A Review of the Abel-Plana Formula

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Introduction

The Abel-Plana formula provides a simple expression for the difference between a sum over discrete values and an integral of a function. Here I give a step-by-step derivation of the formula, following [1] and filling in the details.

Proof of the Formula

Lemma 1 *Let f be entire, and suppose* $\lim_{h\to\infty}\int_{a\pm ih}^{b\pm ih}[g(z)\pm f(z)]\,dz = 0$. *Then we have*

$$\int_a^b f(z)\,dz \;=\; \pi i\sum_k Res_{z_k}g(z) \;-\; \frac{1}{2}\int_{-i\infty}^{+i\infty}[g(u)+\operatorname{sgn}(\mathrm{Im}z)f(u)]_{u=a+z}^{u=b+z}\,dz.$$

Proof: By the residue theorem,

$$\int_{C_h} g(z)\,dz \;=\; 2\pi i\sum_k Res_{z_k}g(z), \tag{1}$$

where the z_k are the poles of g contained in C_h. Let C_h be the rectangle with vertices at $a\pm ih$, $b\pm ih$, and let C_h^+ and C_h^- be the upper and lower halves of C_h. Then

$$\begin{aligned}\textstyle\int_{C_h} g(z)\,dz &= \int_{C_h^+} g(z)\,dz + \int_{C_h^+} f(z)\,dz + \int_{C_h^-} g(z)\,dz - \int_{C_h^-} f(z)\,dz - \int_{C_h^+} f(z)\,dz + \int_{C_h^-} f(z)\,dz \\ &= \int_{C_h^+}[g(z)+f(z)]\,dz + \int_{C_h^-}[g(z)-f(z)]\,dz - \int_{C_h^+} f(z)\,dz + \int_{C_h^-} f(z)\,dz. \end{aligned} \tag{2}$$

Now let $D_h^\pm$ denote the path $C_h^\pm$ closed by $\pm[a,b]$. Since f is analytic, Cauchy's integral theorem implies

$$\int_{D_h^\pm} f(z)\,dz \;=\; 0$$

and

$$\textstyle\int_{D_h^-} f(z)\,dz \;-\; \int_{D_h^+} f(z)\,dz \;=\; \left[\int_{C_h^-} f(z)\,dz \;-\; \int_a^b f(z)\,dz\right] - \left[\int_{C_h^+} f(z)\,dz \;+\; \int_a^b f(z)\,dz\right] \;=\; 0$$

$$\Rightarrow \int_{C_h^-} f(z)\,dz - \int_{C_h^+} f(z)\,dz = 2\int_a^b f(z)\,dz. \tag{3}$$

We write

$$\int_{C_h^+} [g(z)+f(z)]\;dz = \int_b^{b+ih} [g(z)+f(z)]\;dz - \int_{a+ih}^{b+ih} [g(z)+f(z)]\;dz - \int_a^{a+ih} [g(z)+f(z)]\;dz$$

and

$$\int_{C_h^-} [g(z)-f(z)]\;dz = -\int_b^{b-ih} [g(z)-f(z)]\;dz + \int_{a-ih}^{b-ih} [g(z)-f(z)]\;dz + \int_a^{a-ih} [g(z)-f(z)]\;dz,$$

or more compactly,

$$\int_{C_h^\pm} [g(z)\pm f(z)]\;dz = \pm\int_0^{\pm ih} [g(u)\pm f(u)]_{u=a+z}^{u=b+z}\;dz \mp \int_{a\pm ih}^{b\pm ih} [g(z)\pm f(z)]\;dz. \tag{4}$$

Combining (1), (2), (3), and (4) gives

$$\begin{aligned} 2\pi i \textstyle\sum_k Res_{z_k} g(z) = & \int_0^{ih} [g(u)+f(u)]_{u=a+z}^{u=b+z}\;dz - \int_0^{-ih} [g(u)-f(u)]_{u=a+z}^{u=b+z}\;dz - \\ & \int_{a+ih}^{b+ih} [g(z)+f(z)]\;dz + \int_{a-ih}^{b-ih} [g(z)-f(z)]\;dz + 2\int_a^b f(z)\,dz, \end{aligned}$$

or

$$\begin{aligned} \textstyle\int_a^b f(z)\,dz = & \pi i \sum_k Res_{z_k} g(z) - \frac{1}{2}\int_{-ih}^{+ih} [g(u)+\mathrm{sgn}(\mathrm{Im} z) f(u)]_{u=a+z}^{u=b+z}\;dz + \\ & \frac{1}{2}\int_{a+ih}^{b+ih} [g(z)+f(z)]\;dz - \frac{1}{2}\int_{a-ih}^{b-ih} [g(z)-f(z)]\;dz. \end{aligned} \tag{5}$$

In the limit $h\to\infty$, the last two terms on the RHS vanish by hypothesis, so we have the desired equation. ∎

Theorem 1 (Generalized Abel-Plana Formula) *Suppose that in addition to the hypothesis of the lemma, we have*

$$\lim_{b\to\infty}\int_b^{b\pm i\infty} [g(z)\pm f(z)]\,dz = 0.$$

Then

$$\lim_{b\to\infty}\left[\int_a^b f(x)\,dx - \pi i\sum_k Res_{z_k} g(z)\right] = \frac{1}{2}\int_{a-i\infty}^{a+i\infty} [g(z)+\mathrm{sgn}(\mathrm{Im} z) f(z)]\,dz.$$

Proof: Follows immediately from the lemma, if we write

$$\begin{aligned} -\tfrac{1}{2}\textstyle\int_{-i\infty}^{+i\infty} [g(u)+\mathrm{sgn}(\mathrm{Im} z) f(u)]_{u=a+z}^{u=b+z} = & -\frac{1}{2}\left\{\int_b^{b+i\infty} [g(z)+f(z)]\,dz - \int_b^{b-i\infty} [g(z)-f(z)]\,dz\right\} \\ & +\frac{1}{2}\int_{a-i\infty}^{a+i\infty} [g(z)+\mathrm{sgn}(\mathrm{Im} z) f(z)]\,dz \end{aligned}$$

and take the limit $b\to\infty$. ∎

Corollary 1 (Abel-Plana Formula) *With the same hypotheses as in the GAPF and for* $0 < a < 1$,

$$\lim_{n\to\infty}\left[\sum_{s=1}^{n} f(s) - \int_a^{n+a} f(x)\,dx\right] = \frac{1}{2i}\int_a^{a-i\infty} f(z)(\cot\pi z - i)\,dz - \frac{1}{2i}\int_a^{a+i\infty} f(z)(\cot\pi z + i)\,dz. \qquad (6)$$

Proof: Let $b = n + a$ and $g(z) = -if(z)\cot\pi z$. Then the GAPF reads

$$\lim_{b\to\infty}\left\{\int_a^{n+a} f(x)\,dx \;-\; \pi i\sum_k Res_{z_k}[-if(z)\cot\pi z]\right\} = \frac{1}{2}\int_{a-i\infty}^{a+i\infty}[-if(z)\cot\pi z + \mathrm{sgn}(\mathrm{Im}z)f(z)]\,dz. \qquad (7)$$

Now $\cot\pi z$ has simple poles at the positive real integers $s = 1, 2, 3, \ldots$, so if f is analytic,

$$\begin{aligned}
\textstyle\sum_k Res_{z_k}[-if(z)\cot\pi z] &= \sum_{s=1}^{n}\lim_{z\to s}(z-s)[-if(z)\cot\pi z] \\
&= -i\sum_{s=1}^{n} f(s)\lim_{z\to s}\frac{(z-s)\cos\pi z}{\sin\pi z} \\
&= -i\sum_{s=1}^{n} f(s)\lim_{z\to s}\frac{(\cos\pi z - \pi(z-s)\sin\pi z}{\pi\cos\pi z} \\
&= -\frac{i}{\pi}\sum_{s=1}^{n} f(s), \qquad (8)
\end{aligned}$$

and the LHS of (7) matches that of the APF (6), up to a minus sign. Write the RHS of (7) as

$$\frac{1}{2}\int_a^{a+i\infty} f(z)(-i\cot\pi z + 1)\,dz - \frac{1}{2}\int_a^{a-i\infty} f(z)(-i\cot\pi z - 1)\,dz,$$

or

$$\frac{1}{2i}\int_a^{a+i\infty} f(z)(\cot\pi z + i)\,dz - \frac{1}{2i}\int_a^{a-i\infty} f(z)(\cot\pi z - i)\,dz,$$

which, again up to a minus sign, matches the RHS of the APF. ■

To put the formula in its usual form, let a go to zero and substitute $t = -iz$. Note that in the limit $a \to 0$ we pick up a half-residue, since $g\,(= -if\cot\pi z)$ has a pole at zero. Therefore the LHS of (6) reads $\frac{f(0)}{2} + \sum_{n=1}^{\infty} f(n) - \int_0^{\infty} f(x)\,dx$. Under the substitution $z = it$, $dz = i\,dt$, the RHS becomes

$$\begin{aligned}
&\frac{1}{2}\int_0^{-\infty} f(it)[\cot(\pi it) - i]\,dt - \frac{1}{2}\int_0^{\infty} f(it)[\cot(\pi it) + i]\,dt \\
&= -\frac{1}{2}\int_0^{\infty} f(-it)[\cot(-\pi it) - i]\,dt - \frac{1}{2}\int_0^{\infty} f(it)[\cot(\pi it) + i]\,dt \\
&= \frac{1}{2}\int_0^{\infty} f(-it)[\cot(\pi it) + i]\,dt - \frac{1}{2}\int_0^{\infty} f(it)[\cot(\pi it) + i]\,dt \\
&= -\frac{1}{2}\int_0^{\infty}[f(it) - f(-it)][\cot(\pi it) + i]\,dt.
\end{aligned}$$

Adding $\frac{f(0)}{2}$ to both sides, we get

$$\boxed{\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x)\,dx = \frac{f(0)}{2} - \frac{1}{2}\int_0^{\infty} [f(it) - f(-it)][\cot(\pi it) + i]\,dt.}$$

This form is useful for manipulating the values taken by f in the sum on the LHS, since the dependence on the poles of the cotangent is explicit. For evaluation, however, it may be convenient to use the identity

$$\cot(\pm\pi it) \pm i = \frac{\mp 2i}{e^{2\pi t} - 1},$$

yielding

$$\boxed{\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x)\,dx = \frac{f(0)}{2} + i\int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1}\,dt.}$$

Conclusion: Application to the Casimir Effect

The expression given by the APF for the difference between a sum and an integral is particularly useful in calculations related to the Casimir effect [1, 2]. In determining Casimir energies, one is interested in the difference between an observable with a discrete spectrum (representing a field in the presence of boundary conditions: metal plates, curled dimensions, etc.), and an observable with a continuous spectrum (representing the free field). This amounts to subtracting an integral from a sum. The APF has the advantage of not introducing an explicit cutoff function, a technique often used to tame the infinities that plague direct summation calculations. This is a useful feature in cases where the introduction of such a cutoff is not necessarily justified, as in those involving boundary conditions imposed by the shape of the space.

References

[1] Aram A. Saharian. The generalized abel-plana formula. applications to bessel functions and casimir effect. *The Abdus Salam International Center for Theoretical Physics*, (14), 2000.

[2] V. M. Mostepanenko and N. N. Trunov. *The Casimir effect and its applications*. Oxford University Press, 1997.