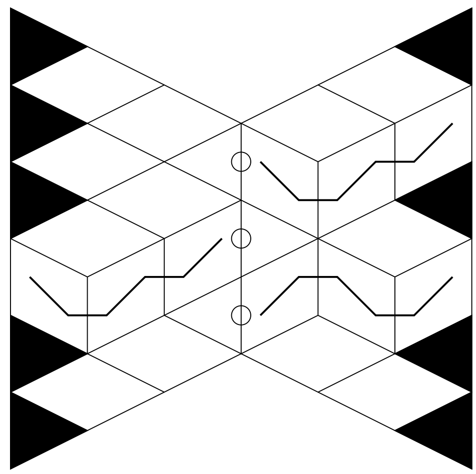

Figure 8: A pair of families of nonintersecting lattice paths on $B_{3,3,3}(2,3;2,1,1)$

We can use the Lindström-Gessel-Viennot theorem to write

$$B_{n-k,k,n-k}(Q;P) = \sum_{(O_2;O_1)} \det(a_{ij})_{i,j=1}^{k_1} \times \det(b_{ij})_{i,j=1}^{k_2}, \tag{19}$$

where a_{ij} is the number of paths from the ith triangle specified by O_1 to the jth triangle specified by P, and b_{ij} is the number of paths from the ith triangle specified by O_2 to the jth triangle specified by Q. Finally, we summarize Eqs. (17), (18), and (19) as

$$AR_{(n-k)\times n}(Q;P) = AD_{n-k} \times B_{n-k,k,n-k}(Q;P) \tag{20}$$

$$AR_{(n-k)\times n}(Q;P) = 2^{(n-k)[(n-k)+1]/2} \sum_{(O_2;O_1)} \det(a_{ij})_{i,j=1}^{k_1} \times \det(b_{ij})_{i,j=1}^{k_2}. \tag{21}$$

6 Closing remarks

We have results relating domino tilings of Aztec diamonds, lozenge tilings of trapezoids and butterflies, and lattice paths whose steps are ASM rows. Corollary 4.1 is particularly suggestive of a connection between domino tilings and lozenge tilings, but more illuminating would be a bijective proof giving a $2^{(n-k)[(n-k)+1]/2}$-to-1 mapping from tilings of Aztec rectangles to tilings of trapezoids. However, it is not obvious that such a mapping exists, since a given tiling of an Aztec rectangle may correspond to more than one ASM-path (and therefore to more than one lozenge tiling of the trapezoid).

It might also be interesting to consider tilings of the entire plane (or half-plane) by dominoes or lozenges. Is it possible to find a relation transforming a tiling of the **Aztec half-plane** (i.e., the union of all unit squares lying above $y = x - 1$, with some squares removed along the lower-right boundary) to a tiling of the **trapezoid half-plane** ($\{(x, y) : x \leq 0\}$, with triangles removed along the right boundary)? The methods used in deriving Corollary 4.1 work locally, so perhaps a transformation can be found even if a bijective proof is not forthcoming.

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