The Ubiquity of Graphs
6/12/2022
Introduction to the Topic

Reference: Harris, Hirst, and Mossinghoff: Combinatorics and Graph Theory

What will we study? Discrete Math
- Graph Theory
- Combinatorics
- Infinite Combinatorics & Graphs

I) Graph Theory

Graphs are a set of vertices with edges between them
The subject originated with the Königsberg Bridge Problem.

1700s—Residents want to walk across bridges, making a route that crosses each bridge exactly once.

1736—Euler, learning about this phenomenon, writes an article about it.
Graphs can be used to model many things.

Example: Four Color Theorem:
4 is the maximum number of colors required to color any map where bordering regions are colored differently.

Example of a 4-colored map

Proposed in 1852, unproven until 1976.
Used over 1000 hours of computer time.

More Examples:
1) Social Networks
2) Roads or Railroads
3) The Internet
4) Games and Tournaments
Ramsey Theory: How many people are required at a gathering so that there must exist either 3 mutual acquaintances or 3 mutual strangers? \( R(3,3) \)

2) Combinatorics

- Enumerative Combinatorics
  - The science of counting
  - The number of possible arrangements of a set of objects, under some constraints

  How many ways to make change for a dollar?
  How many ways to put \( n \) guests at \( k \) tables?

  Binomial Coefficients / Permutations
  Generating Functions

Example: How many \( k \)-element subsets of an \( n \)-element set?
Given by \( k \)th coefficient of
(1+x)^n

- Existential Combinatorics
  - Studies problems concerning the existence of arrangements that possess some specified property

**Ex: Pigeonhole Principle:**
If more than n objects are distributed among n containers, then some container must contain more than one object

- Constructive Combinatorics
  - The design and study of algorithms for creating arrangements with special properties

**Ex: Combinatorial Geometry**

**Sylvester's Problem:**
Given \( n \geq 3 \) points in the plane which don't all lie on the same line, must there exist a line
that passes through exactly two of them?

3) Infinite Combinatorics and Graphs

What happens when the vertex set is infinite in a graph?
Can we "count" infinities? Are there different sizes of infinity?
Graph Theory

Def: A graph consists of a set $V$ and a set $E$:
- Elements of $V$ are called vertices
- Elements of $E$, called edges, are unordered pairs of vertices

We usually stipulate that $V$ & $E$ are finite

Ex: $V = \{a, b, c, d, e, f, g, h\}$
$E = \{(a, d), (a, e), (b, c), (b, e), (b, g), (c, f), (c, d), (d, g), (g, h)\}$
$G = (V, E)$

Can be represented visually:

![Graph](image)

FIGURE 1.2. A visual representation of the graph $G$. 
When thinking of a graph, you should always think about it visually like above.

We can alter our definitions in various ways:

1) If we let $E$ consist of ordered pairs of vertices, we obtain a directed graph.

![FIGURE 1.3. A digraph.](image)

2) We get a multigraph if we allow repeated elements in our set of edges ($E$ becomes a multiset).

![FIGURE 1.4. A multigraph.](image)
(3) We get a **pseudograph** if we allow "loops" e.g. vertices may connect to themselves

![Pseudograph Diagram](image)

**FIGURE 1.5. A pseudograph.**

(4) We get a **hypergraph** if we let our edges be arbitrary subsets of vertices (rather than just pairs)

![Hypergraph Diagram](image)

**FIGURE 1.6. A hypergraph with 7 vertices and 5 edges.**

We will usually focus on **finite, simple graphs**; those without loops or edges.
The vertex set of a graph $G$ is denoted $V(G)$, and the edge set denoted $E(G)$. Denote edge between $u,v$ as $uv$

**Def:**
1. The **order** of a graph $G$ is the cardinality of $V$.
2. The **size** of $G$ is $|E|$.
3. If $u,v \in V$ and $uv \in E$, then $u$ and $v$ are adjacent. $u$ and $v$ are the end vertices of $uv$.
4. If $uv \notin E$, then $u$ and $v$ are nonadjacent.
5. If an edge $e$ has $v$ as an end vertex, we say that $v$ is incident with $e$.

**Diagram:**

- **Order:** 8
- **Size:** 9
- **Adjacent to $b$:** $e, f, g$
- **Incident to $eb$:** $e, f, b, g$
Def: (1) The **open neighborhood** of vertex \( v \), denoted by \( N(v) \), is
\[
N(v) = \{ x \in V | v x \in E \}
\]
(2) The **closed neighborhood** of \( v \), denoted \( N[v] \), is \( \{v\} \cup N(v) \)
(3) For arbitrary \( S \subseteq V \), let
\[
N(S) = \bigcup_{s \in S} N(s)
\]
\[
N[S] = S \cup N(S)
\]
be the **open** and **closed neighborhoods** of \( S \)

\[
N(f) = \{ c, d \} \quad N[f] = \{ f, c, d \}
\]

\( S = \{ a, b, g \} \)

\[
N(S) = \{ e, d, c, g, b, h \} \quad N[S] = \{ e, d, c, g, b, h \}
\]
\[N(a) = \{ e, d \} \quad a \]
\[N(b) = \{ e, g \} \quad c \]
\[N(g) = \{ b, h \} \]

\[\]
Def: (1) The degree of $v \in V$, denoted $\deg(v)$, is the number of edges incident with $v$.

(2) The maximum degree of $G$, denoted $\Delta(G)$, is
\[
\Delta(G) = \max \{ \deg(v) \mid v \in V \}
\]

(3) The minimum degree $\delta(G)$ is
\[
\delta(G) = \min \{ \deg(v) \mid v \in V \}
\]

Rmk: In simple graphs, $\deg(v) = |N(v)|$.

\begin{align*}
\deg(a) &= 3 \\
\Delta(G) &= 4 \\
\delta(G) &= 1
\end{align*}
Q: How many people at Columbia have an odd number of friends?

Thm: In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently, the number of vertices with odd degree is even.

PF: Let $S = \sum_{v \in V} \deg(v)$. Since each edge is exactly 2 vertices, when counting $S$ we count each edge exactly twice. Thus $S = 2|E|$.

Since $S$ is always even, the number of vertices with odd degree is always even, for else $S$ would be odd. □

Partial Answer: An even number of people have an odd number of friends.

$V = \{\text{students at Columbia}\}$

$E = \{uv \mid u \text{ is friends with } v\}$
Examples:

(1) Complete Graph of order $n$, $K_n$

![Complete Graphs](image)

Every vertex adjacent to every other vertex.

Q: How many edges in $K_n$?

$$\text{deg}(v) = n-1$$

$$\sum_v \text{deg}(v) = n(n-1)$$

$$|E| = \frac{n(n-1)}{2}$$

(2) Empty graph on $n$ vertices $E_n$

$$E = \emptyset$$

![Empty Graph](image)

(3) Complement of $G$, $\overline{G}$

$$V(\overline{G}) = V(G)$$

$$E(\overline{G}) = \text{all edges not present in } G$$
Regular graphs are graphs with every vertex the same degree.

\[ K_n \text{ is regular with degree } n-1 \]

A graph \( H \) is a subgraph of \( G \) is \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \).
Walks & Connectivity

Def: (1) A **walk** is a sequence of vertices \( V_1, V_2, \ldots, V_k \) s.t. \( V_i V_{i+1} \in E \) for all \( i \).

We call it a \( V_1-V_k \) walk, and \( V_1, V_k \) are the end vertices of the walk.

(2) If the edges in a walk are distinct it is a **trail**.

(3) If the vertices are distinct it is a **path**.

(4) The **length** of a walk is its number of edges, counting repetitions.

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acfcbd is a walk of length 5
bacbd is a trail but not a path
dgbacf is a path and a trail
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Every path is a trail, but not every trail is a path.

(different vertices \( \Rightarrow \) different edges)
Defn 1) A cycle is a path with an edge $v_kv_1$.

2) A circuit is a trail that begins and ends at the same vertex.

More examples of graphs

$C_n =$ cycle on $n$ vertices

FIGURE 1.7.

FIGURE 1.15. The graph $C_7$. 
\[ P_n \] \text{ path on } n \text{ vertices}

\[ \bullet \bullet \bullet \bullet \bullet \bullet \]

FIGURE 1.16. The graph \( P_6 \).

**Def:** A graph is \textit{connected} if every pair of vertices can be joined by a path.

Each maximal connected piece of a graph is called a \textit{connected component}.

\[ G_1 \quad G_2 \quad G_3 \]

FIGURE 1.9. Connected and disconnected graphs.

Connected: Yes \quad No \quad No

\# Components: 1 \quad 3 \quad 2
I'll end with a few cool graph theoretic proofs of popular theorems.

Source: V. Yegnanarayan, "Graph Theory to Pure Mathematics: Some Illustrative Examples"

**Fermat's Little Theorem:** Let $p$ be a prime and $a \in \mathbb{N}$ such that $p \nmid a$. Then $a^{p} - a$ is divisible by $p$.

**Proof:** Let $G = (V, E)$, where

$V = \text{all sequences } (a_1, \ldots, a_p) \text{ s.t. } 1 \leq a_i \leq q,$

with $a_i \neq a_j$ for some $i \neq j$.

Then $|V| = a^{p} - a$.

For all $u = (u_1, \ldots, u_p) \in V$, let $uv \in E$ iff.

$v = (u_p, u_1, \ldots, u_{p-1})$. Then each vertex

has degree 2; have $v, u$ and $uv_1$.

Hence each component of $G$ is a cycle of

length $p$. Then the number of components

is $(a^{p} - a)/p$.

Since $(a^{p} - a)/p$ is a whole number, $p \mid a^{p} - a, \Box$

Ex: $a = 2, p = 3$

$2^{3} - 2 = 6$

$2(112)$

$2(121)$

$2(122)$

$2(212)$
**Def:** A graph is bipartite if its vertex set can be partitioned into two sets $X$ and $Y$ such that every edge has one end vertex in $X$ and the other in $Y$.

![Bipartite Graphs](image)

**Theorem (Cantor-Shröder-Bernstein):** Let $A, B$ be two sets. If there is an injective mapping $f: A \to B$ and an injective mapping $g: B \to A$, then there is a bijection between $A$ and $B$.

**Proof:** Assume $A \cap B = \emptyset$. We will define a bipartite graph $G$ with partition $A$ and $B$ and $V = A \cup B$. Let $xy \in E$ iff, either $f(x) = y$ or $g(y) = x$ for $x \in A, y \in B$.

Since $f, g$ injective, $1 \leq \deg(v) \leq 2$ for all $v \in V$.

\[
\begin{align*}
  & f(x) = y, \quad g(y) = x \\
  & f(x_1) = y, \quad g(y) = x_2
\end{align*}
\]
Decompose $G$ into connected components:
3 types,
1) one-way infinite path
    \[
    x_0 - x_1 - x_2 - \cdots
    \]
    (cannot be finite, for else contradicts injectivity)

2) Cycle of even length with >2 vertices
    \[
    x_1 \xrightarrow{f(x_1)} y_1
    \]
    \[
    g(x_1) \xleftarrow{g(y_1)} y_1
    \]
    \[
    x_2 \xrightarrow{f(x_2)} y_2
    \]

3) Edge
    \[
    f(x_1) = y_1
    \]
    \[
    g(y_1) = x_1
    \]

For each component, there is a set of edges $E$, s.t. each vertex in the component is incident to exactly one of these edges.
Using these edges, construct a bijection $A \rightarrow B$.
Define $f'(x) = y$ iff. $x y \in E$.
$f'$ is injective and surjective by condition above.$\square$
Ex: If $f, g$ are inverses, get case 3.

Ex: Case 1 is $\mathbb{Z} \rightarrow 2\mathbb{Z}, 2\mathbb{Z} \rightarrow \mathbb{Z}$.