Continuous Choice of Summands in Convolution Sums on a Lie Group

Undergraduate Thesis
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Introduction

In this paper we enhance upon a result obtained by Dixmier and Malliavin in their paper [1]. There they prove the following result (with notation modified from the original for better clarity in what follows):

Theorem 1 (Factorization) *Let G be a Lie group, V a neighbourhood of the identity in G, and $u \in \mathcal{D}(G)$. Then u is a finite sum of functions of the form $v_1 \star v_2$, where $v_1, v_2 \in \mathcal{D}(G)$, $supp(v_1) \subset V$, $supp(v_2) \subset supp(u)$.*

The theorem we prove is

Theorem 2 (Continuous Factorization) *Let G be a Lie group, V a neighbourhood of the identity in G, and $u \in \mathcal{D}(G)$. Then u is a finite sum of functions of the form $v_1 \star v_2$, where $v_1, v_2 \in \mathcal{D}(G)$, $supp(v_1) \subset V$, $supp(v_2) \subset supp(u)$, and where v_1 and v_2 depend continuously on u in the space $\mathcal{D}(G)$.*

This result can be translated to a useful result concerning a continuous representation of G on a complete metrizable space (see [1]). A different, weaker statement with a similar purpose has been recently used in [2]. For the origins of the related Theorem1 the reader can also consult [3] and [4]. A reference for notation and standard facts from functional analysis is [5]. I wish to thank Prof. Jacquet who suggested the problem and whose remarks and advice I greatly appreciate.

To recall, the space $\mathcal{D}(\mathbb{R})$ consists of the infinitely differentiable functions with compact support and with bounded derivatives of all orders. We note that convergence in $\mathcal{D}(\mathbb{R})$ means the following: the sequence of functions $\phi_1, \phi_2, \dots, \phi_l, \dots$ converges to a function ϕ iff all the ϕ_i have support inside a fixed compact set and for any k, given $\epsilon > 0$, there exits L, such that for $l > L$ we have $|D^k \phi_l - D^k \phi| < \epsilon$ (here D^k denotes the k-th derrivative). Thus continuous dependence $\mathcal{D}(\mathbb{R})$ can be determined by this concept for convergence. More generally one considers the space $\mathcal{D}(G)$ for any differentiable manifold G.

The construction in [1] is such that the number of terms in the factorization is bounded by a constant depending only on the dimension of G.

Preliminary construction

As in [1] consider a real function $\phi(x)$ where

$$\phi(x) = \prod_{k=0}^{+\infty} (1 + \frac{x^2}{\lambda_k^2}), \tag{1}$$

and $\lambda_0, \lambda_1, \ldots$ are a subsequence of $(1, 2, \ldots, 2^k, \ldots)$. This function can be extended to an entire function on $\mathbb{C}$, and if we denote $1/\phi$ by $\chi(x)$, then χ is a meromorphic function with simple poles which decreases faster than 1 over any polynomial. Calculations in [1] show that the Fourier transform

$$\psi = \int_{-\infty}^{+\infty} e^{-i\pi xy} \chi dx \tag{2}$$

is a Schwartz function, $\psi \in \mathcal{S}(\mathbb{R})$, and thus $\chi \in \mathcal{S}(\mathbb{R})$. Furthermore the following estimate is obtained for $\epsilon > 0$ *independently of the sequence* λ

$$sup_{y>\epsilon} |\frac{d^n \psi}{dy^n}| < (2\pi)^n \sum_{j=0}^{+\infty} 2^{(n+1)j} e^{-2\pi 2^j} = P_n. \tag{3}$$

Now we give a new, stronger, version of the key Lemma 1 used in [1]:

Lemma 1 *Let $(\beta_0, \beta_1, \ldots,)$ be a sequence of positive numbers. Then there exists a sequence of positive numbers $(\gamma_0, \gamma_1, \ldots)$ and functions $g \in \mathcal{D}(\mathbb{R})$, $h \in \mathcal{D}(\mathbb{R})$ of support inside $[\epsilon, -\epsilon]$ for any given $\epsilon > 0$ such that:*

(i) $\gamma_n \leq \beta_n$ for $n \geq 1$

(ii) $\sum_{n=0}^{p} (-1)^n \gamma_n \delta^{2n} \star g \longrightarrow \delta + h$ in $\mathcal{E}'(\mathbb{R})$ as $p \longrightarrow +\infty$

(iii) The sequences γ so produced satisfy the following additional condition. Say that for the sequence $\beta = (\beta_0, \beta_1, \ldots)$ we have a sequence of sequences β^l:

$$\big(\beta^1 = (\beta_0^1, \beta_1^1, \ldots), \beta^2 = (\beta_0^2, \beta_1^2, \ldots), \ldots\big)$$

which converge pointwise to β, i.e. given $N \in \mathbb{Z}$ and given $\epsilon > 0$ there exists L such that for $l > L$ we have

$$|\beta_i^l - \beta_i| < \epsilon \text{ for all } i \leq N \text{ and all } l \geq L.$$

Then if we denote by γ^l the sequence corresponding to β^l produced by (i) and (ii) the sequences γ^l converge pointwise to γ in the above sense.

(iv) If the sequences β^l converge to β as in (iii) and h_l, g_l correspond to β^l while g, h correspond to β, then the sequences h_l and g_l converge to h and g respectively in $\mathcal{D}(\mathbb{R})$.

Proof of Lemma This is modelled after the proof in [1]; however a number of additional constructions have been made in order to achieve the last two conditions. Fix a function $\omega \in \mathcal{D}(\mathbb{R})$ which is even, equal to 1 on $[-2, 2]$, and with support contained in $[-3, 3]$, and denote $g = \psi\omega$ where ψ is as in (2). (Note that for any $\epsilon > 0$ by an appropriate alternative choice of ω we can assure that g has support in $[-\epsilon, \epsilon]$; the choice of support will not affect further statements.) From (3) there exists a sequence $(P_0, P_1, \ldots)$ of positive numbers such that independently of the sequence λ (as in (1)) we have

$$\sup_{y\geq 1}|\frac{d^n\omega}{dy^n}| \leq P_n. \tag{4}$$

We will inductively and explicitly construct numbers γ_i^j in the following manner. Let $\gamma_0^i = 1$. Then consider the finite product

$$(1+\frac{x^2}{\lambda_0^2})\ldots(1+\frac{x^2}{\lambda_{k-1}^2}) = \sum_{n\geq 0}\gamma_n^{k-1}x^{2n}$$

Denote

$$C_n = inf(\beta_n, \frac{1}{n^2P_{2n}}, \frac{1}{n^2P_{2n+1}}, \ldots, \frac{1}{n^2P_{2n+n}}, 1) \tag{5}$$

Suppose that a choice of $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ has been made such that for all $n \leq k-1$ we have in the above finite sum

$$\gamma_n^{k-1} < C_n \text{ for all } n \leq k-1.$$

At the next k^{th} step we want to choose λ_k so that for the new finite sum we similarly have

$$\gamma_n^k < C_n \text{ for all } n \leq k. \tag{6}$$

Since we have

$$\gamma_n^k = \frac{1}{\lambda_k^2}\gamma_{n-1}^{k-1} + \gamma_n^{k-1},$$

the above requirement means

$$\frac{1}{\lambda_k^2}\gamma_{n-1}^{k-1} < C_n - \gamma_n^{k-1}, \text{ or since by induction } C_n > \gamma_n^{k-1},$$

$$\lambda_k^2 > \frac{\gamma_{n-1}^{k-1}}{C_n - \gamma_n^{k-1}} \text{ for all } n \leq k. \tag{7}$$

Make the following explicit choice for λ_k: it is the smallest number in the sequence $(1, 2, \ldots, 2^i, \ldots)$ which satisfies (7) and is greater than λ_{k-1}. This choice can clearly can be made and it constitutes our explicit inductive construction. Visually we obtain the following diagram.

$$
\begin{array}{ccccccc}
1 & & & & & & \\
 & \searrow & & & & & \\
1 & & \gamma_1^1 & & & & \\
 & \searrow & \downarrow & \searrow & & & \\
1 & & \gamma_1^2 & & \gamma_2^2 & & \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
1 & & \gamma_1^3 & & \gamma_2^3 & & \gamma_3^3 \\
 & & \ldots & & & &
\end{array}
$$

If we now consider the limit

$$\phi(x) = \sum_{n=0}^{\infty} \gamma_n x^{2n} = \lim_{k\to\infty} \prod_{i=0}^{k} (1 + \frac{x^2}{\lambda_i^2}) \tag{8}$$

from the uniform convergence of the left hand and the Taylor expansion we conclude that the numbers in the n^{th} column of the diagram tend to a limit and this limit is precisely the coefficient γ_n on the left hand side of (8). So our diagram has the form:

$$
\begin{array}{cccccccc}
1 & & & & & & & \\
 & \searrow & & & & & & \\
1 & & \gamma_1^1 & & & & & \\
 & \searrow & \downarrow & \searrow & & & & \\
1 & & \gamma_1^2 & & \gamma_2^2 & & & \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & & \text{fig. 1} \\
1 & & \gamma_1^3 & & \gamma_2^3 & & \gamma_3^3 & \\
 & & \ldots & & & & & \\
 & & \ldots & & & & & \\
 & & \ldots & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & \\
1 & & \gamma_1 & & \gamma_2 & \ldots & &
\end{array}
$$

(i) The condition is satisfied by construction because (5) and taking limit in (6) imply that

$$\gamma_n < inf(\beta_n, \frac{1}{n^2 P_{2n}}, \frac{1}{n^2 P_{2n+1}}, \ldots, \frac{1}{n^2 P_{2n+n}}, 1) \tag{9}$$

(ii) To construct the function h we consider the functions

$$\theta_p = \sum_{n=0}^{p} (-1)^n \gamma_n \frac{\delta^{(2n)}}{(2\pi)^{2n}} \star g, \tag{10}$$

with support in $[-3,3]$. We show these functions converge in $\mathcal{E}'(\mathbb{R})$ to a distribution $\delta + h$ where $h \in \mathcal{D}(\mathbb{R})$. It is sufficient to look at the restrictions of θ_p on $(-2,2),(1,4),(3,+\infty)$. First $\theta_p = 0$ in $(3,+\infty)$.

We show that in $(-2,2)$

$$\theta_{p|(-2,2)} = (\sum_{n=0}^{p}(-1)^n\gamma_n\frac{\delta^{(2n)}}{(2\pi)^{2n}} \star \psi)_{|(-2,2)},$$

converges to δ in $\mathcal{D}'(\mathbb{R})$. This is because in $(-2,2)$ $g = \psi$ and by construction in (1) for all $x \in \mathbb{R}$ one has

$$0 \leq (\sum_{n=0}^{p}\gamma_n x^{2n})\chi(x) \leq 1, \text{ and}$$

$$(\sum_{n=0}^{\infty}\gamma_n x^{2n})\chi(x) \;= 1.$$

Therefore $(\sum_{n=0}^{p}\gamma_n x^{2n})\chi(x) \to 1$ in $\mathcal{S}'(\mathbb{R})$ for $p \to \infty$, so

$$(\sum_{n=0}^{p}(-1)^n\gamma_n x^{2n})\frac{\delta^{(2n)}}{(2\pi)^{2n}} \star \psi \to \delta \text{ in } \mathcal{S}'(\mathbb{R}) \text{ when } p \to +\infty.$$

Finally considering $(1,4)$ we see that for $y \geq 1$ due to (9), (4), and (3)

$$|\gamma_n\frac{\delta^{(2n+k)}}{(2\pi)^{2n}} \star g| \leq \gamma_n P_{2n+k} < \frac{1}{n^2} \text{ for } n \geq k, \tag{11}$$

so the sum

$$\sum_{n=0}^{\infty}(-1)^n\gamma_n\frac{\delta^{(2n)}}{(2\pi)^{2n}} \star g$$

converges to the desired limit h in $\mathcal{E}((1,4))$ because (11) ensures that termwise differentiation preserves absolute convergence.

Before we proceed note that by this construction the support of h will be included in that of g (so it will also be arbitrarily small).

(iii) Consider the sequences β^l from the condition: for each of these our construction produces an inductive diagram as the one in fig.1 in which we denote the dependence on l by an upper left index:

$$\begin{array}{ccccccc}
1 & & & & & & \\
 & \searrow & & & & & \\
1 & & {}^l\gamma_1^1 & & & & \\
 & \searrow & \downarrow & \searrow & & & \\
1 & & {}^l\gamma_1^2 & & {}^l\gamma_2^2 & & \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
1 & & {}^l\gamma_1^3 & & {}^l\gamma_2^3 & & {}^l\gamma_3^3 \\
 & & \dots & & & & \\
 & & \dots & & & & \\
 & & \dots & & & & \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & {}^l\gamma_1 & & {}^l\gamma_2 & \dots &
\end{array}$$

We will now need two facts: first the table stabilizes as l increases (Claim 1), and second the columns converge uniformly in l (Claim 2).

Claim 1 *With the above notation for a fixed n and a given K we can choose L such that for $l > L$ we have*

$$ {}^l\gamma_n^k = \gamma_n^k \text{ for all } k \leq K. $$

Proof We proceed inductively. Suppose we achieved this for K-1 with some L' for $n = 0, 1, \dots, K-1$. Then the tables for β and $\beta^{L'}$ coincide up to the $(K-1)^{st}$ row. But then notice that choosing the K^{th} row is equivalent to choosing λ_K, and that choice will be different for β and β^l $(l > L')$ only due to the difference between C_n and ${}^lC_n$ which is in turn either zero or equal to difference between β_n and ${}^l\beta_n$. That last difference by our assumption can be made arbitrarily small if we choose L big enough, so clearly since our choice of λ_K is discrete we can arrange L big enough so that $\lambda_K =^l \lambda_K$ for $l \geq L$. This proves the claim. $\square$

Claim 2 *With the above notation for a given n and given $\epsilon > 0$ we can choose K, so that for all l we have*

$$ |{}^l\gamma_n - {}^l\gamma_n^i| < \epsilon \text{ for } i > K. $$

Proof Notice that by construction

$$ {}^l\gamma_n^k = \frac{1}{{}^l\lambda_k^2}{}^l\gamma_{n-1}^{k-1} +^l \gamma_n^{k-1} < \frac{1}{2^{2k}}{}^lC_{n-1} + {}^l\gamma_n^{k-1} < \frac{1}{2^{2k}} + {}^l\gamma_n^{k-1} $$

since ${}^lC_{n-1} < 1$. So at each step going down a column the next term is bigger at most by $\frac{1}{2^{2k}}$ (and notice that the sequence in the column is increasing). Then choose K so that

$$ \sum_{i=K}^{\infty} \frac{1}{2^{2i}} < \epsilon. $$

This guarantees that
$$|{}^l\gamma_n -{}^l \gamma_n^i| < \epsilon \text{ for } i > K,$$
which is proves the calim. □

Suppose now that as in the condition (iii) we are given a positive $N \in \mathbb{Z}$, and a real $\epsilon > 0$. We combine the results of Claim 1 and Claim 2. From Claim 2 it follows that for a fixed l we can choose K large enough so that

$$|{}^l\gamma_n - {}^l\gamma_n^i| < \epsilon/2 \text{ for } i > K \text{ and for all } n \leq N.$$

Now with this choice of K, from Claim 1 we can choose L large enough so that for $l > L$ we have
$${}^l\gamma_n^k = \gamma_n^k \text{ for all } k \leq K.$$
This implies that now we have a statement independent of l, i.e.

$$|{}^l\gamma_n - {}^l\gamma_n^i| < \epsilon/2 \text{ for } i > K, n \leq N, \text{ and all } l > L$$

and so

$$|{}^l\gamma_n - \gamma_n| = |{}^l\gamma_n - {}^l\gamma_n^K + {}^l\gamma_n^K - \gamma_n| < \epsilon/2 + \epsilon/2 = \epsilon \text{ for } n \leq N, \text{ and all } l > L,$$

which completes the proof of (iii).

(iv) Consider first the function g. By construction $g = \omega\psi$ where ω is a fixed function. So the problem is reduced to showing that the sequence $\{\psi_l\}$ corresponding to $\{\beta^l\}$ converges to ψ (which corresponds to β). As in (1) and (2) we also have corresponding sequences $\{\phi_l\}$ and $\{\chi_l\}$ which are needed in the construction of the $\{\psi_l\}$.

Say we are given a number $a > 0$, and a number $\epsilon > 0$. On the interval $[-a, a]$ we consider the difference

$$|\phi(x) - \phi_l(x)| = |\sum_{n=0}^{\infty} \gamma_n x^{2n} - \sum_{n=0}^{\infty} {}^l\gamma_n x^{2n}|.$$

Since it is bounded independently of l by an uniformly convergent sequence on [-a,a] we can choose N so that

$$|\sum_{n>N}^{\infty} (\gamma_n x^{2n} - {}^l\gamma_n x^{2n})| < \epsilon/2 \text{ for } x \in [-a, a] \text{ and all } l.$$

Because the $\{\gamma_l\}$ satisfy condition (iii) of the Lemma we can also choose L such that
$$|\sum_{n=0}^{N} (\gamma_n x^{2n} - {}^l\gamma_n x^{2n})| < \epsilon/2 \text{ for } x \in [-a, a] \text{ and all } l > L.$$

These two observations imply that for $l > L$ we have

$$|\phi(x) - \phi_l(x)| < \epsilon \text{ when } x \in [-a, a].$$

Now we observe that

$$|\chi(x) - \chi_l(x)| = |\frac{1}{\phi(x)} - \frac{1}{\phi_l(x)}| = |\frac{\phi(x) - \phi_l(x)}{\phi(x)\phi_l(x)}| < |\phi(x) - \phi_l(x)|$$

since for all x $\phi(x) > 1$ and $\phi_l(x) > 1$. This allows us to conclude that with the same choice of L we get

$$|\chi(x) - \chi_l(x)| < \epsilon \text{ for } l > L \text{ and } x \in [-a, a]. \tag{12}$$

Now fix k and given $\epsilon > 0$ consider the difference (here D^k is the k^{th} derivative)

$$|D^k\psi(y) - D^k\psi_l(y)| = |D^k \int_{-\infty}^{+\infty} e^{2\pi ixy}(\chi(x) - \chi_l(x))dx|$$
$$= |\int_{-\infty}^{+\infty} (2\pi x)^k e^{2\pi ixy}(\chi(x) - \chi_l(x))dx| < \int_{-\infty}^{+\infty} |(2\pi x)^k(\chi(x) - \chi_l(x))|dx.$$

Notice that for $x > 1$

$$\chi(x) = \frac{1}{1 + \gamma_1 x^2 + \ldots \gamma_{k+1}x^{2k+2} + \ldots} < \frac{1}{\gamma_{k+1}x^{2k+2}} \text{ and}$$

$$\chi_l(x) = \frac{1}{1 + {}^l\gamma_1 x^2 + \ldots {}^l\gamma_{k+1}x^{2k+2} + \ldots} < \frac{1}{{}^l\gamma_{k+1}x^{2k+2}}.$$

Then

$$|\chi(x) - \chi_l(x)| < \frac{1}{x^{2k+2}}(|\frac{1}{\gamma_{k+1}}| + |\frac{1}{{}^l\gamma_{k+1}}|) < \frac{1}{x^{2k+2}}(2|\frac{1}{\gamma_{k+1}}| + |\frac{1}{\gamma_{k+1}} - \frac{1}{{}^l\gamma_{k+1}}|).$$

But now by the Lemma we can choose L_1 so that $|\frac{1}{\gamma_{k+1}} - \frac{1}{{}^l\gamma_{k+1}}| < 1$ for $l > L_1$ (because $\gamma_{k+1} > 0$ is fixed and the function $1/y$ is continuous for $y > 0$). This implies that because of absolute convergence of the integral we can choose $a > 0$ such that for $l > L_1$ (with C independent of l)

$$|\int_{|x|>a} (2\pi x)^k e^{2\pi ixy}(\chi(x) - \chi_l(x))dx| < \int_{|x|>a} |(2\pi)^k x^k \frac{C}{x^{2k+2}}|dx = \tag{13}$$
$$= \int_{|x|>a} |(2\pi)^k \frac{C}{x^{k+2}}|dx < \epsilon/2$$

Now making use of (12) we can choose L_2 such that for $l > L_2$ and $x \in [-a, a]$ we have $|\chi(x) - \chi_l(x)| < \epsilon_1$ where $\epsilon_1 < \frac{\epsilon/2}{2a(2\pi a)^k}$ which will imply that

$$|\int_{-a}^{+a} (2\pi x)^k e^{2\pi i x y}(\chi(x) - \chi_l(x))dx| < \epsilon/2 \text{ for } l > L_2. \tag{14}$$

Thus (13) and (14) imply that for $l > \max(L_1, L_2)$

$$|D^k\psi(y) - D^k\psi_l(y)| = |\int_{-\infty}^{+\infty} (2\pi x)^k e^{2\pi i x y}(\chi(x) - \chi_l(x))dx| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which is what we wanted to prove.

Lastly we consider the function h. From the results obtained for (10) we have that by construction h is

$$h(x) = \sum_{n=0}^{\infty}(-1)^n \gamma_n \frac{1}{(2\pi)^{2n}} D^{2n} g(x)$$

and the sum can be differentiated any number of times by (11). So if we fix k, and let $\epsilon > 0$ then we will have

$$\begin{aligned} |D^k h - D^k h_l| &= |D^k \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2\pi)^{2n}} \gamma_n D^{2n} g - D^k \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2\pi)^{2n}} {}^l\gamma_n D^{2n} g_l| = \\ &= |\sum_{n=0}^{\infty} \frac{1}{(2\pi)^{2n}} (\gamma_n D^{2n+k} g - {}^l\gamma_n D^{2n+k} g_l)| \end{aligned}$$

1) Since by (11) the above sum is bounded by an absolutely convergent series uniformly in l, we can choose N such that for all l

$$|\sum_{n>N}^{\infty} \frac{1}{(2\pi)^{2n}} (\gamma_n D^{2n+k} g - {}^l\gamma_n D^{2n+k} g_l)| < \epsilon/2$$

2) We now need to consider the front end of the series which is less than:

$$|\sum_{n=0}^{N} (\gamma_n D^{2n+k} g - {}^l\gamma_n D^{2n+k} g_l)|.$$

Choose a number $\epsilon_1 < \frac{\epsilon}{6N}$. Because (iii) is satisfied we can choose L_1 such that

$$|{}^l\gamma_n - \gamma_n| =< \epsilon/2 \text{ for } n \leq N, \text{ and all } l > L_1.$$

Also choose $\epsilon_2 < \frac{\epsilon}{6\sup|D^{2n+k}g|}$ for $n \leq N$; from the just proved convergence of the $\{g_l\}$ it follows that we can choose $L > L_1$ such that for $n \leq N$

$$|D^{2n+k} g - D^{2n+k} g_l| < \epsilon_2 \text{ for } l > L.$$

With these choices we have

$$
\begin{aligned}
&{}^{l}\gamma_n = \gamma_n + \delta_1 \text{ where } |\delta_1| < \epsilon_1, \\
&D^{2n+k}g = D^{2n+k}g_l + \delta_2 \text{ where } |\delta_2| < \epsilon_2.
\end{aligned}
$$

This then means that for $l > L$

$$
\begin{aligned}
& |\sum_{n=0}^{N}(\gamma_n D^{2n+k}g - {}^{l}\gamma_n D^{2n+k}g_l)| \\
= \; & |\sum_{n=0}^{N}(\gamma_n D^{2n+k}g - \gamma_n D^{2n+k}g - \delta_1 D^{2n+k}g - \delta_2\gamma_n - \delta_1\delta_2)| \leq \\
\leq \; & |\sum_{n=0}^{N}\delta_1 D^{2n+k}g| + |\sum_{n=0}^{N}\delta_2\gamma_n| + |\sum_{n=0}^{N}\delta_1\delta_2| \leq \\
\leq \; & N\frac{\epsilon}{6N} + N\frac{\epsilon}{6N} + N\frac{\epsilon}{6N} \leq \frac{\epsilon}{2}
\end{aligned}
$$

From 1) and 2) it follows that for $l > L$

$$|D^k h - D^k h_l| = \sum_{n=0}^{\infty}\frac{1}{(2\pi)^{2n}}(\gamma_n D^{2n+k}g - {}^{l}\gamma_n D^{2n+k}g_l)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which is what we wanted to prove.

This completes the proof of the Lemma 2. $\square$

Proof of Continuous Factorization

For convenience we restate the result:

Theorem 3 (Continuous Factorization) *Let G be a Lie group, V a neighbourhood of the identity in G, and $u \in \mathcal{D}(G)$. Then u is a finite sum of functions of the form $v_1 \star v_2$, where $v_1, v_2 \in \mathcal{D}(G)$, $supp(v_1) \subset V$, $supp(v_2) \subset supp(u)$, and where v_1 and v_2 depend continuously on u in the space $\mathcal{D}(G)$.*

Proof We first consider the special case when $G = \mathbb{R}$. The desired choice is achieved by choosing a sequence $\beta = (\beta_0, \beta_1, \ldots, \beta_n, \ldots)$ in the following manner:

$$\beta_n = inf\Big(\frac{1}{2^n}, \frac{1}{2^n M_{2n}}, \frac{1}{2^n M_{2n+1}}, \ldots, \frac{1}{2^n M_{2n+n}}, 1\Big) \tag{15}$$

where M_k denotes $\sup_{\mathbb{R}}|D^k u|$ (and D^k is the k^{th} derivative). From the Lemma we get a corresponding choice of a sequence γ the members of which satisfy $\gamma_n \leq \beta_n$, and

$$\sum_{n=0}^{p}(-1)^n\gamma_n\delta^{(2n)} \star g \to \delta + h \text{ in } \mathcal{E}'(\mathbb{R}) \text{ when } p \to \infty, \text{ or}$$

$$g \star \sum_{n=0}^{p} (-1)^n \gamma_n \delta^{(2n)} \star u \to u + h \star u \text{ in } \mathcal{E}'(\mathbb{R}) \text{ when } p \to \infty. \quad (16)$$

Now notice that by the choice (15)

$$|f| = |D^k(\sum_{n=0}^{\infty}(-1)^n \gamma_n D^{2n} u)| < \sum_{n=0}^{\infty} |\gamma_n D^{2n+k} u| < \sum_{n=0}^{\infty} |\frac{1}{2^n M_{2n+k}} M_{2n+k}| < \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty.$$

Therefore $f \in \mathcal{D}(\mathbb{R})$, and from (16) we obtain the decomposition

$$u = g \star f - h \star u.$$

Now suppose that u_l are a sequnce of functions converging to u. With the choice (15) it is easy to see that the sequences β^l converge pointwise to the sequence β (in the sense of (iii) in the Lemma). In turn (iii) of the Lemma implies pointwise convergence of the corresponding sequences γ^l. Finally, (iv) implies that g and h depend continuously on u in $\mathcal{D}(\mathbb{R})$.

So it remains to consider f. But from the expression for f it follows that the argument which we used for g in the proof of the Lemma can be used in this case, too. Namely:

1) Using (15) we see that independently of l

$$\begin{aligned} |\textstyle\sum_{2n+k<3n}^{\infty} & (\gamma_n D^{2n+k} u - {}^l\gamma_n D^{2n+k} u_l)| < \\ < & \sum_{2n+k<3n}^{\infty} |\gamma_n D^{2n+k} u| + \sum_{2n+k<3n}^{\infty} |{}^l\gamma_n D^{2n+k} u_l| < \\ < & \sum_{2n+k<3n}^{\infty} \frac{1}{2^n} + \sum_{2n+k<3n}^{\infty} \frac{1}{2^n}; \end{aligned}$$

so we can choose N such that

$$|\sum_{n>N}^{\infty} (\gamma_n D^{2n+k} u - {}^l\gamma_n D^{2n+k} u_l)| < \epsilon/2.$$

2) We now need to consider the front end

$$|\sum_{n=0}^{N} (\gamma_n D^{2n+k} u - {}^l\gamma_n D^{2n+k} u_l)|.$$

Here exactly as for g we get that for $l > L$

$$|\sum_{n=0}^{N} (\gamma_n D^{2n+k} u - {}^l\gamma_n D^{2n+k} u_l)| < \epsilon/2.$$

Combining 1) and 2) we get the desired result, i.e. that f depends continuously on u.

This completes the argument for $G = \mathbb{R}$.

Now consider the case of any lie group G. The construction generalizes the one for $G = \mathbb{R}$. If $x_1, x_2, \ldots, x_m$ is a basis for the Lie algebra of G then the map ζ

$$(t_1, t_2, \ldots, t_m) \to (\exp t_1 x_1)(\exp t_2 x_2) \ldots (\exp t_m x_m)$$

when restriced to $(-1, 1)^m$ in $\mathbb{R}^m$ is a diffeomorphism to an open neighbourhood Ω of G. Let σ be a left Haar measure on G and σ_Ω its restriction to Ω. If $(w_1, w_2, \ldots)$ is a basis for the enveloping algebra W of the Lie algebra of G then if $w \in W$, $w \star u$ defines a differential operator $D_w(u)$. Now denote

$$\sup_{s\in G} |w_i \star x_1^{2n} \star u(s)| = M_{2n,i}.$$

As before choose positive $(\beta_0, \beta_1, \ldots)$ such that

$$\beta_n = \inf\Big(\frac{1}{2^n}, \frac{1}{2^n M_{2n,1}}, \frac{1}{2^n M_{2n,2}}, \ldots, \frac{1}{2^n M_{2n,n}}, 1\Big) \tag{17}$$

Then the Lemma as before produces a corresponding sequence $(\beta_0, \beta_1, \ldots)$. As for $\mathbb{R}$ we obtain functions g and h such that

$$\sum_{n=0}^{p} (-1)^n \gamma_n \delta^{(2n)} \star g \to \delta + h \text{ in } \mathcal{E}'(\mathbb{R}) \text{ when } p \to \infty.$$

The map $t_1 \to expt_1 x_1$ transforms the measures $g(t_1)dt_1, h(t_1)dt_1$ on $\mathbb{R}$ into measures μ, ν on G and

$$\mu \star \sum_{n=0}^{p} (-1)^n \gamma_n x_1^{2n} = \sum_{n=0}^{p} (-1)^n \gamma_n x_1^{2n} \star \mu \to \delta_e + \nu$$

in $\mathcal{E}'(G)$ when $p \to +\infty$. Therefore

$$\mu \star \sum_{n=0}^{p} (-1)^n \gamma_n x_1^{2n} \star u \to u + \nu \star u$$

in $\mathcal{E}'(G)$ when $p \to +\infty$. But due to (17)

$$\sum_{n=0}^{\infty} (-1)^n \gamma_n x_1^{2n} \star u = f \tag{18}$$

where f is in $\mathcal{D}(G)$. Thus

$$u = \mu \star f - \nu \star u \tag{19}$$

Now let u_l be a sequence converging to u in $\mathcal{D}(G)$. Then the sequences β^l converge to β pointwise and by the Lemma as before the fucntions g_l and h_l converge to g and h in $\mathcal{D}(\mathbb{R})$ or g and h depend continuously on u. We also notice that exactly as in the case $G = \mathbb{R}$ for any $\epsilon > 0$ for l large enough

$$|D_{w_k} f - D_{w_k} f_l| = |\sum_{n=0}^{\infty}((-1)^n \gamma_n w_k \star x_1^{2n} \star u) - (-1)^n \gamma_n w_k \star x_1^{2n} \star u_l)| < \epsilon.$$

This establishes that f depends continuously on u in $\mathcal{D}(G)$. Now from (19) it follows that u is a sum of two functions of the form $\xi \star \eta$ where $\eta \in \mathcal{D}(G)$ with $\mathrm{supp}(\eta) \in \mathrm{supp}(u)$ and ξ is the image of a measure $s(t_1)dt_1$ on $\mathbb{R}$ with $s \in \mathcal{D}(\mathbb{R})$ and $\mathrm{supp}(s) \in [-\epsilon, \epsilon]$ where the functions $s(t)$ and η depend continuously on u.

Applying iteratively this argument for all the m elements of the Lie algebra we obtain that u is a finite sum of functions of the form $\xi_1 \star \xi_2 \star \ldots \star \xi_m \star \eta$ where $\eta \in \mathcal{D}(G)$ with $\mathrm{supp}(\eta) \in \mathrm{supp}(u)$ and ξ_i is the image of a measure $s_i(t_1)dt_1$ on $\mathbb{R}$ with $s_i \in \mathcal{D}(\mathbb{R})$ and $\mathrm{supp}(s_i) \in [-\epsilon, \epsilon]$. Also s_i and η depend continuously on u.

Now $\xi_1 \star \xi_2 \star \ldots \star \xi_m$ is the image of $\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_m$ under the product map $G \times G \times G \ldots \times G \to G$ and therefore $\xi_1 \star \xi_2 \star \ldots \star \xi_m$ is the image under ζ of the measure

$$s_1(t_1)s_2(t_2)\ldots s_m(t_m)dt_1 dt_2 \ldots dt_m$$

on $\mathbb{R}^m$. The function $s_1(t_1)s_2(t_2)\ldots s_m(t_m)$ is in $\mathcal{D}(\mathbb{R})$ with support in $[-\epsilon, \epsilon]$ and depends continuously on u as a product of functions which depend continuously on u. Since the image of the restriction $dt_1 dt_2 \ldots dt_m|_{(1,1)^m}$ is the product of σ_Ω and a function in $\mathcal{E}(\Omega)$ $\xi_1 \star \xi_2 \star \ldots \star \xi_m$ is the product of σ and function in $\mathcal{D}(\mathbb{G})$ continuously depending on u. For ϵ sufficiently small this finction has support in V.

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