Fiber Bundles, The Hopf Map and Magnetic Monopoles

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1 Preliminaries

Definition 1 *An n-dimension differentiable manifold is a topological space $\mathcal{X}$ with a differentiable structure that is Hausdorff and second countable.*

Example 1 $\mathbb{R}^n$, S^n, T^n, *etc...*

Definition 2 *A diffeomorphism between two differentiable manifolds, M and N is a smooth, bijective map $\phi : M \to N$ with a smooth inverse.*

Definition 3 *A topological group G is a topological space such that the product and inverse operations are continuous maps.*

Definition 4 *A Lie group G is a topological group such that the product and inverse operations are smooth maps.*

Example 2 $\mathbb{R}$, *O(n), SO(n), U(n), SU(n), etc...*

Counterexample 1 $\mathbb{Q} \subset \mathbb{R}$ *with the subspace topology is a topological group, but not a Lie group.*

2 Fiber Bundles

Definition 5 *A Fiber Bundle is defined as the following:*

1. *Differentiable manifolds B and E, called the Base Space and the Total Space, respectively.*
2. *A topological space F, called the Fiber, or Typical Fiber (typically a differentiable manifold).*
3. *A surjection $\pi : E \to B$ called the Projection.*
4. *A topological group G called the structure group (typically a Lie group).*

5. *A family of diffeomorphisms $\Psi = \{\psi_\alpha\}_{\alpha \in I}$ such that given an open cover $\{U_\alpha\}_{\alpha \in I}$ of B, for all $x \in B$ there exists a $U_\alpha \ni x$ and a $\psi_\alpha \in \Psi$ so that*

$$\psi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$$

called Trivializations of the bundle and

$$\pi(\psi_\alpha(x, y)) = x, \quad for\ all \quad (x, y) \in U_\alpha \times F$$

The set $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ is called a coordinate representation for E. For all coordinate representations,

$$\psi_{\alpha,x} : F \to F_x = \{y \in E | \pi(y) = x\},$$

$\psi_{\alpha,x}(y) = \psi_\alpha(x, y)$ is bijective for all $y \in F$ and $x \in U_\alpha$

6. *A set of maps $\{t_{ij}\}$ such that if $x \in U_\alpha \cap U_\beta$, $t_{\alpha\beta} = \psi_{\alpha,x}^{-1} \circ \psi_{\beta,x}$ is given by an element $g \in G$ and $\psi_\beta(x, y) = \psi_\alpha(x, t_{\alpha\beta}(x)y)$. These are called the Transition functions. We require that*

$$t_{\alpha\alpha}(x) = Id_x, \quad x \in U_\alpha,$$

$$t_{\alpha\beta}(x) = t_{\beta\alpha}^{-1}(x), \quad x \in U_\alpha \cap U_\beta,$$

$$t_{\alpha\beta}(x) \cdot t_{\beta\gamma}(x) = t_{\alpha\gamma}(x), \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

If all the t_{ij}'s are identity maps, then $E = B \times F$ is called a trivial bundle.

Often in literature fiber bundles are simply denoted by $F - E - B$, $\pi : E \to B$ or $E \xrightarrow{\pi} B$.

Definition 6 *A Section of a fiber bundle $\pi : E \to B$ is a map $s : B \to E$ such that $\pi \circ s = Id_B$*

Example 3 *A cylinder is an example of a trivial bundle. The base space is given by S^1 and the typical fiber is given by some interval of $\mathbb{R}$, say $[-1, 1]$. The transition functions are thus given by the identity.*

If instead of taking the structure group to be the identity, we realize a structure with non-trivial topology.

Example 4 *A Möbius band is the simplest example of a non-trivial fiber bundle. Again, the base space is given by S^1 and the typical fiber is given by $[-1, 1] \subset \mathbb{R}$. The transition functions are now given by $t_{12} : t \mapsto -t$ It is easy to see that the structure group is then $\mathbb{Z}/2$.*

This is a special example in that rarely if ever do we see a discrete group for a structure group. An analogous example to the cylinder and Möbius band are the torus and the Klein bottle. The torus, $S^1 \times S^1$, is a trivial bundle in which the base space is a circle and the fiber over each point is a circle. The Klein bottle is a nontrivial circle bundle.

Example 5 *A vector bundle is a fiber bundle having a vector space V as its fiber and $GL(V)$, or one of its subgroups, as a structure group.*

Example 6 *The tangent bundle is the most common example of a vector bundle.*

To see this, lets begin with a definition for the tangent bundle.

Definition 7 *Consider a smooth compact submanifold $M \subset \mathbb{R}^n$ of dimension m. At each point $x \in M$ attach a copy of $\mathbb{R}^m$ tangential to M. This is the tangent space of M at x, denoted T_xM. Then the tangent bundle is the disjoint union of the tangent spaces, i.e.*

$$TM = \coprod_{x \in M} T_xM = \bigcup_{x \in M} \{x\} \times T_xM.$$

So elements of TM can be defined as (x, v) where $x \in M$ and $v \in T_xM$.

To see that this is indeed a fiber bundle we can first observe that there is a natural projection $\pi : TM \to M$ defined by $(x, v) \mapsto x$ (i.e. π takes T_xM to x). So the fibers are given by the T_xM's. To each chart U_α on M with coordinates $(x^1_\alpha, ..., x^m_\alpha)$ there is a corresponding chart $\tilde{U}_\alpha$ with coordinates $(x^1_\alpha, ..., x^m_\alpha, v^1_\alpha, ..., v^m_\alpha)$ where $v = \dot{x}$ is a tangent vector to M along some curve in M. The transition functions between charts have the form

$$x^i_\beta = x^i_\beta(x_\alpha), \quad v^j_\beta = \frac{\partial x^j_\alpha}{\partial x^i_\beta}(x_\alpha) v^i_\alpha$$

We can observe that the structure group is given by $GL(m, \mathbb{R})$.

Example 7 *A covering space is another example of a fiber bundle. In this case the fiber is discrete and the structure group is a factor group of $\pi_1(B)$.*

Definition 8 *A bundle $\pi : E \to B$ in which $F = G$, where G is a Lie group that has a smooth right action of E such that*

1. *The action is free ($e \cdot g = e \Leftrightarrow g = e$).*
2. *The action preserves the fibers.*

then $\pi : E \to B$ is a principal G-bundle, often denoted $P(B, G)$.

Example 8 *S^n is a 2-fold cover of $\mathbb{R}P^n$. The action of $O(1)$ on S^n gives it the structure of a principal $O(1)$-bundle over $\mathbb{R}P^n$;*

$$O(1) - S^{n+1}_{\mathbb{R}} - \mathbb{R}P^n.$$

Similarly,

$$U(1) - S^{n+1}_{\mathbb{C}} - \mathbb{C}P^n$$

is a principal $U(1)$ bundle. For the remainder of the talk we will be focusing on a principal $U(1)$ bundle.

3 The Hopf Map

The Hopf map shows us that S^3 is a principal $U(1)$-bundle over S^2. Recall that,

$$S^3 = \{(x_1, ..., x_4) | \sum_{i=1}^{4} x_i^2 = 1\}$$

and for $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$

$$S^3 \cong S^1_{\mathbb{C}} = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1\}$$

We define the Hopf map $\pi : S^3 \to S^2$ by

$$\xi_1 = 2(x_1 x_3 + x_2 x_4),$$

$$\xi_2 = 2(x_2 x_3 - x_1 x_4),$$

$$\xi_3 = (x_1)^2 + (x_2)^2 - (x_3)^2 - (x_4)^2$$

To verify that the ξ's indeed parametrize S^2 we can observe that

$$(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = ((x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2)^2 = 1$$

Now, take (X, Y) to be coordinates given by stereographic projection from the north pole, N, onto $U_S = S^2 - N$. Then,

$$(X, Y) = (\frac{\xi_1}{1 - \xi_3}, \frac{\xi_2}{1 - \xi_3}).$$

If we observe the complex plane through the equator, $Z = X + iY$ is in the unit circle. With a little algebra, we find that

$$Z = \frac{\xi_1 + i\xi_2}{1 - \xi 3} = \frac{x_1 + ix_2}{x_3 + ix_4} = \frac{z_1}{z_2}, \quad \xi \in U_S$$

We can, of course, proceed similarly for the coordinates (U, V) of $U_N = S^2 - S$. Then,

$$(U, V) = (\frac{\xi_1}{1 + \xi_3}, \frac{-\xi_2}{1 + \xi_3}),$$

and therefore

$$W = U + iV = \frac{\xi_1 - i\xi_2}{1 + \xi 3} = \frac{x_3 + ix_4}{x_1 + ix_2} = \frac{z_2}{z_1}, \quad \xi \in U_N.$$

Observe that on $U_N \cap U_S,\ Z = \frac{1}{W}$, and that (z_1, z_2) is invariant under $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$ for $\lambda \in U(1)$ since $|\lambda| = 1$ and $(\lambda z_1, \lambda z_2) \in S^3$ Now, we can observe the fiber bundle structure as follows. Define local trivializations

$$\psi_S^{-1} : \pi^{-1}(U_S) \to U_S \times U(1)$$

by

$$(z_1, z_2) \mapsto (\frac{z_1}{z_2}, \frac{z_2}{|z_2|})$$

and

$$\psi_N^{-1} : \pi^{-1}(U_N) \to U_N \times U(1)$$

by

$$(z_1, z_2) \mapsto (\frac{z_2}{z_1}, \frac{z_1}{|z_1|})$$

Note that these maps are well defined, for instance, on U_S, $z_2 \neq 0$, so both $\frac{z_1}{z_2}$ and $\frac{z_2}{|z_2|}$ are nonsingular. On the equator, $\xi_3 = 0$, $|z_1| = |z_2| = \frac{1}{\sqrt{2}}$ so the trivializations on the equator are given by,

$$\psi_S^{-1} : (z_1, z_2) \mapsto (\frac{z_1}{z_2}, \sqrt{2}z_2)$$

and

$$\psi_N^{-1} : (z_1, z_2) \mapsto (\frac{z_2}{z_1}, \sqrt{2}z_1)$$

The transition function on $U_N \cap U_S$ is then

$$t_{NS}(\xi) = \frac{\sqrt{2}z_1}{\sqrt{2}z_2} = \xi_1 + i\xi_2 \in U(1)$$

As we go around the equator, we see that $t_{NS}(\xi)$ makes one lap around the unit circle in the complex plane. Therefore, this bundle is of homotopy class 1 of $\pi_1(U(1)) = \mathbb{Z}$, which as we will see describes a monopole of unit strength. Notice that we cannot find a global triviality for S^3. To see this observe that $\pi_1(S^2 \times S^1) = \pi_1(S^2) \oplus \pi_1(S^1) \cong \mathbb{Z} \neq 0 = \pi_1(S^3)$. For a different perspective, one can similarly define the Hopf map

$$\pi : S^1_{\mathbb{C}} \to \mathbb{C}P^1$$

by

$$(z_1, z_2) \mapsto [(z_1, z_2)] = \{\lambda(z_1, z_2) | \lambda \in \mathbb{C} - \{0\}\}.$$

This map takes points $\lambda(z_1, z_2) \in S^3$ with $|\lambda| = 1$ to single points in $\mathbb{C}P^1$.

Play video 7 on $http://dimensions-math.org/Dim\_reg\_AM.htm$.

4 Magnetic Monopoles

Recall Gauss' Law for magnetism,

$$\nabla \cdot \mathbf{B} = 0$$

or in integral form

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0.$$

These equations imply that the magnetic field is given by a solenoidal vector field. What if we redifine Gauss' law as

$$\nabla \cdot \mathbf{B} = 4\pi g \tag{1}$$

where g is the magnetic charge? This equation then has the solution

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3} = -\nabla\frac{g}{r}. \tag{2}$$

This implies that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

is no longer valid because,

$$\oint (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int \nabla \cdot (\nabla \times \mathbf{A}) \cdot d\mathbf{r} = 0.$$

But this is a contradiction since (1) implies that

$$\oint \mathbf{B} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{B} \cdot d\mathbf{r} = 4\pi g.$$

But we know that as we move away from the origin, $\mathbf{B}$ has no divergence, so we must find a vector potential $\mathbf{B}$ that obeys (3) almost everywhere. For simplicity, let us find a vector potential that fails to hold on a line. Following Dirac, we can take this line to be $z > 0$. Again, we take advantage of Stokes' theorem,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = -\int_S \mathbf{B} \cdot d\mathbf{S}, \tag{4}$$

where C is a circle of constant θ on a sphere of constant radius r about the origin, and S is the lower part of the sphere bounded above by C. Then, the surface integral (4) easily follows,

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \frac{g}{r^2} 2\pi r^2 (1 + \cos\theta).$$

We can then take as a solution of (4) to be,

$$\mathbf{A} = A_\phi \hat{\phi},$$

with

$$A_\phi = -\frac{g(1 + \cos\theta)}{r \sin\theta}.$$

Now we let $\theta \to 0$ and notice that the singularity is only on the z axis and

$$\oint_{C'} \mathbf{A} \cdot d\mathbf{r} = \int_{S'} (\nabla \times \mathbf{A}) \cdot d\mathbf{S'} = -4\pi g,$$

where C' is the infinitesimal circle, and S' the infinitesimal area. We can observe from this that the singularity of $\nabla \times \mathbf{A}$ is indeed on the axis and can conclude that the magnetic field is given by

$$\mathbf{B} = \nabla \times \mathbf{A} + 4\pi g \delta(x)\delta(y)\theta(z)\hat{\mathbf{z}},$$

where

$$\theta(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0. \end{cases}$$

This seems like a pretty janky way to describe a monopole, but fortunately this singularity can be avoided if we abandon the use of a single vector potential. This is where our old friend Hopf gives us a hand. But first, I left out a big story about connections on principal bundles. An important thing to note is that a principal bundle is what physicists refer to as a gauge, and that locally a connection one form is refered to as a gauge potential. The nicest case of this is given by $U(1)$ gauge potentials which describe electromagnetism. We will use the following facts about connections, the proofs of which can be found in "Geometry, Topology and Physics" by M Nakahara.

1. Let $P(M, G)$ be a principal bundle, $\{U_i\}$ be an open cover of M and σ_i be a local section defined on each U_i. Given a $Lie(G)$ valued one form $\mathcal{A}_i$ on U_i and a local section $\sigma_i : U_i \to \pi^{-1}(U_i)$, there exists a connection one form (called an Ehresmann connection) $\omega \in Lie(G) \otimes T^*M$ such that $\mathcal{A}_i = \sigma_i^* \omega$.

2. The compatability condition for these is given by

   $$\mathcal{A}_j = t_{ij}^{-1} \mathcal{A}_i t_{ij} + t_{ij}^{-1} dt_{ij}.$$

   This is what physicists refer to as a gauge transformation.

The dirac monopole is defined on $\mathbb{R}^3 - \{0\}$, which is homotopy equivalent to S^2, and is described by a principal $U(1)$-bundle $P(S^2, U(1))$. We can cover S^2 by two charts,

$$U_N \equiv \{(\theta, \phi) | 0 \leq \theta \leq \frac{1}{2}\pi + \epsilon\} \quad U_S \equiv \{(\theta, \phi) | \frac{1}{2}\pi - \epsilon \leq \theta \leq \pi\}$$

with θ and ϕ polar coordinates. Let ω be an Ehresmann connection on $P(S^2, U(1))$ and σ_N,σ_S to be local sections on U_N,U_S, respectively. Then we can define local gauge potentials (Wu-Yang)

$$\mathcal{A}_N = \sigma_N^* \omega \quad \mathcal{A}_S = \sigma_S^* \omega$$

by

$$\mathcal{A}_N = g(1 - \cos\theta) d\phi \quad \mathcal{A}_N = -g(1 + \cos\theta) d\phi$$

Now take the transition function t_{NS} defined on the equator $U_N \cap U_S$. We can write

$$t_{NS} = exp[i\varphi(\phi)] \quad \varphi : S^1 \to \mathbb{R}.$$

Then the gauge potentials are related by the compatability condition

$$\mathcal{A}_N = t_{NS}^{-1}\mathcal{A}_S t_{NS} + t_{NS}^{-1} dt_{NS}, \tag{5}$$

which gives us

$$d\varphi = \mathcal{A}_N - \mathcal{A}_S = 2gd\phi. \tag{6}$$

Now, as ϕ goes around the equator, $\varphi(\phi)$ has the range

$$\Delta\phi \equiv \int d\varphi = \int_0^{2\pi} 2gd\phi = 4\pi g.$$

So for t_{NS} to be defined uniquely, it must be an integer multiple of 2π,

$$\frac{\Delta\phi}{2\pi} = 2g \in \mathbb{Z}.$$

This is the quantization condition for magnetic monopoles. But now let us see how monopoles give us electric charge quantization. Consider a particle with mass m and charge e moving in the field of a sufficiently massive magnetic monopole. We can write Schrödinger's equation for the particle's wave function as

$$\frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 \left|\psi(\mathbf{r})\right\rangle = E\left|\psi(\mathbf{r})\right\rangle.$$

It can be shown that under the gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda,$$

the wavefunction transforms as

$$\left|\psi(\mathbf{r})\right\rangle \to exp[\frac{ie\Lambda}{\hbar c}]\left|\psi(\mathbf{r})\right\rangle \ , \ \Lambda = 2g\phi.$$

We saw by equations (5) and (6) that $\mathcal{A}_N - \mathcal{A}_S = 2gd\phi = \nabla 2g\phi$. If ψ^N and ψ^S are wavefunctions defined on U_N and U_S respectively, then they are related by

$$\left|\psi^S(\mathbf{r})\right\rangle = exp[\frac{ie\Lambda}{\hbar c}]\left|\psi^N(\mathbf{r})\right\rangle$$

Now, if we fix θ and observe the wavefunctions as they go from $\phi = 0$ to $\phi = 2\pi$ we note that the wavefunction is required to be single valued, therefore

$$\frac{2eg}{\hbar c} = n, \quad n \in \mathbb{Z}$$

This tells us that if a single monopole exists, then all electric charges are quantized!

5 References

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