UMS Talk : Some aspects of Geometric Quantization

Thibault Langlais

March 27, 2019

Introduction

The two main (non-relativistic) physical theories are Hamiltonian and Quantum mechanics. A given system then admits a classical and a quantum description, which are very different in the way they are formulated. The aim of quantization is to relate these apparently different description ; more precisely, given a Hamiltonian system, one can try to build the corresponding quantum system: this process is called quantization. Many question arise: which axioms must define quantization? Can we give a systematic construction? Are all Hamiltonian systems quantizable? Are all quantizations of a given system equivalents? These questions are still essentially open, though there are now many systematic quantization processes (with there advantages and downsides). In this talk, we will develop from scratch the theory of Hamiltonian system, with the aim to show that most algebraic structure involved in quantum mechanics are already present in the classical setup, and give the axioms that defines a quantization. In the last part, I give a very brief and fairly partial overview of the difficulties that appear when one actually tries to build a quantization project, but the reader interested in details can find many good references that give much more details and precise statements.

1 Hamiltonian mechanics

1.1 From Newton to Hamilton

The motion of a one dimensional particle is given by Newton's law : $m\frac{d^2x}{dt^2} = -\partial V(x)$. This equation being second order, it is more convenient to change of variables to get a first order equation : let $q = x$, $p = m\frac{dx}{dt}$ and $H(q,p) = \frac{p^2}{2m} + V(q)$ (called the Hamiltonian function, which is most cases corresponds to the energy of the system). Newton's equation is equivalent to :

$$\begin{cases} \frac{dq}{dt} = \frac{\partial H}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial H}{\partial q} \end{cases}$$

which is called Hamilton-Jacobi equation. q represents the position of the particle, and p its momentum. The motion of the point (q,p) in $\mathbf{R}^2$ follows the flow of the vector field $X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$. Note that the divergence of X_H vanishes, so that its flow preserves the standard area form $\omega_0 = dp \wedge dq$ on $\mathbf{R}^2$. A direct computation also shows that $\frac{d}{dt}H(q(t),p(t)) = 0$: we get the well-known fact that the energy is a constant of motion. Also note a few things about the area form :

1. $\omega_0 = -d(p \wedge dq)$ is an exact 2-form; in particular it is closed

2. ω_0 is non-degenerate

3. $\omega_0(X_H, \cdot) = dH$

1.2 Symplectic Geometry

Definition 1.1. Let M be a manifold. A *symplectic form* $\omega \in \Omega^2(M)$ is a 2-form such that:

(i) ω is closed (i.e. $d\omega = 0$)

(ii) for each $x \in M$, ω_x is non-degenerate.

The pair (M, ω) is called a symplectic manifold. If moreover ω is exact, (M, ω) is called an exact symplectic manifold.

Remark. 1. Non-degeneracy implies that M is even-dimensional.

2. Say ω is non-degenerate amounts the same as to say that ω^n (where $2n$ is the dimension of M) never vanishes, i.e. this is a volume form. In particular, any symplectic manifold is orientable.

3. Non-degeneracy is also equivalent to the fact that for each $x \in M$ the map $T_xM \longrightarrow T_x^*M,\ X \longmapsto \omega(X, \cdot)$ is an isomorphism.

Example. 1. On $\mathbf{R}^{2n}$ the standard form is $\omega_0 = -d(p_1dq_1 + ... + p_ndq_n) = dq_1 \wedge dp_1 + ... + dq_n \wedge dp_n$. A noticeable fact is that any symplectic form is equal to the standard form in good local coordinates (Darboux theorem). In particular, unlike Riemannian Geometry, there are no local symplectic invariants.

2. If X is any manifold, its cotangent bundle $\pi : T^*X \longmapsto X$ has a tautological 1-form $(\lambda_{can})_{(x,\alpha)} = \pi^*\alpha$. The canonial symplectic form on T^*X is $\omega_{can} = -d\lambda_{can}$. Under the identification $T^*\mathbf{R}^n = \mathbf{R}^{2n}$ this is just the standard form.

Note that there are many symplectic manifolds which are not cotangent bundles: for instance, the standard form ω_0 on $\mathbf{R}^2$ projects onto a well-defined 2-form on the 2-torus $\mathbf{R}^2/\mathbf{Z}^2$, but the canonical 1-form pdq is not globally defined. Actually no symplectic form on a compact manifold can be exact, otherwise the integral of ω^n would vanish, which contradicts non-degeneracy.

Definition 1.2. Let (M, ω) be a symplectic manifold. A *symplectomorphism* is a diffeomorphism $\phi : M \longrightarrow M$ that preserves ω, that is, $\phi^*\omega = \omega$.

Suppose now that we are given a 1-parameter group of diffeomorphisms ϕ_t on M. It is generated by a vector field X defined by $\frac{d}{dt}\phi_t = X \circ \phi_t$. Recall that the Lie derivative of a differential form α with respect to X is defined by $\mathcal{L}_X\alpha = (\frac{d}{dt}\phi_t^*\alpha)_{t=0}$. By Lie-Cartan formula, $\mathcal{L}_X = d \circ i_X + i_X \circ d$ where i_X is the derivation of $\Omega(M)$ (of degree -1) defined by evaluating a k-form at X on the first variable. Another useful formula is the following : $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$. ϕ_t is a 1-parameter group of symplectomorphisms if and only if $\frac{d}{dt}\phi_t\omega = 0$, which amounts the same as $\mathcal{L}_X\omega = 0$. Together with Lie-Cartan formula and the the fact that ω is closed, we find the condition $d(i_X\omega) = 0$, i.e. the 1-from $i_X\omega$ be closed.

Definition 1.3. A vector field $X \in \mathcal{X}(M)$ is called *symplectic* if $i_X\omega$ is closed. We denote by $\mathcal{X}_{symp}(M)$ the set of symplectic vector fields.

A we just saw, an integrable symplectic vector field generate a 1-parameter group of symplectomorphisms. A particular case is when $i_X\omega$ is exact.

Definition 1.4. A vector field X such that $i_X\omega$ is exact is called *hamiltonian*. We denote by $\mathcal{X}_{ham}(M)$ the set of such vector fields.

Then there exists a function H such that $i_X\omega = dH$. Conversely, using non-degeneracy, for each smooth function H on M, there is a unique vector field X_H on M such that $i_X\omega = dH$. We say that X_H is the hamiltonian vector field generated by H. generated by H. The flow of X_H is called the hamiltonian flow generated by H.

Recall that the Lie bracket of two vector fields X, Y is defined by its action on a smooth function f by $[X,Y]\cdot f = X\cdot Y\cdot f - Y\cdot X\cdot f$. This bracket is bilinear, antisymmetric and satisfies the Jacobi identity :

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$$

Then $\mathcal{X}(M)$ has a structure of Lie algebra.

Proposition 1.1. *$\mathcal{X}_{symp}(M)$ and $\mathcal{X}_{ham}(M)$ are subalgebras of $\mathcal{X}(M)$.*

Proof. Let $X,Y\in\mathcal{X}_{symp}(M)$. Then :

$$\begin{aligned} i_{[X,Y]}\omega &= \mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega \\ &= \mathcal{L}_X i_Y\omega \\ &= di_X i_Y\omega + i_X di_Y\omega \\ &= di_X i_Y\omega \\ &= d(w(Y,X)) \end{aligned}$$

So we find that $[X,Y]$ is the hamiltonian vector field generated by the function $\omega(Y,X)$. □

1.3 Hamiltonian mechanics

Definition 1.5. A *Hamiltonian system* is a triple (M,ω,H) where (M,ω) is a symplectic manifold and H a smooth function on M.

The study of the dynamics of a hamiltonian system is the study of the hamiltonian flow generated by H, which we denote by ϕ_t^H. We already know by construction that this flow leaves ω invariant. It also leaves H invariant since :

$$\frac{d}{dt}H\circ\phi_t^H = dH(X_H\circ\phi_t^H) = \omega(X_H\circ\phi_t^H, X_H\circ\phi_t^H) = 0$$

Then the integral curves of ϕ_t^H lie in the level sets of H.

Example. On $\mathbf{R}^2$ with the area form $dq\wedge dp$, let $H = p^2/2+q^2/2$ the hamiltonian function of the harmonic oscillator. It generates a vector field $X_H = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$. Its trajectories are circles centered at the origin, which indeed are the level sets of H.

Let us think in terms of physical interpretation for a moment. The symplectic manifold M represents the space of states of a physical system, it is called the *phase space*. A function on M represents a physical, measurable quantity. We call them *observables*. For example, the hamiltonian function represents the energy, and on $\mathbf{R}^2$ with the standard form, $(q,p) \mapsto q$ represents the position and $(q,p) \mapsto p$ represents the momentum. The geometric setup we have adopted shows that observables have a dual nature : they generate a vector field and a (local) flow which leaves ω invariant. So one could say that observables "act" on the phase space. This echoes with the formalism of quantum mechanics where, as we shall see, observables are defined as operators on the space of states...

Definition 1.6. Let $f, g \in \mathcal{C}^\infty(M)$. The Poisson bracket of f and g is defined by $\{f,g\} = \omega(X_g, X_f)$.

Proposition 1.2. *(i)* $\{f,g\} = dg(X_f) = -df(X_g)$

(ii) $\{\cdot,\cdot\}$ *is e Lie bracket, i.e., it is bilinear, antisymmetric and satisfies the Jacobi identity*

(iii) $X_{\{f,g\}} = [X_f, X_g]$

(iv) Any symplectomorphism ϕ *preserves the Poisson bracket, in the sense that* $\{f,g\} \circ \phi = \{f \circ \phi, g \circ \phi\}$

Proof. (i) follows from the definition

(ii) check by yourself !

(iii) Recall the proof of Proposition 1.1 and the identity $i_{[X,Y]}\omega = d\omega(\ddot{Y}, X)$ which holds for any two symplectic vector fields. With $X = X_f$ and $Y = X_g$ this is exactly what we want.

(iv) First let us prove that $X_{f\circ\phi} = (d\phi)^{-1} X_f \circ \phi$. Indeed one can write, using $\phi^*\omega = \omega$:

$$\begin{aligned} \omega_x((d\phi)_x^{-1} X_f \circ \phi(x), \cdot) &= \omega_{\phi(x)}(X_f \circ \phi(x), (d\phi)_x \cdot) \\ &= \omega_{\phi(x)}(X_f \circ \phi(x), \cdot) \circ (d\phi)_x \\ &= (df)_{\phi(x)} \circ (d\phi)_x = d(f \circ \phi)_x \\ &= \omega_x(X_{f\circ\phi}(x), \cdot) \end{aligned}$$

Now we have :

$$\omega_x((d\phi)_x^{-1} X_g(\phi(x)), (d\phi)_x^{-1} X_f(\phi(x))) = \omega_{\phi(x)}(X_g(\phi(x)), X_f(\phi(x)))$$

But the let hand side is $\{f \circ \phi, g \circ \phi\}(x)$ and the right hand side is $\{f,g\} \circ \phi(x)$, which completes the proof.

□

Remark. It follows from the proposition that $f \in \mathcal{C}^\infty(M) \longmapsto X_f \in \mathcal{X}_{ham}(M)$ is a surjective Lie algebra homomorphism, with kernel the locally constant functions on M.

So far we have just given a nice structure to $\mathcal{C}^\infty(M)$, but the signification of the Poisson bracket is still obscur. Recall the signification of the Lie bracket on vector fields: $[X,Y]$ vanishes if and only if the (local) flows generated by X and Y commute. So now we wonder what $\{f,g\} = 0$ mean.

Proposition 1.3. *Let (M, ω, H) be a Hamiltonian system and $f \in \mathcal{C}^\infty(M)$. Define $f_t = f \circ \phi_t^H$. Then :*

$$\frac{d}{dt} f_t = \{H, f_t\} = \{H, f\} \circ \phi_t^H$$

In particular, the flow of H preserves f if and only if $\{H, f\} = 0$.

Proof. $\frac{d}{dt} f_t = (df)_{\phi_t^H}(X_H \circ \phi_t^H) = \{H, f\} \circ \phi_t^H = \{H, f_t\}$ □

Now it is time again for some physical interpretation. An observable f represents a measurable quantity, and $\phi_t^H(x)$ is the state of the system at time t with initial condition x at time $t = 0$. Thus f_t is the quantity f measured on the system at time t if we consider that the states don't evolve. Then we can either consider that the observable f is fixed and that the system evolves by the flow of H, either consider that the states are fixed and that observables evolve by the equation $\frac{d}{dt} f_t = \{H, f_t\}$. For those who have followed a course in quantum mechanics, this is similar as to work in Shrödinger picture (fixed obserbables) or in Heisenberg picture (fixed states). In both cases, we somehow forget about the underlying space of states to study the structure of the space of observables. Let us keep this in mind.

Another striking property is the symmetry of the condition $\{H, f\} = 0$. Using the dual nature of observables, any function f is the generator of a 1-parameter group of symplectomorphisms (which is not always globally well-defined, but still), $\{H, f\} = 0$ is also the condition for the hamiltonian flow generated by f to leave H invariant. If we dare call a transformation of the system which leaves both the symplectic form ω and the hamiltonian function H invariants a symmetry of (M, ω, H), we find that any conserved quantity is associated to a 1-parameter group of symmetry of the system. This is precisely the *Noether principle*, which states that there is a one-to-one correspondance between conserved quantities and 1-parameter groups of symmetry. Actually, we only showed a weaker statement, that is, any conserved quantity generates a (local) 1-parameter group of symmetry. The converse question, i.e. whether a given group of symmetry is generated by conserved quantities, is a much more complicated matter (the interested reader should take a look at what is a moment map, the answer is related to the cohomology of the considered symmetry group).

2 Quantum mechanics

2.1 Axioms

In this part, we want to give a few axioms that define quantum formalism and state the properties that quantization must satisfy.

1. A state is represented by a vector ψ of a complex Hilbert space $\mathcal{H}$ (sometimes it is also required that $\mathcal{H}$ be separable)

2. Observables are represented by hermitian operators A acting on $\mathcal{H}$. Its eigenvalues are to be interpreted as possible outcomes for measures on the system, and the average outcome in a normalized state ψ is given by $\langle \psi | A\psi \rangle$.

3. States evolve by Shrödinger's equation $i\partial_t \psi_t = 2\pi H \psi_t$ where H is the quantum Hamiltonian. Integrating the equation we get $\psi = e^{-2i\pi t H}\psi_0$, and so the corresponding evolution on the space of observables:

4. $A_t = e^{2i\pi tH} A e^{-2i\pi tH}$ and by differentiating this identity we get the formula $\frac{d}{dt} A_t = 2i\pi[H, A_t]$

Note that $2i\pi[A, B] = 2i\pi(AB - BA)$ is a Lie bracket on the space of operators in $\mathcal{H}$ and the set of hermitian operators is closed under this bracket, i.e., it is a Lie algebra for this Lie bracket, just as is $\mathcal{C}^\infty(M)$ with the Poisson bracket. Moreover the equation $\frac{d}{dt} f_t = \{H, f_t\}$ is definitely very similar to $\frac{d}{dt} A_t = 2i\pi[H, A_t]$.

2.2 Quantization

The point I would like to emphasize in this talk is the following. Recall that what we want to do is associate a quantum model (that is, a Hilbert space, and a set of quantum observables) to a given Hamiltonian system. There is definitely no clear link (if any link at all) between the symplectic manifold representing the phase space in classical mechanics and the quantum Hilbert space. Similarly, one cannot really relate Newton of Hamilton-Jacobi equation to Shrödinger's. But the geometric setup we developed for Hamiltonian systems highlights deep similarities in the structure of the space of observables. It is the natural to build quantization by associating to each classical observable a quantum observables in a way that preserves the structure of the spaces of observables. A first naive definition of quantization can be the following:

Definition 2.1. Let (M, ω) be a symplectic manifold. A *quantization* of (M, ω) is the data of a Hilbert space $\mathcal{H}$ together with a Lie algebra homomorphism from the space of smooth functions on M with the Poisson bracket onto the space of hermitian operators on $\mathcal{H}$ endowed with the Lie bracket $2i\pi[\cdot, \cdot]$.

We will denote $\mathrm{Op} : \mathcal{C}^\infty(M) \longrightarrow \mathcal{L}(\mathcal{H})$ the quantization. (If you wonder why, I don't know, it should stand for *observable*, but the standard notation is Op and not Ob.) Actually one moment's thought shows that this definition is incomplete, since it doesn't ensure any correspondance between the equations $\frac{d}{dt} f_t = \{H, f_t\}$ and $\frac{d}{dt} A_t = 2i\pi[H, A_t]$. Actually, it doesn't ensure that $\mathrm{Op}(f_t) = \mathrm{Op}(f)_t$, which is certainly what we want in order to relate classical and quantum evolution. We require then that $\mathrm{Op}(f_t) = \mathrm{Op}(f \circ \phi_t^H) = e^{2i\pi tH} \mathrm{Op}(f) e^{-2i\pi tH}$.

Remark. If we think in terms of symmetry, we actually want something stronger. The operator $e^{-2i\pi tH}$ acts by time translation on the space $\mathcal{H}$. So it is the quantum analogous of the flow ϕ_t^H on the classical phase space. Now we could consider a bigger group of symmetry of the phase space (say for instance a Lie group acting by symplectomorphisms on the classical phase space, but in any specific context the relevant concept for symmetry group can be adapted), and we would like to build a quantum analogous of this action. So in a more general setup, we want to have an unitary operator $\mathrm{U}(\phi)$ for any element $\phi \in G$ the symmetry group, such that for all functions we have $\mathrm{U}(\phi)^{-1} \mathrm{Op}(f) \mathrm{U}(\phi) = \mathrm{Op}(f \circ \phi)$. Moreover, we ask the map U to be a projective representation (from a physical point of view, this is related to the fact that states are determined up to a multiplicative constant; from a more mathematical perspective, we usually start with a representation of the Lie algebra of G, but if G is not simply connected, such a representation does not always integrate to a group representation).

2.3 Difficulties

When it comes to actually give quantization construction, many technical difficulties arise. The first one is that usually the Hilbert space $\mathcal{H}$ is infinite dimensional and the observables cannot be represented by bounded operators. Even in the simple case of a one dimensional particle, the Hilbert space is $L^2(\mathbf{R})$ and the operators position and impulsion are represented respectively by by $Q\psi(q) = q\psi(q)$ and $P\psi(q) = \frac{1}{2i\pi}\frac{\partial}{\partial q}\psi(q)$. Those operators are densely defined (for instance they can be defined on the space of Schwarz functions). This is a problem, but not a very annoying one. First, there is a spectral theory for unbounded hermitian operators, and at least in the case of the free particle, one can solve the Schrödinger's equation on the space of Schwarz functions, and check that the evolution operator preserves the L^2-norm. Thus by density, even if the position and impulsion operators and the Hamiltonian H are not well defined operators, $e^{-2i\pi Ht}$ is a well defined isometry of $L^2(\mathbf{R})$. (This is exactly the same thing as for the definition of the Fourier transform on L^2: we define it on a good dense subspace and extend it to the whole space by unitarity). Moreover, if we forget about the underlying space $\mathcal{H}$ and only care about the algebraic structure of the space of observables, there is a symbolic calculus the gives that space of operators a structure of Lie algebra and we can indeed get a somewhat abstract representation.

Another problem comes from the condition $\mathrm{Op}(f \circ \phi_t^H) = e^{2i\pi tH}\mathrm{Op}(f)e^{-2i\pi tH}$ which is usually not satisfied. But this is not the end of the story, and all this work wasn't done in vain. Actually throughout this talk we set $h = 1$ (h being the Planck constant) in Shrödinger's equation, which is usually taken written in the form $i\hbar\partial_t\psi_t = H\psi_t$ and leads to an evolution operator $e^{-iHt/\hbar}$. Then we would like that $e^{iHt/\hbar}\mathrm{Op}(f)e^{-iHt/\hbar} = \mathrm{Op}(f \circ \phi_t^H)$. Of course this is not exactly satisfied (where is the fun otherwise?), but one can required that it be satisfied in the limit $\hbar$ goes to 0, ie $e^{iHt/\hbar}\mathrm{Op}(f)e^{-iHt/\hbar} - \mathrm{Op}(f \circ \phi_t^H) \longrightarrow 0$ when $\hbar$ goes to 0. This limit is called *semi-classical* limit, and the study of this limit can be justified in two ways. The first way comes from physics: it is well known that quantum effects become important when $\hbar$ is of the same order of magnitude as the typical quantities of the system. In other words, we expect that the quantum evolution approaches the classical evolution when $\hbar$ goes to 0. Of course this is not a precise mathematical statement, however this is a belief that leads to many interesting developments, like micro-local analysis, which are importants in the study of PDEs. A more pragmatic way to justify the study of this limit is that when $\hbar$ is fixed, the study of the behavior is quite difficult to carry out, but it is much easier to get asymptotic properties in the semiclassical limit. Furthermore, there are different possible quantization, but it turns out that they are all equivalent in this limit, that is, the asymptotic properties of quantization does not depend on the choice of quantization process.

References

[MS94] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Oxford University Press, 1994.

[Por01] Mason A. Porter. An introduction to quantum chaos, 2001.

[Woo97] N. M. J. Woodhouse. *Geometric Quantization.* Oxford University Press, 1997.