Mathematics W4051y
Basic Topology

Answers to Midterm Exam #2

November 17, 2003

1. Compact subsets of Hausdorff spaces are closed, so A and B are closed in X. Hence $A \cap B$ is closed in X. But closed subsets of compact spaces are compact, so $A \cap B \subset A$ is compact.

2. The inclusion $i : S^n \to \mathbf{R}^{n+1}$ is continuous, so $f \times i : S^n \to \mathbf{R} \times \mathbf{R}^{n+1}$ is continuous. Also, the scalar multiplication map $m : \mathbf{R} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ given by $m(t, v) = tv$ is continuous from calculus, so $g = m \circ (f \times i) : S^n \to \mathbf{R}^{n+1}$ is continuous. But $C = g(S^n)$, and $S^n \subset \mathbf{R}^{n+1}$ is closed (since $S^n = h^{-1}(\{1\})$ where $h(v) = \|v\|$ is continuous, say) and obviously bounded, so compact by Heine-Borel. So C is the image of a compact set under a continuous function, hence compact.

3. If $\|v_n\|$ is bounded, say $\|v_n\| \leq M$ for all n, then the whole sequence is contained in the closed bounded set $\overline{B}_M(0)$. By the Heine-Borel theorem, this is compact, hence sequentially compact since $\mathbf{R}^n$ is a metric space. Therefore there exists a convergent subsequence.

4. This is best regarded as multiplication of a unit complex number z by $-1 = e^{i\pi}$. A good choice of homotopy is then clearly $F : S^1 \times I \to S^1$ given by $F(z, t) = ze^{i\pi t}$. To write this in real terms, take $z = x + iy$ and $e^{i\pi t} = \cos \pi t + i \sin \pi t$ and multiply using $i^2 = -1$ to get $ze^{i\pi t} = x \cos \pi t - y \sin \pi t + i(y \cos \pi t + x \sin \pi t)$. That is, in real coordinates

$$F(x, y, t) = (x \cos \pi t - y \sin \pi t, y \cos \pi t + x \sin \pi t).$$

This is the restriction to $S^1 \times I$ of a function $\mathbf{R}^3 \to \mathbf{R}^2$ which is continuous from calculus, hence F is continuous.

5. Let $p : Y \to X$ be a covering map, $y_0 \in Y$, $x_0 = p(y_0)$. For any continuous path $\gamma : I \to X$ such that $\gamma(0) = x_0$, there exists a unique continuous path $\tilde{\gamma} : I \to Y$ such that $\tilde{\gamma}(0) = y_0$ and $p \circ \tilde{\gamma} = \gamma$.

6. The fundamental groups at any basepoint are 1, $\mathbf{Z}$, and 1 respectively, and $\mathbf{Z}$, being infinite, is definitely not isomorphic to 1, so $\mathbf{R}^2 \setminus \{0\}$ is not homeomorphic to the others. But $\mathbf{R}^2$ is noncompact and S^2 is compact by Heine-Borel, and a compact space cannot be homeomorphic to a noncompact one, since continuous functions take compact sets to compact sets.