Introduction to Mathematics: Week 1 Handout

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1 Mathematical Writing and Logic

1.1 Nomenclature and Abbreviation

Mathematical proofs use certain vocabulary to structure the arguments and communicate more effectively with any reader who is familiar with the language. We'll start by familiarizing ourselves with common terms:

- Theorem (Thm): A substantial mathematical statement which has been proven to be true.
- Corollary (Cor): A consequence of a theorem which follows either immediately from it or from the theorem combined with other established facts.
- Lemma: A smaller statement that needs to be proved as an intermediate step to proving a theorem.
- Remark (Rmk): A note from the author.
- Definition (Def): The meaning of a new term.
- Proof (Pf): A series of logical steps that shows that establishes the validity of a claim.
- Example (Ex/E.g)
- Exercise (Ex.): An example that the author leaves to the reader to work out on their own.
- Nota Bene (N.B.): "Note well" or take note.
- Such that (s.t.)
- Left hand side (LHS): The expression on the left hand side of an inequality or equation.

- Right hand side (RHS): The expression on the right hand side of an inequality or equation.

- Without loss of generality (WLOG): An assumption that does not sacrifice the applicability of a proof to all cases.

- The following are equivalent (TFAE): All of the following statements imply each other (i.e. they are all if and only if with each of the others).

It is often convenient to shorten mathematical writing by replacing certain common logical words with symbols. Below is a list of common symbols you should familiarize yourself with:

Symbol	Meaning
$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique
$\Leftrightarrow$	if and only if (iff)
$\Rightarrow$	LHS implies RHS
$\Leftarrow$	RHS implies LHS
s.t. , $\mid$, :	such that

While these symbols are essential to know, it is generally best to avoid abusing shorthand notation. A proof should read like a paragraph rather than a series of symbols. Presenting your arguments this way helps the reader's understanding, and it also helps you avoid mistakes.

1.2 Propositional Logic

Mathematics is built upon basic logical statements, which are either true or false but not both. We will denote statements as P or Q, and when these statements depend on some value x, we will denote them $P(x)$ and $Q(x)$. For example, the statement P = "all dogs have white fur" is false, and P = "5 is an odd integer" is true.

In logic, there are operations for combining and modifying statements. If P and Q are statements, then we define:
not, : 'not P'

- True when P is false.

- False when P is true.

and , $\wedge$: 'P and Q'

- True when P and Q are both true.

- False when P is false, Q is false, or both P and Q are false.

or $\vee$: 'P or Q'

- True when Q is true or Q is true or both P and Q are true.
- False when both P and Q are false.

, **if... then**, $\Rightarrow$: 'if P then Q'

- True when P and Q are both true or P is false.
- False when P is true and Q is false.

If and only if (iff), $\Leftrightarrow$: 'P iff Q'

- True when P and Q are both true or false.
- False when one of P or Q is true and the other is false.

Converse, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

- True when Q and P are both true or Q is false.
- False when Q is true and P is false.

Exercise 1. Consider the following propositions:

1. $P(x) =$ "x is a perfect square", $Q =$ "x is an integer". Is $P \Rightarrow Q$ true? What about the converse? Is $P \Leftrightarrow Q$ true? Is (not P) $\Rightarrow$ (not Q) true? How can you determine that from your previous conclusions?

2. $P(x) =$ "x has at least 3 distinct prime factors", $Q(x) =$"$x \geq 20$". Which of $P \Rightarrow Q$, $Q \Rightarrow P$, $P \Leftrightarrow Q$ are true?

3. $P(x) =$ "x is even", $Q(x) =$ "x is odd", $R(x) =$"x equals $2n+1$ for some integer n", $S(x) =$ "x equals $2m$ for some integer m". Write 2 statements using P, Q, R, S, AND, OR, and NOT.

1.3 Summary of Logic - Truth Tables

- $\neg P$

P	$\neg P$
T	F
F	T

- P **and** Q

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

- P **or** Q

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

- $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- $P \iff Q$

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

1.4 Basics of Set Theory

The language of set theory is ubiquitous throughout mathematics, primarily because it is used to formulate mathematical statements. We will not be giving a full introduction to set theory here, as it will be covered in your classes.

The most common symbols you'll see are:

- $\{\ \}$ set brackets, and the elements between them are the elements of the set
- $\in$ is an element in (some set)
- $\subset$ subset, LHS is a proper subset of the RHS, i.e. every element on the LHS is also in the RHS, but not all elements in the RHS are in the LHS

- $\not\subset$ LHS is not a subset of the RHS
- $\subseteq$ LHS is a subset of the RHS and may be equal to the RHS
- $\subsetneq$ LHS is a subset of the RHS but is not equal to the whole RHS

N.B.: conventions for $\subset$ and $\subsetneq$ largely vary person to person. Some people use $\subset$ as a catch-all meaning either $\subset$ or $\subseteq$, and some use $\subset$ and $\subsetneq$ interchangeably, so be careful to check what the meaning is in any context.

Some common sets that you've probably already met are

- $\mathbb{N} = \{0, 1, 2, ...\}$
- $\mathbb{Z} = \{..., -3, -2, \check{}1, 0, 1, 2, 3, ...\}$
- $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$
- $\mathbb{R} = (-\infty, \infty)$

Exercise 2. Which of the following statements are true?

- $\mathbb{Z} \not\subset \mathbb{Q}$
- $\mathbb{N} \subsetneq \mathbb{Z}$

Exercise 3. State $A \subsetneq B$ and $A \not\subset B$ in terms of elements (i.e. $a \in A$...)

Remark. In sets, neither order nor multiplicity matters. That is,

$$\{a, b, b\} = \{b, a, b\} = \{a, b\}$$

Exercise 4. Determine if the following sets are equal, and if not which one is a subset of the other:

1. $A = \{1, 2, 2, 3, 4\}$, $B = \{2, 1, 3, 2, 4\}$
2. $A = \{..., -20, -10, 0, 10, 20, ...\}$, $B = \{n \in \mathbb{Z} | n = 10m$ for some $m \in \mathbb{Z}\}$
3. $\mathbb{N}$, $\mathbb{Z}$
4. $A = \{x \in \mathbb{R} \mid x^2 = 1\}$, $B = \{x \in \mathbb{R} \mid x^3 = 1\}$
5. $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, $C = \{1, 1, 2, 2, 3, 3\}$

Remark. Two common tasks in a proof are:

1. Given a set X with two subsets $A, B \subset X$, show that $A \subset B$. This is equivalent to saying $a \in A \Rightarrow a \in B$, so these proofs often begin with assuming that a is an element of A and then showing that a is also in B

2. Given a set X with $A, B \subset X$, show that $A = B$. A common way to do this is to show that both inclusions $A \subset B$ and $B \subset A$ hold, and then since the only way that both can be contained in each other is if they are equal, this shows that $A = B$

1.5 Creating New Sets

Given a set A and a propositional statement P, we can define

$$B := \{a \in A \mid P(a)\}$$

Then B is a subset of A, meaning it's contained in A but may not be all of A if not all elements of A satisfy the condition P, and we write $N \subset A$.

The subsets of A themselves form an important set called the "power set" of A and denoted by,

$$P(A) = \{B \mid B \subset A\}$$

1.6 Quantifiers

There are two basic quantifiers: for all ($\forall$) and there exists ($\exists$). Respectively, these help us form universal and existential statements.
An existential statement is of the form:

$$\exists x \in D : Q(x)$$

Whereas a universal statement is of the form:

$$\forall x \in D : Q(x)$$

Where $Q(x)$ is some propositional statement needing to be proven.

Exercise 5. Write the following statements in symbolic form:

1. There is a number which when multiplied with any other number, results in the original number.

2. Every natural number has a unique integer which is its negative.

3. For every real number, there is another real number such that their product is 1.

4. For any choice of rational number, there is a unique rational number such that their sum is zero.

Exercise 6. Rewrite the following statements into English and determine if they are true or false:

1. $\forall x \in \mathbb{N} : x \;\;$ is even
2. $\forall t \in \{1, 3, 5, 7, 11\} : x$ is prime
3. $\forall x \in \{2, 4, 6, 8, 10\} : 2x < 10$
4. $\exists y \in \mathbb{Z}$ such that $y^2 = y$
5. $\forall x \in \mathbb{N} : \exists y \in \mathbb{Z} : y = -x$
6. $\exists x \in \mathbb{N} : \forall y \in \mathbb{Z} : y = -x$

2 Proofs

2.1 What does a proof look like?

A proof is a series of statements, each of which logically follows from what has already been established. It is an explanation which convinces the reader that a statement is true, and if done well, helps them understand why it is true.

When writing a proof, the core steps are:

1. Write out any assumptions and facts that have already been established.
2. Write out what the goal of the proof is, i.e. what statement you are aiming to establish.
3. Derive a series of statements which are logically consistent that arrive at the desired statement.

At the end of a proof, there are several conventions to indicate that the proof has been concluded. The most formal of these is "Q.E.D", which stands for "Quod Erat Demonstrandum", or "which had to be demonstrated" in Latin. However, especially if you‘re not writing a paper, there are several more commonly used symbols that are better fit for homework, exams, etc.:

$$\square, \blacksquare, \boxtimes$$

Some common mistakes to watch out for when writing proofs are:

- Arguing from examples: just because it‘s true for some cases doesn't mean it's true for all cases, so make sure that the proof works for all possibilities.
- Using the same variable or letter for two different things.
- Circular reasoning: x is true because y is true since x is true.

- Mixing up what is known and what needs to be shown — make sure you clearly state your assumptions, axioms, and theorems versus the conclusions you're aiming to show.
- If vs. if and only if: make sure each implies arrow is showing a logical step.
- Using any vs some.

The basic types of proof that we will cover are:

1. **Direct proof**: Suppose that P holds, and show how to obtain Q.
2. **Proof by contrapositive**: Provide a direct proof of $\neg Q \Rightarrow \neg P$.
3. **Proof by contradiction**: Suppose that P holds and Q fails, and derive a contradiction.
4. **Proof by cases**: Group the possible x such that $P(x)$ into subsets which all behave equivalently under $P(x)$, and then provide a proof for each case that $Q(x)$ holds
5. **Proof by induction**: Divide the proposition into smaller claims of the form $P(n)$ for each positive integer n. Establish the base case $P(1)$ (or whatever n the base case occurs at). Then show that the implication "$P(n)$ implies $P(n+1)$" holds for every positive integer n.

2.2 Proving Existential vs. Universal Statements

In a proof of an existential statement, you are trying to show existence of an element which satisfies the statement within the set of possible elements that could satisfy the claim, since

$$\exists x \in D : Q(x) \iff Q(x) \text{ is true for some (at least one) } x \in D.$$

In a proof of a universal statement, you need to show that the statement holds for all possible elements that it may hold for:

$$\forall x \in D, P(x) \iff P(x) \text{ holds for every } x \in D.$$

Exercise 7. Describe what you need to do to complete the following proofs:

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y = x^{-1}$
2. $\forall n \in \mathbb{N}$, $n \leq n \cdot m$ for any $m \geq 1$
3. $\exists \frac{a}{b} \in \mathbb{Q}$ such that $\frac{a}{b} \geq \frac{b}{a}$.

N. B. When combined, the order of quantifies matters: consider the statement

$$\forall x \in X, \exists y \in Y : P(x, y)$$

This is not equivalent to

$$\exists y \in Y : \forall x \in X : P(x, y)$$

This is because in the first statement, y may be dependent on x, but in the second statement the same y must work for all x.

Exercise 8. Come up with an example of this, such as "for all integers n, there is an integer m such that $n + m = 0$", which is not the same as "there exists an integer m such that for all integers n, $n + m = 0$" (which is false).

Proposition 1. Negating a statement switches $\forall$ and $\exists$. In the case of a universal statement, we get

$$\neg(\forall x \in X, P(x))$$

is equivalent to

$$\exists x \in X : \neg P(x)$$

and if we have an existential statement, negating gives us

$$\neg(\exists x \in X : P(x))$$

is equivalent to

$$\forall x \in X, \neg P(x).$$

Exercise 9. What happens when we negate:

1. $\forall x \in D : P(x)$
2. $\exists x \in D : P(x)$
3. P and Q
4. P or Q
5. $P \Rightarrow Q$
6. $P \iff Q$

2.3 Direct Proofs

The logical structure of a direct proof for $P \Rightarrow Q$ is:

- Assume that P is true
- Using P, show that Q is true

Definition 2. A natural number n is *even* if $\exists k \in \mathbb{N}$ such that $n = 2k$. We say n is *odd* if $\exists k \in \mathbb{N}$ so that $n = 2k - 1$.

Exercise 10. Prove that the sum of any two even integers is even.

Exercise 11. For any positive integer n, there exist n consecutive positive composite integers.

Exercise 12. Prove that if $m, n \in \mathbb{Z}$ such that m is even and n is odd, then $m + n - 2$ is odd.

Exercise 13. If m is odd $\Rightarrow m^2 + 1$ is even.

Exercise 14. Decide which of the following are valid proofs of the statement: If ab is an even number, then a or b is even.

Proof (a). Suppose by contradiction that a and b are odd, i.e. $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$. Then

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned}$$

Therefore ab is odd and we get a contradiction. □

Proof (b). Assume that a or b is even: WLOG, assume a is even (as the case where b is even will be the same). Then by definition, a = 2k, and we have

$$ab = (2k)b = 2(kb).$$

Therefore ab is even. □

Proof (c). Suppose that ab is even but a and b are both odd, i.e. $ab = 2n, a = 2k + 1$, and $b = 2j + 1$ for some integers n, k, and j. Then

$$\begin{aligned} 2n &= (2k+1)(2j+1) \\ &= 4kj + 2k + 2j + 1. \end{aligned}$$

Hence, $n = 2kj + k + j + \frac{1}{2}$ But since $2kj + k + j$ is an integer, this equation says that the integer n is not an integer, which is impossible and so we have a contradiction. Thus ab even $\Rightarrow a$ and b are not both odd. □

Proof (d). Let ab be an even number, say $ab = 2n$, and a be an odd number, so $a = 2k + 1$. Then

$$2n = 2kb + b.$$

Now if we subtract $2kb$ from both sides we obtain

$$2n - 2kb = b.$$

By the distributive property,

$$2(n - kb) = b.$$

Therefore b must be even. □

3 Proofs/Answers

Exercise 1.

- $P \Rightarrow Q$ is true. The converse $Q \Rightarrow P$ is false. $P \iff Q$ is false. Also, not $P \Rightarrow$ not Q is false.
- $P \Rightarrow Q$ is true, $Q \Rightarrow P$ is false, and $P \iff Q$ is false.

□

Exercise 2.

- False
- False

□

Exercise 3. $A \subsetneq B$: $a \in A \Rightarrow a \in B$ and $\exists b \in B$ such that $b \notin A$
$A \not\subset B$: $\exists a \in A$ such that $a \notin B$ □

Exercise 4.

1. $A = B$
2. $A = B$
3. $\mathbb{N} \subset \mathbb{Z}$
4. $B \subset A$
5. $A = B = C$

□

Exercise 5.

1. $\exists x : \forall y : xy = x$
2. $\forall x \in \mathbb{N} : \exists! y \in \mathbb{N} : x + y = 0$
3. $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : xy = 1$
4. $\forall x \in \mathbb{Q} : \exists! y \in \mathbb{Q} : x + y = 0$

$\square$

Exercise 6.

1. all natural numbers are even: false
2. each of 1,3,5,7,11 are prime: false
3. the double of each of 2,4,6,8,10 is less than 10: false
4. there exists an integer which squares to itself: true
5. every natural number has an integer which is its negative: true
6. there exists a natural number such that every integer is its negative: false

$\square$

Exercise 9.

1. $\exists x \in D : \neg P(x)$
2. $\forall x \in D : \neg P(x)$
3. $\neg P$ or $\neg Q$
4. $\neg P$ and $\neg Q$
5. P and $\neg Q$
6. $(P$ and $\neg Q)$ or $(\neg P$ or $\neg Q)$

$\square$

Exercise 10. We start by formalizing the statement:

$$\forall m, n \in \mathbb{Z}, m \text{ and } n \text{ even } \Rightarrow m + n \text{ even.}$$

Suppose m and n are any even integers, say $m = 2r$ and $n = 2s$ for some $r, s \in \mathbb{Z}$. Then

$$m + n = 2r + 2s = 2(r + s)$$

by the distributive property. Since $r + s$ is an integer, say $r + s = k$, we then have that $m + n = 2k$ for $k \in \mathbb{Z}$. Therefore, $m + n$ is even.

$\square$

Exercise 11. We will prove the statement by constructing such a sequence for any n $\in \mathbb{Z}$. Consider the sequence of integers

$$(n+1)!+2,\ (n+1)!+3, \dots,\ (n+1)!+n+1$$

Note that every integer in the sequence is composite because k divides $(n+1)!+k$ for all k such that $2 \leq k \leq n+1$. This completes the proof as we have found such a sequence. □

Remark. The proof above is what is known as a constructive proof: you prove existence by explicitly constructing the elements which satisfy the statement.

Exercise 12.

Assume m is even and n is odd. By definition, this means that $m = 2k$ and $n = 2p+1$ for some $k, p \in \mathbb{Z}$. We need to show that $m+n-2$ is odd, i.e. $m+n-2 = 2q+1$ for some $q \in \mathbb{Z}$. Using our definitions, we write

$$m+n-2 = 2k + (2p+1) - 2$$

By the distributivity and associativity,

$$m+n = 2(k+p)-1 = 2(k+p)-1+2-2 = 2(k+p)+1-2 = 2(k+p-1)+1$$

since addition commutes in $\mathbb{Z}$. So if we define $q = k+p-1 \in \mathbb{Z}$, we see that

$$m+n-2 = 2q+1$$

and thus $m+n-2$ is odd.

□

Exercise 13. Suppose m is odd. Then by definition, $m = 2k+1$ for some $k \in \mathbb{Z}$. Then $m^2+1 = (2k+1)^2+1 = 4k^2+4k+2 = 2(2k^2+2k+1)$, and since $(2k^2+2k+1)$ is an integer, we have that m^2+1 is even. □

Exercise 14.

1. Valid proof by contrapositive.
2. Invalid, this is a proof of the converse.
3. Valid (but highly inelegant) proof by contradiction.
4. Valid direct proof.

□