Asymptotic Dimension

Friday, July 17, 2020 9:35 PM

Today — Dimension

① Why 'number of coordinates' is not an extensible notion of dimension

② Discrete-valued spaces ($\mathbb{Z}, \mathbb{Z}^2$) and their geometric structure inside continuous-valued spaces

③ Topology & Covering Dimension

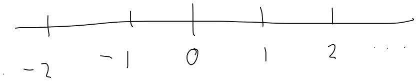
④ Asymptotic Dimension (i.e. dimension at large scales)

Motivation & Definitions

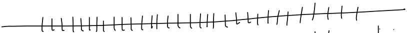
- Consider $\mathbb{Z}$ at different scales

i.e. let's 'zoom in' & 'zoom out' on $\mathbb{Z}$.

"Reasonable scale" $\mathbb{Z}$



"Zoom out" $\mathbb{Z}$

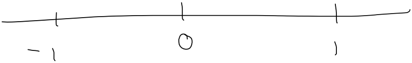


infinitely zoom out → resembles structure of $\mathbb{R}$, in some ways.

↳ looks more and more like a line.

If we say a line has dim 1, @ large scales dim $\mathbb{Z}$ looks like 1.

“Zoom in” $\mathbb{Z}$



In what we see in our zoom, the closer we look, the more $\mathbb{Z}$ looks like a point.

If we say a point has dim 0, @ small scales dim $\mathbb{Z}$ looks like 0.

Goal: Formalize this notion

Establish a good & extensible notion of dimension for discrete-valued spaces, & at different scales.

Asymptotic Dimension: A notion of dimension at large scales.

Topic 1

Why 'number of coordinates' is not an extensible notion of dimension

In some spaces, we like to think of dimension in terms of the # of coordinates we use to describe a space.

e.g. surface of earth:
(longitude, latitude)

ocean:
(x, y, depth)

Issue: What if we have a curve on which we can define our position in terms of one coordinate, but the curve fills up a space we think of as two-dimensional?

Peano, 1890 "Space-Filling Curve"

Start with [0,1], the unit line.

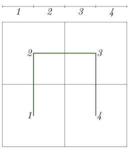


Fig. 1.

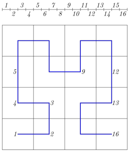


Fig. 2.

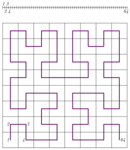
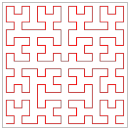
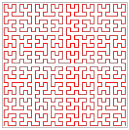
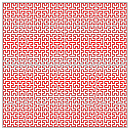


Fig. 3.







Continue iterating infinitely → touches every point of the unit square.

So, let's reject '# of coordinates' as a measure of dimension.

Topic 2

Discrete-valued spaces ($\mathbb{Z}$, $\mathbb{Z}^2$) and their geometric structure inside continuous-valued spaces

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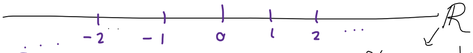
Let's consider the geometry of $\mathbb{Z}$ in relation to $\mathbb{R}$.

Intuitively, let's say that "things that look like lines should have dimension 1".



as a number line

$\mathbb{Z}$, a discrete set (doesn't look like a line)

Macro-level similarities between $\mathbb{R}$ & $\mathbb{Z}$:

we wander off to $-\infty$ moving left and ∞ moving right

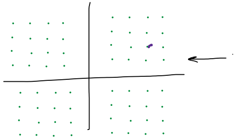
- 2 directions:

$\xleftarrow{\text{negative}}$ & $\xrightarrow{\text{positive}}$ to consider

Since $\mathbb{Z}$ is 'captured in essence' by $\mathbb{R}$, we motivate that we want $\dim(\mathbb{Z}) = 1$ (at a macro level).

Let's suppose this to be true for now.

$\mathbb{Z}^2$ sitting in $\mathbb{R}^2$:



$$\dim(\mathbb{R}^2) = \dim(\mathbb{R}) + \dim(\mathbb{R})$$
$$2 = 1 + 1$$

We want the same to hold for $\mathbb{Z}^2$:

$$\dim(\mathbb{Z}^2) = \dim(\mathbb{Z}) + \dim(\mathbb{Z}) = 1 + 1 = 2.$$

Geometry of the space:

$\mathbb{Z}^2$ sits in the Euclidean plane $\mathbb{R}^2$ much in the same way $\mathbb{Z}$ sits inside $\mathbb{R}$.

Coordinates?

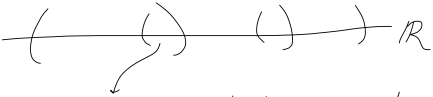
- 2 coords in $\mathbb{Z}^2$? (x, y) s.t. $x, y \in \mathbb{Z}$

But also, $\exists$ a 1:1 correspondence $\mathbb{Z}^2 \longrightarrow \mathbb{N}$, so can describe the position of a point in $\mathbb{Z}^2$ w/ one coord too.

Yet again, we cannot rely on coordinates to think about dimension!

$\mathbb{R}$: Dimensionality in terms of how open sets/intervals overlap

$\mathbb{R} \to$ for a cover with finite-length open intervals, it's impossible to have disjoint intervals.



we will have points that meet 2 intervals, in a minimal cover.

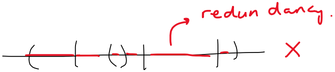
We say: "The minimal cover of $\mathbb{R}$ has order 2"

We said $\dim(\mathbb{R}) = 1$. This is (order of min. cover) $- 1$.

Extend to $\mathbb{Z}$:

Want (order of min. cover) $- 1 = \dim(\mathbb{Z})$ too.

But, we can't have such overlapping open covers.



Let's find an analog.

Analog in $\mathbb{Z}$ – Our Idea.

· Have a fixed scale r, $r > 1$ by a lot.

· Job: color the integers with the fewest number of colors possible s.t.:

① The # of consecutive $x \in \mathbb{Z}$ of the same color is always smaller than some fixed number.

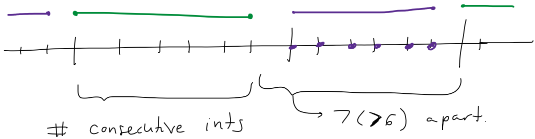
② If $m, n \in \mathbb{Z}$ have the same color and $|m-n| \le r$, then all ints between m and n are the same color.

Exercise: For $\mathbb{Z}$, this can be done w/ 2 colors, but cannot be done w/ 1 color.

Think of order = # colors in cover.

$$(\text{order of cover}) - 1 = 2 - 1 = 1$$
$$= \dim(\mathbb{Z}) \text{ proposed}$$

e.g. $r = 6$ (in reality, should be larger.)



# consecutive ints of same color is < 7, a fixed num.

Topic 3

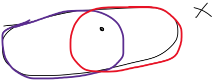
Topology & Covering Dimension

We want a notion of dimension that is independent of a process of "choosing coordinates" → Topology can help us!

Covering Dimension — Some Definitions

Def. (order) A collection $\mathcal{U}$ of subsets of a space X is said to have order $n+1$ if there is some point in X that lies in exactly $n+1$ sets from $\mathcal{U}$ and there is no point that lies in $n+2$ sets from $\mathcal{U}$.

↳ similar to our def'n of order for $\mathbb{R}$ from before



Def (refinement) A collection V of subsets is said to refine the collection $\mathcal{U}$ if every set in V is a subset of some set from $\mathcal{U}$.

Def (open cover) A collection $\mathcal{U}$ of subsets of X is an open cover of a space X if every set $U \in \mathcal{U}$ is open and if every $x \in X$ is contained in some $U \in \mathcal{U}$

↳ Note: You can choose the open sets in the open cover.

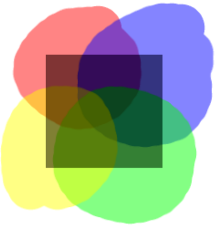
Def (Covering Dimension)

The (covering) dimension of a metric space X does not exceed n

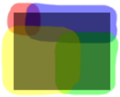
$(\dim X \leq n)$

if for every open cover $\mathcal{U}$ of X there is a refinement V with order at most $n+1$.

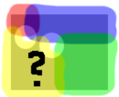
Example.



A cover of □ of order 4.



A refinement of our cover of order 3.



An order 2 refinement is not possible.

Using the def'n of covering dimension for our example:

Since □ has an open cover with a refinement of order $3 = n+1$, its covering dimension cannot exceed 2, i.e. $\dim(\square) \leq 2$.

Topic 4

Asymptotic Dimension
(i.e. dimension at large scales)

Building up to asymptotic dimension: Dimension 0 at fixed scales

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A metric space X is said to have dimension 0 at scale r if it can be expressed as a union $X = \bigcup X_i$ where:

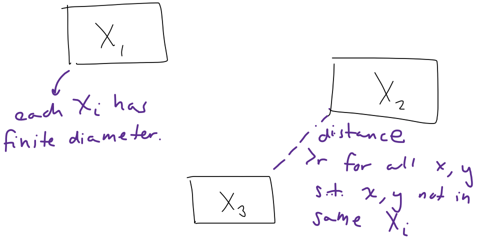
① $\sup\{\operatorname{diam}(X_i)\} < \infty$

i.e. finite size subsets, "uniformly bounded"

② $\inf\{d(x, x') \mid x \in X_i, x' \in X_j\} > r$ whenever $i \neq j$.

i.e. all pairs of points from 2 different subsets have distance $> r$ from each other, "r-disjoint" i.e. X has dim 0 at scale r if it admits a uniformly bounded cover by r-disjoint sets.

Example $X = X_1 \cup X_2 \cup X_3$



each X_i has finite diameter.

distance $> r$ for all x, y s.t. x, y not in same X_i

So we say $X = X_1 \cup X_2 \cup X_3$ has dimension 0 at fixed scale r.

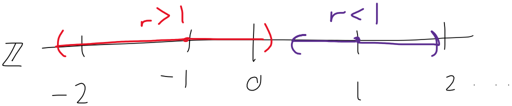
Revisiting $\mathbb{Z}$ – Changing the Scale

- Considering $\mathbb{Z}$ as a metric space with distance metric $d(m, n) = |m-n|$, we see $\mathbb{Z}$ has:

  - dim 0 for every scale $r < 1$
    ↳ Subsets: the singleton sets $X_i = \{i\}$

dim > 0 for every scale $r \geq 1$.

↳ fails at "r-disjoint" condition for dim 0.



Agrees with our earlier intuition:

↳ @ smaller scales $\mathbb{Z}$ looks like a discrete collection of points → 0-dim

↳ @ larger scales $\mathbb{Z}$ no longer looks 0-dim.

Defining Asymptotic Dimension

Def. (Asymptotic Dimension)

The asymptotic dimension of a metric space X does not exceed n

$(\operatorname{asdim} X \leq n)$

if for each $r > 0$, $\exists$ subsets

$X_0, X_1, \ldots, X_n$ with $X = X_0 \cup X_1 \cup \cdots \cup X_n$

and for each i, X_i has dim 0

at scale r.

Intuition asdim $X \leq n$ if for every r, the entire space X can be painted with splotches of $n+1$ colors of paint s.t. the splotches are

- uniformly bounded
- r-disjoint.

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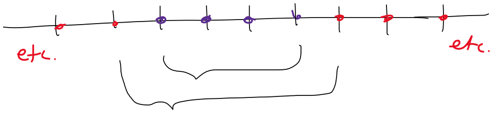
asdim $X = n$: means asdim $X \leq n$
and asdim $X \nleq n-1$

Recall our splotches for $\mathbb{Z}$:

asdim $\mathbb{Z} \leq 1$

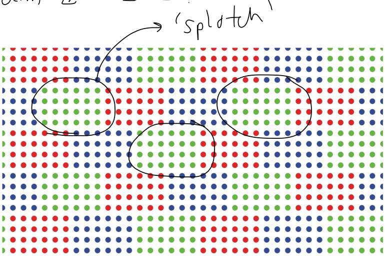
X_0 = red

X_1 = purple



Splotches for $\mathbb{Z}^2$

asdim $\mathbb{Z}^2 \leq 2$



- X_0 = green
- X_1 = red
- X_2 = blue

Usefulness of asdim

- Asymptotic dimension is used in a 1998 paper of Yu, which proved that if G is a finitely generated group of finite homotopy type s.t. asdim$(G) < \infty$, then G satisfies the Novikov conjecture (one of the most important unsolved problems in topology).

End of Content

We accomplished the following:

- developed a definition of the dimension of discrete-valued spaces & other spaces at different scales.
- considered the downsides, benefits, & extensibility of different approaches to thinking about dimension.