Abstract

We continue to pursue our goal to describe the set Spec(n) and the equivalence relation defined on it. First, we will define Cont(n) — the set of content vectors of length n — under the motivation to further restrict possible vectors that can be in Spec(n). We will then define the Young graph and some related notions, and show that there is a bijection between Cont(n) and the set of Young tableaux Tab(n) which also preserves the equivalence class in each set. This discussion will not only enable us to fulfill our goal stated above but also help to obtain an explicit model of representations of Sn and derive the formula for their characters.

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1 Review of the Plan

1.1 Basic Definitions

Definition 1.1. The Young-Jucys-Murphy elements \( X_1, X_2, \ldots, X_n \), or YJM-elements, are the elements of \( \mathbb{C}[S_n] \):
\[
X_i = (1, i) + (2, i) + \cdots + (i - 1, i)
\]
In particular, \( X_1 = 0 \).

Definition 1.2. The Coxeter generators \( s_1, \ldots, s_{n-1} \) are the elements of \( S_n \):
\[
s_i = (i, i+1)
\]

Definition 1.3. The Young Basis \( \mathcal{Y} \) is the union of all Gelfand-Tsetlin basis of all irreducible representations of \( S_n \):
\[
\mathcal{Y} = \biguplus_{\lambda \in S_n^\wedge} \{v_T\}_\lambda
\]

Recall that the YJM-elements consist a set of generators of the Gelfand-Tsetlin algebra, which is the algebra of all operators diagonal in the Gelfand-Tsetlin basis. Hence the Young basis is a common eigenbasis of the YJM-elements, which enables us to make the following definition:

Definition 1.4. For any element \( v \in \mathcal{Y} \), the weight of \( v \), denoted as \( \alpha(v) \), is the vector
\[
\alpha(v) = (a_1, \ldots, a_n) \in \mathbb{C}^n
\]
where \( a_i \) is the eigenvalue of \( X_i \) on \( v \).

Since each \( v \in \mathcal{T} \) is determined uniquely by the eigenvalues of the elements of \( GZ(n) \) on \( v \), the map \( v \mapsto \alpha(v) \) is a one-to-one correspondence, the inverse of which we will denote as \( \alpha \mapsto v_\alpha \).

Definition 1.5. The spectrum of the YJM-elements, or \( \text{Spec}(n) \) is defined as
\[
\text{Spec}(n) = \{\alpha(v) \mid v \in \mathcal{Y}\}
\]
Also, define an equivalence relation \( \sim \) as
\[
\alpha \sim \beta, \quad \alpha, \beta \in \text{Spec}(n)
\]
if \( v_\alpha \) and \( v_\beta \) belong to the same irreducible representation of \( S_n \).

Clearly there is a natural bijection between \( \text{Spec}(n) \) and the Young basis \( \mathcal{Y} \), as well as \( \text{Spec}(n)/\sim \) and the set of irreducible representations \( S_n^\wedge \).
1.2 The Plan

Under this context, our plan to analyze the irreducible representations of \( S_n \) is as follows:

1. Describe the set \( \text{Spec}(n) \)
2. Describe the equivalence relation \( \sim \)
3. Calculate the matrix elements in the Young basis
4. Calculate the characters of irreducible representations

Recall the following result proved in Ryan’s talk:

**Proposition 1.6.** Let

\[
\alpha = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \in \text{Spec}(n)
\]

Then \( a_1 = 0, a_i \in \mathbb{Z} \) and

1. \( a_i \neq a_{i+1} \) for all \( i \)
2. if \( a_{i+1} = a_i \pm 1 \), then \( s_i \cdot v_\alpha = \pm v_\alpha \)
3. if \( a_{i+1} \neq a_i \pm 1 \), then

\[
\alpha' = s_i \cdot \alpha = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \text{Spec}(n)
\]

and \( \alpha' \sim \alpha \).

2 Content Vectors

We aim to nearly complete the first two steps of the plan outlined above.

2.1 Motivation and Definition

It can be expected that the transpositions \( s_i \) in the third case of Proposition 1.6 will play an important role in further investigating what values \( \text{Spec}(n) \) can take. We call them (i.e. \( s_i \)'s such that \( a_{i+1} \neq a_i \pm 1 \)) *admissible transpositions.* Indeed, the only hindrance of transporting the value of \( a_i \) to other positions by these transpositions are the entries that takes the values \( a_i \pm 1 \) (if they exist), and one can be motivated to inspect them. Under this motivation we state the following definition:

**Definition 2.1.** A vector \( \alpha = (a_1, \ldots, a_n) \in \mathbb{C}^n \) is a **content vector** if \( \alpha \) satisfies the following conditions:

1. \( a_1 = 0 \)
2. for all \( q > 1 \), if \( a_q > 0 \), then \( a_i = a_q - 1 \) for some \( i < q \); and if \( a_q < 0 \), then \( a_i = a_q + 1 \) for some \( i < q \)
3. if \( a_p = a_q = a \) for some \( p < q \), then

\[
\{a - 1, a + 1\} \subseteq \{a_{p+1}, \ldots, a_{q-1}\}
\]
Also, define $\text{Cont}(n)$ as the set of content vectors of length $n$, with an equivalence relation $\approx$ defined as

$$\alpha \approx \beta, \quad \alpha, \beta \in \text{Cont}(n)$$

if $\alpha$ is a permutation of $\beta$\(^1\).

**Remark.** It follows from easy induction that $\text{Cont}(n) \subset \mathbb{Z}^n$.

### 2.2 The Theorem

As wanted, it turns out that $\text{Spec}(n)$ is a subset of $\text{Cont}(n)$. To prove this, we need the following lemma.

**Lemma 2.2.** Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $a_i = a_{i+2} = a_{i+1} \mp 1$ for some $i$ — that is, $\alpha$ contains a fragment of the form $(a, a \pm 1, a)$. Then $\alpha \notin \text{Spec}(n)$.

**Proof.** Assume $\alpha \in \text{Spec}(n)$. By Proposition 1.6, $s_i v_\alpha = \pm v_\alpha$ and $s_{i+1} v_\alpha = \mp v_\alpha$. This implies $s_i s_{i+1} s_i v_\alpha = \mp v_\alpha$ and $s_{i+1} s_i s_{i+1} v_\alpha = \pm v_\alpha$, contradicting the identity $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$\(^2\).

Now we proceed to prove the main theorem.

**Theorem 2.3.** $\text{Spec}(n) \subset \text{Cont}(n)$.

**Proof.** Let $\alpha = (a_1, \ldots, a_n) \in \text{Spec}(n)$. Denote the three conditions in Definition 2.1 as (1), (2), (3), and we show $\alpha$ satisfies each of these conditions. Since $X_1 = 0$, we have $a_1 = 0$, and (1) is true.

We prove that (2) and (3) hold for $\alpha$ by induction on $n$. The case $n = 1$ and $n = 2$ are straightforward, so let $n \geq 3$ and assume the induction hypothesis that the statement is true for $n - 1$. Since $(a_1, \ldots, a_{n-1}) \in \text{Spec}(n-1)$, it suffices to prove when $q = n$ in both conditions.

Suppose $a_n > 0$. If $a_{n-1} = a_n - 1$ then (2) immediately follows. If $a_{n-1} = a_n + 1$, there exists $i < n - 1$ such that $a_i = a_{n-1} - 1 = a_n$. Using induction hypothesis again, there exists $j < i$ such that $a_j = a_i - 1 = a_{n-1} - 1$. If $a_{n-1} \neq a_n + 1$, then $(a_1, \ldots, a_{n-2}, a_n) \in \text{Spec}(n)$, and applying the induction hypothesis to $(a_1, \ldots, a_{n-2}, a_n) \in \text{Spec}(n-1)$ shows that there exists some $i < n - 1$ that $a_i = a_{n-1} - 1$, so in all cases (2) is true. The case $a_n < 0$ can be proven exactly the same way, except that all the signs are reversed.

Now assume that $a_p = a_n = a$ for some $p < n$, and that $a-1 \notin \{a_{p+1}, \ldots, a_{n-1}\}$. We may assume that $p$ is taken to be the largest possible index, so that $a \notin \{a_{p+1}, \ldots, a_{n-1}\}$. If $a+1$ occurs twice or more times in the set $\{a_{p+1}, \ldots, a_{n-1}\}$, $a$ should also occur in it by induction hypothesis, so $a + 1$ occurs at most once in $\{a_{p+1}, \ldots, a_{n-1}\}$. Then $\{a_{p+1}, \ldots, a_{n-1}\}$ only contains the numbers different

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\(^1\)In the reference text, this equivalence relation $\approx$ is defined as $\alpha \approx \beta$ if $\beta$ is an admissible permutation (i.e. the product of finite number of admissible transpositions) of $\alpha$, and this is a mistake. We will soon see that these two definitions are compatible.

\(^2\)Since $s_i s_{i+1} s_i$ is a 3-cycle, $(s_i s_{i+1} s_i)^3 = \text{id}$, so $s_i s_{i+1} s_i s_{i+1} = s_{i+1} s_i s_{i+1} = s_{i+1} s_i s_{i+1}$. 

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from $a-1, a, a+1$ with at most one possible exception that is equal to $a+1$. Hence, by applying a finite number of admissible transpositions, we obtain an element in $Spec(n)$ that contains a fragment $(a, a)$ or $(a, a+1, a)$, which are both impossible by Proposition 1.6 and Lemma 2.2. So $a-1 \in \{a_p+1, \ldots, a_n-1\}$, and $a+1 \in \{a_p+1, \ldots, a_n-1\}$ can be proven exactly the same way except that all the signs are reversed. □

3 The Young Graph

We now introduce a combinatorial object that visualizes the information of $Cont(n)$ (and thus helps us to work on it more easily).

3.1 Definitions

Definition 3.1. A **Young diagram** is a finite stack of boxes arranged in rows and columns in the way so that the first cell of each row lies in the first column and the lengths of rows are in non-increasing order.

![Young Diagram](image)

Figure 1: An example of a Young diagram.

We often denote the Young diagram as a list of the number of boxes in each row. For example, the above Young diagram is denoted $(5, 4, 1)$. In this sense, the set of Young diagrams with $n$ boxes is in a natural bijection with the set of partitions of $n$.

Definition 3.2. A **Young graph** $Y$ is a simple directed graph such that its vertices are the Young diagrams and two vertices $\nu, \eta$ are joined by a directed edge if $\nu \subset \eta$ and $\eta/\nu$ is a single box, in which case we write $\nu \nearrow \eta$.

The first five layers (excluding the empty set) of the Young graph are shown in the next page.

Definition 3.3. Given a box $\Box \in \nu$ in a Young diagram, the number

$$c(\Box) = (x\text{-coordinate of } \Box) - (y\text{-coordinate of } \Box)$$

is called the **content** of $\Box$. Here, the coordinates are defined so that the boxes in the $i$th column have $x$-coordinates equal to $i-1$ and the boxes in the $j$th row have $y$-coordinates equal to $j-1$.

Definition 3.4. A **Young tableau** or **standard tableau** is a path in $Y$ from $\emptyset$ to a certain vertex (i.e. a Young diagram). We denote $\text{Tab}(\nu)$ the set of
Young tableaux from $\emptyset$ to $\nu$. Also, let $Tab(n)$ the set of Young tableaux from $\emptyset$ to a Young diagram with $n$ boxes, that is,

$$Tab(n) = \bigcup_{|\nu|=n} Tab(\nu)$$

A convenient way to represent a path $T \in Tab(\nu)$

$$\emptyset = \nu_0 \nearrow \cdots \nearrow \nu_n = \nu$$

is to label the boxes $\nu_1/\nu_0, \ldots, \nu_n/\nu_{n-1}$ of $\nu_n$ by numbers $1, \ldots, n$, respectively.\footnote{Those who are familiar with the notion of Young tableau as an arbitrary labeling of $n$ boxes in the Young diagram by $1, \ldots, n$ might be feeling a bit awkward, since our definition only induces the labelings such that the entries in each row and each column are increasing. Such tableau that satisfies this condition is called \textit{standard} in the general context (as the alternative terminology of Definition 3.4 shows). In this particular talk, every Young tableau we'll discuss will be considered \textit{standard}.}

### 3.2 Application to Analysis of $Spec(n)$

The following proposition writes the information of content vectors in terms of the notions related to the Young graph.

**Proposition 3.5.** Let

$$T = \nu_0 \nearrow \cdots \nearrow \nu_n \in Tab(n).$$

The mapping

$$T \mapsto (c(\nu_1/\nu_0), \ldots, c(\nu_n/\nu_{n-1}))$$
is a bijection between $\text{Tab}(n)$ and $\text{Cont}(n)$. Also, for any $\alpha, \beta \in \text{Cont}(n)$, we have $\alpha \approx \beta$ if and only if the corresponding paths in this bijection have the same end (i.e. they are the Young tableaux of the same Young diagram).

**Proof.** Take arbitrary $T = \nu_0 \nearrow \cdots \nearrow \nu_n \in \text{Tab}(n)$ and for $i \in \{1, \cdots, n\}$, denote the box in $\nu_i/\nu_{i-1}$ as $(x_i, y_i)$, where $x_i$ and $y_i$ are $x$-coordinate and $y$-coordinate of the box, respectively. Note that for every $i \in \{1, \cdots, n\}$ and a nonnegative pair of integers $(p, q)$ such that $(p, q) < (x_i, y_i)$ (i.e. $p \leq x_i$, $q \leq y_i$, and $(p, q) \neq (x_i, y_i)$), there exists a positive integer $j < i$ such that $(x_j, y_j) = (p, q)$. In fact, the sequence of sets of boxes $\emptyset = \eta_0 \subset \eta_1 \subset \cdots \subset \eta_n$ such that $\eta_i/\eta_{i-1} = \{(u_i, v_i)\}$ gives a Young tableau if and only if for every $i \in \{1, \cdots, n\}$ and a nonnegative pair of integers $(p, q)$ such that $(p, q) < (u_i, v_i)$, there exists a positive integer $j < i$ such that $(u_j, v_j) = (p, q)$. 

Finally, we prove the second part of the proposition. If $\alpha, \beta \in \text{Cont}(n)$ satisfy $\alpha \approx \beta$ (i.e. $\alpha$ is a permutation of $\beta$), let $S$ be the multiset of entries of $\alpha$. Obviously $S$ is also the multiset of entries of $\beta$. Suppose $t$ is any integer, and appears $n$ times in $S$. By the proof of first part of the proposition above, the Young
diagrams of the corresponding Young tableaux $T_\alpha, T_\beta$ contain the smallest $n$ pairs $(z_i, w_i)$ of nonnegative integers such that $c(z_i, w_i) = t$, and doesn’t contain any other pair $(z', w')$ such that $c(z', w') = t$. Since this is true for every integer $t$, the diagrams should be equal. Conversely, if the corresponding final diagrams are equal, $\alpha \approx \beta$ since the multiset of entries of a content vector is equal to the multiset of values of the content of the boxes in the corresponding Young diagram.

Now consider the transpositions on the Young tableau that switches the labels of two boxes whose labels are consecutive integers. This operation produces a Young tableau if and only if the two boxes are located at different rows and different columns, which is equivalent to the fact that the content values of the two boxes doesn’t have a difference $\pm 1$ (due to the fact that the two boxes have consecutive labels). In the bijection above, this corresponds exactly to the admissible transpositions in the content vector. Hence admissible transpositions are transpositions in the Young tableau that switches the boxes with consecutive labels and preserves the Young tableau structure.

**Proposition 3.6.** Any two Young tableaux $T_1, T_2 \in Tab(\nu)$ with diagram $\nu$ can be obtained from each other by a sequence of admissible transpositions. In other words, if $\alpha, \beta \in Cont(n)$ and $\alpha \approx \beta$, then $\beta$ can be obtained from $\alpha$ by a sequence of admissible transpositions.

**Proof.** Induct on $n$; base case $n = 1$ is easy, so assume $n \geq 2$ and the statement is true for $n - 1$. It suffices to show that we can transform any Young tableau $T \in Tab(\nu), \nu = (\nu_1, \cdots, \nu_k)$ to the tableau $T_0$ with the same diagram and labeled with horizontal monotone numeration (i.e. the box $(0, 0)$ is labeled 1, $(1, 0)$ is labeled 2, $\cdots$, $(\nu_1 - 1, 0)$ is labeled $\nu_1$, $(0, 1)$ is labeled $\nu_1 + 1$, $\cdots$, $(\nu_2 - 1, 1)$ is labeled $\nu_1 + \nu_2$, $(0, 2)$ is labeled $\nu_1 + \nu_2 + 1$, $\cdots$). Consider the box $B = (\nu_k - 1, k - 1)$ — the last box of the last row or $\nu$ — and let $i$ the label of this box in $T$. Since all other boxes that share the same row or column with $B$ have both coordinates smaller than or equal to those of $B$, they are labeled with an integer less than $i$. Hence $(i, i + 1)$ is an admissible transposition, and we obtain another tableau after switching the labels $i$ and $i + 1$ in the corresponding boxes. After this operation, $B$ is labeled $i + 1$, and $(i + 1, i + 2)$ is an admissible transposition in this new tableau. Repeating this $n - i$ times, we obtain a tableau with $B$ labeled as $n$. Since the label of $B$ in $T_0$ is also $n$, by inductive hypothesis $T_0$ can be obtained from this tableau by a sequence of admissible transpositions.

**Corollary 3.7.** If $\alpha \in Spec(n), \beta \in Cont(n)$, and $\alpha \approx \beta$, then $\beta \in Spec(n)$ and $\alpha \sim \beta$.

**Proof.** By Proposition 3.6, $\beta$ can be obtained from $\alpha$ by a sequence of admissible transpositions. Hence by (3) of Proposition 1.6 $\beta$ is in the same equivalence class of $\alpha$ in $Spec(n)$.
4 What’s Next?

It turns out that $\text{Spec}(n) = \text{Cont}(n)$, the corresponding equivalence relations $\sim$ and $\approx$ coincide, and the Young graph $\mathbb{Y}$ is the branching graph of the symmetric groups when each Young diagram is looked upon as its corresponding irreducible representation (We are almost done with proving this!). This completes the task of describing the set $\text{Spec}(n)$ and the equivalence relation $\sim$ in our plan. We will go on to obtain an explicit model of representations of $S_n$ and sketch the derivation of the formula for their characters.