INTRODUCTION TO PROOFS: SESSION 4

In the sequel, everything happens in $\mathbb{R}$.

1. Limits of Sequences

First, we recall the definition of a sequence.

Definition. (sequence) A **sequence** (in $\mathbb{R}$) is a function $a : \mathbb{N} \to \mathbb{R}$.

More often than not, we will notate a sequence $a : \mathbb{N} \to \mathbb{R}$ not as a function, but as a family of real numbers indexed by $\mathbb{N}$. Specifically, we denote a by $\{a_n\}_{n=1}^{\infty}$, where $a_n = a(n)$ for each $n \in \mathbb{N}$. When the entire sequence can be inferred from the first few terms, sometimes we will even write only the first few terms. For example, we might write $\{1, 1, \dots\}$ for the constant sequence $a_n = 1$.

Example 1. Here are some examples of sequences in $\mathbb{R}$.

(1) $a_n = n$, which yields $\{0, 1, 2, 3, 4, \dots\}$.
(2) $a_n = n^2$, which yields $\{1, 4, 9, 16, \dots\}$.
(3) $a_n = \frac{1}{n}$, which yields $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.
(4) $a_n = [1 - (-1)^n]/2$, which yields $\{1, 0, 1, 0, 1, \dots\}$.

We want to know how the sequence behaves in the long run, i.e., as n tends to ∞. After all, a sequence can do any number of things. It might converge to a finite value, it might tend to one extreme or another, or it might oscillate indefinitely.

Definition (limit of a_n as n tends to ∞)**.** We say L is the **limit** of $\{a_n\}_{n=1}^{\infty}$ as n tends to ∞, or equivalently,

$$\lim_{n\to\infty} a_n = L,$$

if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq N$.

Proving that a real number L is actually the limit of a function is as easy as 1,2,3! Just follow the recipe below.

(1) Let $\epsilon > 0$ be given.
(2) Conjure up a suitable N. This N should probably depend on ϵ in some way.
(3) Verify that $|a_n - L| < \epsilon$ for all $n \geq N$.

Along the way, you might find the following facts useful to keep in mind.

Theorem 1. The following statements are equivalent and true.

(i) If a, b are real numbers and $a > 0$, then exists $n > 0$ such that $na > b$.
(ii) For every $x \in \mathbb{R}$, there exists n such that $n \leq x < n + 1$.
(iii) For every $x > 0$, there exists $n > 0$ such that $\frac{1}{n} \leq x$.

Exercise 1. Prove that $\lim_{n\to\infty} 1 = 1$.

Exercise 2. Prove that $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$.

2. Limits of Functions I

Exercise 3. Consider the function

$$f(x) = \begin{cases} -x^2 & \text{if } x \leq -2, \\ x-1 & \text{if } -2 \leq 1, \\ \log(x) & \text{if } x > 1. \end{cases}$$

For which a does the limit $\lim_{x\to a} f(x)$ exist?

3. Characterizing nearness

Definition (open interval)**.** An **open interval** is a set of the form

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

for some pair of real numbers $a < b$.

Open intervals (and more generally open neighborhoods) help us characterize what it means for two points to be "as close as possible" to one another.

Definition (open ball)**.** The open ball centered at a point x of radius r is the set

$$B_r(x) = \{y \in \mathbb{R} : |x - y| < r\}.$$

Notice that open balls are open intervals.

Exercise 4. For each function f and each pair of real numbers c, r listed below, draw the graph of $f(x)$ and highlight the set $f^{-1}(B_r(c))$ on the x-axis. Next, write each set $f^{-1}(B_r(c))$ as the union of open intervals.

(a) $f(x) = x$, $c = 4$, $r = 1$
(b) $f(x) = x^2$, $c = 1$, $r = 1/2$
(c) $f(x) = 1/x$, $c = 0$, $r = 1$

4. Limits of Functions II

Definition (limit of $f(x)$ as x tends to a)**.** We say L is the **limit** of $f(x)$ as x tends to a, or equivalently,

$$\lim_{x\to a} f(x) = L,$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Proving that a real number L is actually the limit of a function is as easy as 1,2,3! Just follow the recipe below.

(1) Let $\epsilon > 0$ be given.
(2) Conjure up a suitable δ. This δ should probably depend on ϵ in some way.
(3) Verify that $|f(x) - L| < \epsilon$ for any x such that $0 < |x - a| < \delta$.

Exercise 5. Prove that $\lim_{x\to 1/2} 2x = 1$.

Exercise 6. Prove that $\lim_{x\to 1/2} 2x \neq 2$.

Of course, the limit of an arbitrary function f at a point a need not exist (as Exercise 7 demonstrates), but when it does, it is unique. We may therefore speak of *the* limit. You will prove this yourself in Exercise 8 below.

Exercise 7. Show that the Heaviside step function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

has no limit at 0.

Exercise 8. Prove that limits are unique.

Exercise 9. Suppose that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Prove that $\lim_{x\to a}[f(x) + g(x)]$ exists, and that

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Does the converse hold?

Exercise 10. Suppose that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Prove that $\lim_{x\to a} f(x)g(x)$ exists, and that

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

Does the converse hold?

5. Continuity of Functions

Definition (continuous function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous** at $a \in \mathbb{R}$ if

$$\lim_{x\to a} f(x) = f(a).$$

The same function f is said to be **continuous** if f is continuous at a for every $a \in \mathbb{R}$.

Exercise 11. Prove that the function $f(x) = 2x$ is continuous.

Exercise 12. Prove that the Heaviside step function is continuous everywhere except for 0.

Exercise 13. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Prove that $f + g$ is continuous.

Exercise 14. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Prove that $f \cdot g$ is continuous.

Remark. The space $\mathcal{C}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ is a ring.