YOUNG-JUCYS-MURPHY ELEMENTS

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Abstract. In this lecture, we will be considering the branching multigraph of irreducible representations of the $S_n$, although the morals of the arguments are applicable to more general cases. We will liberally apply criteria about the centralizer of a subrepresentation to show that the restriction of an irreducible representation to a subrepresentation has simple multiplicity, which will show that the branching graph of irreducible representations of $S_n$ is in fact simple. We will then define the Young-Jucys-Murphy elements in $\mathbb{C}[S_n]$, show that they in fact generate the Gelfand-Tsetlin algebra, and see how they relate to the GZ-basis.

N.B. We will assume that all vector spaces are finite-dimensional over $\mathbb{C}$.

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1. Set-up

Recall the following:

Let $\{1\} = G_0 < G_1 < G_2 < \cdots$ be a chain of finite groups, and denote $\hat{G} = \{\text{irreps of } G\}$ (irreducible = no $G$-invariant subspaces).

For each $\rho \in \hat{G}_n$, $V^\rho$ decomposes into a direct sum of irreps $\mu \in \hat{G}_{n-1}$ with multiplicities $m_\mu = \dim \text{Hom}_G(V^\mu, V^\rho)$.

$$V^\rho = \bigoplus_{\mu \in \hat{G}_{n-1}} (V^\mu)^{m_\mu}.$$  

The branching graph is the directed multigraph whose vertices are elements of $\bigsqcup_{k \geq 0} \hat{G}_k$, with $\hat{G}_n$ called the $n$th level.

Two vertices $\mu \in \hat{G}_{n-1}$ and $\rho \in \hat{G}_n$ are connected by $k$ directed edges if $k = \dim \text{Hom}_{G_{n-1}}(V^\mu, V^\rho)$. 


Write \( \mu \nrightarrow \rho \) if \( \mu \) and \( \rho \) are connected, i.e. if \( V^\mu \) is a factor in the decomposition of \( V^\rho \).

Call the branching graph simple if all multiplicities are either 0 or 1, in which case

\[
V^\rho = \bigoplus_T V_t
\]

\( V_t := \mathbb{C}, \; t \in T = \{ \text{increasing paths } t = \{ \rho_0 \nrightarrow \rho_1 \nrightarrow \cdots \nrightarrow \rho_n = \rho \}, \; \rho_i \in \widehat{G}_i \} \)

Choose units \( v_t \in V_t, \forall t \in T \), then the GZ-basis of \( V^\rho := \{ v_t : t \in T \} \)

Remark 1.1. What this looks like is choosing some generator of \( V^{(1)} \), and tracking its image under each path in \( V^\rho \).

Definition 1.2. The Gelfand-Tsetlin algebra \( GZ_n \) is the algebra generated by the centers \( Z_1 \subset \mathbb{C}[G_1], \ldots, Z_n \subset \mathbb{C}[G_n] \); that is, \( GZ_n = < Z_1, \cdots, Z_n > \)

Proposition 1.3. \( GZ_n \) is the maximal commutative subalgebra in \( \mathbb{C}[G_n] \) when the branching graph is simple, and consists of operators that are diagonal in the GZ-basis.

Remark 1.4. We will eventually be looking at how \( GZ_n \) acts on each irrep.

We also recall the following lemma which will be very handy:

Lemma 1.5. ("Criteria"): Let \( M \) be a semisimple finite-dimensional \( \mathbb{C} \)-algebra, \( N \subset M \) a subalgebra. The centralizer \( Z_N(M) \) is commutative if and only if, for any \( \rho \in \hat{M} \), the restriction \( \text{Res}^N_M V^\rho \) of an irrep of \( M \) to \( N \) has simple multiplicities.

As well, keep in mind that irreps (i.e. irreducible representations) of a finite group \( G \) \( \hookrightarrow \mathbb{C}[G] \) - modules.

N.B. From now on, take \( G_n = S_n \).

2. Assorted Lemmas, Tidbits, and Facts

We will be looking at the branching graph of irreps of \( \mathbb{C}[S_n] \) to study the branching graph of \( S_n \). We can do this because we have a bijection:

\[
\text{Hom}_{\text{groups}}(G, GL(n, \mathbb{C})) \cong \text{Hom}_{\text{unital, algebras}}(\mathbb{C}[G], \text{Mat}_{n \times n}(\mathbb{C}))
\]

To see why this is, if you have any homomorphism on the elements of \( G \), you can extend it to \( \mathbb{C}[G] \), and likewise any homomorphism of \( \mathbb{C}[G] \) can be restricted to the basis \( G \). We also need to think about invertibility, but will hand-wave that for now.

While \( S_n \) has a trivial centre, the group algebra \( \mathbb{C}[S_n] \) does not, so we can get more information by working with it.

Note the following fact from elementary algebra:

Fact 2.1. Conjugation preserves cycle type, and if \( \sigma = (i_1, \cdots, i_k) \), then

\[
\tau \sigma \tau^{-1} = (\tau(i_1), \cdots, \tau(i_k))
\]

Proof. Consult any algebra textbook ever written. \( \square \)
As we will be proving results about the centres and centralizers of \( \mathbb{C}[S_n] \), let’s think about what the centre \( Z_n := Z(\mathbb{C}[S_n]) = \{ z \in \mathbb{C}[S_n] : \text{zy = yz} \ \forall y \in \mathbb{C}[S_n] \} \) looks like:

**Proposition 2.2.** Let \( z \in Z_n \), \( z = \sum_{g \in S_n} c_g g \). For any \( g \in S_n \), \( h \in S_n \), \( hgh^{-1} = z \). Since conjugation will permute the \( g_i \)'s within the conjugacy class \([g]\) (which consists of all permutations of a particular cycle type), we must have that \( c_g \)'s are all the same, i.e.

\[
z = \sum_{[g]} c_{[g]} \sum_{g_i \in [g]} g_i
\]

So, we can describe \( Z_n \) as:

\[
Z_n = \{ Z_{\lambda} | \lambda \vdash n \}
\]

where

\[
Z_{\lambda} = \sum_{\sigma \in S_n \text{ with cycle type } \lambda} \sigma
\]

**Lemma 2.3.** (Lemma 1) Every \( g \in S_n \) is conjugate to \( g^{-1} \in S_n \), i.e. \( \exists h \in S_n \) s.t. \( g^{-1} = hgh^{-1} \). Moreover, we can find such an \( h \in S_{n-1} \).

**Proof.** Clearly every permutation \( \in S_n \) is conjugate to its inverse, since if \( \sigma = (i_1, ..., i_n) \), then \( \sigma^{-1} = (i_n, i_{n-1}, ..., i_1) \), \( \rho \sigma \rho^{-1} = (\rho(i_1), ..., \rho(i_n)) \), choose \( \rho \) s.t. \( \rho(i_1, ..., i_n) = (i_n, i_{n-1}, ..., i_1) \).

For \( g \in S_n \), let \( g' \in S_{n-1} \) be the induced permutation in \( S_{n-1} \). Take \( h \in S_{n-1} \) that conjugates \( g' \) and \( g'^{-1} \), so \( g'^{-1} = hgh^{-1} \). Then \( h \) has the fixed point \( n \), so extended to \( S_n \), it satisfies \( g^{-1} = hgh^{-1}. \) \( \square \)

Before the next lemma, it’s time for another definition:

**Definition 2.4.** An \textit{involution algebra}, or a \( *-\text{algebra} \), is an algebra \( A \) with a map (called an involution) \( *: A \rightarrow A \) satisfying:

\begin{enumerate}
  \item \( (a^*)^* = a \)
  \item \( (ab)^* = b^*a^* \)
  \item \( (\lambda a + b)^* = \lambda a^* + b^* \)
\end{enumerate}

Call \( a^* \) the \textit{conjugate} or \textit{adjoint} of \( a \).

**Example 2.5.** \( \mathbb{C} \) is a \( * \)-algebra over \( \mathbb{R} \) with \( * = \text{conjugation} \).

**Definition 2.6.** In an \( *-\text{algebra} \) \( A \), call \( a \in A \) \textit{normal} if \( a \) commutes with its conjugate, i.e. \( aa^* = a^*a \), and call a \textit{self-adjoint} if \( a = a^* \).

**Lemma 2.7.** (Lemma 2) Let \( A \) be an \( *-\text{algebra} \) over \( \mathbb{C} \). Then

\begin{enumerate}
  \item \( A \) is commutative \iff all of its elements are normal.
  \item If every real element is self-adjoint, then \( A \) commutative.
\end{enumerate}

**Proof.** (1) \( \Rightarrow \): Trivial

\( \Leftarrow \): Suppose \( aa^* = a^*a \ \forall a \in A \), and denote \( A_{\text{sa}} = \{ a \in A : a \text{ self-adjoint} \} \). \( A \) can be decomposed as \( A = A_{\text{sa}} + iA_{\text{sa}} \) (by properties of \( * \)). If \( a, b \in A_{\text{sa}} \), then \( a = a^* \) and \( b = b^* \), so \( (a + ib)^* = a^* + ib^* = a - ib \). But \( (a + ib) \) normal \( \Rightarrow (a + ib)(a - ib) = (a - ib)(a + ib) \Rightarrow ab = ba \), i.e. \( a \) and \( b \) commute. Hence \( A \) commutative.
(2) Let \( A_\mathbb{R} \) be the real subalgebra of \( A \), i.e. \( A = \mathbb{C} \otimes A_\mathbb{R} \), and assume all \( a \in A \) are self-conjugate (i.e. \( a = a^* \forall a \in A \)). Then \( a, b \in A_\mathbb{R} \Rightarrow ab = (ab)^* = b^*a^* = ba \), so \( A_\mathbb{R} \) commutative, but then so is \( A = \mathbb{C} \otimes A_\mathbb{R} \).

\[\square\]

3. Centralizers and Centres

**Theorem 3.1.** The centralizer \( Z_{\mathbb{C}[S_{n-1}]}(\mathbb{C}[S_n]) =: Z_{n-1}(n) \) of \( \mathbb{C}[S_{n-1}] \) in \( \mathbb{C}[S_n] \) is commutative.

**Proof.** By lemma 2, we have reduced this problem to checking that every real element of the centralizer \( Z_{n-1}(n) \subseteq \mathbb{C}[S_n] \) is self-adjoint.

Let \( \{ g_i : g_i \in S_n \} \) be a basis for \( \mathbb{C}[S_n] \), \( f = \sum_i c_i g_i \in Z_{n-1}(n) \) \( c_i \)'s \( \in \mathbb{R} \).

What does self-adjoint look like in \( \mathbb{C}[S_n] \)? The adjoint of \( g \), viewing \( g \) as a function \( g : \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n] \), can also be categorized as the function \( g^* : \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n] \) such that \(< ga, b > = < a, g^*b > \forall a, b \in \mathbb{C}[S_n] \). Take the inner product \(< g_i, g_j > = \delta_{ij} \). Then \(< g_i, g_j > = < gg_i, g_j > \), and by def. of adjoint we have \(< gg_i, g_j > = < g_i, g^*g_j > = < g_i, g_i^{-1}g_j > \Rightarrow g^* = g_i^{-1} \).

So, \((\sum_i c_i g_i)^* = \sum_i c_i g_i^{-1} = \sum_i c_i g_i^{-1} f \in Z_{n-1}(n) \) commutes with all \( h \in S_{n-1} \), so since the expression for \( f \) is unique, this expression is invariant under conjugation by \( h \in S_{n-1} \), \( f \rightarrow hfh^{-1} = f \). By lemma 1, we can choose \( h_i \) s.t. \( h_i g_i h_i^{-1} = g_i^{-1} \), so \( \sum_i c_i g_i \mapsto \sum_i c_i g_i^{-1} \). But since \( hfh^{-1} = f \), that means that \( c_i g_i \) a summand \( \Rightarrow c_i g_i^{-1} \) a summand, so \( f^* = f \). \[\square\]

In light of this, we have the following:

**Theorem 3.2.** A finite-dimensional *-algebra over \( \mathbb{R} \), \( B \) a *-subalgebra. Let \( G = \{ g_i \} \) be a basis of \( A \) that is closed under * and s.t. \( \forall i, \exists \) orthogonal \( h_i \in B \) (\( b^* = b^{-1} \)) s.t. \( b_i g_i h_i^{-1} = g_i^{-1} \). Then \( Z_B(A) \) is commutative, and hence the restriction of any irrep from \( A \) to \( B \) has simple multiplicity.

In particular, if \( A \) is a group algebra of a finite group \( G \) and \( B \) of \( H \leq G \), where \( A \) has basis \( G \), then \( \forall g \in G, \exists h \in H \), \( g^* = g^{-1} \) s.t. \( h^{-1}g'h = g \), and if we can take \( g' = g \), then \( Z_B(A) \) is commutative, and hence the restriction of any irrep from \( A \) to \( B \) has simple multiplicity.

**Remark 3.3.** The proof we just gave for the simplicity of the spectrum (i.e. the branching of the multigraph) didn’t require any knowledge of the representations themselves, only elementary algebraic properties of the group.

4. Young-Jucys-Murphy Elements

**Remark 4.1.** We can also look at the branching through analyzing the centralizers of the group algebras, so we’re going to develop a more detailed description of the centralizer \( Z_{n-1}(n) \) and its relation to \( GZ_n \).

**Definition 4.2.** The Young – Jucys – Murphy elements (henceforth abbreviated as \( YJM \) elements) are

\[X_i = (1i) + (2i) + \ldots + ((i-1)i) \in \mathbb{C}[S_n] \]

**Remark 4.3.** \( X_i = \sum \) (transpositions in \( S_i \)) - \( \sum \) (transpositions in \( S_{i-1} \)), so \( X_i \) is the difference of an element in \( Z(i) \) and \( Z(i-1) \), so \( X_i \in GZ_i \); in particular, the \( X_i \)'s commute.

**Proposition 4.4.** \( X_k \notin Z_k \) for any \( k \).
**Proof.** \( Z_k = \text{span}\{ \sum_{\sigma \in S_n} \sigma|\lambda \text{ a partition of } k \} \), so for cycle type \( \lambda = 2,1,1,\ldots,1 \), any element \( \in Z_n \) consisting of 2-cycles must be expressed in terms of all the 2-cycles \( \in S_n \).

\[ \square \]

**Theorem 4.5.** The centre \( Z_n \subset \mathbb{C}[S_n] \) is a subalgebra of the one generated by the centre \( Z_{n-1} \subset \mathbb{C}[S_{n-1}] \) and \( X_n \):

\[ Z_n \subset < Z_{n-1}, X_n > \]

**Proof.** (Sketch)

(1) Show all classes of one-cycle type permutations lie in \( < Z_{n-1}, X_n > \):

\[
X_n = \sum_{i=1}^{n-1} \text{X}_{ij} = \sum_{i \neq j, i=1}^{n} \text{X}_{ij} - \sum_{i \neq j, j=1}^{n-1} \text{X}_{ij},
\]

where the first sum \( \in < Z_{n-1}, X_n > \) and the second \( \in Z_{n-1} \). Note that \( X_n^2 = \sum_{i \neq j, j=1}^{n} (i j n) = \sum_{i \neq j, j=1}^{n-1} (i j n) + 1 \), so \( X_n^2 + \sum_{i \neq j, j=1}^{n-1} (i j k) = \sum_{i \neq j, k, i=1}^{n} (i j k) \in < Z_{n-1}, X_n > \).

In a similar fashion, we can show that the sum \( X_n^k + \sum_{i \neq j, k, \ldots, i=1}^{n} (i_1 \cdots i_k) \in < Z_{n-1}, X_n > \), so that all one-cycle type classes in \( Z_n \) lie in \( < Z_{n-1}, X_n > \).

(2) Apply the general theorem that \( Z_n \) is generated by the classes of one-cycle type permutations (i.e. these permutations are the multiplicative generators) to conclude that \( Z_n \subset < Z_{n-1}, X_n > \).

\[ \square \]

**Corollary 4.6.** The algebra \( GZ_n \) is generated by the YJM elements, i.e.

\[ GZ_n = < X_1, \ldots, X_n > \]

**Proof.** By definition, \( GZ_n = < Z_1, \ldots, Z_n > \). We will proceed by induction:

\( GZ_2 = \mathbb{C}[S_2] = < X_1 = 0, X_2 >= \mathbb{C} \). Now, assume by induction that \( GZ_{n-1} = < X_1, \ldots, X_{n-1} > \).

\( \bigcup \): Clearly \( GZ_n \subset < GZ_{n-1}, X_n > \) since \( GZ_{n-1} \subset GZ_n \) and \( X_n \in Z_n \).

\( \subset \): By the previous theorem, \( Z_n \subset < Z_{n-1}, X_n > \subset < GZ_{n-1}, X_n > \).

\[ \square \]

**Proposition 4.7.** \( X_k \notin Z_k \) for any \( k \).

**Proof.** \( Z_k = \text{span}\{ \sum_{\sigma \in S_n} \sigma|\lambda \leftarrow k \} \), so for cycle type \( \lambda = 2,1,1,\ldots,1 \), any element \( \in Z_n \) consisting of 2-cycles must be expressed in terms of all the 2-cycles \( \in S_n \).

\[ \square \]

**Theorem 4.8.** The centralizer \( Z_{n-1}(n) \) is generated by the centre \( Z_{n-1} \subset \mathbb{C}[S_n] \) and the YJM-element \( X_n \):

\[ Z_{n-1}(n) = < Z_{n-1}, X_n > \]

**Proof.** Let’s consider a basis for \( Z_{n-1}(n) \). This will be the union of the basis of \( Z_{n-1} \), along with classes of the form:

\[
\{ (i_1,1 \cdots i_1,k_1-1)n) \cdots (i_{n,1} \cdots i_{n,k_n}) | k_1, \ldots, k_n \in \mathbb{N} \}
\]

where \( i_{j,k} \in \{ 1, \ldots, n-1 \} \) and the cycle lengths \( k_1, \ldots, k_n \) over possible partitions of \( n \). This is because conjugating by any element \( h \in Z_{n-1} \) must fix any \( z \in Z_{n-1}(n) \), and so

\[
h(\sum_{i=1}^{n} (i_1,1 \cdots i_1,k_1-1)n) = \sum_{i=1}^{n} (h(i_1,1) \cdots h(i_{n,1} \cdots i_{n,k_n}) = \sum_{i=1}^{n} (h(i_1,1) \cdots h(i_{n,1} \cdots i_{n,k_n})
\]

\[ \square \]
Since \( n \) must stay in the cycle of length \( k_1 \) and \( h \) permutes the rest of the elements. As in the proof showing that \( Z_n \subset <Z_{n-1}, X_n> \), we can take the sum of these classes with the corresponding classes:

\[
\{ \sum (i_1, \ldots, i_{k_1}) (i_2, \ldots, i_{k_2}) \cdots (i_{n-1}, \ldots, i_n) | k_1, \ldots, k_j \rightarrow n \}
\]

from \( Z_{n-1} \) to obtain classes in \( Z_n \), which shows that a basis of \( Z_{n-1} \) can be obtained via a linear combination of elements in the bases of \( Z_{n-1} \) and \( Z_n \):

\[
Z_{n-1}(n) \subset <Z_{n-1}, Z_n>
\]

But we already proved that \( Z_n \subset <Z_{n-1}, X_n> \), so

\[
Z_{n-1}(n) \subset <Z_{n-1}, X_n>
\]

Conversely, \( Z_{n-1} \subset Z_{n-1}(n) \) and \( X_n \) commutes with all elements \( \in Z_{n-1} \), so we also have

\[
Z_{n-1}(n) \supset <Z_{n-1}, X_n>
\]

\[\square\]

5. Simplicity of Branching

**Theorem 5.1.** (Main) The branching of the chain \( \mathbb{C}[S_1] \subset \cdots \subset \mathbb{C}[S_n] \) is simple, hence the same is true for \( S_1 < \ldots < S_n \).

**Proof.** \( Z_{n-1}(n) \subset <Z_{n-1}, X_n> \subset GZ_n \) commutative \( \Rightarrow \) any restriction from \( \mathbb{C}[S_n] \) to \( \mathbb{C}[S_{n-1}] \) is simple by “Criteria”. \( \square \)

**Corollary 5.2.** \( GZ_n \) is the maximal commutative subalgebra of \( \mathbb{C}[S_n] \), and in each irrep of \( S_n \), the GZ-basis is determined up to scalar factors (as a consequence of Schur’s lemma).

**Definition 5.3.** The union of GZ-bases of irreps \( \in \hat{S}_n \) is called the Young basis.

**Proposition 5.4.** Let \( v \) be a vector in the Young basis of some irrep, and denote the weight of \( v \):

\[
\alpha(v) = (a_1, \ldots, a_n) \in \mathbb{C}^n
\]

be the eigenvalues of \( X_1, \ldots, X_n \) on \( v \). Denote the spectrum of YJM-elements as

\[
\text{Spec}(n) = \{ \alpha(v) | v \in \text{Youngbasis} \}
\]

Then \( \alpha \in \text{Spec}(n) \) determines \( v \) up to scalar multiplication, and we can easily see that \( |\text{Spec}(n)| = \sum_{\lambda \in S_n} \dim(\lambda) = \dim(GZ_n) \) i.e. \( \dim(GZ_n) = \) the sum of pairwise non-isomorphic irreps

**Proposition 5.5.** We also have a bijection between \( \text{Spec}(n) \) and the set of paths in the branching graph, and a natural equivalence relation on \( \text{Spec}(n) \): if \( v_\alpha \) and \( v_\beta \) the corresponding vectors in the Young basis corresponding to weights \( \alpha \) and \( \beta \in \text{Spec}(n) \), then \( \alpha \beta \) if the paths \( T_\alpha \) and \( T_\beta \) have the same end, i.e. \( v_\alpha \) and \( v_\beta \) are in the same irrep, so that \( |\text{Spec}(n)| = |\hat{S}_n| \)

**Remark 5.6.** Later, we will see how these eigenvectors can be found from combinatorial data from Young tableaux.

6. Takeaways

1. YJM-elements \( X_n = (1n) + \ldots + ((n-1)n) \)
2. \( GZ_n = <Z_1, \ldots, Z_n> = <X_1, \ldots, X_n> \)
3. The branching of the multigraph of \( \mathbb{C}[S_n] \) and therefore that of \( S_n \) is simple
4. The spectrum of the YJM-elements determines the branching of the graph