# UNIQUENESS OF $\mathbb{CP}^n$

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Abstract. We give an exposition of some results of Hirzebruch, Kodaira and Yau about the uniqueness of the Kähler structure of complex projective space. No originality is claimed.

**Theorem 1** (Hirzebruch, Kodaira [HK], Yau [Y1]). *If a Kähler manifold $M$ is homeomorphic to $\mathbb{CP}^n$ then $M$ is biholomorphic to it.*

**Theorem 2** (Yau [Y1]). *If a compact complex surface $M$ is homotopy equivalent to $\mathbb{CP}^2$ then it is biholomorphic to it.*

*Proof of Theorem 1.* The fact that $M$ is Kähler gives us the Hodge decomposition on cohomology, which we will use repeatedly. From the hypothesis we see that

$$H^2(M,\mathbb{Z}) \cong \mathbb{Z}, \quad H^1(M,\mathbb{C}) \cong 0 \cong H^{0,1}(M),$$

$$H^2(M,\mathbb{C}) \cong \mathbb{C} \cong H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M),$$

and since $H^{2,0}(M) \cong H^{0,2}(M)$, we see that they are both zero, while $H^{1,1}(M) \cong \mathbb{C}$. Then the exponential exact sequence gives that the map

$$c_1 : \mathrm{Pic}(M) \to H^2(M,\mathbb{Z}) \cong \mathbb{Z},$$

the first Chern class, is an isomorphism. Pick a Kähler form $\tilde{\omega}$. Its cohomology class $[\tilde{\omega}]$ lies in $H^2(M,\mathbb{R}) \cong \mathbb{R}$ so we can rescale $\tilde{\omega}$ to get another Kähler form $\omega$ whose cohomology class generates $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$. On $\mathbb{CP}^n$ a generator $\alpha$ of $H^2(M,\mathbb{Z})$ satisfies $\langle \alpha^{\smile n}, [\mathbb{CP}^n]\rangle = \pm 1$, and since $\omega$ is Kähler we have that $\int_M \omega^n = 1$. Since $c_1$ is an isomorphism, there exists $L \to M$ a holomorphic line bundle whose first Chern class is $[\omega]$ and $L$ admits a Hermitian metric whose curvature is $\omega$. So $L$ is positive and hence ample, and $M$ is projective. Moreover the classes $[\omega^k] \in H^{k,k}(M)$ are nonzero for $1 \leq k \leq n$, and as above the Hodge decomposition implies that $H^{p,q}(M) = 0$ if $p \neq q$. This gives that $\chi(M,\mathcal{O}) = 1$. Recall the following definition: if $F \to M$ is a real vector bundle, then its Pontrjagin classes are defined to be $p_i(F) = (-1)^i c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(M,\mathbb{Z})$. If $F = TM$ we just write $p_i(M)$. Now we need the following theorem.

**Theorem 3** (Novikov [N]). *The rational Pontrjagin classes of a closed smooth manifold are invariant under homeomorphism.*

In particular if $f : M \to \mathbb{CP}^n$ is the given homeomorphism, and if $p_i(M) \in H^{4i}(M,\mathbb{R})$ are the real Pontrjagin classes, then $f^*p_i(\mathbb{CP}^n) = p_i(M)$ for all $i$.

(notice that if $f$ is assumed to be a diffeomorphism then this is obvious, so we don't need Novikov's Theorem in that case). If $H$ denotes the hyperplane class on $\mathbb{CP}^n$ then it is well-known ([MS]) that

$$p_i(\mathbb{CP}^n) = \binom{n+1}{i} H^{2i}.$$

Moreover the fact that $f$ is a homeomorphism implies that $f^*H$ is a generator of $H^2(M,\mathbb{Z})$ and so $f^*H = \pm[\omega]$. Putting these together we get

$$p_i(M) = \binom{n+1}{i} [\omega^{2i}]. \tag{1}$$

The Hirzebruch-Riemann-Roch theorem gives

$$\chi(M,\mathcal{O}) = \int_M e^{\frac{c_1(M)}{2}} \hat{A}(M) = \int_M e^{\frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1},$$

where the second equality follows from the fact that $\hat{A}(M)$ can be written in terms of the Pontrjagin classes of $M$, which are given by (1). We need to determine $c_1(M)$. The reduction mod 2 of $c_1(M)$ is the second Stiefel-Whitney class $w_2(M) \in H^2(M,\mathbb{Z}_2)$, which is a topological invariant. Hence it is equal to $w_2(\mathbb{CP}^n)$ which is $c_1(\mathbb{CP}^n)$ mod 2, that is $n+1$ mod 2. On the other hand we have $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{Z}$, and so $\lambda = n+1+2s$ for some $s \in \mathbb{Z}$. We get

$$\chi(M,\mathcal{O}) = \int_M e^{\frac{n+1+2s}{2}\omega} \left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} = \int_M e^{s\omega} \left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1}.$$

It is easily seen that the coefficient of $\omega^n$ in the power series inside the integral is $\binom{n+s}{n}$, with the usual meaning when $s < 0$. Since $\chi(M,\mathcal{O}) = 1$, we get

$$\binom{n+s}{n} = 1,$$

which can be rewritten as

$$n! = (s+1)\cdots(s+n).$$

If $n$ is odd this implies that $s = 0$, while if $n$ is even, $s$ is either 0 or $-n-1$. But we saw that $c_1(M) = (n+1+2s)[\omega]$ and so if $n$ is odd we get $c_1(M) = (n+1)[\omega]$, while if $n$ is even, $c_1(M)$ is either $(n+1)[\omega]$ or $-(n+1)[\omega]$. If $c_1(M) = (n+1)[\omega]$ then $c_1(K_M) = -c_1(M) = -(n+1)c_1(L)$ and so $K_M = -(n+1)L$. Then Serre duality gives $H^k(M,L) \cong H^{n-k}(M, K_M - L)$ and $K_M - L = -(n+2)L$ is negative, so $H^k(M,L) = 0$ if $k > 0$ by Kodaira vanishing. Hence

$$\begin{aligned}\dim H^0(M,L) = \chi(M,L) &= \int_M e^{c_1(L)+\frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} \\ &= \int_M e^{\omega} \left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1} = n+1.\end{aligned}$$

Then the following lemma, whose proof is postponed, gives that $M$ is biholomorphic to $\mathbb{CP}^n$.

**Lemma 1** (Theorem 1.1 in [KO])**.** *If $L$ is a positive line bundle on $M$ with $\int_M c_1^n(L) = 1$ and $\dim H^0(M, L) = n + 1$ then $M$ is biholomorphic to $\mathbb{CP}^n$.*

We can then assume that $n$ is even (so $n \geq 2$) and that $c_1(M) = -(n+1)[\omega]$, which says that $K_M$ is positive. By the Aubin-Yau's Theorem [A, Y2] we know that $M$ then admits a unique Kähler-Einstein metric with constant Ricci curvature equal to $-1$, that is a Kähler metric $\omega_{KE}$ such that

$$\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}.$$

We have the following

**Lemma 2.** *If $(M, \omega)$ is a Kähler-Einstein manifold of complex dimension $n \geq 2$, so that $\mathrm{Ric}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$, then we have*

$$\left(\frac{2(n+1)}{n} c_2(M) - c_1^2(M)\right) \cdot [\omega]^{n-2} \geq 0, \tag{2}$$

*with equality iff $\omega$ has constant holomorphic sectional curvature.*

*Proof.* The tensor

$$R^0_{i\bar{j}k\bar{\ell}} = R_{i\bar{j}k\bar{\ell}} - \frac{\lambda}{n+1}(g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}})$$

vanishes iff $\omega$ has constant holomorphic sectional curvature. Its norm square is easily computed as

$$|\mathrm{Rm}^0|^2 = |\mathrm{Rm}|^2 + \frac{\lambda^2}{(n+1)^2}(2n^2 + 2n) - \frac{4\lambda}{n+1}R.$$

The assumption $R_{i\bar{j}} = \lambda g_{i\bar{j}}$ gives $R = \lambda n$ and $|\mathrm{Ric}|^2 = \lambda^2 n$. Then

$$|\mathrm{Rm}^0|^2 = |\mathrm{Rm}|^2 - \frac{2\lambda^2 n}{n+1}.$$

On the other hand if $\Omega_i^j = \sqrt{-1}R^j_{ik\bar{\ell}} dz^k \wedge d\bar{z}^\ell$ denote the curvature forms, then Chern-Weil theory says that $\frac{1}{2\pi}\mathrm{Ric}(\omega) = \frac{1}{2\pi}\sum_i \Omega_i^i = \frac{\sqrt{-1}}{2\pi} R_{k\bar{\ell}} dz^k \wedge d\bar{z}^\ell$ is a closed form that represents $c_1(M)$, while the form $\frac{1}{4\pi^2}\mathrm{tr}(\Omega \wedge \Omega) = \frac{1}{4\pi^2}\sum_{k,i} \Omega_i^k \wedge \Omega_k^i = \frac{(\sqrt{-1})^2}{4\pi^2} R^k_{ip\bar{q}} R^i_{kr\bar{s}} dz^p \wedge d\bar{z}^q \wedge dz^r \wedge d\bar{z}^s$ represents $c_1^2(M) - 2c_2(M)$. Then we can compute that

$$\begin{aligned} n(n-1)\mathrm{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} &= \sum_{p \neq r} (R^k_{ip\bar{p}} R^i_{kr\bar{r}} - R^k_{ip\bar{r}} R^i_{kr\bar{p}})\omega^n \\ &= \sum_{p,r} (R^k_{ip\bar{p}} R^i_{kr\bar{r}} - R^k_{ip\bar{r}} R^i_{kr\bar{p}})\omega^n \\ &= (|\mathrm{Ric}|^2 - |\mathrm{Rm}|^2)\omega^n = (\lambda^2 n - |\mathrm{Rm}|^2)\omega^n. \end{aligned}$$

Hence

$$|\mathrm{Rm}^0|^2\frac{\omega^n}{n(n-1)} = -\mathrm{tr}(\Omega\wedge\Omega)\wedge\omega^{n-2} + \lambda^2\left(\frac{1}{n-1} - \frac{2}{(n+1)(n-1)}\right)$$
$$= -\mathrm{tr}(\Omega\wedge\Omega)\wedge\omega^{n-2} + \frac{\lambda^2}{n+1}.$$

Now notice that

$$\frac{1}{4\pi^2}\int_M \lambda^2\omega^n = \frac{1}{4\pi^2}\int_M (\lambda\omega)^2\wedge\omega^{n-2} = c_1^2(M)\cdot[\omega]^{n-2},$$

and so

$$\frac{1}{n(n-1)4\pi^2}\int_M |\mathrm{Rm}^0|^2\omega^n = \left(2c_2(M) - \left(1-\frac{1}{n+1}\right)c_1^2(M)\right)\cdot[\omega]^{n-2},$$

which is what we want. □

We claim that equality does in fact hold in our case. This will finish the proof, since then $M$ would have constant negative holomorphic sectional curvature, and so it would be biholomorphic to the unit ball in $\mathbb{C}^n$ (see e.g. Theorem IX.7.9 in [KN]), which is impossible.

We already know that $c_1^2(M) = (n+1)^2[\omega^2]$. To compute $c_2(M)$ we notice that by definition $p_1(M) = p_1(TM) = -c_2(TM\otimes\mathbb{C})$. But $TM\otimes\mathbb{C}\cong TM\oplus\overline{TM}$ and the Chern classes satisfy $c_k(\overline{TM}) = (-1)^k c_k(TM)$, so

$$\begin{aligned} p_1(M) &= -c_2(TM\oplus\overline{TM}) = -c_2(TM) - c_2(\overline{TM}) - c_1(TM)\cdot c_1(\overline{TM}) \\ &= -2c_2(M) + c_1^2(M). \end{aligned} \tag{3}$$

Putting this together with (1) we get

$$2c_2(M) = (n+1)^2[\omega^2] - (n+1)[\omega^2] = n(n+1)[\omega^2],$$

and thus equality holds in (2). □

*Proof of Theorem 2.* Let's denote by $\tau(M)$ the index of $M$, which is a topological invariant (up to sign). Then $\tau(M) = \pm\tau(\mathbb{CP}^2) = \pm 1$. Recall Hirzebruch's Signature Theorem:

$$\tau(M) = \frac{1}{3}\int_M p_1(M).$$

But from (3) we get

$$\frac{1}{3}\int_M (c_1^2(M) - 2c_2(M)) = \pm 1,$$

and Chern-Gauss-Bonnet's Theorem gives

$$\int_M c_2(M) = \chi(M) = \chi(\mathbb{CP}^2) = 3,$$

and so

$$\int_M c_1^2(M) = 3(2\pm 1) > 0.$$

A theorem of Kodaira [Ko] then says that $M$ is projective. As before we see that $\chi(M, \mathcal{O}) = 1$ and then Riemann-Roch gives

$$\chi(M, \mathcal{O}) = \frac{K_M^2 + \chi(M)}{12} = \frac{K_M^2 + 3}{12},$$

which gives $\int_M c_1^2(M) = K_M^2 = 9$ (so in fact $\tau(M) = 1$). Let $\omega$ be as before, then $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{Z}$. Then we have that $\lambda = \pm 3$, and these are exactly the same cases as in Theorem 1. If $\lambda = 3$, we need to check that $\dim H^0(M, L) = 3$. But we have $K_M = -3L$ and $K_M \cdot L = -3$ so Riemann Roch gives

$$\chi(M, L) = 1 + \frac{L^2 - K_M \cdot L}{2} = 3.$$

Serre duality and Kodaira vanishing give

$$H^1(M, L) \cong H^1(M, K_M - L) = 0,$$

because $K_M - L = -4L$ is negative, and also

$$H^2(M, L) \cong H^0(M, K_M - L) = 0.$$

So $\chi(M, L) = \dim H^0(M, L) = 3$. Then the proof continues as in Theorem 1. $\square$

*Proof of Lemma 1.* Let $(\varphi_1, \ldots, \varphi_{n+1})$ be a basis of $H^0(M, L)$ and let $D_j = \{\varphi_j = 0\}$ be the corresponding divisors (they are nonempty, because otherwise $L$ would be trivial, and so it would have $\dim H^0(M, L) = 1$). Define $V_n = M$ and

$$V_{n-k} = D_1 \cap \cdots \cap D_k$$

for $1 \leq k \leq n$.

**Lemma 3.** *For each $0 \leq r \leq n$ we have that*
*1. $V_{n-r}$ is irreducible, of dimension $n - r$ and Poincaré dual to $c_1^r(L)$*
*2. The sequence*

$$0 \to \mathrm{Span}(\varphi_1, \ldots, \varphi_r) \to H^0(M, L) \to H^0(V_{n-r}, L)$$

*is exact, where the last map is given by restriction.*

*Proof.* The proof is by induction on $r$, the case $r = 0$ being obvious. Assuming that 1 and 2 hold for $r-1$, we see that $V_{n-r+1}$ is irreducible and that $\varphi_r$ is not identically zero on it. Hence $V_{n-r} = \{x \in V_{n-r+1} \mid \varphi_r(x) = 0\}$ is an effective divisor on $V_{n-r+1}$ and so it can be expressed as a sum of irreducible subvarieties of dimension $n - r$. Since $c_1^{r-1}(L)$ is dual to $V_{n-r+1}$ and $c_1(L)$ is dual to $D_r$ we see that $c_1^r(L)$ is dual to $V_{n-r}$. If $V_{n-r}$ were reducible, then $V_{n-r} = V' + V''$ and so

$$1 = \int_M c_1^n(L) = \int_M c_1^r(L) \cdot c_1^{n-r}(L) = \int_{V_{n-r}} c_1^{n-r}(L)$$
$$= \int_{V'} c_1^{n-r}(L) + \int_{V''} c_1^{n-r}(L).$$

But since $L$ is positive, the last two term are both positive integers, and this is a contradiction. Thus 1 is proved. As for 2, the restriction exact sequence

$$0 \to \mathcal{O}_{V_{n-r+1}} \to \mathcal{O}_{V_{n-r+1}}(L) \to \mathcal{O}_{V_{n-r}}(L) \to 0,$$

gives

$$0 \to H^0(V_{n-r+1}, \mathcal{O}) \to H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L),$$

where the first map is given by multiplication by $\varphi_r$. This means that the kernel of the restriction map $H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L)$ is spanned by $\varphi_r$. This together with the statement in 2 for $r-1$ proves 2 for $r$. □

Now we apply Lemma 3 with $r = n$ and see that $V_0$ is a single point and that $\varphi_{n+1}$ doesn't vanish there. So given any point of $M$ there is a section of $L$ that doesn't vanish there (i.e. $L$ is base-point-free). Then we can define a holomorphic map $f : M \to \mathbb{CP}^n$ by sending $x$ to $\{\varphi \in H^0(M, L) \mid \varphi(x) = 0\}$. This is a hyperplane in $H^0(M, L) \cong \mathbb{C}^{n+1}$ and so gives a point in $\mathbb{CP}^n$. If $y \in \mathbb{CP}^n$ corresponds to a hyperplane, which is spanned by some sections $(\varphi_1, \ldots, \varphi_n)$, then $f(x) = y$ iff $\varphi_1(x) = \cdots = \varphi_n(x) = 0$. Again Lemma 3 with $r = n$ says that $x = V_0$ exists and is unique, and so $f$ is a bijection. □

## References


[A] Aubin, T. *Équations du type Monge-Ampère sur les variétés kähleriennes compactes*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 3, Aiii, A119–A121.

[HK] Hirzebruch, F., Kodaira, K. *On the complex projective spaces*, J. Math. Pures Appl. **36** (1957), 201–216.

[KN] Kobayashi, S., Nomizu, K. *Foundations of differential geometry, Vol. II*, John Wiley & Sons, 1969.

[KO] Kobayashi, S., Ochiai, T. *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. **13** (1973), 31–47.

[Ko] Kodaira, K. *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798.

[MS] Milnor, J.W., Stasheff, J.D. *Characteristic classes*, Princeton University Press, 1974.

[N] Novikov, S.P. *Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds. I*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 1373–1388.

[Y1] Yau, S.-T. *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

[Y2] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411.



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