Bordered Sutured Floer Homology

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ABSTRACT

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We investigate the relationship between two versions of Heegaard Floer homology for 3–manifolds with boundary—the sutured Floer homology of Juhasz, and the bordered Heegaard Floer homology of Lipshitz, Ozsváth, and Thurston.

We define a new invariant called Bordered sutured Floer homology which encompasses these two invariants as special cases. Using the properties of this new invariant we prove a correspondence between the original bordered and sutured homologies.

In one direction we prove that for a 3–manifold Y with connected boundary $F = \partial Y$, and sutures $\Gamma \in \partial Y$, we can compute the sutured Floer homology $SFH(Y)$ from the bordered invariant $\widehat{CFA}(Y)_{\mathcal{A}(F)}$. The chain complex $SFC(Y, \Gamma)$ defining SFH is quasi-isomorphic to the derived tensor product $\widehat{CFA}(Y) \boxtimes \widehat{CFD}(\Gamma)$ where ${}_{\mathcal{A}(F)}\widehat{CFD}(\Gamma)$ is a module associated to Γ.

In the other direction we give a description of the bordered invariants in terms of sutured Floer homology. If F is a closed connected surface, then the boredered algebra $\mathcal{A}(F)$ is a direct sum of certain sutured Floer complexes. These correspond to the 3–manifold $(F \setminus D^2) \times [0,1]$, where the sutures vary in a finite collection. Similarly, if Y is a connected 3–manifold with boundary $\partial Y = F$, the module $\widehat{CFA}(Y)_{\mathcal{A}(F)}$ is a direct sum of sutured Floer complexes for Y where the sutures on ∂Y vary over a finite collection. The multiplication structure on $\mathcal{A}(F)$ and the action of $\mathcal{A}(F)$ on $\widehat{CFA}(Y)$ correspond to a natural gluing map on sutured Floer homology. (Further work of the author shows that this map coincides with the one defined by Honda, Kazez, and Matić, using contact topology and open book decompositions).

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Chapter 1

Introduction

In the 1980s Freedman's work on topological 4–manifolds [Fre82] and Donaldson's work on smooth 4–manifolds [Don83] showed that there is a huge gap between these two categories. The primary tool for distinguishing these categories are certain invariants which can distinguish different smooth structures on the same topological manifold.

Two such numerical invariants were developed based on gauge theory—Donaldson theory [DK90] and Seiberg-Witten theory [Wit94]. In addition to their success in the study of 4–manifolds, these invariants fit into a more general framework, suited to the study of 3–manifold topology—that of a *topological quantum field theory* or TQFT. The philosophy of a TQFT is that there is a functor from the category with objects smooth closed n–manifolds, and morphisms smooth $(n+1)$–dimensional cobordisms, i.e. $(n+1)$–dimensional manifolds with boundary, to an appropriate algebraic category. In the gauge theory case, this means that to a closed 3–manifold one associates a graded abelian group, and to a 4–dimensional cobordism one associates a morphism between such groups.

For the gauge theoretic invariants described above, the corresponding 3–dimensional theories are *instanton Floer homology*, or *HI*, first developed by Floer [Flo88], and *monopole Floer homology*, or *HM*, developed by Kronheimer and Mrowka [KM07]. In a different but related direction, Ozsváth and Szabó developed *Heegaard Floer homology*, or *HF* [OS04d, OS04c, OS06]—another TQFT-like invariant of 3 and 4–manifolds, with a more topological flavor—instead of gauge theory it is defined using Heegaard splittings of 3–manifolds, and holomorphic curves. Despite their different origins, the monopole and Heegaard Floer theories were recently shown to be the same for 3–manifolds—by Kutluhan, Lee, and Taubes [KLT10], and independently by Colin, Ghiggini, and Honda [CGH11] (for one version—the "hat" theory). They are also conjectured to be the same for 4–manifolds.

A TQFT can sometimes be expanded to a so-called *extended TQFT*—a functor from the 2–category of closed n–manifolds, $(n+1)$–manifolds with boundary, and $(n+2)$–manifolds with codimension–2 corners, to a suitable algebraic 2–category. (One can go even further, working with k–categories, and manifolds with corners of codimension k). It is then natural to ask if the Floer theories can be extended in this fashion to invariants of surfaces, 3–manifolds with boundary, and 4–manifolds with corners. In the case of Heegaard Floer homology there has been progress toward this goal in two different directions.

In one direction, *knot Floer homology*, or *HFK* was developed by Ozsváth and Szabó [OS04b, OS08a], and independently by Rasmussen [Ras03], as a version of *HF* for knots in a 3–manifold. This theory is powerful enough to detect the Seifert genus of a knot [OS04a], the Thurston norm of its complement [OS08b], and whether the knot is fibered [Ghi08, Ni07]. While not strictly an invariant of 3–manifolds with boundary, *HFK* can be regarded as associated to the complement of a knot. Further in this direction Juhász introduced *sutured Floer homology*, or *SFH* [Juh06].

Sutured manifolds were first introduced by Gabai in his study of foliations on 3–manifolds [Gab83]. A sutured manifold is a 3–dimensional manifold-with-boundary Y, equipped with a collection of decorations Γ on its boundary, called *sutures* (the collection of sutures is also sometimes called a *dividing set*). Juhász defined the sutured Floer homology *SFH*, as an invariant of (Y,Γ), and showed it generalizes both $\widehat{HF}$ and $\widehat{HFK}$ (specific versions of the *HF* and *HFK* theories). It has some elements of the desired TQFT-like structure. For example, for a sutured manifold (Y,Γ) and a properly embedded surface F in Y, there is a *sutured decomposition* of Y along F into a new, possibly disconnected sutured manifold (Y',Γ'). Juhász showed that, under certain assumptions on F, the homology $SFH(Y',\Gamma')$ is a direct summand of $SFH(Y,\Gamma)$ [Juh08]. He used this fact to give new proofs and generalizations of some of the earlier properties of *HF* and *HFK* described above.

Sutured Floer homology is also well suited to the study of contact topology—Honda, Kazez, and Matić used it to define an invariant of contact 3–manifolds manifolds with convex boundary [HKM09]. They also constructed a contact cobordism map for *SFH* [HKM08], which can alternatively be interpreted as follows. Suppose $\mathit{SFH}(Y_1, \Gamma_1)$ and $\mathit{SFH}(Y_2, \Gamma_2)$ are two sutured manifolds, and a surface with boundary F can be identified with subsets of ∂Y_1 and ∂Y_2, with opposite orientations, such that the sutures $\Gamma_1 \cap F$ and $\Gamma_2 \cap F$ are appropriately matched. One can construct a sutured manifold $(Y_1 \cup_F Y_2, \Gamma')$ by gluing. Then there is a homomorphism

$$\mathit{SFH}(Y_1, \Gamma_1) \otimes \mathit{SFH}(Y_2, \Gamma_2) \to \mathit{SFH}(Y_1 \cup_F Y_2, \Gamma').$$

However, it is generally neither injective, nor surjective. It is also hard to relate the source and target groups. Thus *SFH* has limitations from the point of view of extended TQFT structure on *HF*.

In a different direction, Lipshitz, Ozsváth, and Thurston introduced *bordered Heegaard Floer homology* [LOT09, LOT10a]. At its current stage, it is a TQFT-like invariant for surfaces and 3–manifolds with boundary. To a closed connected surface F, equipped with a handle decomposition, it associates a differential graded, or DG, algebra $\mathcal{A}(F)$. In the most basic form, to a 3–manifold Y with boundary $\partial Y = F$, one associates (a homotopy equivalence class of) an $\mathcal{A}_\infty$–module $\widehat{\mathit{CFA}}(Y)$ over $\mathcal{A}(F)$, or alternatively (a homotopy equivalence class of) a DG-module $\widehat{\mathit{CFD}}(Y)$ over $\mathcal{A}(-F)$. If Y_1 and Y_2 are two 3–manifolds with boundaries $\partial Y_1 = F$, and $\partial Y_2 = -F$, then there is a pairing theorem:

$$H_*\left(\widehat{\mathit{CFA}}(Y_1) \,\widetilde{\otimes}_{\mathcal{A}}\, (F)\widehat{\mathit{CFD}}(Y_2)\right) \cong \widehat{\mathit{HF}}(Y_1 \cup_F Y_2).$$

More generally, to a connected cobordism Y between two closed connected surfaces F_1 and F_2 (equipped with a framed arc connecting the two boundary components), one can associate a bimodule $\widehat{\mathit{CFDA}}(Y)$ over the two algebras $\mathcal{A}(F_1)$ and $\mathcal{A}(F_2)$. This construction is functorial, in the sense that if Y_1 and Y_2 are two cobordisms, from F_1 to F_2, and from F_2 to F_3, respectively, the associated bimodules are related in the following way:

$$\widehat{\mathit{CFDA}}(Y_1 \cup_F Y_2) \simeq \widehat{\mathit{CFDA}}(Y_1) \,\widetilde{\otimes}\, \widehat{\mathit{CFDA}}(Y_2).$$

Thus, there are currently two separate constructions that generalize $\widehat{HF}$ to 3–manifolds with boundary—bordered Floer homology, and sutured Floer homology. At first glance, they are defined very differently, using different types of auxiliary data, and behave differently. It is natural to ask if they are related to each, and how. Do they contain the same information, or not? The goal of this thesis is to give a conclusive answer to this question.

1.1 Results about *SFH* and bordered Floer homology

Our main results concern the relationship between bordered Floer homology, and sutured Floer homology. In short, if we include some gluing homomorphisms for *SFH*, of the type Honda, Kazez, and Matić define, the two theories become essentially equivalent. This can be broken up into two parts.

The first part concerns the way to get *SFH* from the bordered invariants, and is expressed in the following theorem:

Theorem 1. *Suppose Y is a connected 3–manifold with connected boundary. With any set of sutures Γ on ∂Y we can associate modules $\widehat{CFA}(\Gamma)$ and $\widehat{CFD}(\Gamma)$ over $\mathcal{A}(\pm\partial Y)$, of the appropriate form, such that the following formula holds.*

$$SFH(Y,\Gamma) \cong H_*(\widehat{CFA}(Y) \,\widetilde{\otimes}\, \widehat{CFD}(\Gamma)) \cong H_*(\widehat{CFA}(\Gamma) \,\widetilde{\otimes}\, \widehat{CFD}(Y)). \tag{1.1}$$

The second part concerns the way we can express the bordered theory in terms of *SFH*. This takes a little more effort to describe. A main ingredient in the construction is a gluing map Ψ on sutured Floer homology, not unlike the one defined by Honda, Kazez, and Matić and discussed above. We will say more about this map in Section 1.3.

Fix a parametrized closed surface F, with bordered algebra $A = \mathcal{A}(F)$. Let F' be F with a disc removed, and let $p, q \in \partial F'$ be two points. We can find $2^{2g(F)}$ distinguished dividing sets on F, which we denote Γ_I for $I \subset \{1, \dots, 2g\}$, and corresponding dividing sets $\Gamma'_I = \Gamma_I \cap F'$ on F'. Let $\Gamma_{I\to J}$ be a dividing set on $F' \times [0,1]$ which is Γ'_I along $F' \times \{0\}$, Γ'_J along $F' \times \{1\}$, and half of a negative Dehn twist of $\{p,q\} \times [0,1]$ along $\partial F' \times [0,1]$.

Theorem 2. *Suppose the surfaces F and F', the algebra A, and the dividing sets Γ_I, Γ'_I, and $\Gamma_{I\to J}$ are as described above. Then there is an isomorphism*

$$H_*(A) \cong \bigoplus_{I,J\subset\{1,\dots,2g\}} SFH(F'\times[0,1],\Gamma_{I\to J}),$$

and the multiplication map μ_2 on $H_*(A)$ can be identified with the gluing map $\Psi_{F'}$, corresponding to gluing two product manifolds $F'\times[0,1]$ and $F'\times[1,2]$ along $F'\times\{1\}$. It maps $SFH(F'\times[0,1],\Gamma_{I\to J})\otimes SFH(F'\times[1,2],\Gamma_{J\to K})$ to $SFH(F'\times[0,2],\Gamma_{I\to K})\cong SFH(F'\times[0,1],\Gamma_{I\to K})$, and sends all other summands to 0.

The module $\widehat{CFA}$ can be similarly described.

Theorem 3. *Suppose Y is a 3–manifold with boundary $\partial Y\cong F$. There is an isomorphism*

$$H_*(\widehat{CFA}(Y)_A) \cong \bigoplus_{I\subset\{1,\dots,2g\}} SFH(Y,\Gamma_I),$$

and the action m_2 of $H_*(A)$ on $H_*(\widehat{CFA}(Y))$ can be identified with the gluing map $\Psi_{F'}$, corresponding to gluing Y and $F'\times[0,1]$ along $F'\times\{0\}\subset F=\partial Y$. It maps $SFH(Y,\Gamma_I)\otimes SFH(F'\times[0,1],\Gamma_{I\to J})$ to $SFH(Y,\Gamma_J)$, and sends all other summands to 0.

1.2 Bordered sutured Floer homology

The proofs of Theorems 1, 2, and 3 use the machinery of *bordered sutured Floer homology* which we develop in the thesis. It is essentially a hybrid of the sutured and bordered Floer homology theories. In the basis of the theory lie the topological notions of a *sutured surface*, and *bordered sutured manifold*. We give a brief outline below, while the precise and in-depth definitions are left for Chapter 3.

For context we give the full definition of a sutured manifold.

Definition 1.2.1. *A* sutured 3–manifold (Y,Γ) *is a 3–manifold Y, with a multi-curve Γ on its boundary, dividing the boundary into a positive and negative region, denoted R_+ and R_-, respectively. We usually impose the conditions that Y has no closed components, and that Γ intersects every component of ∂Y.*

We can introduce analogous notions one dimension lower.

Definition 1.2.2. *A* sutured surface (F, Λ) *is a surface F, with a 0–manifold $\Lambda \subset \partial F$, dividing the boundary ∂F into a positive and negative region, denoted S_+ and S_-, respectively. Again, we impose the condition that F has no closed components, and that Λ intersects every component of ∂F.*

Definition 1.2.3. *A* sutured cobordism (Y, Γ) *between two sutured surfaces (F_1, Λ_1) and (F_2, Λ_2) is a cobordism Y between F_1 and F_2, together with a collection of properly embedded arcs and circles*

$$\Gamma \subset \partial Y \setminus (F_1 \cup F_2),$$

dividing $\partial Y \setminus (F_1 \cup F_2)$ into a positive and negative region, denoted R_+ and R_-, respectively, such that $R_\pm \cap F_i = S_\pm(F_i)$, for $i = 1, 2$. Again, we require that Y has no closed components, and that Γ intersects every component of $\partial Y \setminus (F_1 \cup F_2)$.

There is a *sutured category* $\mathcal{S}$ whose objects are sutured surfaces, and whose morphisms are sutured cobordisms. The identity morphisms are cobordisms of the form $(F \times [0,1], \Lambda \times [0,1])$, where (F, Λ) is a sutured surface. As a special case, sutured manifolds are the morphisms from the empty surface $(\varnothing, \varnothing)$ to itself.

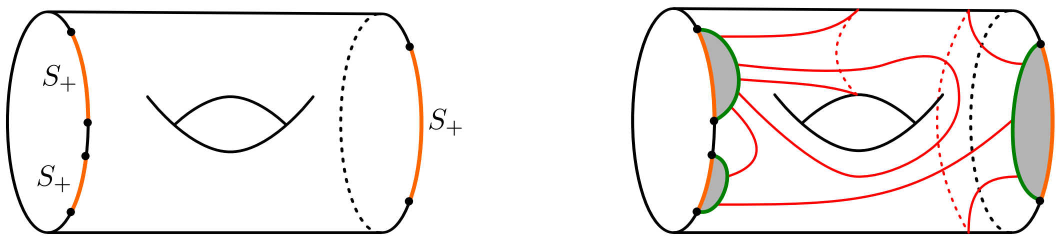
We cannot directly define invariants for the sutured category, and we need impose a little extra structure.

Definition 1.2.4. *An* arc diagram *is a relative handle diagram for a 2–manifold with corners, where the bottom and top boundaries are both 1–manifolds with no closed components.*

Definition 1.2.5. *A* parametrized *or* decorated sutured surface *is a sutured surface (F, Λ) with a handle decomposition given by an arc diagram $\mathcal{Z}$, expressing F as a cobordism from S_+ to S_-.*

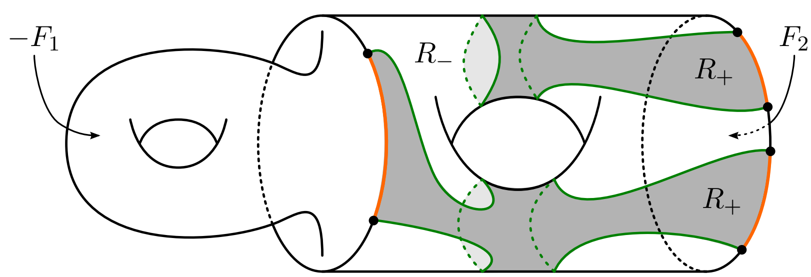
A parametrized *or* decorated sutured cobordism *is a sutured cobordism (Y, Γ) from (F_1, Λ_1) to (F_2, Λ_2), such that (F_i, Λ_i) is parametrized by an arc diagram $\mathcal{Z}_i$, for $i = 1, 2$.*

Examples of a sutured surface and its decorated version are given in Figure 1. A sutured cobordism and its decorated version are given in Figure 2. We visualize the handle decomposition coming from an arc diagram by drawing the cores of the 1–handles.

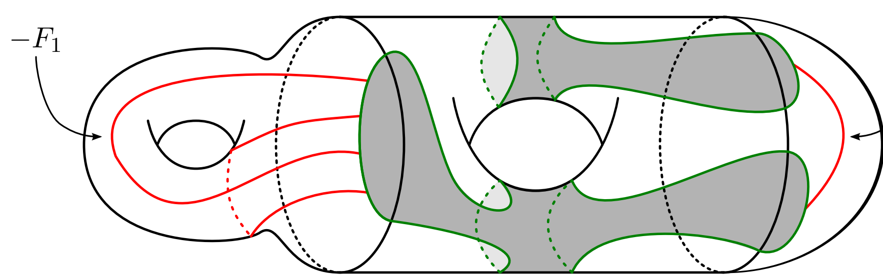


(a) Unparametrized. (b) Parametrized by an arc diagram.

Figure 1: A sutured surface (F, Λ). The sutures Λ are denoted by dots, while the positive region $S_+ \subset \partial F$ is colored in orange.



(a) Unparametrized.



(b) Parametrized (and with smoothed corners).

Figure 2: A sutured cobordism (Y, Γ) from a once punctured torus to a disc. The sutures Γ are colored in green, while the positive region $R_+ \subset \partial Y$ is shaded.

The *decorated sutured category* $\mathcal{SD}$ is a category whose objects are decorated sutured surfaces—or alternatively their arc diagrams—and whose morphisms are decorated sutured cobordisms. Note that all decorations on the sutured identity $(F \times [0,1], \Lambda \times [0,1])$ are isomorphisms, while the ones where the two parametrizations on $F \times \{0\}$ and $F \times \{1\}$ agree are the identity morphisms in $\mathcal{SD}$. In particular, any two parametrizations of the same sutured surface are isomorphic, and the forgetful functor $\mathcal{Z} \mapsto F(\mathcal{Z})$ is an equivalence of categories.

Sutured cobordisms have another, slightly different topological interpretation. For a sutured cobordism (Y, Γ) from (F_1, Λ_1) to (F_2, Λ_2), we can smooth its corners, and set $\Gamma' = \Gamma \cup S_+(F_1) \cup S_+(F_2)$. This turns (Y, Γ') into a regular sutured manifold. Therefore, we can think of a sutured cobordism as a sutured manifold, with two distinguished subsets F_1 and F_2 of its boundary.

Applying the same procedure to the decorated versions of sutured cobordisms, we come up with the notion of *bordered sutured manifolds*, defined more precisely in Chapter 3.

Definition 1.2.6. *A* bordered sutured manifold $(Y, \Gamma, \mathcal{Z})$ *is a sutured manifold* (Y, Γ)*, with a distinguished subset* $F \subset \partial Y$*, such that* $(F, \partial F \cap \Gamma)$ *is a sutured surface, parametrized by the arc diagram* $\mathcal{Z}$*.*

Any bordered sutured manifold $(Y, \Gamma, \mathcal{Z}_1 \cup \mathcal{Z}_2)$, where $\mathcal{Z}_i$ parametrizes $(F_i, \partial F_i \cap \Gamma)$ gives a decorated sutured cobordism $(Y, \Gamma \setminus (F_1 \cup F_2))$ from $-F_1$ to F_2, and vice versa.

The power of the theory comes from the existence of several invariants. To any arc diagram $\mathcal{Z}$—or alternatively decorated sutured surface parametrized by $\mathcal{Z}$—we associate a differential graded algebra $\mathcal{A}(\mathcal{Z})$, which is a subalgebra of some strand algebra, as defined in [LOT09].

These algebras behave nicely under disjoint union. If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are arc diagrams, then $\mathcal{A}(\mathcal{Z}_1 \cup \mathcal{Z}_2) \cong \mathcal{A}(\mathcal{Z}_1) \otimes \mathcal{A}(\mathcal{Z}_2)$.

To a bordered sutured manifold $(Y, \Gamma, \mathcal{Z})$ we associate a right $\mathcal{A}_\infty$–module $\widehat{BSA}(Y, \Gamma)$ over $\mathcal{A}(\mathcal{Z})$, and a left differential graded module $\widehat{BSD}_M(Y, \Gamma)$ over $\mathcal{A}(-\mathcal{Z})$.

Generalizing this construction, let (F_1, Λ_1) and (F_2, Λ_2) be two sutured surfaces, which are parametrized by the arc diagrams $\mathcal{Z}_1$ and $\mathcal{Z}_2$, respectively. To any sutured cobordism

(Y,Γ) between them we associate (a homotopy equivalence class of) an $\mathcal{A}_\infty$ $\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)$–bimodule, denoted $\widehat{BSDA}_M(Y,\Gamma)$. This specializes to $\widehat{BSA}(Y,\Gamma)$, respectively $\widehat{BSD}_M(Y,\Gamma)$, when F_1, respectively F_2 is empty, or to the sutured chain complex $SFC(Y,\Gamma)$, when both are empty.

Definition 1.2.7. *Let $\mathcal{D}$ be the category whose objects are differential graded algebras, and whose morphisms are the graded homotopy equivalence classes of $\mathcal{A}_\infty$–bimodules over any two such algebras. Composition is given by the derived tensor product $\widetilde{\otimes}$. The identity is the homotopy equivalence class of the algebra considered as a bimodule over itself.*

Theorem 4. *The invariant $\widehat{BSDA}_M$ respects compositions of decorated sutured cobordisms. Explicitly, let $(Y_1,\Gamma_1,-\mathcal{Z}_1\cup\mathcal{Z}_2)$ and $(Y_2,\Gamma_2,-\mathcal{Z}_2\cup\mathcal{Z}_3)$ be two bordered sutured manifolds, representing decorated sutured cobordisms from $\mathcal{Z}_1$ to $\mathcal{Z}_2$, and from $\mathcal{Z}_2$ to $\mathcal{Z}_3$, respectively. Then there are graded homotopy equivalences*

$$\widehat{BSDA}_M(Y_1,\Gamma_1)\,\widetilde{\otimes}_{\mathcal{A}(\mathcal{Z}_2)}\,\widehat{BSDA}_M(Y_2,\Gamma_2)\simeq\widehat{BSDA}_M(Y_1\cup Y_2,\Gamma_1\cup\Gamma_2). \tag{1.2}$$

Specializing to $\mathcal{Z}_1=\mathcal{Z}_3=\varnothing$, we get

$$\widehat{BSA}(Y_1,\Gamma_1)\,\widetilde{\otimes}_{\mathcal{A}(\mathcal{Z}_2)}\,\widehat{BSDD}(Y_2,\Gamma_2)\simeq SFC(Y_1\cup Y_2,\Gamma_1\cup\Gamma_2). \tag{1.3}$$

Theorem 5. *The invariant $\widehat{BSDA}_M$ respects the identity. In other words, if $(Y,\Gamma,-\mathcal{Z}\cup\mathcal{Z})$ is the identity cobordism from $\mathcal{Z}$ to itself, then $\widehat{BSDA}_M(Y,\Gamma)$ is graded homotopy equivalent to $\mathcal{A}(\mathcal{Z})$ as an $\mathcal{A}_\infty$–bimodule over itself.*

Together, Theorems 4 and 5 imply that $\mathcal{A}$ and $\widehat{BSDA}_M$ form a functor.

Corollary 6. *The invariants $\mathcal{A}$ and $\widehat{BSDA}_M$ give a functor from $\mathcal{SD}$ to $\mathcal{D}$, inducing a (non-unique) functor from the equivalent category $\mathcal{S}$ to $\mathcal{D}$. In particular, if $\mathcal{Z}_1$ and $\mathcal{Z}_2$ parametrize the same sutured surface, then $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$ are isomorphic in $\mathcal{D}$. In other words, there is an $\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)$ $\mathcal{A}_\infty$–bimodule providing an equivalence of the derived categories of $\mathcal{A}_\infty$–modules over $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$.*

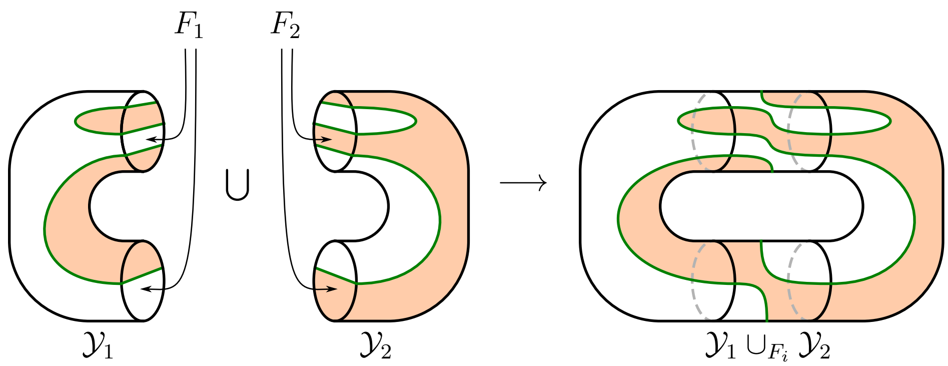


Figure 3: Gluing two solid balls along $F = D^2 \cup D^2$, to obtain a solid torus. The R_+ regions have been shaded.

1.3 Gluing and joining

After defining the bordered sutured invariants in Part I, we use them in Part II to define the gluing map Ψ mentioned above.

Suppose (Y_1, Γ_1) and (Y_2, Γ_2) are two sutured manifolds. We say that we can glue them if there are subsets F_1 and F_2 of their boundaries, where F_1 can be identified with the mirror of F_2, such that the multicurve $\Gamma_1 \cap F_1$ is identified with $\Gamma_2 \cap F_2$, preserving the orientations on Γ_i. This means that the regions R_+ and R_- on the two boundaries are interchanged. We will only talk of gluing in the case when F_i have no closed components, and all components of ∂F_i intersect the dividing sets Γ_i.

Definition 1.3.1. *Suppose (Y_1, Γ_1), (Y_2, Γ_2), F_1 and F_2 are as above. The* gluing *of (Y_1, Γ_1) and (Y_2, Γ_2)* along *F_i is the sutured manifold $(Y_1 \cup_{F_i} Y_2, \Gamma_{1+2})$. The dividing set Γ_{1+2} is obtained from $(\Gamma_1 \setminus F_1) \cup_{\partial F_i} (\Gamma_2 \setminus F_2)$ as follows. Along each component f of ∂F_i the orientations of Γ_1 and Γ_2 disagree. We apply the minimal possible positive fractional Dehn twist along f that gives a consistent orientation.*

An illustration of gluing is given in Figure 3. We define a gluing map Ψ on *SFH* corresponding to this topological construction.

Theorem 7. *Let (Y_1, Γ_1) and (Y_2, Γ_2) be two balanced sutured manifolds, that can be glued along F. Then there is a well defined map*

$$\Psi_F \colon \mathit{SFH}(Y_1, \Gamma_1) \otimes \mathit{SFH}(Y_2, \Gamma_2) \to \mathit{SFH}((Y_1, \Gamma_1) \cup_F (Y_2, \Gamma_2)),$$

satisfying the following properties:

1. *Symmetry: The map Ψ_F for gluing Y_1 to Y_2 is equal to that for gluing Y_2 to Y_1.*

2. *Associativity: Suppose that we can glue Y_1 to Y_2 along F_1, and Y_2 to Y_3 along F_2, such that F_1 and F_2 are disjoint in ∂Y_2. Then the order of gluing is irrelevant:*

$$\Psi_{F_2} \circ \Psi_{F_1} = \Psi_{F_1} \circ \Psi_{F_2} = \Psi_{F_1 \cup F_2}.$$

3. *Identity: Given a dividing set Γ on F, there is a dividing set Γ' on $F \times [0,1]$, and an element $\Delta_\Gamma \in SFH(F \times [0,1], \Gamma')$, satisfying the following. For any sutured manifold (Y, Γ'') with $F \subset \partial Y$ and $\Gamma'' \cap F = \Gamma$, there is a diffeomorphism $(Y, \Gamma'') \cup_F (F \times [0,1], \Gamma') \cong (Y, \Gamma'')$. Moreover, the map $\Psi_F(\cdot, \Delta_\Gamma)$ is the identity of $SFH(Y, \Gamma'')$.*

The gluing construction and the gluing map readily generalize to a more general *join* construction, and *join map*, which are 3–dimensional analogs. Suppose that (Y_1, Γ_1) and (Y_2, Γ_2) are two sutured manifolds, and F_1 and F_2 are subsets of their boundaries, satisfying the conditions for gluing. Suppose further that the diffeomorphism $F_1 \to F_2$ extends to $W_1 \to W_2$, where W_i is a compact codimension–0 submanifold of Y_i, and $\partial W_i \cap \partial Y_i = F_i$. Instead of gluing Y_1 and Y_2 along F_i, we can *join* them along W_i.

Definition 1.3.2. *The* join of (Y_1, Γ_1) and (Y_2, Γ_2) along W_i *is the sutured manifold*

$$((Y_1 \setminus W_1) \cup_{\partial W_i \setminus F_i} (Y_2 \setminus W_2), \Gamma_{1+2}),$$

where the dividing set Γ_{1+2} is constructed exactly as in Definition 1.3.1. We denote the join by $(Y_1, \Gamma_1) \Cup_{W_i} (Y_2, \Gamma_2)$.

An example of a join is shown in Figure 4. Notice that if W_i is a collar neighborhood of F_i, then the notions of join and gluing coincide. That is, the join operation is indeed a generalization of gluing. In fact, throughout the thesis we work almost exclusively with joins, while only regarding gluing as a special case.

Theorem 8. *There is a well-defined join map*

$$\Psi_W \colon SFH(Y_1, \Gamma_1) \otimes SFH(Y_2, \Gamma_2) \to SFH((Y_1, \Gamma_1) \Cup_W (Y_2, \Gamma_2)),$$

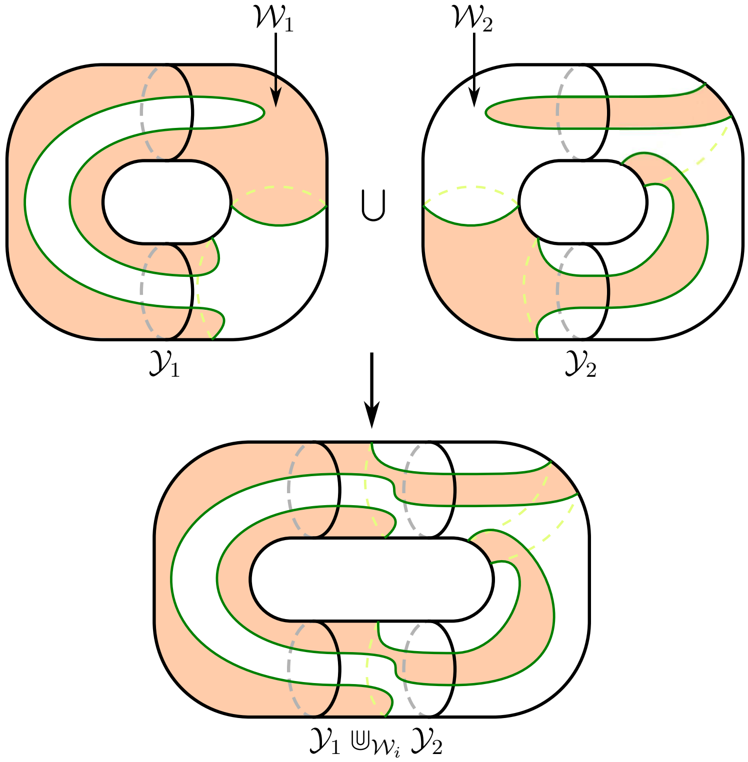


Figure 4: Join of two solid tori along $D^2 \times [0,1]$, to obtain another solid torus. The R_+ regions have been shaded.

satisfying properties of symmetry, associativity, and identity, analogous to those listed in Theorem 7.

The join map is constructed as follows. We cut out W_1 and W_2 from Y_1 and Y_2, respectively, and regard the complements as bordered sutured manifolds. The join operation corresponds to replacing W_1 and W_2 by an interpolating piece $\mathcal{TW}_{F,+}$. We define a map between the bordered sutured invariants, from the product $\widehat{BSA}(W_1) \otimes \widehat{BSA}(W_2)$ to the bimodule $\widehat{BSAA}(\mathcal{TW}_{F,+})$. We show that for an appropriate choice of parametrizations, the modules $\widehat{BSA}(W_1)$ and $\widehat{BSA}(W_2)$ are duals, while $\widehat{BSAA}(\mathcal{TW}_{F,+})$ is the dual of the bordered algebra for F. The map is then an $\mathcal{A}_\infty$–version of the natural pairing between a module and its dual. The proof of invariance and the properties from Theorems 7 and 8 is purely algebraic. Most of the arguments involve $\mathcal{A}_\infty$–versions of standard facts in commutative algebra.

The proofs of Theorems 2 and 3 involve several steps. First, we find a manifold whose bordered sutured invariant is the bordered algebra, as a bimodule over itself. Second, we find manifolds whose bordered sutured invariants are all possible simple modules over the algebra. Finally, we compute the gluing map Ψ explicitly in several cases.

1.4 Further applications

Besides the results described in the current thesis, bordered sutured Floer homology has a number of further applications.

One of these applications is to define a functorial Heegaard Floer invariant for tangles, which reduces to knot Floer homology in the case of a closed knot or link [Zar11a]. This is analogous to the situation in Khovanov homology, where such tangle invariants have existed for some time [Kho02]. This may give new insights into the structure of *HFK*.

Another application involves the relation of bordered Floer homology, and the category of contact structures on a thickened surface. In [Zar11b] we prove that the gluing map Ψ defined here is actually equivalent to the contact cobordism maps from [HKM08]. This allows us prove yet one more correspondence—that of the bordered algebra $\mathcal{A}(F)$ and certain isotopy classes of tight contact structures on $(F \setminus D^2) \times [0,1]$.

A third application involves computing direct limits of sutured Floer homology groups. In [EVVZ11], John Etnyre, Shea Vela-Vick, and the author prove that the minus version of knot Floer homology, $HFK^{-}(Y,K)$ for a knot K in a three-manifold Y is the direct limit of certain sutured Floer homology groups (which normally only see the hat version of Heegaard Floer homology). This is related to the study of Legendrian and transverse knots.

There are other speculative applications, that we hope will materialize in the future.

In contrast to the gluing map, there is no analog of the join map in the setting of Honda, Kazez, and Matić. However, there is a natural pair-of-pants cobordism

$$Z_W \colon (Y_1,\Gamma_1) \sqcup (Y_2,\Gamma_2) \to (Y_1,\Gamma_1) \Cup_W (Y_2,\Gamma_2).$$

Juhász defines a cobordism map F_{Z_W} in that situation using counts of holomorphic triangles [Juh10]. We conjecture that the join map Ψ_W is equivalent to F_{Z_W}.

One potential application is to use bordered sutured Floer homology to express Heegaard Floer homology in an axiomatic way, by breaking manifolds into simple enough building blocks. This could provide another more conceptual approach to the equivalence between the different Floer theories (*HF*, *HM*, and embedded contact homology, or *ECH*).

In another direction, Theorems 2 and 3 suggest an approach towards defining a bordered theory that corresponds to the plus or minus versions of Heegaard Floer homology—so far that has been elusive. The two theorems proved here tell us that current, or hat, version of bordered Floer homology is really about surfaces with boundary. We conjecture that a similar construction involving $SFH(F \times [0,1])$, where F is a closed surface would in fact provide the desired theory, which can be used to compute $HF^{\pm}(Y)$ for a closed manifold Y.

1.5 Organization

The thesis is separated into two parts.

Part I defines bordered sutured manifolds and their invariants. The first few chapters are devoted to the topological constructions. First, in Chapter 2 we define arc diagrams, and how they parametrize sutured surfaces, as well as the $\mathcal{A}_\infty$–algebra associated to an arc diagram. In Chapter 3 we define bordered sutured manifolds, and in Chapter 4 we define the Heegaard diagrams associated to them.

The next few chapters define the invariants and give their properties. In Chapter 5 we talk about the moduli spaces of curves necessary for the definitions of the invariants. In Chapter 7 we give the definitions of the bordered sutured invariants $\widehat{BSDD}$ and $\widehat{BSA}$, and prove Eq. (1.3) from Theorem 4. In Chapter 8 we extend the definitions and properties to the bimodules $\widehat{BSDA}_M$, and sketch the proof of the rest of Theorem 4, as well as Theorem 5. The gradings are defined together for all three invariants on the diagram level in Chapter 6.

A lot of the material in these chapters is a reiteration of analogous constructions and definitions from [LOT09], with the differences emphasized. The reader who is encountering bordered Floer homology for the first time can skip most of that discussion on the first reading, and use Theorems 7.5.1, 7.5.2 and 8.5.2 as definitions.

Chapter 10 gives some examples of bordered sutured manifolds and computations of their invariants. The reader is encouraged to read this section first, or immediately after Chapter 4. The examples can be more enlightening than the definitions, which are rather involved.

Finally, Chapter 9 gives several applications of the new invariants, in particular proving Theorem 1, and giving a new proof of the surface decomposition theorem of Juhász.

Part II defines the gluing a join maps and derives some of their properties. We start by introducing in more detail the topological constructions of the gluing join operations in Chapter 11. In Chapter 12 we also discuss how definitions of $\widehat{BSA}$ and $\widehat{BSD}$ involving only α–arcs can be extended to diagrams using β–arcs. Chapter 12.4 contains computations of several $\widehat{BSA}$ invariants needed later.

We define the join map in Chapter 13, on the level of chain complexes. The same section contains the proof that it descends to a unique map on homology. In the following Chapter 14 we prove the properties from Theorems 7 and 8. Finally, Chapter 15 contains the statement and the proof of a slightly more general version of Theorems 2 and 3.

Throughout the thesis make use of a diagrammatic calculus to compute $\mathcal{A}_\infty$–morphisms, which greatly simplifies the arguments. Appendix A contains a brief description of this calculus, and the necessary algebraic assumptions. Appendix B gives an overview of $\mathcal{A}_\infty$–bimodules in terms of the diagrammatic calculus, as they are used in the paper.

Part I

Bordered sutured invariants

Chapter 2

The algebra associated to a parametrized surface

The invariants defined by Lipshitz, Ozsváth and Thurston in [LOT09] work only for connected manifolds with one closed boundary component. These were extended in [LOT10a] to manifolds with two or more closed boundary components.

In our construction we parametrize surfaces with boundary, and possibly many connected components. This class of surfaces and of their allowed parametrizations is much wider, so we need to expand the algebraic constructions describing them. We discuss below the generalized definitions and discuss the differences from the purely bordered setting.

2.1 Arc diagrams and sutured surfaces

We start by generalizing the definition of a pointed matched circle in [LOT09].

Definition 2.1.1. *An* arc diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ *is a triple consisting of a collection* $\mathbf{Z} = \{Z_1, \dots, Z_l\}$ *of oriented line segments, a collection* $\mathbf{a} = \{a_1, \dots, a_{2k}\}$ *of distinct points in* $\mathbf{Z}$*, and a matching of* $\mathbf{a}$*, i.e. a 2–to–1 function* $M\colon \mathbf{a} \to \{1, \dots, k\}$*. Write* $|Z_i|$ *for* $\#(Z_i \cap \mathbf{a})$*. We will assume* $\mathbf{a}$ *is ordered by the order on* $\mathbf{Z}$ *and the orientations of the individual segments. We allow* l *or* k *to be 0.*

We impose the following condition, called non-degeneracy*. After performing oriented surgery on the 1–manifold* $\mathbf{Z}$ *at each 0–sphere* $M^{-1}(i)$*, the resulting 1–manifold should have no closed components.*

Definition 2.1.2. *We can sometimes consider* degenerate arc diagrams *which do not satisfy the non-degeneracy condition. However, we will tacitly assume all arc diagrams are non-degenerate, unless we specifically say otherwise.*

Remark. The pointed matched circles of Lipshitz, Ozsváth and Thurston correspond to arc diagrams where $\mathbf{Z}$ has only one component. The arc diagram is obtained by cutting the matched circle at the basepoint.

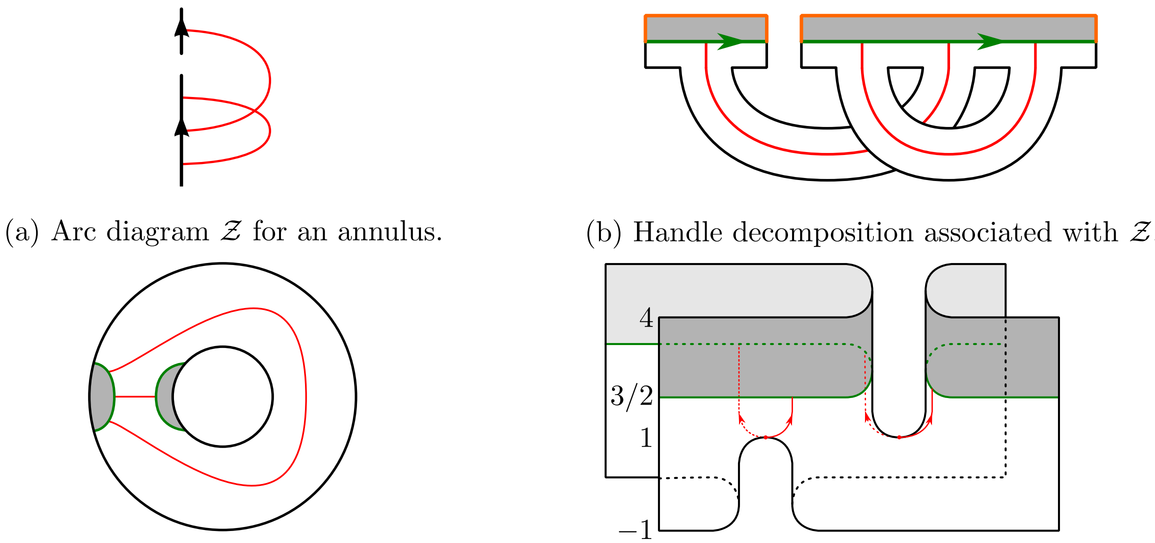
We can interpret $\mathcal{Z}$ as an upside-down handlebody diagram for a sutured surface $F(\mathcal{Z})$, or just F. It will often be convenient to think of F as a surface with corners, and we will use these descriptions interchangeably.

To construct F we start with a collection of rectangles $Z_i \times [0,1]$ for $i = 1, \dots, l$. Then attach 1–handles at $M^{-1}(i) \times \{0\}$ for $i = 1, \dots, k$. Thus $\chi(F) = l - k$, and F has no closed components. Set $\Lambda = \partial\mathbf{Z} \times \{1/2\}$, and $S_+ = \mathbf{Z} \times \{1\} \cup \partial\mathbf{Z} \times [1/2, 1]$. Such a description uniquely specifies F up to isotopy fixing the boundary.

Remark. The non-degeneracy condition on $\mathcal{Z}$, is equivalent to the condition that any component of ∂F intersects Λ. Indeed, the effect on the boundary of adding the 1–handles is surgery on $\mathbf{Z} \times \{0\}$. If $\mathcal{Z}$ is non-degenerate, this surgery produces no new closed components, and F is indeed a sutured surface.

Alternatively, instead of a handle decomposition we can consider a Morse function on F. Whenever we talk about Morse functions, a (fixed) choice of Riemannian metric is implicit.

Definition 2.1.3. *A* $\mathcal{Z}$–compatible Morse function *on* F *is a self-indexing Morse function* $f\colon F \to [-1, 4]$*, such that the following conditions hold. There are no index–0 or index–2 critical points. There are exactly* k *index–1 critical points and they are all interior. The gradient of* f *is tangent to* $\partial F \setminus f^{-1}(\{-1, 4\})$*. The preimage* $f^{-1}([-1, -1/2])$ *is isotopic to a collection of rectangles* $[0,1] \times [-1, -1/2]$ *such that* f *is projection on the second factor.*



(a) Arc diagram $\mathcal{Z}$ for an annulus.

(b) Handle decomposition associated with $\mathcal{Z}$.

(c) Annulus parametrized by $\mathcal{Z}$.

(d) Morse function compatible with $\mathcal{Z}$.

Figure 5: Arc diagram for an annulus, and three different views of parametrization.

Similarly, $f^{-1}([3/2,4])$ *is isotopic to a collection of rectangles* $[0,1]\times[3/2,4]$ *such that* f *is projection on the second factor.*

Furthermore, we can identify $f^{-1}(\{3/2\})$ *with* $\mathbf{Z}$ *such that the unstable manifolds of the* i*–th index–1 critical point intersect* $\mathbf{Z}$ *at* $M^{-1}(i)$*. We require that the orientation of* $\mathbf{Z}$ *and* ∇f *form a positive basis everywhere.*

Clearly, a compatible Morse function and a handle decomposition as above are equivalent. Examples of an arc diagram, and the different ways we can interpret its parametrization of a sutured surface, are given in Figure 5. A slightly more complicated example of an arc diagram, corresponding to the parametrization in Figure 1b, is given in Figure 6.

There is one more way to describe the above parametrization. Recall that a *ribbon graph* is a graph with a cyclic ordering of the edges incident to any vertex. An embedding of a ribbon graph into a surface will be considered *orientation preserving* if the ordering of the edges agrees with the positive direction on the unit tangent circle of the vertex in the surface.

Definition 2.1.4. *Let* F *be a sutured surface obtained from an arc diagram* $\mathcal{Z}$ *as above. The* ribbon graph associated to $\mathcal{Z}$ *is the ribbon graph* $G(\mathcal{Z})$ *with vertices* $\partial\mathbf{Z}\cup\mathbf{a}$*, and edges the components of* $\mathbf{Z}\setminus\mathbf{a}$ *and the cores of the 1–handles, which we denote* e_i *for* $i=1,\dots,k$*.*

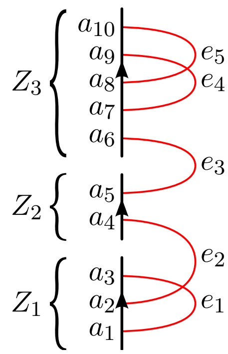


Figure 6: An arc diagram $\mathcal{Z}$ for a twice punctured torus, and its graph $G(\mathcal{Z})$.

The cyclic ordering is induced from the orientation of F.

In these terms, F is parametrized by $\mathcal{Z}$ if we specify an orientation preserving proper embedding $G(\mathcal{Z}) \hookrightarrow F$, such that F deformation retracts onto the image.

Remark. When we draw an arc diagram $\mathcal{Z}$ we are in fact drawing its graph $G(\mathcal{Z})$. An example, with all elements of the graph denoted, is given in Figure 6.

2.2 The algebra associated to an arc diagram

Recall the definition of the strands algebra from [LOT09].

Definition 2.2.1. *The* strands algebra $\mathcal{A}(n, k)$ *is a free* $\mathbb{Z}/2$*–module with generators of the form* $\mu = (T, S, \phi)$*, where* S *and* T *are* k*–element subsets of* $\{1, \dots, n\}$*, and* $\phi\colon S \to T$ *is a non–decreasing bijection. (We think of* ϕ *as a collection of* strands *from* S *to* T*.) Denote by* $\mathrm{inv}(\mu) = \mathrm{inv}(\phi)$ *the number of inversions of* ϕ*, i.e. the elements of* $\mathrm{Inv}(\mu) = \{(i, j) : i, j \in S, i < j, \phi(i) > \phi(j)\}$*.*

Multiplication is given by

$$(S, T, \phi) \cdot (U, V, \psi) = \begin{cases} (S, V, \psi \circ \phi) & \text{if } T = U \text{ and } \mathrm{inv}(\phi) + \mathrm{inv}(\psi) = \mathrm{inv}(\psi \circ \phi), \\ 0 & \text{otherwise.} \end{cases}$$

The differential on (S, T, ϕ) *is given by the sum of all possible ways to "resolve" an inversion, i.e. switch* $\phi(i)$ *and* $\phi(j)$ *for some inversion* $(i, j) \in \mathrm{Inv}(\mu)$*.*

Next, we consider the larger *extended strands algebra*

$$A(n_1,\dots,n_l;k)=\bigoplus_{k_1+\cdots+k_l=k}\mathcal{A}(n_1,k_1)\otimes\cdots\otimes\mathcal{A}(n_l,k_l).$$

We will slightly abuse notation and think of elements of $\mathcal{A}(n_i,k_i)$ as functions acting on subsets of $\{(n_1+\cdots+n_{i-1})+1,\dots,(n_1+\cdots+n_{i-1})+n_i\}$ instead of $\{1,\dots,n_i\}$. This allows us to identify $\mathcal{A}(n_1,\dots,n_l;k)$ with a subalgebra of $\mathcal{A}(n_1+\cdots+n_l,k)$.

We will sometimes talk about the direct sums $\mathcal{A}(n)=\mathcal{A}(n,0)\oplus\cdots\oplus\mathcal{A}(n,n)$, and $\mathcal{A}(n_1,\dots,n_l)=\mathcal{A}(n_1,\dots,n_l;0)\oplus\cdots\oplus\mathcal{A}(n_1,\dots,n_l;n_1+\cdots+n_l)$.

The definition of $\mathcal{A}(\mathcal{Z},i)$ as a subalgebra of $\mathcal{A}(|Z_1|,\dots,|Z_l|;i)$ below is a straightforward generalization of the definition of the algebra associated to a pointed matched circle in [LOT09]. There is, however, a difference in notation. In [LOT09] $\mathcal{A}(\mathcal{Z},0)$ denotes the middle summand and negative summand indices are allowed. Here, $\mathcal{A}(\mathcal{Z},0)$ is the bottom summand, and we only allow non-negative indices.

For any i–element subset $S\subset\{1,\dots,2k\}$, there is an idempotent $I(S)=(S,S,\mathrm{id}_S)\in\mathcal{A}(|Z_1|,\dots,|Z_l|,i)$. For an i–element subset $s\subset\{1,\dots,k\}$, a *section* S of s is an i–element set $S\subset M^{-1}(s)$, such that $M|_S$ is injective. To each s there is an associated idempotent

$$I(s)=\sum_{S\text{ is a section of }s}I(S).$$

Consider triples of the form (S,T,ψ), where $S,T\subset\{1,\dots,2k\}$, $\psi\colon S\to T$ is a strictly increasing bijection. Consider all possible sets $U\subset\{1,\dots,2k\}$ disjoint from S and T, and such that $S\cup U$ has i elements. Let

$$a_i(S,T,\psi)=\sum_{U\text{ as above}}(S\cup U,T\cup U,\psi_U)\in\mathcal{A}(|Z_1|,\dots,|Z_l|;i),$$

where $\psi_U|_T=\psi$, and $\psi_U|_U=\mathrm{id}_U$. In the language of strands, this means "to a set of moving strands add all possible consistent collections of stationary (or horizontal) strands".

Let $\mathcal{I}(\mathcal{Z},i)$ be the subalgebra generated by $I(s)$ for all i–element sets s, and let $I=\sum_s I(s)$ be their sum. Let $\mathcal{A}(\mathcal{Z},i)$ be the subalgebra generated by $\mathcal{I}(\mathcal{Z},i)$ and all elements of the form $I\cdot a_i(S,T,\psi)\cdot I$. The latter form a basis over $\mathbb{Z}/2$.

All elements (S,T,ϕ) considered have the property that $M|_S$ and $M|_T$ are injective.

Definition 2.2.2. *The* algebra associated *with the arc diagram* $\mathcal{Z}$ *is*

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=0}^{k} \mathcal{A}(\mathcal{Z}, i),$$

which is a module over

$$\mathcal{I}(\mathcal{Z}) = \bigoplus_{i=0}^{k} \mathcal{I}(\mathcal{Z}, i).$$

To any element of $\mu = (S, T, \phi) \in \mathcal{A}(|Z_1|, \ldots, |Z_l|)$ we can associate its *homology class* $[\mu] \in H_1(\mathbf{Z}, \mathbf{a})$, by setting

$$[\mu] = \sum_{i \in S} [l_i],$$

where l_i is the positively oriented segment $[a_i, a_{\phi(i)}] \subset \mathbf{Z}$. For any two homogeneous elements $\mu, \mu' \in \mathcal{A}(|Z_1|, \ldots, |Z_l|)$ we have $[\mu \cdot \mu'] = [\mu] + [\mu']$, unless $\mu \cdot \mu' = 0$. Similarly, $[\partial\mu] = [\mu]$, unless $\partial\mu = 0$. Since the homology class of (S, T, ϕ) only depends on the moving strands, any element of the form $I \cdot a_i(S, T, \psi) \cdot I \in \mathcal{A}(\mathcal{Z}, i)$ is homogeneous. Thus we can talk about the homology classes of basis elements in $\mathcal{A}(\mathcal{Z})$.

Remark. With a collection $\mathcal{Z}_1, \ldots, \mathcal{Z}_p$ of arc diagrams we can associate their *union* $\mathcal{Z} = \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_p$, where $\mathbf{Z} = \mathbf{Z}_1 \sqcup \cdots \sqcup \mathbf{Z}_p$, preserving the matching on each piece.

There are natural identifications, of algebras

$$\mathcal{A}(\mathcal{Z}) = \bigotimes_{i=1}^{p} \mathcal{A}(\mathcal{Z}_i),$$

and of surfaces

$$F(\mathcal{Z}) = \bigsqcup_{i=1}^{p} F(\mathcal{Z}_i).$$

2.3 Reeb chord description

We give an alternative interpretation of the strands algebra $\mathcal{A}(\mathcal{Z})$.

Given an arc diagram $\mathcal{Z}$ with k arcs, there is a unique positively oriented contact structure on the 1–manifold $\mathbf{Z}$, while the 0–manifold $\mathbf{a} \subset \mathbf{Z}$ is Legendrian. There is a family of *Reeb chords* in $\mathbf{Z}$, starting and ending at $\mathbf{a}$ and positively oriented. For a Reeb chord ρ we will denote its starting and ending point by ρ^- and ρ^+, respectively. Moreover, for a collection $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_n\}$ of Reeb chords as above, we will write $\boldsymbol{\rho}^- = \{\rho_1^-, \dots, \rho_n^-\}$, and $\boldsymbol{\rho}^+ = \{\rho_1^+, \dots, \rho_n^+\}$.

Definition 2.3.1. *A collection $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_n\}$ of Reeb chords is p–*completable *if the following conditions hold:*

1. *$\rho_i^- \neq \rho_i^+$ for all $i = 1, \dots, n$.*
2. *$M(\rho_1^-), \dots, M(\rho_n^-)$ are all distinct.*
3. *$M(\rho_1^+), \dots, M(\rho_n^+)$ are all distinct.*
4. *$\#(M(\boldsymbol{\rho}^-) \cup M(\boldsymbol{\rho}^+)) \leq k - (p - n)$.*

Condition (4) guarantees that there is at least one choice of a $(p-n)$–element set $s \subset \{1, \dots, k\}$, disjoint from $M(\boldsymbol{\rho}^-)$ and $M(\boldsymbol{\rho}^+)$. Such a set is called *a p–completion* or just *completion* of $\boldsymbol{\rho}$. Every completion of $\boldsymbol{\rho}$ defines an element of $\mathcal{A}(\mathcal{Z}, p)$:

Definition 2.3.2. *For a p–completable collection $\boldsymbol{\rho}$ and a completion s, their* associated element *in $\mathcal{A}(\mathcal{Z}, p)$ is*

$$a(\boldsymbol{\rho}, s) = \sum_{S \text{ is a section of } s} (\boldsymbol{\rho}^- \cup S, \boldsymbol{\rho}^+ \cup S, \phi_S),$$

where $\phi_S(\rho_i^-) = \rho_i^+$, for $i = 1, \dots, n$, and $\phi|_S = \mathrm{id}_S$.

Definition 2.3.3. *The* associated element *of $\boldsymbol{\rho}$ in $\mathcal{A}(\mathcal{Z}, p)$ is the sum over all p–completions:*

$$a_p(\boldsymbol{\rho}) = \sum_{s \text{ is a } p\text{-completion of } \boldsymbol{\rho}} a(\boldsymbol{\rho}, s).$$

If $\boldsymbol{\rho}$ is not p–completable, we will just set $a_p(\boldsymbol{\rho}) = 0$. We will also sometimes use the complete sum

$$a(\boldsymbol{\rho}) = \sum_{p=0}^{k} a_p(\boldsymbol{\rho}).$$

The algebra $\mathcal{A}(\mathcal{Z}, p)$ is generated over $\mathcal{I}(\mathcal{Z}, p)$ by the elements $a_p(\boldsymbol{\rho})$ for all possible p–completable $\boldsymbol{\rho}$. Algebra multiplication of such associated elements corresponds to certain concatenations of Reeb chords.

We can define the *homology* class $[\rho] \in H_1(\mathbf{Z}, \mathbf{a})$ in the obvious way, and extend to a set of Reeb chords $\boldsymbol{\rho} = \{\rho_1, \ldots, \rho_n\}$, by taking the sum $[\boldsymbol{\rho}] = [\rho_1] + \cdots + [\rho_n]$. It is easy to see that $[a(\boldsymbol{\rho}, s)] = [\boldsymbol{\rho}]$, and in particular it doesn't depend on the completion s.

2.4 Grading

There are two ways to grade the algebra $\mathcal{A}(\mathcal{Z})$. The simpler is to grade it by a nonabelian group $\mathrm{Gr}(\mathcal{Z})$, which is a $\frac{1}{2}\mathbb{Z}$–extension of $H_1(\mathbf{Z}, \mathbf{a})$. This group turns out to be too big, and does not allow for a graded version of the pairing theorems. For this a subgroup $\underline{\mathrm{Gr}}(\mathcal{Z})$ of $\mathrm{Gr}(\mathcal{Z})$ is necessary, that can be identified with a $\frac{1}{2}\mathbb{Z}$–extension of $H_1(F(\mathcal{Z}))$. Unfortunately, there is no canonical way to get a $\underline{\mathrm{Gr}}(\mathcal{Z})$–grading on $\mathcal{A}(\mathcal{Z})$.

Remark. Our notation differs from that in [LOT09]. In particular, our grading group $\mathrm{Gr}(\mathcal{Z})$ is analogous to the group $G'(\mathcal{Z})$ used by Lipshitz, Ozsváth and Thurston, while $\underline{\mathrm{Gr}}(\mathcal{Z})$ corresponds to their $G(\mathcal{Z})$. Moreover, our grading function gr corresponds to their gr′, while $\underline{\mathrm{gr}}$ corresponds to gr.

We start with the $\mathrm{Gr}(\mathcal{Z})$–grading. Suppose $\mathbf{Z} = \{Z_1, \ldots, Z_l\}$. We will define a grading on the bigger algebra $\mathcal{A}(|Z_1|, \ldots, |Z_l|)$ that descends to a grading on $\mathcal{A}(\mathcal{Z})$.

First, we define some auxiliary maps.

Definition 2.4.1. *Let $m\colon H_0(\mathbf{a}) \times H_1(\mathbf{Z}, \mathbf{a}) \to \frac{1}{2}\mathbb{Z}$ be the map defined by counting local multiplicities. More precisely, given the positively oriented line segment $l = [a_i, a_{i+1}] \subset Z_p$, set*

$$m([a_j], [l]) = \begin{cases} \frac{1}{2} & \text{if } j = i, i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and extend linearly to all of $H_0(\mathbf{a}) \times H_1(\mathbf{Z}, \mathbf{a})$.

Definition 2.4.2. *Let $L\colon H_1(\mathbf{Z}, \mathbf{a}) \times H_1(\mathbf{Z}, \mathbf{a}) \to \frac{1}{2}\mathbb{Z}$, be*

$$L(\alpha_1, \alpha_2) = m(\partial(\alpha_1), \alpha_2),$$

where ∂ is the connecting homomorphism in homology.

The group $\mathrm{Gr}(\mathcal{Z})$ is defined as a central extension of $H_1(\mathbf{Z}, \mathbf{a})$ by $\frac{1}{2}\mathbb{Z}$ in the following way.

Definition 2.4.3. *Let* $\mathrm{Gr}(\mathcal{Z})$ *be the set* $\frac{1}{2}\mathbb{Z} \times H_1(\mathbf{Z}, \mathbf{a})$*, with multiplication*

$$(a_1, \alpha_1) \cdot (a_2, \alpha_2) = (a_1 + a_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2).$$

For an element $g = (a, \alpha) \in \mathrm{Gr}(\mathcal{Z})$ *we call* a *the* Maslov component*, and* α *the* homological component *of* g*.*

Note that if $\mathbf{Z}$ has just one component Z_1 and $|Z_1| = n$, then this grading group is the same as the group $G'(n)$ defined in [LOT09, Section 3]. In general, if $\mathbf{Z} = \{Z_1, \ldots, Z_l\}$, as a set

$$G'(|Z_1|) \times \cdots \times G'(|Z_l|) \cong \left(\frac{1}{2}\mathbb{Z}\right)^l \times H_1(\mathbf{Z}, \mathbf{a}),$$

since $H_1(Z_1, \mathbf{a} \cap Z_1) \oplus \cdots \oplus H_1(Z_l, \mathbf{a} \cap Z_l) \cong H_1(\mathbf{Z}, \mathbf{a})$. Adding the Maslov components together induces a surjective homomorphism

$$\sigma \colon G'(|Z_1|) \times \cdots \times G'(|Z_l|) \to \mathrm{Gr}(\mathcal{Z}).$$

We can now define the grading $\mathrm{gr} \colon \mathcal{A}(|Z_1|, \ldots, |Z_l|) \to \mathrm{Gr}(\mathcal{Z})$.

Definition 2.4.4. *For an element* $a = (S, T, \phi)$ *of* $\mathcal{A}(|Z_1|, \ldots, |Z_l|)$*, set*

$$\iota(a) = \mathrm{inv}(\phi) - m(S, [a]),$$
$$\mathrm{gr}(a) = (\iota(a), [a]).$$

Breaking up a into its components $a = (a_1, \ldots, a_l) \in \mathcal{A}(|Z_1|) \oplus \cdots \oplus \mathcal{A}(|Z_l|)$, we see that $\mathrm{gr}(a) = \sigma(\mathrm{gr}'(a_1), \ldots, \mathrm{gr}'(a_l))$.

Therefore, we can apply the results about G' and gr' from [LOT09] to deduce the following proposition.

Proposition 2.4.5. *The function* gr *is indeed a grading on* $\mathcal{A}(|Z_1|, \ldots, |Z_l|)$*, with the same properties as* G' *on* $\mathcal{A}(n)$*. Namely, the following statements hold.*

1. *Under* gr*,* $\mathcal{A}(|Z_1|, \ldots, |Z_l|)$ *is a differential graded algebra, where the differential drops the grading by the central element* $\lambda = (1, 0)$*.*

2. *For any completable collection of Reeb chords $\boldsymbol{\rho}$, the grading of $a(\boldsymbol{\rho}, s)$ as an element of $\mathcal{A}(|Z_1|, \dots, |Z_l|)$ does not depend on the completion s.*

3. *For any completable collection $\boldsymbol{\rho}$, the element $a(\boldsymbol{\rho})$ is homogeneous.*

4. *The grading gr descends to $\mathcal{A}(\mathcal{Z})$.*

Proof. The proof of (1) follows from the corresponding statement for gr', after noticing that the differential on $\mathcal{A}(|Z_1|, \dots, |Z_l|)$ is defined via the Leibniz rule, and the differentials on the individual components drop one Maslov component by 1, while keeping all the rest fixed.

The rest of the statements then follow analogously to those for gr' in [LOT09]. □

2.5 Reduced grading

We can now define the refined grading group $\underline{\mathrm{Gr}}(\mathcal{Z})$. Recall that the surface $\mathcal{F}(\mathcal{Z})$ retracts to the graph $G(\mathcal{Z})$, consisting of the segments $\mathbf{Z}$, and the arcs $E = \{e_1, \dots, e_k\}$, such that $\mathbf{Z} \cap E = \mathbf{a}$. From the long exact sequence for the pair (G, E) we know that the following piece is exact.

$$0 \to H_1(G) \to H_1(G, E) \to H_0(E)$$

The differential $\partial\colon H_1(G, E) \to H_0(E)$ can be identified with the composition $M_* \circ \partial\colon H_1(\mathbf{Z}, \mathbf{a}) \to H_0(E)$, and $H_1(F) = H_1(G)$ can be identified with $\ker \partial \subset H_1(\mathbf{Z}, \mathbf{a})$. The identification can also be seen by adding the arcs e_i to cycles in $(\mathbf{Z}, \mathbf{a})$ to obtain cycles in $G = \mathbf{Z} \cup E$. This induces a map $\partial'\colon\ \mathrm{Gr}(\mathcal{Z}) \to H_0(E)$, and the kernel $\underline{\mathrm{Gr}}(\mathcal{Z}) = \ker \partial'$ is just the subgroup of $\mathrm{Gr}(\mathcal{Z})$, consisting of elements with homological component in $\ker \partial \cong H_1(F)$.

Proposition 2.5.1. *Under the identification $\ker \partial = H_1(F)$, the group $\underline{\mathrm{Gr}}(\mathcal{Z})$ can be explicitly described as a central extension of $H_1(F)$ by $\frac{1}{2}\mathbb{Z}$, with multiplication law*

$$(a_1, [\alpha_1]) \cdot (a_2, [\alpha_2]) = (a_1 + a_2 + \#(\alpha_1 \cap \alpha_2), [\alpha_1] + [\alpha_2]),$$

where $a_1, a_2 \in \frac{1}{2}\mathbb{Z}$, and α_1 and α_2 are curves in F, and $\#(\alpha_1 \cap \alpha_2)$ is the signed intersection number, according to the orientation of F.

Proof. First, notice that the intersection pairing is well-defined, as it is, via Poincáre duality, just the pairing $\langle \cdot \cup \cdot, [F, \partial F]\rangle$ on $H^1(F, \partial F)$. The remaining step is to show that under the identification $\ker \partial = H_1(F)$, this agrees with the pairing L on $H_1(\mathbf{Z}, \mathbf{a})$. This can be seen by starting with line segments on $\mathbf{Z}$ and arcs in E, pushing the arcs on E away from each other in the $2^{\#E}$ possible ways. One can then count that ± 1 contributions to L always give rise to an intersection point, while $\pm 1/2$ contributions create an intersection point exactly half of the time. □

In fact, for any generator $a \in \mathcal{A}(\mathcal{Z})$ with starting and ending idempotents I_s and I_e, respectively, $\partial'(\mathrm{gr}(a)) = I_e - I_s$, if we think of the idempotents as linear combinations of the e_i. Therefore, for any a with $I_e = I_s$, $\mathrm{gr}(a)$ is already in $\underline{\mathrm{Gr}}$, and in general it is "almost" there. At this point we would like to find a retraction $\mathrm{Gr} \to \underline{\mathrm{Gr}}$ and use this to define the refined grading. However this fails even in simple cases. For instance, when $\mathcal{Z}$ is an arc diagram for a disc with several sutures, $\underline{\mathrm{Gr}}(\mathcal{Z}) = \frac{1}{2}\mathbb{Z}$ is abelian, as $H_1(F)$ vanishes, while the commutator of $\mathrm{Gr}(\mathcal{Z})$ is $\mathbb{Z} \subset \underline{\mathrm{Gr}}(\mathcal{Z})$, and there can be no retraction, even if we pass to $\mathbb{D}$–coefficients.

The solution is to assign a grading to $\mathcal{A}(\mathcal{Z})$ with values in $\underline{\mathrm{Gr}}(\mathcal{Z})$, depending on the starting and ending idempotents. First, note that the generating idempotents come in sets of *connected components*, where I is connected to J if and only if $I - J$ is in the image of ∂', or equivalently in the kernel of $H_0(E) \to H_0(F)$. These connected components correspond to the possible choices of how many arcs are occupied in each connected component of $F(\mathcal{Z})$.

Definition 2.5.2. *A* grading reduction *r for $\mathcal{Z}$ is a choice of a* base idempotent *I_0 in each connected component, and a choice $r(I) \in \partial'^{-1}(I - I_0)$ for any $I \in [I_0]$.*

Definition 2.5.3. *Given a grading reduction r, define the* reduced grading

$$\underline{\mathrm{gr}}_r(a) = r(I_s) \cdot \mathrm{gr}(a) \cdot r(I_e)^{-1} \in \underline{\mathrm{Gr}}(\mathcal{Z}),$$

for any generator $a \in \mathcal{A}(\mathcal{Z})$ with starting and ending idempotents I_s and I_e, respectively. When unambiguous, we write simply $\underline{\mathrm{gr}}(a)$.

For any elements a and b, such that $a \cdot b$, or even $a \otimes b$ is nonzero, the r–terms in $\underline{\mathrm{gr}}$ cancel, and $\underline{\mathrm{gr}}(a \otimes b) = \underline{\mathrm{gr}}(a \cdot b) = \underline{\mathrm{gr}}(a) \cdot \underline{\mathrm{gr}}(b)$. Since $\left\langle \frac{1}{2}\mathbb{Z}, 0 \right\rangle$ is in the center, there is still a well-defined $\mathbb{Z}$–action by $\lambda = \langle 1, 0 \rangle$, and $\underline{\mathrm{gr}}(\partial a) = \lambda^{-1}\underline{\mathrm{gr}}(a)$. Therefore, $\underline{\mathrm{gr}}$ is indeed a grading.

Notice that for any a with $I_s = I_e$, $\underline{\mathrm{gr}}(a)$ is the conjugate of $\mathrm{gr}(a) \in \underline{\mathrm{Gr}}$ by $r(I_s)$. In particular, the homological part of the grading is unchanged, and whenever it vanishes, the Maslov component is also unchanged.

Remark. Given a set of Reeb chords $\boldsymbol{\rho}$, the element $a(\boldsymbol{\rho}) \in \mathcal{A}(\mathcal{Z})$ is no longer homogeneous under $\underline{\mathrm{gr}}$. Indeed, $\underline{\mathrm{gr}}(a(\boldsymbol{\rho}, s))$ depends on the completion s.

2.6 Orientation reversals

It is sometimes useful to compare the arc diagrams $\mathcal{Z}$ and $-\mathcal{Z}$ and the corresponding gradings. Recall that $-\mathcal{Z}$ and $\mathcal{Z}$ differ only by the orientation of $\mathbf{Z}$. Consequently, the homology components $H_1(\pm\mathbf{Z}, \mathbf{a})$ in $\mathrm{Gr}(\pm\mathcal{Z})$ can be identified, while their canonical bases are opposite in order and sign. In particular, the pairings $L_{\pm\mathcal{Z}}$ are opposite from each other. Therefore $\mathrm{Gr}(\mathcal{Z})$ and $\mathrm{Gr}(-\mathcal{Z})$ are anti-isomorphic, via the map fixing both the Maslov and homological components.

Similarly, $F(\pm\mathcal{Z})$ differ only in orientation, the homological components $H_1(F)$ can be naturally identified while the intersection pairings are opposite from each other. Thus $\underline{\mathrm{Gr}}(\mathcal{Z})$ and $\underline{\mathrm{Gr}}(-\mathcal{Z})$ are also anti-homomorphic, via the map that fixes both components, which agrees with the restriction of the corresponding map on $\mathrm{Gr}(\mathcal{Z})$.

Thus, left actions by $\mathrm{Gr}(\mathcal{Z})$ or $\underline{\mathrm{Gr}}(\mathcal{Z})$ naturally correspond to right actions by $\mathrm{Gr}(-\mathcal{Z})$ or $\underline{\mathrm{Gr}}(-\mathcal{Z})$, respectively, and vice versa.

Chapter 3

Bordered sutured 3–manifolds

In this section—and for most of the rest of the thesis—we will be working from the point of view of bordered sutured manifolds, as sutured manifolds with extra structure. We will largely avoid the alternative description of decorated sutured cobordisms.

3.1 Sutured manifolds

Definition 3.1.1. *A* divided surface (S, Γ) *is a closed surface F, together with a collection $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of pairwise disjoint oriented simple closed curves on F, called* sutures*, satisfying the following conditions.*

Every component B of $F \setminus \Gamma$ has nonempty boundary (which is the union of sutures). Moreover, the boundary orientation and the suture orientation of ∂B either agree on all components, in which case we call B a positive region*, or they disagree on all components, in which case we call B a* negative region*. We denote by $R_+(\Gamma)$ or R_+ (respectively $R_-(\Gamma)$ or R_-) the closure of the union of all positive (negative) regions.*

Notice that the definition doesn't require F to be connected, but it requires that each component contain a suture.

Definition 3.1.2. *A divided surface (F, Γ) is called* balanced *if $\chi(R_+) = \chi(R_-)$.*

*It is called k–*unbalanced *if $\chi(R_+) = \chi(R_-) + 2k$, where k could be positive, negative or 0. In particular 0–unbalanced is the same as balanced.*

Notice that since F is closed, and $\chi(S) = \chi(R_+) + \chi(R_-)$, it follows that $\chi(R_+) - \chi(R_-)$ is always even.

Now we can express the balanced sutured manifolds of [Juh06] in terms of divided surfaces.

Definition 3.1.3. *A* balanced sutured manifold (Y, Γ) *is a 3–manifold Y with no closed components, such that $(\partial Y, \Gamma)$ is a balanced divided surface.*

We can extend this definition to the following.

Definition 3.1.4. *A k*–unbalanced sutured manifold (Y, Γ) *is a 3–manifold Y with no closed components, such that $(\partial Y, \Gamma)$ is a k–unbalanced divided surface.*

Although our unbalanced sutured manifolds are more general than the balanced ones of Juhász, they are still strictly a subclass of Gabai's general definition in [Gab83]. For example, he allows toric sutures, while we do not.

3.2 Bordered sutured manifolds

In this section we describe how to obtain a bordered sutured manifold from a sutured manifold, by parametrizing part of its boundary.

Definition 3.2.1. *A* bordered sutured manifold $(Y, \Gamma, \mathcal{Z}, \phi)$ *consists of the following.*

1. *A sutured manifold (Y, Γ).*

2. *An arc diagram $\mathcal{Z}$.*

3. *An orientation preserving embedding $\phi\colon G(\mathcal{Z}) \hookrightarrow \partial Y$, such that $\phi|_{\mathbf{Z}}$ is an orientation preserving embedding into Γ, and $\phi(G(\mathcal{Z}) \setminus \mathbf{Z}) \cap \Gamma = \varnothing$. It follows that each arc e_i embeds in R_-.*

Note that a closed neighborhood $\overline{\nu(G(\mathcal{Z}))} \subset \partial Y$ can be identified with the parametrized surface $F(\mathcal{Z})$. We will make this identification from now on.

An equivalent way to give a bordered sutured manifold would be to specify an embedding $F(\mathcal{Z}) \hookrightarrow \partial Y$, such that the following conditions hold. Each 0–handle of F intersects Γ in a single arc, while each 1–handle is embedded in $\mathrm{Int}(R_-(\Gamma))$.

Proposition 3.2.2. *Any bordered sutured manifold* $(Y, \Gamma, \mathcal{Z}, \phi)$ *satisfies the following condition, called* homological linear independence*.*

$$\pi_0(\Gamma \setminus \phi(\mathbf{Z})) \to \pi_0(\partial Y \setminus F(\mathcal{Z})) \text{ is surjective.} \tag{3.1}$$

Proof. Indeed, Eq. (3.1) is equivalent to Γ intersecting any component of $\partial Y \setminus F$. But Γ already intersects any component of ∂Y. Any component of $\partial Y \setminus F$ is either a component of ∂Y, or has common boundary with F. The non-degeneracy condition on $\mathcal{Z}$ guarantees that any component of ∂F hits Γ. □

Remark. If we want to work with degenerate arc diagrams (which give rise to degenerate sutured surfaces) we can still get well-defined invariants, as long as we impose homological linear independence on the manifolds. However, in that case there is no category, since the identity cobordism from a degenerate sutured surface to itself *does not* satisfy homological linear independence.

3.3 Gluing

We can glue two bordered sutured manifolds to obtain a sutured manifold in the following way.

Let $(Y_1, \Gamma_1, \mathcal{Z}, \phi_1)$, and $(Y_2, \Gamma_2, -\mathcal{Z}, \phi_2)$ be two bordered sutured manifolds. Since ϕ_1 and ϕ_2 are embeddings, and $G(-\mathcal{Z})$ is naturally isomorphic to $G(\mathcal{Z})$ with its orientation reversed, there is a diffeomorphism $\phi_1(G(\mathcal{Z})) \to \phi_2(G(-\mathcal{Z}))$ that can be extended to an orientation reversing diffeomorphism $\psi\colon F(\mathcal{Z}) \to F(-\mathcal{Z})$ of their neighborhoods. Moreover, $\psi|_{\Gamma_1}\colon \Gamma_1 \cap F(\mathcal{Z}) \to \Gamma_2 \cap F(-\mathcal{Z})$ is orientation reversing.

Set $Y = Y_1 \cup_\psi Y_2$, and $\Gamma = (\Gamma_1 \setminus F(\mathcal{Z})) \cup (\Gamma_2 \setminus F(-\mathcal{Z}))$. By homological linear independence on Y_1 and Y_2, the sutures Γ on Y intersect all components of ∂Y, and (Y, Γ) is a sutured manifold.

More generally, we can do partial gluing. Suppose $(Y_1, \Gamma_1, \mathcal{Z}_0 \cup \mathcal{Z}_1, \phi_1)$ and $(Y_2, \Gamma_2, -\mathcal{Z}_0 \cup \mathcal{Z}_2, \phi_2)$ are bordered sutured. Then

$$(Y_1 \cup_{F(\mathcal{Z}_0)} Y_2, (\Gamma_1 \setminus F(\mathcal{Z}_0)) \cup (\Gamma_2 \setminus F(-\mathcal{Z}_0)), \mathcal{Z}_1 \cup \mathcal{Z}_2, \phi_1|_{G(\mathcal{Z}_1)} \cup \phi_2|_{G(\mathcal{Z}_2)})$$

is also bordered sutured.

Chapter 4

Heegaard diagrams

4.1 Diagrams and compatibility with manifolds

Definition 4.1.1. *A* bordered sutured Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}, \psi)$ *consists of the following data:*

1. *A surface with boundary* Σ.
2. *An arc diagram* $\mathcal{Z}$.
3. *An orientation reversing embedding* $\psi\colon G(\mathcal{Z}) \hookrightarrow \Sigma$, *such that* $\psi|_{\mathbf{Z}}$ *is an orientation preserving embedding into* $\partial\Sigma$, *while* $\psi|_{G(\mathcal{Z})\setminus\mathbf{Z}}$ *is an embedding into* $\mathrm{Int}(\Sigma)$.
4. *The collection* $\boldsymbol{\alpha}^a = \{\alpha_1^a, \dots, \alpha_k^a\}$ *of arcs* $\alpha_i^a = \psi(e_i)$.
5. *A collection of simple closed curves* $\boldsymbol{\alpha}^c = \{\alpha_1^c, \dots, \alpha_n^c\}$ *in* $\mathrm{Int}(\Sigma)$, *which are disjoint from each other and from* $\boldsymbol{\alpha}^a$.
6. *A collection of simple closed curves* $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_m\}$ *in* $\mathrm{Int}(\Sigma)$, *which are pairwise disjoint and transverse to* $\boldsymbol{\alpha} = \boldsymbol{\alpha}^a \cup \boldsymbol{\alpha}^c$.

We also require that $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \to \pi_0(\Sigma \setminus \boldsymbol{\alpha})$ *and* $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \to \pi_0(\Sigma \setminus \boldsymbol{\beta})$ *be surjective. We call this condition* homological linear independence *since it is equivalent to each of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *being linearly independent in* $H_1(\Sigma, \mathbf{Z})$.

Homological linear independence on diagrams is the key condition required for admissibility and avoiding boundary degenerations.

Definition 4.1.2. *A* boundary compatible Morse function *on a bordered sutured manifold* $(Y, \Gamma, \mathcal{Z}, \phi)$ *is a self-indexing Morse function* $f\colon Y \to [-1, 4]$ *(with an implicit choice of Riemannian metric* g*) with the following properties.*

1. *The parametrized surface* $F(\mathcal{Z}) = \overline{\nu(G(\mathcal{Z}))}$ *is totally geodesic,* ∇f *is parallel to* F*, and* $f|_F$ *is a* $\mathcal{Z}$*–compatible Morse function.*
2. *A closed neighborhood* $N = \overline{\nu(\Gamma \setminus \mathcal{Z})}$ *is isotopic to* $(\Gamma \setminus \mathcal{Z}) \times [-1, 4]$*, such that* f *is projection on the second factor (and* $f(\Gamma) = 3/2$*).*
3. $f^{-1}(-1) = \overline{R_-(\Gamma) \setminus (N \cup F)}$*, and* $f^{-1}(4) = \overline{R_+(\Gamma) \setminus (N \cup F)}$*.*
4. f *has no index–0 or index–3 critical points.*
5. *The are no critical points in* $\partial Y \setminus F$*, and the index–1 critical points for* F *are also index–1 critical points for* Y*.*

See Figure 8a for a schematic illustration.

From a boundary compatible Morse function f we can get a bordered sutured Heegaard diagram by setting $\Sigma = f^{-1}(3/2)$, and letting $\boldsymbol{\alpha}$ be the intersection of the stable manifolds of the index–1 critical points with Σ, and $\boldsymbol{\beta}$ be the intersection of the unstable manifolds of the index–2 critical points with Σ. Note that the internal critical points give $\boldsymbol{\alpha}^c$ and $\boldsymbol{\beta}$, while the ones in $F \subset \partial Y$ give $\boldsymbol{\alpha}^a$. We notice that $\mathbf{Z} \subset F \cap \Sigma$ and $\boldsymbol{\alpha}^a$ form an embedding $\psi\colon G(\mathcal{Z}) \to \Sigma$. Homological linear independence for the diagram follows from that of manifold.

Definition 4.1.3. *A diagram as above is called a* compatible bordered sutured Heegaard diagram *to* f*.*

Proposition 4.1.4. *Compatible diagrams and boundary compatible Morse functions are in a one-to-one correspondence.*

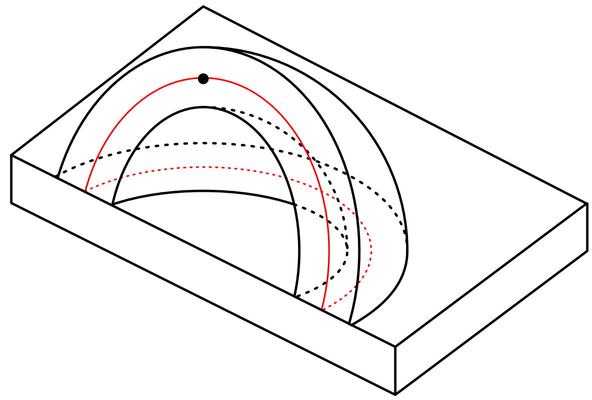


Figure 7: Half of a 2–handle attached along an arc. Its critical point and two incoming gradient flow lines are in the boundary.

Proof. We need to give an inverse construction. Start with a bordered sutured diagram $\mathcal{H}$, and construct a bordered sutured manifold in the following way. To $\Sigma \times [1,2]$ attach 2–handles at $\boldsymbol{\alpha}_i^c \times \{1\}$, and at $\boldsymbol{\beta}_i \times \{2\}$. Finally, at $\boldsymbol{\alpha}_i^a \times \{1\}$ attach "halves of 2–handles". These are thickened discs $D^2 \times [0,1]$ attached along an arc $a \times \{1/2\} \subset \partial D^2 \times \{1/2\}$. (See Figure 7.) Then Γ is $\partial\Sigma \times \{3/2\}$, and $F(\mathcal{Z})$ is $\mathbf{Z} \times [1,2]$, together with the "middles" of the partial handles, i.e. $(\partial D^2 \setminus a) \times [0,1]$. To such a handle decomposition on the new manifold Y corresponds a canonical boundary compatible Morse function f. Note that attaching the half-handles has no effect topologically, but adds boundary critical points. □

Proposition 4.1.5. *Any sutured bordered manifold has a compatible diagram in the above sense. Moreover, any two compatible diagrams can be connected by a sequence of moves of the following types:*

1. *Isotopy of the circles in $\boldsymbol{\alpha}^c$ and $\boldsymbol{\beta}$, and isotopy, relative to the endpoints, of the arcs in $\boldsymbol{\alpha}^a$.*

2. *Handleslide of a circle in $\boldsymbol{\beta}$ over another circle in $\boldsymbol{\beta}$.*

3. *Handleslide of any curve in $\boldsymbol{\alpha}$ over a circle in $\boldsymbol{\alpha}^c$.*

4. *Stabilization.*

Proof. For the proof of this proposition we will modify our definition of a compatible Morse function, to temporarily "forget" about F.

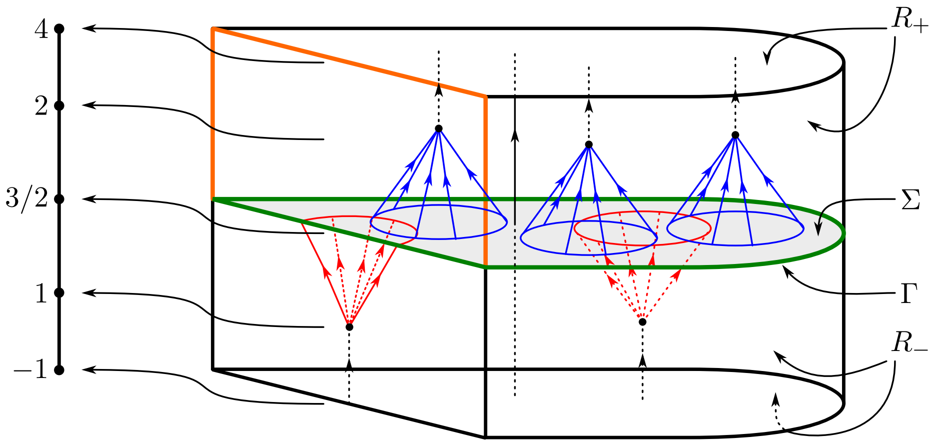
A *pseudo boundary compatible Morse function* f for the bordered sutured manifold $(Y, \Gamma, \mathcal{Z}, \phi)$ is a boundary compatible Morse function for the manifold $(Y, \Gamma, \varnothing, \varnothing \hookrightarrow \partial Y)$ (which is just a standard Morse function for the sutured manifold (Y, Γ), in the sense of [Juh06]), with some additional conditions. Namely, we require that $f^{-1}([-1, 3/2]) \cap \phi(e_i)$ consist of two arcs (at the endpoints of $\phi(e_i)$), tangent to ∇f. We also require that $\phi(G(\mathcal{Z}))$ be disjoint from the unstable manifolds of index–1 critical points.

Such Morse functions are in 1–to–1 correspondence with compatible diagrams by the following construction. As usual, $\Sigma = f^{-1}(3/2)$, while $\boldsymbol{\alpha}^c$ and $\boldsymbol{\beta}$ are the intersections of Σ with stable, respectively unstable, manifolds for index–1 and index–2 critical points. On the other hand, $\boldsymbol{\alpha}_i^a$ is the intersection of Σ with the gradient flow from e_i. Since the flow avoids index–1 critical points, $\boldsymbol{\alpha}^a$ is disjoint from $\boldsymbol{\alpha}^c$. See Figure 8 for a comparison between the two types of Morse functions.

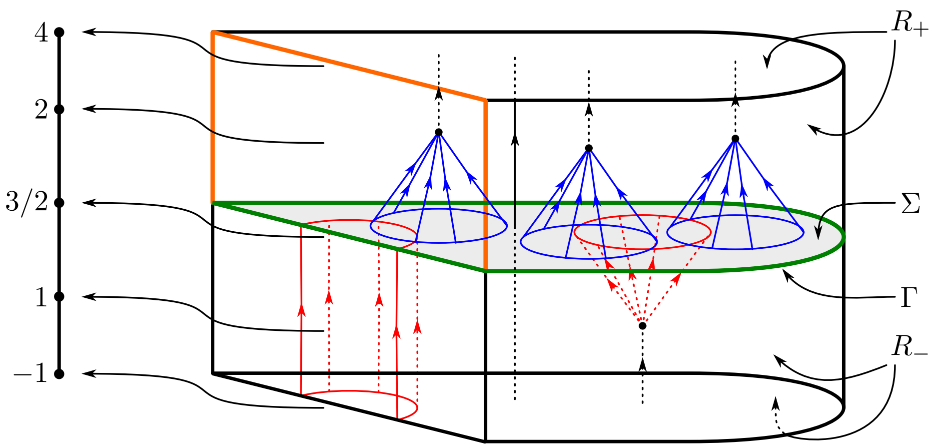
The backwards construction is the same as for true boundary-compatible Morse functions, except we do not attach the half 2–handles at $\boldsymbol{\alpha}^a \times \{1\}$, and instead just set $e_i = \alpha_i^a \times \{1\} \cup \partial\alpha_i^a \times [1, 3/2]$.

This alternative construction allows us to use standard results about sutured manifolds. In particular, [Juh06, Propositions 2.13—2.15] imply that (Y, Γ) has a compatible Morse function, and hence Heegaard diagram, and any two compatible diagrams are connected by Heegaard moves. Namely, there is a family f_t of Morse functions, which for generic t corresponds to an isotopy, and for a finite number of critical points corresponds to a index–1, index–2 critical point creation, (i.e. stabilization of the diagram), or a flowline between critical points of the same index (handleslides between circles in $\boldsymbol{\alpha}^c$ or between circles in $\boldsymbol{\beta}$).

Since the stable manifold of any index–1 critical point intersects R_- at a pair of points, we can always perturb f to get a pseudo-compatible diagram for $(Y, \Gamma, \mathcal{Z}, \phi)$. Any two such diagrams are connected by a sequence of sutured Heegaard moves (ignoring $\boldsymbol{\alpha}^a$). For generic t, a sutured compatible f_t is also pseudo bordered sutured compatible. At non-generic t, there is a flow from some point on e_i to an index–1 critical point. This corresponds to sliding



(a) A true boundary compatible Morse function. There is one boundary critical point giving rise to $\boldsymbol{\alpha}^a$.



(b) A pseudo boundary compatible Morse function. There is one arc in $f^{-1}(-1)$ giving rise to $\boldsymbol{\alpha}^a$.

Figure 8: Comparison of a boundary compatible and pseudo boundary compatible Morse functions. Several internal critical points are given in each, with gradient flowlines, giving rise to $\boldsymbol{\alpha}^c$ and $\boldsymbol{\beta}$.

α_i over the corresponding circle in $\boldsymbol{\alpha}^c$, so we must add those to the list of allowed Heegaard moves. $\square$

4.2 Generators

Definition 4.2.1. *A* generator *for a bordered sutured diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a collection $\mathbf{x} = (x_1, \dots, x_g)$ of intersection points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$, such that there is exactly one point on each α^c circle, exactly one point on each β circle, and at most one point on each α^a arc.*

The set of all generators for $\mathcal{H}$ is denoted $\mathcal{G}(\mathcal{H})$ or $\mathcal{G}$.

As a degenerate case, when $\#\boldsymbol{\beta} = \#\boldsymbol{\alpha}^c = 0$, we will let $\mathcal{G}$ contain a single element, which is the empty collection $\mathbf{x} = ()$.

Notice that if $\mathcal{G}$ is nonempty, then necessarily $g = \#\boldsymbol{\beta} \geq \#\boldsymbol{\alpha}^c$. We call g the *genus* of $\mathcal{H}$. Moreover, exactly $p = g - \#\boldsymbol{\alpha}^c$ many of the α^a arcs are occupied by each generator. Let $o(\mathbf{x}) \subset \{1, \dots, k\}$ denote the set of occupied α^a arcs, and $\overline{o}(\mathbf{x}) = \{1, \dots, k\} \setminus o(\mathbf{x})$ denote the set of unoccupied arcs.

Remark. If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a bordered sutured diagram compatible with a p–unbalanced bordered sutured manifold, then exactly p many α^a arcs are occupied by each generator for $\mathcal{H}$.

Indeed, let $g = \#\boldsymbol{\beta}$, and $h = \#\boldsymbol{\alpha}$. By the construction of a compatible manifold, $R_-(\Gamma)$ is diffeomorphic to Σ after surgery at each α^c circle, while $R_+(\Gamma)$ is diffeomorphic to Σ after surgery at each β circle. But surgery on a surface at a closed curve increases its Euler characteristic by 2. Therefore, the manifold is $(g - h)$–unbalanced.

4.3 Homology classes

Later we will look at pseudoholomorphic curves that go "between" two generators. We can classify such curves into homology classes as follows.

Definition 4.3.1. *For given generators* $\mathbf{x}$ *and* $\mathbf{y}$*, the* homology classes *from* $\mathbf{x}$ *to* $\mathbf{y}$*, denoted by* $\pi_2(\mathbf{x},\mathbf{y})$*, are the elements of*

$$H_2(\Sigma\times[0,1]\times[0,1],(\boldsymbol{\alpha}\times\{1\}\times[0,1])\cup(\boldsymbol{\beta}\times\{0\}\times[0,1])\\ \cup(\mathbf{Z}\times[0,1]\times[0,1])\cup(\mathbf{x}\times[0,1]\times\{0\})\cup(\mathbf{y}\times[0,1]\times\{1\})),$$

which map to the relative fundamental class of $\mathbf{x}\times[0,1]\cup\mathbf{y}\times[0,1]$ *under the boundary homomorphism, and collapsing the rest of the boundary.*

There is a product map $*\colon \pi_2(\mathbf{x},\mathbf{y})\times\pi_2(\mathbf{y},\mathbf{z})\to\pi_2(\mathbf{x},\mathbf{z})$ given by concatenation at $\mathbf{y}\times[0,1]$. This product turns $\pi_2(\mathbf{x},\mathbf{x})$ into a group, called the group of *periodic classes at* $\mathbf{x}$.

Definition 4.3.2. *The* domain *of a homology class* $B\in\pi_2(\mathbf{x},\mathbf{y})$ *is the image*

$$[B]=\pi_{\Sigma*}(B)\in H_2(\Sigma,\mathbf{Z}\cup\boldsymbol{\alpha}\cup\boldsymbol{\beta}).$$

We interpret it as a linear combination of regions in $\Sigma\setminus(\boldsymbol{\alpha}\cup\boldsymbol{\beta})$*. We call the coefficient of such a region in a domain* D *its* multiplicity*.*

The domain of a periodic class is a periodic domain*.*

We can split the boundary $\partial[B]$ into pieces $\partial^\partial B\subset\mathbf{Z},\partial^\alpha B\subset\boldsymbol{\alpha}$, and $\partial^\beta B\subset\boldsymbol{\beta}$. We can interpret $\partial^\partial B$ as an element of $H_1(\mathbf{Z},\mathbf{a})$.

Definition 4.3.3. *The set of* provincial homology classes *from* $\mathbf{x}$ *to* $\mathbf{y}$ *is the kernel* $\pi_2^\partial(\mathbf{x},\mathbf{y})$ *of* $\partial^\partial\colon\ \pi_2(\mathbf{x},\mathbf{y})\to H_1(\mathbf{Z},\mathbf{a})$*.*

The periodic classes in $\pi_2^\partial(\mathbf{x},\mathbf{x})$ *are* provincial periodic class *and their domains are* provincial periodic domains*.*

The groups of periodic classes reduce to the much simpler forms

$$\pi_2(\mathbf{x},\mathbf{x})\cong H_2(\Sigma\times[0,1],\mathbf{Z}\times[0,1]\cup\boldsymbol{\alpha}\times\{0\}\cup\boldsymbol{\beta}\times\{1\}),$$
$$\pi_2^\partial(\mathbf{x},\mathbf{x})\cong H_2(\Sigma\times[0,1],\boldsymbol{\alpha}^c\times\{0\}\cup\boldsymbol{\beta}\times\{1\}).$$

Since 2–handles and half-handles are contractible, these two groups are isomorphic to $H_2(Y,F)$ and $H_2(Y)$, respectively, by attaching the cores of the handles.

4.4 Admissibility

As usual in Heegaard Floer homology, in order to get well defined invariants, we need to impose certain admissibility conditions on the Heegaard diagrams. Like in [LOT09], there are two different notions of admissibility.

Definition 4.4.1. *A bordered sutured Heegaard diagram is called* admissible *if every nonzero periodic domain has both positive and negative multiplicities.*

A diagram is called provincially admissible *if every nonzero provincial periodic domain has both positive and negative multiplicities.*

Proposition 4.4.2. *Any bordered sutured Heegaard diagram can be made admissible by performing isotopy on $\boldsymbol{\beta}$.*

Corollary 4.4.3. *Any bordered sutured 3–manifold has an admissible diagram, and any two admissible diagrams are connected, using Heegaard moves, through admissible diagrams.*

The analogous statement holds for provincially admissible diagrams.

Since admissible diagrams are also provincially admissible, the second part of the argument trivially follows from the first. The first part, on the other hand, follows from Proposition 4.4.2, by taking any sequence of diagrams connected by Heegaard moves, and making all of them admissible, through a consistent set of isotopies.

Proof of Proposition 4.4.2. The proof is analogous to those for bordered manifolds and sutured manifolds. We use the isomorphism from the previous section between periodic domains and $H_2(Y, F)$.

Notice that $H_1(\Sigma, \partial\Sigma \setminus \mathbf{Z})$ maps onto $H_1(Y, \partial Y \setminus F)$, and therefore pairs with $H_2(Y, F)$ and periodic domains. Find a basis for $H_1(\Sigma, \partial\Sigma \setminus \mathbf{Z})$, represented by pairwise disjoint properly embedded arcs $a_1, \ldots, a_m$. We can always do that since every component of Σ hits $\partial\Sigma \setminus \mathbf{Z}$. Cutting Σ along such arcs will give a collection of discs, each of which contains exactly one component of $\mathbf{Z}$ in its boundary.

We can do finger moves of $\boldsymbol{\beta}$ along each a_i, and along a push off b_i of a_i, in the opposite direction. This ensures that there are regions, for which the multiplicities of any periodic domain D are equal to its intersection numbers with a_i and b_i, which have opposite signs. Suppose D has a nonzero region, and pick a point p in such a region. By homological linear independence p can be connected to $\partial\Sigma \setminus \mathbf{Z}$ in the complement of $\boldsymbol{\alpha} \cup \mathbf{Z}$, as well as in the complement of $\boldsymbol{\beta}$. Connecting these paths gives a cycle in $H_1(\Sigma, \partial\Sigma \setminus \mathbf{Z})$, which pairs non trivially with D. Since the a_i span this group, at least one of them pairs non trivially with D, which means D has negative multiplicity in some region. $\square$

4.5 Spin^c–structures

Recall that a Spin^c*–structure* on an n–manifold is a lift of its principal $SO(n)$–bundle to a $\mathrm{Spin}^c(n)$–bundle. For 3–manifolds there is a useful reformulation due to Turaev (see [Tur97]). In this setting, a Spin^c–structure $\mathfrak{s}$ on the 3–manifold Y is a choice of a non vanishing vector field v on Y, up to *homology*. We say that two vector fields are *homologous* if they are homotopic outside of a finite collection of disjoint open balls.

Given a trivialization of TY, a vector field v on Y can be thought of as a map $v\colon Y \to S^2$. This gives an identification of the set $\mathrm{Spin}^c(Y)$ of all Spin^c–structures with $H^2(Y)$ via $\mathfrak{s}(v) \mapsto v^*(\omega)$, where $\omega = \mathrm{PD}([\mathrm{pt}]) \in H^2(S^2)$ is the top-dimensional cohomology class. The identification depends on the trivialization of TY, but only by an overall shift. This means that $\mathrm{Spin}^c(Y)$ is naturally an affine space over $H^2(Y)$.

Given a fixed vector field v_0 on a subspace $X \subset Y$, we can define the space of *relative* Spin^c*–structures* $\mathrm{Spin}^c(Y, X, v_0)$, or just $\mathrm{Spin}^c(Y, X)$ in the following way. A relative Spin^c–structure is a vector field v on Y, such that $v|_X = v_0$, considered up to homology in $Y \setminus X$. If $\mathrm{Spin}^c(Y, X, v_0)$ is nonempty, it is an affine space over $H^2(Y, X)$.

To a Spin^c–structure $\mathfrak{s}$ in $\mathrm{Spin}^c(X)$ or $\mathrm{Spin}^c(Y, X, v_0)$, represented by a vector field v, we can associate its *Chern class* $c_1(\mathfrak{s})$, which is just the first Chern class $c_1(v^\perp)$ of the orthogonal complement subbundle $v^\perp \subset TY$.

With a generator in a Heegaard diagram we will associate two types of Spin^c–structures. Let $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ be a generator. Fix a boundary-compatible Morse function f (and appropriate metric). The vector field ∇f vanishes only at the critical points of f. Each intersection point in $\mathbf{x}$ lies on a gradient trajectory connecting an index–1 to an index–2 critical point. If we cut out a neighborhood of that trajectory, we can modify the vector field inside to one that is non vanishing (the two critical points have opposite parity). For any unoccupied α^a arc, the corresponding critical point is in $F \subset \partial Y$. We can therefore modify the vector field in its neighborhood to be non vanishing. Call the resulting vector field $v(\mathbf{x})$.

Notice that $v_0 = v(\mathbf{x})|_{\partial Y \setminus F} = \nabla f|_{\partial Y \setminus F}$ does not depend on $\mathbf{x}$, while $v(\mathbf{x})|_{\partial Y} = v_{o(\mathbf{x})}$ only depends on $o(\mathbf{x})$. Moreover, under a change of the Morse function or metric (even for different diagrams), v_0 and $v_{o(\mathbf{x})}$ can only vary inside a contractible set. Therefore the corresponding sets $\mathrm{Spin}^c(Y, \partial Y \setminus F, v_0)$ and $\mathrm{Spin}^c(Y, \partial Y, v_{o(\mathbf{x})})$, respectively, are canonically identified. Thus we can talk about $\mathrm{Spin}^c(Y, \partial Y \setminus F)$ and $\mathrm{Spin}^c(Y, \partial Y, o)$, where $o \subset \{1, \ldots, k\}$, as invariants of the underlying bordered sutured manifold. This justifies the following definition.

Definition 4.5.1. *Let $\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}^{\mathrm{rel}}(\mathbf{x})$ be the relative* Spin^c*–structures induced by $v(\mathbf{x})$ in* $\mathrm{Spin}^c(Y, \partial Y \setminus F)$ *and* $\mathrm{Spin}^c(Y, \partial Y, o(\mathbf{x}))$*, respectively.*

We can separate the generators into Spin^c classes. Let

$$\mathcal{G}(\mathcal{H}, \mathfrak{s}) = \{\mathbf{x} \in \mathcal{G}(\mathcal{H}) : \mathfrak{s}(\mathbf{x}) = \mathfrak{s}\},$$
$$\mathcal{G}(\mathcal{H}, o, \mathfrak{s}^{\mathrm{rel}}) = \{\mathbf{x} \in \mathcal{G}(\mathcal{H}) : o(\mathbf{x}) = o, \mathfrak{s}^{\mathrm{rel}}(\mathbf{x}) = \mathfrak{s}^{\mathrm{rel}}\}.$$

The fact that the invariants split by Spin^c structures is due to the following proposition.

Proposition 4.5.2. *The set $\pi_2(\mathbf{x}, \mathbf{y})$ is nonempty if and only if $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y})$. The set $\pi_2^{\partial}(\mathbf{x}, \mathbf{y})$ is nonempty if and only if $o(\mathbf{x}) = o(\mathbf{y})$ and $\mathfrak{s}^{\mathrm{rel}}(\mathbf{x}) = \mathfrak{s}^{\mathrm{rel}}(\mathbf{y})$.*

Proof. This proof is, again, analogous to those for bordered and for sutured manifolds.

To each pair of generators $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$, we associate a homology class $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y, F)$. We do that by picking 1–chains $a \subset \boldsymbol{\alpha}$, and $b \subset \boldsymbol{\beta}$, such that $\partial a = \mathbf{y} - \mathbf{x} + \mathbf{z}$, where $\mathbf{z}$ is a 0–chain in $\mathbf{Z}$, and $\partial b = \mathbf{y} - \mathbf{x}$, and setting $\epsilon(\mathbf{x}, \mathbf{y}) = [a - b]$. We can interpret $a - b$ as a set of properly embedded arcs and circles in (Y, F) containing all critical points.

The vector fields $v(\mathbf{x})$ and $v(\mathbf{y})$ differ only in a neighborhood of $a - b$. One can see that in fact $\mathfrak{s}(\mathbf{y}) - \mathfrak{s}(\mathbf{x}) = \mathrm{PD}([a - b]) = \mathrm{PD}(\epsilon(\mathbf{x}, \mathbf{y}))$. On the other hand, we can interpret $\epsilon(\mathbf{x}, \mathbf{y})$

as an element of

$$H_1(\Sigma \times [0,1], \boldsymbol{\alpha} \times \{0\} \cup \boldsymbol{\beta} \times \{1\} \cup \mathbf{Z} \times [0,1]) \cong \ H_1(Y, F).$$

In particular, $\pi_2(\mathbf{x}, \mathbf{y})$ is nonempty, if and only if there is a 2–chain in $\Sigma \times [0,1]$ with boundary which is a representative for $\epsilon(\mathbf{x}, \mathbf{y})$ in the relative group above. This is equivalent to $\epsilon(\mathbf{x}, \mathbf{y}) = 0 \in H_1(Y, F)$. This proves the first part of the proposition.

The second one follows analogously, noticing that we can pick a path $a - b$, such that $a \subset \boldsymbol{\alpha}$, if and only if $o(\mathbf{x}) = o(\mathbf{y})$, and in that case $\pi_2^{\partial}(\mathbf{x}, \mathbf{y})$ is nonempty if and only if $\epsilon^{\mathrm{rel}}(\mathbf{x}, \mathbf{y}) = [a - b] = 0 \in H_1(Y)$, while $\mathfrak{s}^{\mathrm{rel}}(\mathbf{y}) - \mathfrak{s}^{\mathrm{rel}}(\mathbf{x}) = PD([a - b]) \in H^2(Y, \partial Y)$. $\square$

4.6 Gluing

We can glue bordered sutured diagrams, similar to the way we glue bordered sutured manifolds.

Let $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ and $\mathcal{H}_2 = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ be bordered sutured diagrams for the manifolds $(Y_1, \Gamma_1, \mathcal{Z}, \phi_1)$ and $(Y_2, \Gamma_2, -\mathcal{Z}, \phi_2)$, respectively. We can identify $\mathbf{Z}$ with its embeddings in $\partial\Sigma_1$ and $\partial\Sigma_2$ (one is orientation preserving, the other is orientation reversing).

Let $\Sigma = \Sigma_1 \cup_{\mathbf{Z}} \Sigma_2$. Each α^a arc in $\mathcal{H}_1$ matches up with the corresponding one in $\mathcal{H}_2$ to form a closed curve in Σ. Let $\boldsymbol{\alpha}$ denote the union of all α^c circles in $\mathcal{H}_1$ and $\mathcal{H}_2$, together with the newly formed circles from all α^a arcs. Finally, let $\boldsymbol{\beta} = \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2$.

Proposition 4.6.1. *The diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is compatible with the sutured manifold $Y_1 \cup_{F(\mathcal{Z})} Y_2$, as defined in Chapter 3.3.*

Proof. The manifolds Y_1 and Y_2 are obtained from $\Sigma_1 \times [1,2]$ and $\Sigma_2 \times [1,2]$, respectively, by attaching 2–handles (corresponding to α^c and β circles), and halves of 2–handles (corresponding to α^a arcs). The surface of gluing F can be identified with the union of $\mathbf{Z} \times [1,2]$ with the middles of the half-handles. Thus, we get a base of $(\Sigma_1 \cup_{\mathbf{Z}} \Sigma_2) \times [1,2]$, with the combined 2–handles from each side. In addition the half-handles glue in pairs to form actual 2–handles, each of which is glued along matching α^a arcs. $\square$

Similarly, we can do partial gluing. If we have manifolds $(Y_1, \Gamma_1, \mathcal{Z}_0 \cup \mathcal{Z}_1, \phi_1)$ and $(Y_2, \Gamma_2, -\mathcal{Z}_0 \cup \mathcal{Z}_2, \phi_2)$ with diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $\mathcal{H}_1 \cup_{\mathbf{Z}_0} \mathcal{H}_2$ is a diagram compatible with the bordered sutured manifold $Y_1 \cup_{F(\mathcal{Z}_0)} Y_2$.

4.7 Nice diagrams

As with the other types of Heegaard Floer invariants, the invariants become a lot easier to compute (at least conceptually) if we work in the category of *nice diagrams*, developed originally by Sarkar and Wang in [SW10].

Definition 4.7.1. *A bordered sutured diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}, \psi)$ is* nice *if every region of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ is either adjacent to $\partial\Sigma \setminus \mathbf{Z}$—in which case we call it a* boundary region*—or is one of the following two types:*

- *A bigon, no sides of which are in $\mathbf{Z}$.*
- *A quadrilateral, at most one of whose sides is in $\mathbf{Z}$.*

Proposition 4.7.2. *Any bordered sutured diagram can be made nice by isotopies of $\boldsymbol{\beta}$, handleslides among the circles in $\boldsymbol{\beta}$, and stabilizations.*

Proof. The proof is a combination of those for bordered and sutured manifolds, in [LOT09] and [Juh08], respectively.

First, we make some stabilizations until every component of Σ contains both α and β curves. Next we do finger moves of β curves until any curve in $\boldsymbol{\alpha}$ intersects $\boldsymbol{\beta}$, and vice versa. Then, we ensure all non boundary regions are simply connected. We do that inductively, decreasing the rank of H_1 relative boundary for each region.

Then, following [LOT09], we do finger moves of some β curves along curves parallel to each component of $\mathbf{Z}$ to ensure that all regions adjacent to some Reeb chord in $\mathbf{Z}$ are rectangles (where one side is in $\mathbf{Z}$, two are in $\boldsymbol{\alpha}^a$, and one is in $\boldsymbol{\beta}$).

Finally, we label all regions by their *distance*, i.e. number of β arcs in $\Sigma \setminus \boldsymbol{\alpha}$ one needs to cross, to get to a boundary region, and by their badness (how many extra corners they have).

We do finger moves of a β arc in a bad region through α arcs, until we hit a boundary, a bigon, or another (or the same) bad region. There are several cases depending on what kind of region we hit, but the overall badness of the diagram decreases, so the algorithm eventually terminates. The setup is such that we can never hit a region adjacent to a Reeb chord, so the algorithm for sutured manifolds goes through for bordered sutured manifolds. $\square$

Chapter 5

Moduli spaces of holomorphic curves

In this section we describe the moduli spaces of holomorphic curves involved in the definitions of the bordered invariants and prove the necessary properties. The definitions and arguments are mostly a straightforward generalization of those in [LOT09, Chapter 5].

5.1 Differences with bordered Floer homology

For the reader familiar with border Floer homology we highlight the similarities and the differences with our definitions.

In the bordered setting of Lipshitz, Ozsváth, and Thurston, there is one boundary component and one basepoint on the boundary. One counts pseudoholomorphic discs in $\Sigma \times [0,1] \times \mathbb{R}$, but in practice one thinks of their domains in Σ. Loosely speaking, the curves that do not hit $\partial\Sigma$ correspond to differentials, the ones that do hit the boundary correspond to algebra actions, while the ones that hit the basepoint are not counted at all.

In the bordered sutured setting, the boundary $\partial\Sigma$ has several components, while some subset $\mathbf{Z}$ of $\partial\Sigma$ is distinguished. We again count pseudoholomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$, and again, those curves that do not hit the boundary correspond to differentials. The novel idea is the interpretation of the boundary. Here the algebra action comes from curves that hit any component of $\mathbf{Z} \subset \partial\Sigma$, while the curves that hit any component of $\partial\Sigma \setminus \mathbf{Z}$ are not counted. In a sense, the set $\partial\Sigma \setminus \mathbf{Z}$ plays the role of the basepoint.

With this in mind, most of the constructions in [LOT09] carry over. Below we describe the necessary analytic constructions.

5.2 Holomorphic curves and conditions

We will consider several variations of the Heegaard surface Σ, namely the compact surface with boundary $\overline{\Sigma} = \Sigma$, the open surface $\mathrm{Int}(\Sigma)$, which can be thought of as a surface with several punctures $\mathbf{p} = \{p_1, \dots, p_n\}$, and the closed surface $\Sigma_{\overline{e}}$, obtained by filling in those punctures. Alternatively, it is obtained from $\overline{\Sigma}$ by collapsing all boundary components to points.

We will also be interested in the surface $\mathbb{D} = [0,1] \times \mathbb{R}$, with coordinates $s \in [0,1]$ and $t \in \mathbb{R}$.

Let ω_Σ be a symplectic form on $\mathrm{Int}(\Sigma)$, such that $\partial\Sigma$ is a cylindrical end, and let j_Σ be a compatible almost complex structure. We can assume that $\boldsymbol{\alpha}^a$ is cylindrical near the punctures in the following sense. There is a neighborhood $U_{\mathbf{p}}$ of the punctures, symplectomorphic to $\partial\Sigma \times (0,\infty) \subset T^*(\partial\Sigma)$, such that j_Σ and $\boldsymbol{\alpha}^a \cap U_{\mathbf{p}}$ are invariant with respect to the $\mathbb{R}$–action on $\partial\Sigma \times (0,\infty)$. Let $\omega_{\mathbb{D}}$ and $j_{\mathbb{D}}$ be the standard symplectic form and almost complex structure on $\mathbb{D} \subset \mathbb{C}$.

Consider the projections

$$\begin{aligned} \pi_\Sigma &\colon \ \mathrm{Int}(\Sigma) \times \mathbb{D} \to \mathrm{Int}(\Sigma), \\ \pi_{\mathbb{D}} &\colon \ \mathrm{Int}(\Sigma) \times \mathbb{D} \to \mathbb{D}, \\ s &\colon \ \mathrm{Int}(\Sigma) \times \mathbb{D} \to [0,1], \\ t &\colon \ \mathrm{Int}(\Sigma) \times \mathbb{D} \to \mathbb{R}. \end{aligned}$$

Definition 5.2.1. *An almost complex structure J on* $\mathrm{Int}(\Sigma) \times \mathbb{D}$ *is called* admissible *if the following conditions hold:*

- *$\pi_{\mathbb{D}}$ is J–holomorphic.*
- *$J(\partial_s) = \partial_t$ for the vector fields ∂_s and ∂_t in the fibers of π_Σ.*

- *The $\mathbb{R}$–translation action in the t–coordinate is J–holomorphic.*
- $J = j_\Sigma \times j_{\mathbb{D}}$ *near* $\boldsymbol{p} \times \mathbb{D}$.

Definition 5.2.2. *A* decorated source $S^{\triangleright}$ *consists of the following data:*

- *A topological type of a smooth surface S with boundary, and a finite number of boundary punctures.*
- *A labeling of each puncture by one of "+", "−", or "e".*
- *A labeling of each e puncture by a Reeb chord ρ in* $\mathbf{Z}$.

Given $S^{\triangleright}$ as above, denote by $S_{\overline{e}}$ the surface obtained from S by filling in all the e punctures.

We consider maps

$$u\colon (S, \partial S) \to (\mathrm{Int}(\Sigma) \times \mathbb{D}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R}))$$

satisfying the following conditions:

1. u is (j, J)–holomorphic for some almost complex structure j on S.
2. $u\colon S \to \mathrm{Int}(\Sigma) \times \mathbb{D}$ is proper.
3. u extends to a proper map $u_{\overline{e}}\colon S_{\overline{e}} \to \Sigma_{\overline{e}} \times \mathbb{D}$.
4. $u_{\overline{e}}$ has finite energy in the sense of Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder [BEH+03].
5. $\pi_{\mathbb{D}} \circ u\colon S \to \mathbb{D}$ is a g–fold branched cover. (Recall that g is the cardinality of $\boldsymbol{\beta}$, not the genus of Σ).
6. At each $+$ puncture q of S, $\lim_{z \to q} t \circ u(z) = +\infty$.
7. At each $-$ puncture q of S, $\lim_{z \to q} t \circ u(z) = -\infty$.
8. At each e puncture q of S, $\lim_{z \to q} \pi_\Sigma \circ u(z)$ is the Reeb chord ρ labeling q.

9. $\pi_\Sigma \circ u \colon S \to \mathrm{Int}(\Sigma)$ does not cover any of the regions of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ adjacent to $\partial\Sigma \setminus \mathbf{Z}$.

10. *Strong boundary monotonicity.* For each $t \in \mathbb{R}$, and each $\beta_i \in \boldsymbol{\beta}$, $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of exactly one point. For each $\alpha_i^c \in \boldsymbol{\alpha}^c$, $u^{-1}(\alpha_i^c \times \{1\} \times \{t\})$ consist of exactly one point. For each $\alpha_i^a \in \boldsymbol{\alpha}^a$, $u^{-1}(\alpha_i^a \times \{1\} \times \{t\})$ consists of *at most* one point.

11. u is embedded.

Under conditions (1)–(9), at each $+$ or $-$ puncture, u is asymptotic to an arc $z \times [0,1] \times \{\pm\infty\}$, where z is some intersection point in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$. If in addition we require condition (10), then the intersection points $x_1, \ldots, x_g$ corresponding to $-$ punctures form a generator $\mathbf{x}$, while the ones $y_1, \ldots, y_g$ corresponding to $+$ punctures form a generator $\mathbf{y}$. We call $\mathbf{x}$ the *incoming* generator, and $\mathbf{y}$ the *outgoing* generator for u.

If we compactify the $\mathbb{R}$ component of $\mathbb{D}$ to include $\{\pm\infty\}$, we get a compact rectangle $\widetilde{\mathbb{D}} = [0,1] \times [-\infty, +\infty]$. Let u be a map satisfying conditions (1)–(10), and with incoming and outgoing generators $\mathbf{x}$ and $\mathbf{y}$. Let $\widetilde{S}$ be S with all punctures filled in by arcs. Then u extends to a map

$$\begin{aligned}\widetilde{u} \colon (\widetilde{S}, \partial\widetilde{S}) \to (\Sigma \times \widetilde{\mathbb{D}}, (\boldsymbol{\alpha} \times \{1\} \times [-\infty, +\infty]) \cup (\boldsymbol{\beta} \times \{0\} \times [-\infty, +\infty]) \\ \cup (\mathbf{Z} \times \widetilde{\mathbb{D}}) \cup (\mathbf{x} \times [0,1] \times \{-\infty\}) \cup (\mathbf{y} \times [0,1] \times \{+\infty\})).\end{aligned}$$

Notice that the pair of spaces on the right is the same as the one in Definition 4.3.1. It is clear that a map u satisfying conditions (1)–(10) has an associated homology class $B = [u] = [\widetilde{u}] \in \pi_2(\mathbf{x}, \mathbf{y})$.

We will also impose an extra condition on the height of the e punctures of S.

Definition 5.2.3. *For a map u from a decorated source $S^{\triangleright}$, and an e puncture q on ∂S, the* height *of q is the evaluation* $\mathrm{ev}(q) = t \circ u_{\overline{e}}(q) \in \mathbb{R}$.

Definition 5.2.4. *Let $E(S^{\triangleright})$ be the set of all e punctures in S. Let $\overrightarrow{P} = (P_1, \ldots, P_m)$ be an ordered partition of $E(S^{\triangleright})$ into nonempty subsets. We say u is $\overrightarrow{P}$–*compatible *if for $i = 1, \ldots, m$ all the punctures in P_i have the same height* $\mathrm{ev}(P_i)$*, and moreover* $\mathrm{ev}(P_i) < \mathrm{ev}(P_j)$ *for $i < j$.*

To a partition $\overrightarrow{P} = (P_1, \dots, P_m)$ we can associate a sequence $\overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P}) = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$ of sets of Reeb chords, by setting

$$\boldsymbol{\rho}_i = \{\rho : \rho \text{ labels } q, q \in P_i\}.$$

Moreover, to any such sequence $\overrightarrow{\boldsymbol{\rho}}$ we can associate a homology class

$$[\overrightarrow{\boldsymbol{\rho}}] = [\boldsymbol{\rho}_1] + \cdots + [\boldsymbol{\rho}_m] \in H_1(\mathbf{Z}, \mathbf{a}),$$

and an algebra element

$$a(\overrightarrow{\boldsymbol{\rho}}) = a(\boldsymbol{\rho}_1) \cdots a(\boldsymbol{\rho}_m).$$

It is easy to see that $[a(\overrightarrow{\boldsymbol{\rho}})] = [\overrightarrow{\boldsymbol{\rho}}]$ (unless $s(\overrightarrow{\boldsymbol{\rho}})$ vanishes). It is also easy to see that for a curve u satisfying conditions (1)–(10) with homology class $[u] = B$, and for any partition $\overrightarrow{P}$ we have $[\overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P})] = \partial^\partial B$.

5.3 Moduli spaces

We are now ready to define the moduli spaces that we will consider.

Definition 5.3.1. *Let $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$ be generators, let $B \in \pi_2(\mathbf{x}, \mathbf{y})$ be a homology class, and let $S^\triangleright$ be a decorated source. We will write*

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright)$$

for the space of curves u with source $S^\triangleright$ satisfying conditions (1)–(10), asymptotic to $\mathbf{x}$ at $-\infty$ and to $\mathbf{y}$ at $+\infty$, and with homology class $[u] = B$.

This moduli space is stratified by the possible partitions of $E(S^\triangleright)$. More precisely, given a partition $\overrightarrow{P}$ of $E(S^\triangleright)$, we write

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$$

for the space of $\overrightarrow{P}$–compatible maps in $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright)$, and

$$\widetilde{\mathcal{M}}^B_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$$

for the space of maps in $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$ that also satisfy (11).

Remark. The definitions in the current section are analogous to those in [LOT09], and a lot of the results in that paper carry over without change. We will cite several of them here without proof.

Proposition 5.3.2. *There is a dense set of admissible J with the property that for all generators $\mathbf{x}$, $\mathbf{y}$, all homology classes $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and all partitions $\overrightarrow{P}$, the spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$ are transversely cut out by the $\overline{\partial}$–equations.*

Proposition 5.3.3. *The expected dimension* $\operatorname{ind}(B, S^\triangleright, \overrightarrow{P})$ *of* $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$ *is*

$$\operatorname{ind}(B, S^\triangleright, P) = g - \chi(S) + 2e(B) + \#\overrightarrow{P},$$

where $e(B)$ is the Euler measure of the domain of B.

It turns out that whether the curve $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$ is embedded depends entirely on the topological data consisting of B, $S^\triangleright$, and $\overrightarrow{P}$. That is, there are entire components of embedded and of non embedded curves. Moreover, for such curves there is another index formula that does not depend on $S^\triangleright$. To give it we need some more definitions.

For a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and a point $z \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$, let $n_z(B)$ be the average multiplicity of $[B]$ at the four regions adjacent to z. Let $n_{\mathbf{x}} = \sum_{x \in \mathbf{x}} n_x(B)$, and $n_{\mathbf{y}} = \sum_{y \in \mathbf{y}} n_y(B)$.

For a sequence $\overrightarrow{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m)$, let $\iota(\overrightarrow{\boldsymbol{\rho}})$ be the Maslov component of the grading $\operatorname{gr}(\boldsymbol{\rho}_1) \cdots \operatorname{gr}(\boldsymbol{\rho}_m)$.

Definition 5.3.4. *For a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and a sequence $\overrightarrow{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m)$ of Reeb chords, define the* embedded Euler characteristic *and* embedded index

$$\chi_{\text{emb}}(B, \overrightarrow{\boldsymbol{\rho}}) = g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\overrightarrow{\boldsymbol{\rho}}),$$
$$\operatorname{ind}(B, \overrightarrow{\boldsymbol{\rho}}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + \#\overrightarrow{\boldsymbol{\rho}} + \iota(\overrightarrow{\boldsymbol{\rho}}).$$

Proposition 5.3.5. *Suppose $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$. Exactly one of the following two statements holds.*

1. *u is embedded and the following equalities hold.*

$$\begin{aligned}\chi(S^{\triangleright}) &= \chi_{\mathrm{emb}}(B, \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P})),\\ \mathrm{ind}(B, S^{\triangleright}, \overrightarrow{P}) &= \mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P})),\\ \widetilde{\mathcal{M}}^{B}_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}) &= \widetilde{\mathcal{M}}^{B}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}).\end{aligned}$$

2. *u is not embedded and the following inequalities hold.*

$$\begin{aligned}\chi(S^{\triangleright}) &> \chi_{\mathrm{emb}}(B, \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P})),\\ \mathrm{ind}(B, S^{\triangleright}, \overrightarrow{P}) &< \mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P})),\\ \widetilde{\mathcal{M}}^{B}_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}) &= \varnothing.\end{aligned}$$

Proof. This is essentially a restatement of [LOT09, Proposition 5.47] □

Each of these moduli spaces has an $\mathbb{R}$–action that is translation in the t factor. It is free on each $\widetilde{\mathcal{M}}^{B}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P})$, except when the moduli space consists of a single curve u, where $\pi_{\mathbb{D}} \circ u$ is a trivial g–fold cover of $\mathbb{D}$, and $\pi_{\Sigma} \circ u$ is constant (so $B = 0$). We say that u is *stable* if it is not this trivial case.

For moduli spaces of stable curves we mod out by this $\mathbb{R}$–action:

Definition 5.3.6. *For given $\mathbf{x}$, $\mathbf{y}$, $S^{\triangleright}$, and $\overrightarrow{P}$, set*

$$\begin{aligned}\mathcal{M}^{B}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}) &= \widetilde{\mathcal{M}}^{B}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P})/\mathbb{R},\\ \mathcal{M}^{B}_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}) &= \widetilde{\mathcal{M}}^{B}_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P})/\mathbb{R}.\end{aligned}$$

5.4 Degenerations

The properties of the moduli spaces which are necessary to prove that the invariants are well defined and have the expected properties, are essentially the same as in [LOT09]. Their proofs also carry over with minimal change. We sketch below the most important results.

To study degenerations we first pass to the space of *holomorphic combs* which are trees of holomorphic curves in $\Sigma \times \mathbb{D}$, and ones that live at *East infinity*, i.e. in $\mathbf{Z} \times \mathbb{R} \times \mathbb{D}$. This is the proper ambient space to work in, to ensure compactness.

The possible degenerations that can occur at the boundary of 1–dimensional moduli spaces of embedded curves are of two types. One is a two story holomorphic building, as usual in Floer theory. The second type consists of a single curve u in $\Sigma \times \mathbb{D}$, with another curve degenerating at East infinity, at the e punctures of u. Those curves that can degenerate at East infinity are of several types, *join* curves, *split* curves, and *shuffle* curves, that correspond to certain operations on the algebra $\mathcal{A}(\mathcal{Z})$. In fact, the types of curves that can appear dictate how the algebra should behave.

There are also corresponding gluing results, that tell us that in the cases we care about, a rigid holomorphic comb is indeed the boundary of a 1–dimensional space of curves. Unfortunately, in some cases the compactified moduli spaces are not compact 1–manifolds. However, we can still recover the necessary result that certain counts of 0–dimensional moduli spaces are even, and thus become 0, when reduced to $\mathbb{Z}/2$.

The only place where significant changes need to be made to the arguments, are in ruling out bubbling and boundary degenerations. The reason for the changes are the different homological assumptions we have made for Σ, $\mathbf{Z}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ in the definition of bordered sutured Heegaard diagrams. We give below the precise statement, and the modified proof. The rest of the arguments are essentially local in nature, and do not depend on these homological assumptions.

Proposition 5.4.1. *Suppose $\mathcal{M} = \mathcal{M}^B(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P})$ is 1–dimensional. Then the following types of degenerations cannot occur as the limit u of a sequence u_j of curves in $\mathcal{M}$.*

1. *u bubbles off a closed curve.*
2. *u has a* boundary degeneration*, i.e. u is a nodal curve that collapses one or more properly embedded arcs in $(S, \partial S)$.*

Proof. For (1) notice that if a closed curve bubbles off, it has to map to $\mathrm{Int}(\Sigma) \times \mathbb{D} \simeq \mathrm{Int}(\Sigma)$ which has no closed components. In particular, $H_2(\mathrm{Int}(\Sigma) \times \mathbb{D}) = 0$, and the bubble will have zero energy.

For (2), assume there is such a degeneration u with source $S^{\triangleright\prime}$. Repeating the argument in [LOT09, Lemma 5.37], if an arc $a \in S^{\triangleright}$ collapses in u, then by strong boundary monotonicity its endpoints ∂a lie on the same arc in $\partial\Sigma$. If b is the arc in $\partial S^{\triangleright\prime}$ connecting them, then $t \circ u$ is constant on b. Therefore, $\pi_{\mathbb{D}} \circ u$ is constant on the entire component T of $S^{\triangleright\prime}$ containing b.

There is a compactification $\overline{T}$ of T, filling in the punctures by arcs, and an induced map $\overline{u}\colon \overline{T} \to \Sigma \times \mathbb{D}$. The image of the boundary $\partial\overline{T}$ under $\overline{u}$ is contained in the two sets $\partial_\alpha = (\boldsymbol{\alpha} \cup \mathbf{Z}) \times \{1\} \times \mathbb{R}$ and $\partial_\beta = \boldsymbol{\beta} \times \{0\} \times \mathbb{R}$. Their projections $t(\partial_\alpha)$ and $\pi_{\mathbb{D}}(\partial_\beta)$ are disjoint, while $\pi_{\mathbb{D}} \circ \overline{u}$ is constant on $\overline{T}$. Thus, $\overline{u}(\partial\overline{T})$ is entirely in ∂_α, or entirely in ∂_β. In particular, we have either a map

$$\pi_\Sigma \circ \overline{u}\colon\ (\overline{T}, \partial\overline{T}) \to (\Sigma, \boldsymbol{\alpha} \cup \mathbf{Z}),$$

or a map

$$\pi_\Sigma \circ \overline{u}\colon\ (\overline{T}, \partial\overline{T}) \to (\Sigma, \boldsymbol{\beta}).$$

By homological linear independence both of the groups $H_2(\Sigma, \boldsymbol{\alpha} \cup \mathbf{Z})$ and $H_2(\Sigma, \boldsymbol{\beta})$ are 0, and $u|_T$ must have zero energy. $\square$

The equivalent statement in the bordered setting is necessary for [LOT09, Proposition 5.32].

Chapter 6

Diagram gradings

In this section we define gradings on the set of generators $\mathcal{G}(\mathcal{H})$ for a given bordered sutured diagram $\mathcal{H}$. More precisely, if $\mathcal{H}$ represents the bordered sutured manifold $(Y, \Gamma, \mathcal{Z})$, for each Spin^c–structure $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y \setminus F(\mathcal{Z}))$ we define grading sets $\mathrm{Gr}(\mathcal{H}, \mathfrak{s})$ and $\underline{\mathrm{Gr}}(\mathcal{H}, \mathfrak{s})$ which have left actions by $\mathrm{Gr}(-\mathcal{Z})$ and $\underline{\mathrm{Gr}}(-\mathcal{Z})$, respectively, and right actions by $\mathrm{Gr}(\mathcal{Z})$ and $\underline{\mathrm{Gr}}(\mathcal{Z})$, respectively. Then we define maps $\mathcal{G}(\mathcal{H}, \mathfrak{s}) \to \mathrm{Gr}(\mathcal{H}, \mathfrak{s})$ and $\mathcal{G}(\mathcal{H}, \mathfrak{s}) \to \underline{\mathrm{Gr}}(\mathcal{H}, \mathfrak{s})$, which are well-defined up to a shift to be made precise below. In the next couple of sections we use these maps to define relative gradings on the bordered sutured invariants.

6.1 Domain gradings

We start by defining a grading on all homology classes in $\pi_2(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}$ and $\mathbf{y}$ generators in $\mathcal{G}(\mathcal{H})$. We will abuse notation and will not distinguish between a given homology class and its associated domain in $H_2(\Sigma, \mathbf{Z} \cup \boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

Definition 6.1.1. *Given a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ define*

$$\mathrm{gr}(B) = (-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), \partial^{\partial} B) \in \mathrm{Gr}(\mathcal{Z}).$$

Given a grading reduction r from $\mathrm{Gr}(\mathcal{Z})$ to $\underline{\mathrm{Gr}}(\mathcal{Z})$, define

$$\underline{\mathrm{gr}}(B) = r(I(o(\mathbf{x}))) \cdot \mathrm{gr}(B) \cdot r(I(o(\mathbf{y})))^{-1} \in \underline{\mathrm{Gr}}(\mathcal{Z}).$$

The basic properties of these gradings, and in fact the reason they are called gradings is that they are compatible with composition of domains. They are also compatible with the indices of moduli spaces.

Proposition 6.1.2. *Given a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$, for any compatible sequence $\overrightarrow{\boldsymbol{\rho}}$ of sets of Reeb chords, we have* $\mathrm{gr}(B) = \lambda^{-\operatorname{ind}(B,\overrightarrow{\boldsymbol{\rho}})+\#\overrightarrow{\boldsymbol{\rho}}} \cdot \mathrm{gr}(\overrightarrow{\boldsymbol{\rho}})$.

For any two domains $B_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $B_2 \in \pi_2(\mathbf{y}, \mathbf{z})$, their concatenation has grading $\mathrm{gr}(B_1 * B_2) = \mathrm{gr}(B_1) \cdot \mathrm{gr}(B_2)$.

Similar statements hold for $\underline{\mathrm{gr}}(B)$.

Proof. For the first statement, recall that $\operatorname{ind}(B, \overrightarrow{\boldsymbol{\rho}}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + \iota(\overrightarrow{\boldsymbol{\rho}}) + \#\overrightarrow{\boldsymbol{\rho}}$, and the homological components of $\mathrm{gr}(\overrightarrow{\boldsymbol{\rho}})$ and $\mathrm{gr}(B)$ are both $\partial^{\partial} B$ for a compatible pair. The second statement follows from the first, using the fact that the index is additive for domains, and λ is central.

For the equivalent statement for $\underline{\mathrm{gr}}$, we just have to use $\underline{\mathrm{gr}}(I_{o(\mathbf{x})} \cdot a(\overrightarrow{\boldsymbol{\rho}}) \cdot I_{o(\mathbf{y})})$, instead of $\underline{\mathrm{gr}}(\overrightarrow{\boldsymbol{\rho}})$ which is not defined, and notice that the reduction terms match up. □

6.2 Generator gradings

We will give a relative grading for the generators in each Spin^c–structure. Here a *relative grading in a G–set* means a map $g\colon \mathcal{G}(\mathcal{H}, \mathfrak{s}) \to A$, where G acts on A, say on the right. Two such gradings g and g' with values in A and A' are equivalent, if there is a bijection $\phi\colon A \to A'$, such that ϕ is G–equivariant, and $g' = \phi \circ g$. The traditional case of a relative $\mathbb{Z}$ or $\mathbb{Z}/n$–valued grading corresponds to $\mathbb{Z}$ acting on its quotient, with the grading map defined up to an overall shift by a constant.

Definition 6.2.1. *For a Heegaard diagram $\mathcal{H}$ and generator $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ define the* stabilizer subgroup $\mathcal{P}(\mathbf{x}) = \mathrm{gr}(\pi_2(\mathbf{x}, \mathbf{x})) \subset \mathrm{Gr}(\mathcal{Z})$. *For any* Spin^c*–structure $\mathfrak{s}$ pick a generator $\mathbf{x}_0 \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$ and let $\mathrm{Gr}(\mathcal{H}, \mathfrak{s})$ be the set of right cosets $\mathcal{P}(\mathbf{x}_0) \backslash \mathrm{Gr}(\mathcal{Z})$ with the usual right $\mathrm{Gr}(\mathcal{Z})$–action. Define the grading $\mathrm{gr}\colon \mathcal{G}(\mathcal{H}, \mathfrak{s}) \to \mathrm{Gr}(\mathcal{H}, \mathfrak{s})$ by $\mathrm{gr}(\mathbf{x}) = \mathcal{P} \cdot \mathrm{gr}(B)$ for any $B \in \pi_2(\mathbf{x}_0, \mathbf{x})$.*

Proposition 6.2.2. *Assuming $\mathcal{G}(\mathcal{H}, \mathfrak{s})$ is nonempty, this is a well-defined relative grading, independent of the choice of $\mathbf{x}_0$, and has the property $\mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(B) = \mathrm{gr}(\mathbf{y})$ for any $B \in \pi_2(\mathbf{x}, \mathbf{y})$.*

Proof. These follow quickly from the fact that concatenation of domains respects the grading. For example, for any two domains B_1, B_2 from $\mathbf{x}_0$ to $\mathbf{x}$, the cosets $\mathcal{P}(\mathbf{x}_0) \cdot \mathrm{gr}(B_i)$ are the same. Independence from the choice of $\mathbf{x}_0$ follows from the fact that $\mathcal{P}(\mathbf{x})$ is a conjugate of $\mathcal{P}(\mathbf{x}_0)$. $\square$

Fixing a grading reduction r, and setting $\underline{\mathcal{P}}(\mathbf{x}) = \underline{\mathrm{gr}}(\pi_2(\mathbf{x}, \mathbf{x})) = r(I_{o(\mathbf{x})}) \cdot \mathcal{P}(\mathbf{x}) \cdot r(I_{o(\mathbf{x})})^{-1}$, we get a reduced grading set $\underline{\mathrm{Gr}}(\mathcal{H}, \mathfrak{s})$ with a right $\underline{\mathrm{Gr}}(\mathcal{Z})$–action, and reduced grading $\underline{\mathrm{gr}}$ on $\mathcal{G}(\mathcal{H}, \mathfrak{s})$ with the same properties as gr.

In light of the discussion in Chapter 2.6, the sets $\mathrm{Gr}(\mathcal{H}, \mathfrak{s})$ and $\underline{\mathrm{Gr}}(\mathcal{H}, \mathfrak{s})$ have left actions by $\mathrm{Gr}(-\mathcal{Z})$ and $\underline{\mathrm{Gr}}(-\mathcal{Z})$, respectively. Keep in mind that for the reduced grading, the reduction term used for acting on $\underline{\mathrm{gr}}(\mathbf{x})$ is $r(I_{\overline{o}(\mathbf{x})})$, corresponding to the complementary idempotent of $\mathbf{x}$.

To define the grading on the bimodules $\widehat{BSDA}$, we will need to take a mixed approach. Given a bordered sutured manifold $(Y, \Gamma, \mathcal{Z}_1 \cup \mathcal{Z}_2)$, thought of as a cobordism from $-\mathcal{Z}_1$ to $\mathcal{Z}_2$, we will use the left action of $\mathrm{Gr}(-\mathcal{Z}_1) \subset \mathrm{Gr}(-(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ and the right action of $\mathrm{Gr}(\mathcal{Z}_2) \subset \mathrm{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$. The two actions commute since the correction term L vanishes on mixed pairs. Moreover, the Maslov components act the same on both sides.

6.3 A simpler description

In the special case when $\mathcal{Z} = \varnothing$ and the manifold is just sutured, the grading takes a simpler form that is the same as the usual relative grading on *SFH*. Recall that the *divisibility* of a Spin^c–structure $\mathfrak{s}$ is the integer $\mathrm{div}(\mathfrak{s}) = \gcd_{\alpha \in H_2(Y)} \langle c_1(\mathfrak{s}), \alpha \rangle$, and that sutured Floer homology groups $SFH(Y, \mathfrak{s})$ are relatively-graded by the cyclic group $\mathbb{Z}/\mathrm{div}(\mathfrak{s})$. (See [Juh06].)

Theorem 6.3.1. *Let $\mathcal{H}$ be a Heegaard diagram for a sutured manifold (Y, Γ), which can also be interpreted as a diagram for the bordered sutured manifold $(Y, \Gamma, \varnothing)$. For any* Spin^c–

structure $\mathfrak{s}$, the grading sets are $\underline{\mathrm{Gr}}(\mathcal{H},\mathfrak{s}) = \mathrm{Gr}(\mathcal{H},\mathfrak{s}) = \frac{1}{2}\mathbb{Z}/\operatorname{div}(\mathfrak{s})$, *with the usual action by* $\underline{\mathrm{Gr}}(\varnothing) = \mathrm{Gr}(\varnothing) = \frac{1}{2}\mathbb{Z}$. *Moreover, the relative gradings* $\mathrm{gr} = \underline{\mathrm{gr}}$ *on* $\mathcal{G}(\mathcal{H},\mathfrak{s})$ *coincide with relative grading on SFH. In particular, only the integer gradings are occupied.*

Proof. The grading on *SFH* is defined in essentially the same way, on a diagram level. There a domain $B \in \pi_2(\mathbf{x},\mathbf{y})$ is graded as $-\operatorname{ind}(B) = -e(B) - n_{\mathbf{x}} - n_{\mathbf{y}} = \mathrm{gr}(B) \in \mathrm{Gr}(\varnothing)$. The rest of definition is exactly the same, with the result that the gradings coincide, except that in the bordered sutured case we start with the bigger group $\frac{1}{2}\mathbb{Z}$, while $\mathrm{gr}(B) = -\operatorname{ind}(B)$ still takes only integer values. □

In general, the grading sets $\mathrm{Gr}(\mathcal{H},\mathfrak{s})$ and $\underline{\mathrm{Gr}}(\mathcal{H},\mathfrak{s})$ can look very complicated, but if we forget some of the structure we can give a reasonably nice description similar to the purely sutured case.

Proposition 6.3.2. *There is a projection map* $\pi\colon \underline{\mathrm{Gr}}(\mathcal{H},\mathfrak{s}) \to \operatorname{im}(i_*\colon H_1(F) \to H_1(Y))$ *with the following properties. Each fiber looks like* $\frac{1}{2}\mathbb{Z}/\operatorname{div}(\mathfrak{s})$, *with the usual translation action by the central subgroup* $(\frac{1}{2}\mathbb{Z},0)$. *Any element of the form* $(*,\alpha)$ *permutes the fibers of* π, *sending* $\pi^{-1}(\beta)$ *to* $\pi^{-1}(\beta + i_*(\alpha))$, *while preserving the* $\frac{1}{2}\mathbb{Z}$*–action.*

Proof. Recall that $\pi_2(\mathbf{x},\mathbf{x})$ is isomorphic to $H_2(Y,F)$ by attaching the cores of 2–handles and half-handles. Inside, the subgroup $\pi_2^{\partial}(\mathbf{x},\mathbf{x})$ of provincial periodic domains is isomorphic to $H_2(Y) \subset H_2(Y,F)$. Similar to the purely sutured case, for provincial periodic domains $e(B) + 2n_{\mathbf{x}}(B) = \langle c_1(\mathfrak{s}), [B]\rangle$. The subgroup

$$\mathcal{P}^{\partial}(\mathbf{x}) = \mathrm{gr}(\pi_2^{\partial}(\mathbf{x},\mathbf{x})) = (\langle c_1(\mathfrak{s}), H_2(Y)\rangle, 0) = (\operatorname{div}(\mathfrak{s})\mathbb{Z}, 0)$$

is central, and therefore $\underline{\mathcal{P}}^{\partial}(\mathbf{x}) = \underline{\mathrm{gr}}(\pi_2^{\partial}(\mathbf{x},\mathbf{x})) \subset \underline{\mathcal{P}}(\mathbf{x})$ is also $(\operatorname{div}(\mathfrak{s})\mathbb{Z},0)$ and central in $\underline{\mathrm{Gr}}(\mathcal{Z})$.

In particular, taking the quotient $\underline{\mathrm{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}^{\partial}$ has the effect of reducing the Maslov component modulo $\operatorname{div}(\mathfrak{s})$. On the other hand, since any two classes B_1 and B_2 with the same ∂^{∂} differ by a provincial domain, $\underline{\mathcal{P}}/\underline{\mathcal{P}}^{\partial}$ is isomorphic to $\operatorname{im}(\partial\colon H_2(Y,F) \to H_1(F)) = \ker(i_*\colon H_1(F) \to H_1(Y))$. If we ignore the Maslov component, passing to $(\underline{\mathcal{P}}/\underline{\mathcal{P}}^{\partial})\backslash(\underline{\mathrm{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}^{\partial}) = \underline{\mathcal{P}}\backslash\underline{\mathrm{Gr}}(\mathcal{Z})$ reduces the homological component $H_1(F)$ modulo $\ker i_*$. Therefore, the new homological component is valued in $H_1(F)/\ker i_* \cong \operatorname{im} i_*$. □

6.4 Grading and gluing

The most important property of the reduced grading is that it behaves nicely under gluing of diagrams. This will later allow us to show that the pairing on $\widehat{BSDA}$ respects the grading. First, we define a grading for a pair of diagrams which can be glued together, and then show it coincides with the grading on the gluing.

Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are diagrams for $(Y_1, \Gamma_1, -\mathcal{Z}_1 \cup \mathcal{Z}_2)$ and $(Y_2, \Gamma_2, -\mathcal{Z}_2 \cup \mathcal{Z}_3)$, respectively, and fix reductions for $\mathrm{Gr}(\mathcal{Z}_1)$, $\mathrm{Gr}(\mathcal{Z}_2)$, and $\mathrm{Gr}(\mathcal{Z}_3)$. Recall that $\underline{\mathrm{Gr}}(\mathcal{H}_1, \mathfrak{s}_1)$ has left and right actions by $\underline{\mathrm{Gr}}(\mathcal{Z}_1)$ and $\underline{\mathrm{Gr}}(\mathcal{Z}_2)$, respectively, while $\underline{\mathrm{Gr}}(\mathcal{H}_2, \mathfrak{s}_2)$ has left and right actions by $\underline{\mathrm{Gr}}(\mathcal{Z}_2)$ and $\underline{\mathrm{Gr}}(\mathcal{Z}_3)$, respectively.

It is easy to see that generators in $\mathcal{H}_1 \cup_{\mathbf{Z}_2} \mathcal{H}_2$ correspond to pairs of generators with complementary idempotents at $\mathcal{Z}_2$, and there are restriction maps on Spin^c–structures, such that $\mathfrak{s}(\mathbf{x}_1, \mathbf{x}_2)|_{Y_i} = \mathfrak{s}(\mathbf{x}_i)$. Let $F_i = F(\mathcal{Z}_i)$. From the long exact homology sequence for the triple $(Y_1 \cup Y_2, F_1 \cup F_2 \cup F_3, F_1 \cup F_3)$ and Poincaré duality, we can see that $\{\mathfrak{s} : \mathfrak{s}|_{Y_i} = \mathfrak{s}_i\}$ is either empty or an affine set over $\mathrm{im}(i_*\colon\ H_1(F_2) \to H_1(Y_1 \cup Y_2, F_1 \cup F_3))$.

Definition 6.4.1. *Let* $\underline{\mathrm{Gr}}(\mathcal{H}_1, \mathcal{H}_2, \mathfrak{s}_1, \mathfrak{s}_2)$ *be the product*

$$\underline{\mathrm{Gr}}(\mathcal{H}_1, \mathfrak{s}_1) \times_{\underline{\mathrm{Gr}}(\mathcal{Z}_2)} \underline{\mathrm{Gr}}(\mathcal{H}_2, \mathfrak{s}_2),$$

i.e. the usual product of the two sets, modulo the relation $(a \cdot g, b) \sim (a, g \cdot b)$ *for any* $g \in \underline{\mathrm{Gr}}(\mathcal{Z}_2)$*. It inherits a left action by* $\underline{\mathrm{Gr}}(\mathcal{Z}_1)$ *and a right action by* $\underline{\mathrm{Gr}}(\mathcal{Z}_3)$*, which commute and where the Maslov components act in the same way.*

Define a grading on $\cup_{\mathfrak{s}|_{Y_i} = \mathfrak{s}_i} \mathcal{G}(\mathcal{H}_1 \cup \mathcal{H}_2, \mathfrak{s})$ *by*

$$\underline{\mathrm{gr}}'(\mathbf{x}_1, \mathbf{x}_2) = [(\underline{\mathrm{gr}}(\mathbf{x}_1), \underline{\mathrm{gr}}(\mathbf{x}_2)] \in \underline{\mathrm{Gr}}(\mathcal{H}_1, \mathcal{H}_2, \mathfrak{s}_1, \mathfrak{s}_2).$$

Theorem 6.4.2. *Assume* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are compatible, i.e. there is at least one* $\mathfrak{s}$ *restricting to each of them. There is a projection from the mixed grading set*

$$\pi\colon\ \underline{\mathrm{Gr}}(\mathcal{H}_1, \mathcal{H}_2, \mathfrak{s}_1, \mathfrak{s}_2) \twoheadrightarrow \mathrm{im}(i_*\colon\ H_1(F_2) \to H_1(Y_1 \cup Y_2, F_1 \cup F_3)),$$

defined up to a shift in the image, with the following properties.

1. *For any two generators* $\mathbf{x}$ *and* $\mathbf{y}$ *with* $\mathfrak{s}(\mathbf{x})|_{Y_i} = \mathfrak{s}(\mathbf{y})|_{Y_i} = \mathfrak{s}_i$*, we have*

$$\mathrm{PD}(\mathfrak{s}(\mathbf{y}) - \mathfrak{s}(\mathbf{x})) = \pi(\underline{\mathrm{gr}}'(\mathbf{x})) - \pi(\underline{\mathrm{gr}}'(\mathbf{y})),$$

   i.e. π *distinguishes* Spin^c*–structures. Moreover, the* $\underline{\mathrm{Gr}}(\mathcal{Z}_1)$ *and* $\underline{\mathrm{Gr}}(\mathcal{Z}_3)$*–actions preserve the fibers of* π*.*

2. *For each* $\mathfrak{s}$*, such that* $\mathfrak{s}|_{Y_i} = \mathfrak{s}_i$*, there is a unique fiber* $\underline{\mathrm{Gr}}_{\mathfrak{s}}$ *of* π*, such that the grading* $\underline{\mathrm{gr}}'|_{\mathcal{G}(\mathcal{H}_1 \cup \mathcal{H}_2, \mathfrak{s})}$ *is valued in* $\underline{\mathrm{Gr}}_{\mathfrak{s}}$*, and is equivalent to* $\underline{\mathrm{gr}}$ *valued in* $\underline{\mathrm{Gr}}(\mathcal{H}_1 \cup \mathcal{H}_2, \mathfrak{s})$*.*

Proof. It is useful to pass to only right actions, as the grading sets were originally defined. We will use $\underline{\mathrm{Gr}}_{ij}$ as shorthand for $\underline{\mathrm{Gr}}(-\mathcal{Z}_i \cup \mathcal{Z}_j)$, for $i, j = 1, 2, 3$, and $\underline{\mathrm{Gr}}_{1223}$ as shorthand for $\underline{\mathrm{Gr}}(-\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup -\mathcal{Z}_2 \cup \mathcal{Z}_3)$. Recall that for $i = 1, 2$, the grading set $\underline{\mathrm{Gr}}(\mathcal{H}_i, \mathfrak{s}_i)$ was defined as the quotient $\underline{\mathcal{P}}(\mathbf{x}_i)\backslash\underline{\mathrm{Gr}}_{i,(i+1)}$ for some $\mathbf{x}_i \in \mathcal{G}(\mathcal{H}_i, \mathfrak{s}_i)$. We can identify $\underline{\mathrm{Gr}}(\mathcal{Z}_2)$ with the subgroup

$$\underline{\mathcal{Q}}' = \left\{((a, \alpha), (-a, -\alpha)) : a \in \frac{1}{2}\mathbb{Z}, \alpha \in H_1(F_2)\right\} \subset \underline{\mathrm{Gr}}_{12} \times \underline{\mathrm{Gr}}_{23}.$$

Note that $\underline{\mathcal{Q}}'$ commutes with $\underline{\mathrm{Gr}}(-\mathcal{Z}_1) \times \underline{\mathrm{Gr}}(\mathcal{Z}_3)$ in $\underline{\mathrm{Gr}}_{12} \times \underline{\mathrm{Gr}}_{23}$, so we can think of the mixed grading set $\underline{\mathrm{Gr}}(\mathcal{H}_1, \mathcal{H}_2, \mathfrak{s}_1, \mathfrak{s}_2)$ as the double quotient

$$(\underline{\mathcal{P}}(\mathbf{x}_1) \times \underline{\mathcal{P}}(\mathbf{x}_2))\backslash(\underline{\mathrm{Gr}}_{12} \times \underline{\mathrm{Gr}}_{23})/\underline{\mathcal{Q}}',$$

with a right action by $\underline{\mathrm{Gr}}(-\mathcal{Z}_1) \times \underline{\mathrm{Gr}}(\mathcal{Z}_3)$.

The Maslov components are central, and we can take the quotients of $\underline{\mathcal{Q}}'$ and $\underline{\mathrm{Gr}}_{12} \times \underline{\mathrm{Gr}}_{23}$ by $\frac{1}{2}\mathbb{Z} = \{((a, 0), (-a, 0))\} \subset \underline{\mathcal{Q}}'$. This has the effect of collapsing the two Maslov components into one. Thus $(\underline{\mathrm{Gr}}_{12} \times \underline{\mathrm{Gr}}_{23})/\frac{1}{2}\mathbb{Z}$ is canonically isomorphic to $\underline{\mathrm{Gr}}_{1223}$, and $\underline{\mathcal{Q}}'/\frac{1}{2}\mathbb{Z}$ is canonically isomorphic to the abelian subgroup $\underline{\mathcal{Q}} = \{(0, \alpha, -\alpha) : \alpha \in H_1(F_2)\} \subset \underline{\mathrm{Gr}}_{22}$. Let $\underline{\mathcal{P}}$ be generated by the images of $\underline{\mathcal{P}}(\mathbf{x}_i) \subset \underline{\mathrm{Gr}}_{i,(i+1)}$ in $\underline{\mathrm{Gr}}_{1223}$, for $i = 1, 2$. We can identify $\underline{\mathrm{Gr}}(\mathcal{H}_1, \mathcal{H}_2, \mathfrak{s}_1, \mathfrak{s}_2)$ with the quotient $\underline{\mathcal{P}}\backslash\underline{\mathrm{Gr}}_{1223}/\underline{\mathcal{Q}}$, which has a right action by $\underline{\mathrm{Gr}}_{13}$. In other words, the mixed grading set has elements of the form $[a] = \underline{\mathcal{P}} \cdot a \cdot \underline{\mathcal{Q}}$, with action $[a] \cdot g = [a \cdot g]$.

Let $\Pi \colon \underline{\mathrm{Gr}}_{1223} \to H_1(F_2)$ be addition of the two $H_1(F_2)$–homological component terms together, and ignoring the rest. Note that Π is surjective, $\underline{\mathcal{Q}} \subset \ker \Pi$, while $\Pi|_{\underline{\mathcal{P}}}$ is the restriction of $\Pi \circ \underline{\mathrm{gr}}$ to

$$\pi_2(\mathbf{x}_1, \mathbf{x}_1) \times \pi_2(\mathbf{x}_2, \mathbf{x}_2) \cong H_2(Y_1, F_1 \cup F_2) \times H_2(Y_2, F_2 \cup F_3)$$
$$\cong H_2(Y_1 \cup Y_2, F_1 \cup F_2 \cup F_3),$$

and coincides with the boundary map

$$\partial\colon\ H_2(Y_1 \cup Y_2, F_1 \cup F_2 \cup F_3) \to H_1(F_1 \cup F_2 \cup F_3, F_1 \cup F_3) \cong H_1(F_2)$$

from the long exact sequence of the triple. Therefore $\Pi(\underline{\mathcal{P}}) = \operatorname{im}\partial = \ker(i_*\colon\ H_1(F_2) \to H_1(Y_1 \cup Y_2, F_1 \cup F_3))$, and Π descends on the cosets to a projection $\pi\colon\ (\underline{\mathcal{P}}\backslash\underline{\mathrm{Gr}}_{1223}/\underline{\mathcal{Q}}) \to H_1(F_2)/\ker i_* \cong \operatorname{im} i_*$. A different choice of $\mathbf{x}_i$ only shifts the homological components, and so the image of π.

To prove (1), we need to check that for any compatible $\mathbf{y}_i \in \mathcal{G}(\mathcal{H}_i, \mathfrak{s}_i)$, the difference $\mathfrak{s}(\mathbf{y}_1, \mathbf{y}_2) - \mathfrak{s}(\mathbf{x}_1, \mathbf{x}_2)$ is the same as $-(\pi(\underline{\mathrm{gr}}'(\mathbf{y}_1, \mathbf{y}_2)) - \pi(\underline{\mathrm{gr}}'(\mathbf{x}_1, \mathbf{x}_2)))$. Suppose $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$. Then the latter difference is $-\pi([\underline{\mathrm{gr}}(B_1), \underline{\mathrm{gr}}(B_2)]) + \pi([0, 0]) = -i_*(h_1 + h_2)$, where h_i is the $H_1(F_2)$ part of the homological component of $\underline{\mathrm{gr}}(B_i)$. Since the $\mathcal{Z}_2$–idempotents of $\mathbf{x}_1$ and $\mathbf{x}_2$ are complementary, as well as those of $\mathbf{y}_1$ and $\mathbf{y}_2$, the reduction terms r cancel, and we can look at $\mathrm{gr}(B_1)$ and $\mathrm{gr}(B_2)$, instead. Therefore $h_1 + h_2 = \partial^{\partial_2} B_1 + \partial^{\partial_2} B_2$, interpreted as an element of $H_1(F_2) \subset H_1(\mathbf{Z}_2, \mathbf{a}_2)$. Here ∂^{∂_2} denotes the $\mathcal{Z}_2$ part of ∂^{∂}. It is indeed in that subgroup, again because of the complementary idempotents.

By the proof of Proposition 4.5.2, we have $\mathfrak{s}(\mathbf{y}) - \mathfrak{s}(\mathbf{x}) = \mathrm{PD}([a - b])$, where a and b are any two 1–chains in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, with $\partial a = \mathbf{y} - \mathbf{x} + \mathbf{z}$ and $\partial b = \mathbf{y} - \mathbf{x}$, where $\mathbf{z}$ is a 0–chain in $\mathbf{Z}_1 \cup \mathbf{Z}_3$. The boundaries of B_1 and B_2 almost give us such chains. Let $a_i = \partial^{\alpha} B_i$, $b_i = \partial^{\beta} B_i$, and $c_i = \partial^{\partial_2} B_i$, as chains. Then $[c_1 + c_2] = h_1 + h_2$, in $H_1(\mathbf{Z}_2, \mathbf{a}_2)$, and $[a_1 + a_2 + b_1 + b_2 + c_1 + c_2] = -[\partial^{\partial_1} B_1 + \partial^{\partial_3} B_2] = 0 \in H_1(Y_1 \cup Y_2, F_1 \cup F_3)$. Notice that we can represent $h_1 + h_2 \in H_1(F_2)$ by the 1–chain $c_1 + c_2 + d$, where d is a sum of some $\boldsymbol{\alpha}^a$–arcs in $\mathcal{H}_1$. Let $a = a_1 + a_2 - d$, and $b = -(b_1 + b_2)$. They have the desired properties, and $[a - b] = [a_1 + a_2 + b_1 + b_2 + c_1 + c_2] - [c_1 + c_2 + d] = 0 - i_*(h_1 + h_2)$. This finishes the proof of the relation between π and the Spin^c–structures in (1). Finally, since $\underline{\mathrm{Gr}}(-\mathcal{Z}_1 \cup \mathcal{Z}_3) \subset \ker \Pi$, its action preserves the fibers of π.

For (2), we know the restriction $\underline{\mathrm{gr}}'|_{\mathcal{G}(\mathcal{H}_1\cup\mathcal{H}_2,\mathfrak{s})}$ takes values in a unique fiber $\underline{\mathrm{Gr}}_{\mathfrak{s}}$. To see that this grading is equivalent to $\underline{\mathrm{gr}}$, we need three results. First, we need to show that the action of $\underline{\mathrm{Gr}}(-\mathcal{Z}_1\cup\mathcal{Z}_3)$ is transitive on $\underline{\mathrm{Gr}}_{\mathfrak{s}}$. Second, we need to show that the stabilizers of $\underline{\mathrm{gr}}'(\mathbf{y}_0)$ and $\underline{\mathrm{gr}}(\mathbf{y}_0)$ are the same for some $\mathbf{y}_0\in\mathcal{G}(\mathcal{H}_1\cup\mathcal{H}_2,\mathfrak{s})$. These two steps show that $\underline{\mathrm{Gr}}_{\mathfrak{s}}$ and $\underline{\mathrm{Gr}}(\mathcal{H}_1\cup\mathcal{H}_2,\mathfrak{s})$ are equivalent as grading sets. Finally, we need to show that for any other $\mathbf{y}\in\mathcal{G}(\mathcal{H}_1\cup\mathcal{H}_2,\mathfrak{s})$, there is at least one $g\in\underline{\mathrm{Gr}}(-\mathcal{Z}_1\cup\mathcal{Z}_3)$, such that $\underline{\mathrm{gr}}(\mathbf{y})=\underline{\mathrm{gr}}(\mathbf{y}_0)\cdot g$ and $\underline{\mathrm{gr}}'(\mathbf{y})=\underline{\mathrm{gr}}'(\mathbf{y}_0)\cdot g$.

For the first part, notice that $\underline{\mathrm{Gr}}(-\mathcal{Z}_1\cup\mathcal{Z}_3)\times\underline{\mathcal{Q}}$ is exactly the kernel of Π, while the reduction to π was exactly by the image of $\underline{\mathcal{P}}$. Therefore, if $\pi([a_1])=\pi([a_2])$, then $\Pi(a_1)=\Pi(p\cdot a_2)$ for some $p\in\mathcal{P}$, so $a_1=p\cdot a_2\cdot g\cdot q$ for some $g\in\underline{\mathrm{Gr}}(-\mathcal{Z}_1\cup\mathcal{Z}_3)$ and $q\in\underline{\mathcal{Q}}$. In other words, $[a_1]=[a_2]\cdot g$.

For the second part, we can assume $(\mathbf{x}_1,\mathbf{x}_2)$ are in $\mathfrak{s}$, and use that as $\mathbf{y}_0$. In this case the stabilizer for $\underline{\mathrm{gr}}'$ is $(\mathcal{P}\cdot H)\cap\underline{\mathrm{Gr}}(-\mathcal{Z}_1\cup\mathcal{Z}_3)$. We may also assume the base idempotents for r are $I_{\bar{o}_1(\mathbf{x}_1)}$ for $\mathcal{Z}_1$, $I_{o_2(\mathbf{x}_1)}=I_{\bar{o}_2(\mathbf{x}_2)}$ for $\mathcal{Z}_2$, and $I_{o_3(\mathbf{x}_3)}$ for $\mathcal{Z}_3$. This ensures $\underline{\mathrm{gr}}=\mathrm{gr}$ for periodic domains at $(\mathbf{x}_1,\mathbf{x}_2)$. This corresponds to the gradings of pairs of periodic classes $B_i\in\pi_2(\mathbf{x}_i,\mathbf{x}_i)$ with $\partial^{\partial_2}B_1+\partial^{\partial_2}B_2=0$, canceling those terms. But such pairs are in 1–to–1 correspondence with periodic class $B\in\pi_2((\mathbf{x}_1,\mathbf{x}_2),(\mathbf{x}_1,\mathbf{x}_2))$. The gradings for such pairs are additive, so the stabilizer of $\underline{\mathrm{gr}}'$ is the same as $\underline{\mathrm{gr}}$.

Finally, to show the relative gradings are the same, pick any $B\in\pi_2(\mathbf{y}_0,\mathbf{y})$. It decomposes into two classes B_i connecting the $\mathcal{H}_i$ components of $\mathbf{y}_0$ and $\mathbf{y}$ with canceling ∂^{∂_2}. Similar to the above discussion, the regular gradings satisfy $\mathrm{gr}(B)=\mathrm{gr}(B_1)\,\mathrm{gr}(B_2)$. The reduction terms match up, so the same holds for $\underline{\mathrm{gr}}$. Thus $\underline{\mathrm{gr}}(B)$ is the grading difference between $\mathbf{y}$ and $\mathbf{y}_0$ for both gradings. $\square$

Chapter 7

One-sided invariants

7.1 Overview

To a bordered sutured manifold $(Y, \Gamma, \mathcal{Z}, \phi)$, we will associate the following invariants. Each of them is defined up to homotopy equivalence, in the appropriate sense.

1. A right $\mathcal{A}_\infty$–module over $\mathcal{A}(\mathcal{Z})$, denoted $\widehat{BSA}(Y, \Gamma, \mathcal{Z}, \phi)$.
2. A left *type D structure* over $\mathcal{A}(-\mathcal{Z})$, denoted $\widehat{BSD}(Y, \Gamma, \mathcal{Z}, \phi)$.
3. A left differential graded module over $\mathcal{A}(-\mathcal{Z})$, which we denote $\widehat{BSDD}(Y, \Gamma, \mathcal{Z}, \phi)$.

7.2 Type D structures

Although we can express all of the invariants and their properties in terms of differential graded modules and $\mathcal{A}_\infty$–modules, from a practical standpoint it is more convenient to use the language of type D structures introduced in [LOT09]. We recall here the definitions and basic properties. To simplify the discussion we will restrict to the case where the algebra is differential graded, and not a general $\mathcal{A}_\infty$–algebra. This is all that is necessary for the present applications.

Remark. Any algebra or module has an implicit action by a base ring, and any usual tensor product $\otimes$ is taken over such a base ring. In the case of the algebra $\mathcal{A}(\mathcal{Z})$ associated with an arc algebra, and any modules over it, the base ring is the idempotent ring $\mathcal{I}(\mathcal{Z})$. We will omit the base ring from the notation to avoid clutter.

Let A be a differential graded algebra with differential μ_1 and multiplication μ_2.

Definition 7.2.1. *A* (left) type D structure *over A is a graded module N over the base ring, with a homogeneous operation*

$$\delta\colon\ N \to (A \otimes N)[1],$$

satisfying the compatibility condition

$$(\mu_1 \otimes \mathrm{id}_N) \circ \delta + (\mu_2 \otimes \mathrm{id}_N) \circ (\mathrm{id}_A \otimes \delta) \circ \delta = 0. \tag{7.1}$$

We can define induced maps

$$\delta_k\colon\ N \to (A^{\otimes k} \otimes N)[k],$$

by setting

$$\delta_k = \begin{cases} \mathrm{id}_N & \text{for } k = 0, \\ (\mathrm{id}_A \otimes \delta_{k-1}) \circ \delta & \text{for } k \geq 1. \end{cases}$$

Definition 7.2.2. *We say a type D structure N is* bounded *if for any $n \in N$, $\delta_k(n) = 0$ for sufficiently large k.*

Given two left type D structures N, N' over A, the space $\mathrm{Hom}(N, A \otimes N')$ of linear maps over the base ring becomes a graded chain complex with differential

$$Df = (\mu_1 \otimes \mathrm{id}_{N'}) \circ f + (\mu_2 \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_A \otimes \delta') \circ f + (\mu_2 \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_A \otimes f) \circ \delta.$$

The grading is given by the grading shifts on homogeneous maps.

Definition 7.2.3. *A* map of type D structures, *from N to N', is a cycle in the above chain complex.*

Two such maps f and g are homotopic *if $f - g = Dh$ for some $h \in \mathrm{Hom}(N, A \otimes N')$, called a* homotopy *from f to g.*

If $f\colon\ N \to A \otimes N'$, and $g\colon\ N' \to A \otimes N''$ are type D structure maps, their composition *$g \circ f\colon\ N \to A \otimes N''$ is defined to be*

$$g \circ f = (\mu_2 \otimes \mathrm{id}_{N''}) \circ (\mathrm{id}_A \otimes g) \circ f.$$

With the above definitions, type D structures over A form a differential graded category. This allows us, among other things, to talk about homotopy equivalences. (In general, for an $\mathcal{A}_\infty$–algebra A, this is an $\mathcal{A}_\infty$–category.)

Let M be a right $\mathcal{A}_\infty$–module over A, with higher $\mathcal{A}_\infty$ actions

$$m_k \colon\ M \otimes A^{\otimes(k-1)} \to M[2-k], \text{ for } k \geq 1.$$

Let N be a left type D structure. We can define a special tensor product between them.

Definition 7.2.4. *Assuming at least one of M and N is bounded, let*

$$M \boxtimes_A N$$

be the graded vector space $M \otimes N$, with differential

$$\partial \colon\ M \otimes N \to (M \otimes N)[1],$$

defined by

$$\partial = \sum_{k=1}^{\infty} (m_k \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \delta_{k-1}).$$

The condition that M or N is bounded guarantees that the sum is always finite. In that case $\partial^2 = 0$ (using $\mathbb{Z}/2$ coefficients), and $M \boxtimes N$ is a graded chain complex.

The most important property of $\boxtimes$, as shown in [LOT10a] is that it is functorial up to homotopy and induces a bifunctor on the level of derived categories.

The chain complex $A \boxtimes N$ is in fact a graded differential module over A, with differential

$$\partial = \mu_1 \otimes \mathrm{id}_N + (\mu_2 \otimes \mathrm{id}_N) \circ \delta,$$

and algebra action

$$a \cdot (b, n) = (\mu_2(a, b), n).$$

In a certain sense working with type D structures is equivalent to working with their associated left modules. In particular, $A \boxtimes \cdot$ is a functor, and $M \mathbin{\widetilde{\otimes}} (A \boxtimes N)$ and $M \boxtimes N$ are homotopy equivalent as graded chain complexes.

7.3 $\widehat{BSD}$ and $\widehat{BSDD}$

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}, \psi)$ be a provincially admissible bordered sutured Heegaard diagram, and let J be an admissible almost complex structure.

We will define $\widehat{BSD}$ as a type D structure over $\mathcal{A}(-\mathcal{Z})$.

Definition 7.3.1. *Fix a relative* Spin^c*–structure* $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y \setminus F)$*. Let* $\widehat{BSD}(\mathcal{H}, J, \mathfrak{s})$ *be the* $\mathbb{Z}/2$ *vector space generated by the set of all generators* $\mathcal{G}(\mathcal{H}, \mathfrak{s})$*. Give it the structure of an* $\mathcal{I}(-\mathcal{Z})$ *module as follows. For any* $\mathbf{x} \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$ *set*

$$I(s) \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } s = \overline{o}(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

We consider only discrete partitions $\overrightarrow{P} = (\{q_1\}, \dots, \{q_m\})$.

Definition 7.3.2. *For* $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$ *define*

$$a_{\mathbf{x},\mathbf{y}} = \sum_{\substack{\mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P}))=1 \\ \overrightarrow{P} \text{ discrete}}} \#\mathcal{M}^B_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}) \cdot a(-P_1) \cdots a(-P_m).$$

We compute $a(-P_i)$, since the Reeb chord ρ_i labeling the puncture q_i is oriented opposite from $-\mathcal{Z}$.

Definition 7.3.3. *Define* $\delta\colon\ \widehat{BSD}(\mathcal{H}, J, \mathfrak{s}) \to \mathcal{A}(-\mathcal{Z}) \otimes \widehat{BSD}(\mathcal{H}, J, \mathfrak{s})$ *as follows.*

$$\delta(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{G}(\mathcal{H})} a_{\mathbf{x},\mathbf{y}} \otimes \mathbf{y}.$$

Note that, $\pi_2(\mathbf{x}, \mathbf{y})$ is nonempty if and only if $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y})$, so the range of δ is indeed correct.

Theorem 7.3.4. *The following statements are true.*

1. $\widehat{BSD}(\mathcal{H}, J, \mathfrak{s})$ *equipped with* δ*, and the grading* $\underline{\mathrm{Gr}}(\mathcal{H}, \mathfrak{s})$*–valued grading* $\underline{\mathrm{gr}}$ *is a type* D *structure over* $\mathcal{A}(-\mathcal{Z})$*. In particular,*

   $$\lambda^{-1} \cdot \underline{\mathrm{gr}}(\mathbf{x}) = \underline{\mathrm{gr}}(a) \cdot \underline{\mathrm{gr}}(\mathbf{y}),$$

   whenever $\delta(\mathbf{x})$ *contains the term* $a \otimes \mathbf{y}$*.*

2. If $\mathcal{H}$ is admissible, $\widehat{BSD}(\mathcal{H}, J, \mathfrak{s})$ is bounded.

3. For any two provincially admissible diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$, equipped with admissible almost complex structures J_1 and J_2, there is a graded homotopy equivalence

$$\widehat{BSD}(\mathcal{H}_1, J_1, \mathfrak{s}) \simeq \widehat{BSD}(\mathcal{H}_2, J_2, \mathfrak{s}).$$

Therefore we can talk about $\widehat{BSD}(Y, \Gamma, \mathcal{Z}, \psi, \mathfrak{s})$ or just $\widehat{BSD}(Y, \Gamma, \mathfrak{s})$, relatively graded by $\underline{\mathrm{Gr}}(Y, \mathfrak{s})$.

Proof. In light of the discussion in Chapter 5.4, the proofs carry over from those for $\widehat{CFD}$ in the bordered case. We sketch the main steps below.

For (1), first we use provincial admissibility to guarantee the sums in the definitions are finite. Indeed, only finitely many provincial domains $B \in \pi_2^{\partial}(\mathbf{x}, \mathbf{y})$ are positive and can contribute. The number of non provincial domains ends up irrelevant, since only finitely many sequences of elements of $\mathcal{A}(-\mathcal{Z})$ have nonzero product.

To show that Eq. (7.1) is satisfied, we count possible degenerations of 1–dimensional moduli spaces, which are always an even number. Two story buildings correspond to the $(\mu_2 \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta) \circ \delta$ term. The degenerations with a curve at East infinity correspond to the $(\mu_1 \otimes \mathrm{id}_N) \circ \delta$ term.

To show that the grading condition is satisfied, recall that $a \otimes \mathbf{y}$ can be a term in $\delta(\mathbf{x})$ only if there is a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and a compatible sequence $\overrightarrow{\boldsymbol{\rho}} = (\{\rho_1\}, \ldots, \{\rho_p\})$ of Reeb chords, such that $\mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}) = 1$, and $a = I_{\overline{o}(\mathbf{x})} \cdot a(-\rho_1) \cdots a(-\rho_p) \cdot I_{\overline{o}(\mathbf{y})}$. We will prove the statement for gr, which allows us to ignore the idempotents at the end. The $\underline{\mathrm{gr}}$–version then follows from using the same reduction terms.

Notice that $\mathrm{gr}_{-\mathcal{Z}}(-\rho_i) = (-1/2, -[\rho_i])$, and so

$$\mathrm{gr}(a) = (-p/2 + \sum_{i<j} L_{-\mathcal{Z}}(-[\rho_i], -[\rho_j]), -[\overrightarrow{\boldsymbol{\rho}}]),$$

which we can also interpret as a $\mathrm{Gr}(\mathcal{Z})$–grading acting on the right. On the other hand, $\mathrm{gr}_{\mathcal{Z}}(\rho_i) = (-1/2, [\rho_i])$, and

$$\mathrm{gr}(\overrightarrow{\boldsymbol{\rho}}) = (-p/2 + \sum_{i<j} L_{\mathcal{Z}}([\rho_i], [\rho_j]), [\overrightarrow{\boldsymbol{\rho}}]).$$

Recall that $L_{\mathcal{Z}}$ and $L_{-\mathcal{Z}}$ have opposite signs, so we have the relation $\mathrm{gr}(\overrightarrow{\boldsymbol{\rho}})\,\mathrm{gr}(a) = \lambda^{-p}$. Thus, we have

$$\begin{aligned}
\mathrm{gr}(a \otimes \mathbf{y}) &= \mathrm{gr}_{-\mathcal{Z}}(a) \cdot \mathrm{gr}(\mathbf{y}) = \mathrm{gr}(\mathbf{y}) \cdot \mathrm{gr}_{\mathcal{Z}}(a) \\
&= \mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(B)\,\mathrm{gr}(a) = \mathrm{gr}(\mathbf{x}) \cdot \lambda^{-\,\mathrm{ind}(B,\overrightarrow{\boldsymbol{\rho}})+\#\overrightarrow{\boldsymbol{\rho}}}\,\mathrm{gr}(\overrightarrow{\boldsymbol{\rho}})\,\mathrm{gr}(a) \\
&= \mathrm{gr}(\mathbf{x}) \cdot \lambda^{-1+p}\lambda^{-p} = \mathrm{gr}(\mathbf{x}) \cdot \lambda^{-1}.
\end{aligned}$$

For (2), we use the fact that with admissibility, only finitely many domains $B \in \pi_2(\mathbf{x}, \mathbf{y})$ are positive, and could contribute to δ_k, for any k. Therefore, only finitely many of the terms of $\delta_k(\mathbf{x})$ are nonzero.

For (3) we use the fact that provincially admissible diagrams can be connected by Heegaard moves. To isotopies and changes of almost complex structure, we associate moduli spaces, depending on a path $(\mathcal{H}_t, J_t)$ of isotopic diagrams and almost complex structures. Counting 0–dimensional spaces gives a type D map $\widehat{BSD}(\mathcal{H}_0, J_0) \to \widehat{BSD}(\mathcal{H}_1, J_1)$. Analogous results to those in Chapter 5 and counting the ends of 1–dimensional moduli spaces show that the map is well defined and is in fact a homotopy equivalence. To handleslides, we associate maps coming from counting holomorphic triangles, which also behave as necessary in this special case.

For invariance of the grading, we show that both in time-dependent moduli spaces, and when counting triangles we can grade domains compatibly. In particular, the stabilizers are still conjugate, and the grading set is preserved. In both cases we count domains with index 0, so the relative gradings of individual elements are also preserved. $\square$

If we ignore Spin^c structures we can talk about the total invariant

$$\widehat{BSD}(Y, \Gamma) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y \setminus F)} \widehat{BSD}(Y, \Gamma, \mathfrak{s}).$$

We define $\widehat{BSDD}$ in terms of $\widehat{BSD}$. The two are essentially different algebraic representations of the same object.

Definition 7.3.5. *Given a bordered sutured manifold* $(Y, \Gamma, \mathcal{Z}, \phi)$*, let*

$$\widehat{BSDD}(Y, \Gamma, \mathfrak{s}) = \mathcal{A}(-\mathcal{Z}) \boxtimes \widehat{BSD}(Y, \Gamma, \mathfrak{s}),$$
$$\widehat{BSDD}(Y, \Gamma) = \mathcal{A}(-\mathcal{Z}) \boxtimes \widehat{BSD}(Y, \Gamma).$$

Remark. Recall that if (Y, Γ) is p–unbalanced, then any generator has p many occupied arcs. However, for $\widehat{BSD}$ the algebra action depends on unoccupied arcs. Therefore, if $\mathcal{Z}$ has k many arcs, then $\widehat{BSD}(Y, \Gamma)$ is in fact a type D structure over $\mathcal{A}(-\mathcal{Z}, k - p)$ only.

7.4 $\widehat{BSA}$

The definition of $\widehat{BSA}$ is similar to that of $\widehat{BSD}$, but differs in some important aspects. In particular, we count a wider class of curves and they are recorded differently.

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}, \psi)$ be a provincially admissible bordered sutured Heegaard diagram, and let J be an admissible almost complex structure.

We define $\widehat{BSA}$ as an $\mathcal{A}_\infty$–module over $\mathcal{A}(\mathcal{Z})$.

Definition 7.4.1. *Fix a relative* Spin^c*–structure* $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y \setminus F)$*. Let* $\widehat{BSA}(\mathcal{H}, J, \mathfrak{s})$ *be the* $\mathbb{Z}/2$ *vector space generated by the set of all generators* $\mathcal{G}(\mathcal{H}, \mathfrak{s})$*. Give it the structure of an* $\mathcal{I}(\mathcal{Z})$ *module by setting*

$$\mathbf{x} \cdot I(s) = \begin{cases} \mathbf{x} & \text{if } s = o(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

For generators $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$, a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and a source $S^\triangleright$ we consider all partitions $\overrightarrow{P} = (P_1, \dots, P_m)$, not necessarily discrete. We also associate to a sequence of Reeb chords a sequence of algebra elements, instead of a product. Let

$$\overrightarrow{a}(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}}) = I(o(\mathbf{x})) \cdot (a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_m)) \cdot I(o(\mathbf{y})).$$

Definition 7.4.2. *For* $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$*,* $\overrightarrow{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$ *define*

$$c_{\mathbf{x},\mathbf{y},\overrightarrow{\boldsymbol{\rho}}} = \sum_{\substack{\overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P}) = \overrightarrow{\boldsymbol{\rho}} \\ \mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}) = 1}} \#\mathcal{M}^B_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P}).$$

Definition 7.4.3. *Define* $m_k\colon \widehat{BSA}(\mathcal{H},J,\mathfrak{s})\otimes\mathcal{A}(\mathcal{Z})^{\otimes(k-1)}\to\widehat{BSA}(\mathcal{H},J,\mathfrak{s})$ *as follows.*

$$m_k(\mathbf{x},a_1,\dots,a_{k-1})=\sum_{\substack{\mathbf{y}\in\mathcal{G}(\mathcal{H})\\ \overrightarrow{a}(\mathbf{x},\mathbf{y},\overrightarrow{\boldsymbol{\rho}})=a_1\otimes\cdots\otimes a_{k-1}}} c_{\mathbf{x},\mathbf{y},\overrightarrow{\boldsymbol{\rho}}}\cdot\mathbf{y}.$$

Theorem 7.4.4. *The following statements are true.*

1. $\widehat{BSA}(\mathcal{H},J,\mathfrak{s})$ *equipped with the actions* m_k *for* $k\geq 1$, *and the* $\underline{\mathrm{Gr}}(\mathcal{H},\mathfrak{s})$*–valued grading* $\underline{\mathrm{gr}}$ *is an* $\mathcal{A}_\infty$*–module over* $\mathcal{A}(\mathcal{Z})$. *In particular,*

$$\underline{\mathrm{gr}}(m_k(\mathbf{x},a_1,\dots,a_{k-1}))=\underline{\mathrm{gr}}(\mathbf{x})\cdot\underline{\mathrm{gr}}(a_1)\cdots\underline{\mathrm{gr}}(a_{k-1})\lambda^{k-2}.$$

2. *If* $\mathcal{H}$ *is admissible,* $\widehat{BSA}(\mathcal{H},J,\mathfrak{s})$ *is bounded.*

3. *For any two provincially admissible diagrams* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *equipped with admissible almost complex structures* J_1 *and* J_2, *there is a graded homotopy equivalence*

$$\widehat{BSA}(\mathcal{H}_1,J_1,\mathfrak{s})\simeq\widehat{BSA}(\mathcal{H}_2,J_2,\mathfrak{s}).$$

Therefore we can talk about $\widehat{BSA}(Y,\Gamma,\mathcal{Z},\psi,\mathfrak{s})$ *or just* $\widehat{BSA}(Y,\Gamma,\mathfrak{s})$, *relatively graded by* $\underline{\mathrm{Gr}}(Y,\mathfrak{s})$.

Proof. The proofs are analogous to those for $\widehat{BSD}$, with some differences. The biggest difference is that we count more domains, so we need to use more results about degenerations.

The other major difference is the grading. Again, we prove the statement for gr, and the one for $\underline{\mathrm{gr}}$ follows immediately. Suppose $\mathbf{y}$ is a term in $m_k(\mathbf{x},a_1,\dots,a_{k-1})$. Then there is a domain $B\in\pi_2(\mathbf{x},\mathbf{y})$ and a compatible sequence $\overrightarrow{\boldsymbol{\rho}}=(\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_{k-1})$ of sets of Reeb chords, such that $\mathrm{ind}(B,\overrightarrow{\boldsymbol{\rho}})=1$, and $a_i=a(\boldsymbol{\rho}_i,s_i)$, for some appropriate completion s_i. In particular, $\mathrm{gr}(a_1)\cdots\mathrm{gr}(a_{k-1})=\mathrm{gr}(\overrightarrow{\boldsymbol{\rho}})$. On the other hand,

$$\begin{aligned}\mathrm{gr}(\mathbf{y})=\mathrm{gr}(\mathbf{x})\cdot\mathrm{gr}(B)=\mathrm{gr}(\mathbf{x})\cdot\lambda^{-\mathrm{ind}(B,\overrightarrow{\boldsymbol{\rho}})+\#\overrightarrow{\boldsymbol{\rho}}}\,\mathrm{gr}(\overrightarrow{\boldsymbol{\rho}})&\\ =\mathrm{gr}(\mathbf{x})\cdot\lambda^{-1+(k-1)}\,\mathrm{gr}(a_1)\cdots\mathrm{gr}(a_{k-1}).&\quad\square\end{aligned}$$

As with $\widehat{BSD}$, if we ignore Spin^c–structures we can talk about the total invariant

$$\widehat{BSA}(Y,\Gamma)=\bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y,\partial Y\setminus F)}\widehat{BSA}(Y,\Gamma,\mathfrak{s}).$$

Remark. As with $\widehat{BSD}$ the only nontrivial algebra action is by a single component of $\mathcal{A}(\mathcal{Z})$. In this case the action depends on occupied arcs. Therefore if (Y, Γ) is p–unbalanced, then $\widehat{BSA}(Y, \Gamma)$ is an $\mathcal{A}_\infty$–module over $\mathcal{A}(\mathcal{Z}, p)$ only.

7.5 Invariants from nice diagrams

For a nice diagram $\mathcal{H}$, the invariants can be computed completely combinatorially, avoiding all discussion of moduli spaces.

Theorem 7.5.1. *Let $\mathcal{H}$ be a nice diagram. Then for any admissible almost complex structure J, the type D structure $\widehat{BSD}(\mathcal{H}, J)$ can be computed as follows. The map $\delta(\mathbf{x})$ counts the following types of curves.*

1. *A source $S^{\triangleright}$ from $\mathbf{x}$ to $\mathbf{y}$, consisting of g bigons with no e punctures, where all but one of the bigons are constant on Σ, while the remaining one embeds as a convex bigon. The interior of the image contains none of the points in $\mathbf{x} \cap \mathbf{y}$. Such a curve contributes $I(\bar{o}(\mathbf{x})) \otimes \mathbf{y}$ to $\delta(\mathbf{x})$.*

2. *A source $S^{\triangleright}$ from $\mathbf{x}$ to $\mathbf{y}$, consisting of $g-2$ bigons, each of which has no e punctures and is constant on Σ, and a single quadrilateral with no e punctures, which embeds as a convex rectangle. The interior of the image contains none of the points in $\mathbf{x} \cap \mathbf{y}$. Such a curve contributes $I(\bar{o}(\mathbf{x})) \otimes \mathbf{y}$ to $\delta(\mathbf{x})$.*

3. *A source $S^{\triangleright}$ from $\mathbf{x}$ to $\mathbf{y}$, consisting of $g-1$ bigons, each of which has no e punctures and is constant on Σ, and a single bigon with one e puncture, which embeds as a convex rectangle, one of whose sides is the Reeb chord $-\rho \subset \mathbf{Z}$ labeling the puncture. The interior of the image contains none of the points in $\mathbf{x} \cap \mathbf{y}$. Such a curves contributes $I(\bar{o}(\mathbf{x}))a(\rho)I(\bar{o}(\mathbf{y})) \otimes \mathbf{y}$ to $\delta(\mathbf{x})$.*

Theorem 7.5.2. *Let $\mathcal{H}$ be a nice diagram. Then for any admissible almost complex structure J, the $\widehat{BSA}(\mathcal{H}, J)$ can be computed as follows.*

The differential $m_1(\mathbf{x})$ counts the following types of regions. (These are the same as cases (1) and (2) in Theorem 7.5.1.)

1. *A source $S^{\triangleright}$ from $\mathbf{x}$ to $\mathbf{y}$, consisting of g bigons with no e punctures, where all but one of the bigons are constant on Σ, while the remaining one embeds as a convex bigon. The interior of the image contains none of the points in $\mathbf{x}\cap\mathbf{y}$. Such a curve contributes $\mathbf{y}$ to $m_1(\mathbf{x})$.*

2. *A source $S^{\triangleright}$ from $\mathbf{x}$ to $\mathbf{y}$, consisting of $g-2$ bigons, each of which has no e punctures and is constant on Σ, and a single quadrilateral with no e punctures, which embeds as a convex rectangle. The interior of the image contains none of the points in $\mathbf{x}\cap\mathbf{y}$. Such a curve contributes $\mathbf{y}$ to $m_1(\mathbf{x})$.*

The algebra action $m_2(\mathbf{x},\cdot)$ counts regions of the type below.

1. *A source $S^{\triangleright}$ from $\mathbf{x}$ to $\mathbf{y}$, consisting of $g-k$ bigons, each of which has no e punctures and is constant on Σ, and a collection of k bigons, each of which has one e puncture and which embeds as a convex rectangle, one of whose sides is the Reeb chord $\rho_i \subset \mathbf{Z}$. The height of all e punctures is the same, the interior of any image rectangle contains none of the points in $\mathbf{x}\cap\mathbf{y}$ and no other rectangles. Such a curve contributes $\mathbf{y}$ to the action $m_2(\mathbf{x}, I(o(\mathbf{x}))\{a(\rho_1,\ldots,\rho_k\})I(o(\mathbf{y}))$.*

In addition, all actions m_k for $k\geq 3$ are zero.

Proof of Theorems 7.5.1 and 7.5.2. The proofs follow the same steps as the ones for nice diagrams in bordered manifolds. By looking at the index formula, and the restricted class of regions, one can show that the only B, $S^{\triangleright}$, and $\overrightarrow{P}$ that have index $\operatorname{ind}(B,S^{\triangleright},\overrightarrow{P})=1$ are of the following two types.

1. S has no e punctures, and consists of $g-1$ trivial components, and one non-trivial bigon component, or $g-2$ trivial component and one non-trivial rectangle component.

2. S has several trivial components, and several bigons with a single e puncture each. Moreover, the partition $\overrightarrow{P}$ consists of only one set.

The extra condition that the embedded index is also 1 (so the moduli space consists of embedded curves), is equivalent to having no fixed points in the interior of a region, and no region contained completely inside another.

For such curves, $\mathcal{M}^B_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P})$ has exactly one element, independent of the almost complex structure J, using for example the Riemann mapping theorem. $\square$

7.6 Pairing theorem

In this section we describe the relationship between the sutured homology of the gluing of two bordered sutured manifolds, and their bordered sutured invariants, proving the second part of Theorem 4.

Recall that bordered sutured invariants are homotopy types of chain complexes, while sutured Floer homology is usually regarded as an isomorphism type of homology groups. However, one can also regard the underlying chain complex as an invariant up to homotopy equivalence. To be precise, we will use *SFH* to denote sutured Floer homology, and *SFC* to denote a representative chain complex defining that homology.

Theorem 7.6.1. *Suppose* $(Y_1, \Gamma_1, \mathcal{Z}, \phi_1)$ *and* $(Y_2, \Gamma_2, -\mathcal{Z}, \phi_2)$ *are two bordered sutured manifolds that glue along* $F = F(\mathcal{Z})$ *to form the sutured manifold* (Y, Γ)*. Let* $\mathfrak{s}_i \in \mathrm{Spin}^c(Y_i, \partial Y_i \backslash F)$ *be relative* Spin^c*–structures for* $i = 1, 2$*. Then there is a graded chain homotopy equivalence*

$$\bigoplus_{\mathfrak{s}|_{Y_i} = \mathfrak{s}_i} \mathit{SFC}(Y, \Gamma, \mathfrak{s}) \simeq \widehat{\mathit{BSA}}(Y_1, \Gamma_1, \mathfrak{s}_1) \boxtimes_{\mathcal{A}(\mathcal{Z})} \widehat{\mathit{BSD}}(Y_2, \Gamma_2, \mathfrak{s}_2),$$

provided that at least one of the modules on the right hand-side comes from an admissible diagram.

To identify the gradings, we use the fact that the combined grading set $\underline{\mathrm{Gr}}(Y_1, \mathfrak{s}_1) \times_{\underline{\mathrm{Gr}}(\mathcal{Z})} \underline{\mathrm{Gr}}(Y_2, \mathfrak{s}_2)$ *distinguishes the individual* Spin^c*–structures* $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y)$ *by their homological components, while the Maslov component agrees with the SFH grading on each* $\mathit{SFC}(Y, \mathfrak{s})$*.*

Corollary 7.6.2. *In terms of modules and derived tensor products, the pairing theorem can be expressed as*

$$\bigoplus_{\mathfrak{s}|_{Y_i} = \mathfrak{s}_i} \mathit{SFC}(Y, \Gamma, \mathfrak{s}) \simeq \widehat{\mathit{BSA}}(Y_1, \Gamma_1, \mathfrak{s}_1) \,\widetilde{\otimes}\, \widehat{\mathit{BSDD}}(Y_2, \Gamma_2, \mathfrak{s}_2).$$

Corollary 7.6.2 is a restatement of Theorem 7.6.1 in purely $\mathcal{A}_\infty$–module language. This allows us to dispose of type D structures entirely. However, in practice, the definition of the derived tensor product $\widetilde{\otimes}$ involves an infinitely generated chain complex, while that of $\boxtimes$ only a finitely generated chain complex (assuming both sides are finitely generated).

Proof of Theorem 7.6.1. We can prove the theorem using nice diagrams, similar to [LOT09, Chapter 8].

Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are nice diagrams for Y_1 and Y_2, respectively. If we glue them to get a diagram $\mathcal{H} = \mathcal{H}_1 \cup_{\mathbf{Z}} \mathcal{H}_2$ for $Y = Y_1 \cup_F Y_2$, then $\mathcal{H}$ is also a nice diagram. Indeed, the only regions that change are boundary regions, which are irrelevant, and regions adjacent to a Reeb chord. In the latter case, two rectangular regions in $\mathcal{H}_1$ and $\mathcal{H}_2$, that border the same Reeb chord, glue to a single rectangular region in $\mathcal{H}$.

Generators in $\mathcal{G}(\mathcal{H})$ correspond to pairs of generators in $\mathcal{G}(\mathcal{H}_1)$ and $\mathcal{G}(\mathcal{H}_2)$ that occupy complementary sets of arcs. Provincial bigons and rectangles in $\mathcal{H}_i$ are also bigons and rectangles in $\mathcal{H}$. The only other regions in $\mathcal{H}$ that contribute to the differential ∂ on SFC are rectangles that are split into two rectangles in $\mathcal{H}_1$ and $\mathcal{H}_2$, each of which is adjacent to the same Reeb chord ρ in $\mathbf{Z}$. Such rectangles contribute terms of the form $(m_2 \otimes \mathrm{id}_{\widehat{BSD}(\mathcal{H}_2)}) \circ (\mathrm{id}_{\widehat{BSA}(\mathcal{H}_1)} \otimes \delta)$. Overall, terms in $\partial : \mathit{SFC}(\mathcal{H}) \to \mathit{SFC}(\mathcal{H})$ are in a one-to-one correspondence with terms in $\partial : \widehat{BSA}(\mathcal{H}_1) \boxtimes \widehat{BSD}(\mathcal{H}_2) \to \widehat{BSA}(\mathcal{H}_1) \boxtimes \widehat{BSD}(\mathcal{H}_2)$.

This shows that there is an isomorphism of chain complexes

$$\mathit{SFC}(\mathcal{H}) \cong \widehat{BSA}(\mathcal{H}_1) \boxtimes \widehat{BSD}(\mathcal{H}_2).$$

The splitting into Spin^c–structures and the equivalence of the gradings follow from Theorems 6.3.1 and 6.4.2, where the latter is applied to $(Y_1, \Gamma_1, -\varnothing \cup \mathcal{Z})$ and $(Y_2, \Gamma_2, -\mathcal{Z} \cup \varnothing)$. $\square$

Chapter 8

Bimodule invariants

As promised in the introduction, we will associate to a decorated sutured cobordism, a special type of $\mathcal{A}_\infty$–bimodule. We will sketch the construction, which closely parallels the discussion of bimodules in [LOT10a]. The reader is encouraged to look there, especially for a careful discussion of the algebra involved.

8.1 Algebraic preliminaries

The invariants we will define have the form of *type DA structures*, which is a combination of a type D structure and an $\mathcal{A}_\infty$–module.

Definition 8.1.1. *Let A and B be differential graded algebras with differential and multiplication denoted ∂_A, ∂_B, μ_A, and μ_B, respectively. A* type DA structure *over A and B is a graded vector space M, together with a collection of homogeneous operations $m_k\colon M \otimes B^{\otimes(k-1)} \to A \otimes M[2-k]$, satisfying the compatibility condition*

$$\sum_{p=1}^{k}(\mu_A \otimes \mathrm{id}_M) \circ (\mathrm{id}_A \otimes m_{k-p+1}) \circ (m_p \otimes \mathrm{id}_{B^{\otimes(k-p)}}) + (\partial_A \otimes \mathrm{id}_M) \circ m_k$$
$$+ \sum_{p=0}^{k-2} m_k \circ (\mathrm{id}_M \otimes \mathrm{id}_{B^{\otimes p}} \otimes \partial_B \otimes \mathrm{id}_{B^{\otimes(k-p-2)}})$$
$$+ \sum_{p=0}^{k-3} m_k \circ (\mathrm{id}_M \otimes \mathrm{id}_{B^{\otimes p}} \otimes \mu_B \otimes \mathrm{id}_{B^{\otimes(k-p-3)}}) = 0,$$

for all $k \geq 0$.

We can also define $m_k^i\colon M \otimes B^{\otimes(k-1)} \to A^{\otimes i} \otimes M[1+i-k]$, such that $m_1^0 = \mathrm{id}_M$, $m_k^0 = 0$ for $k > 1$, $m_k^1 = m_k$, and m_k^i is obtained by iterating m_*^1:

$$m_k^i = \sum_{j=0}^{k-1} (\mathrm{id}_{A^{\otimes(i-1)}} \otimes m_{j+1}) \circ (m_{k-j}^{i-1} \otimes \mathrm{id}_{B^{\otimes j}}).$$

In the special case where A is the trivial algebra $\{1\}$, this is exactly the definition of a right $\mathcal{A}_\infty$–module over B. In the case when B is trivial, or we ignore m_k^i for $k \geq 2$, this is exactly the definition of a left type D structure over A. In that case m_1^i corresponds to δ_i.

We will use some notation from [LOT10a] and denote a type DA structure over A and B by ${}^AM_B$. In the same vein, a type D structure over A is ${}^AM$, and a right $\mathcal{A}_\infty$–module over B is M_B. We can extend the tensor $\boxtimes$ to type DA structures as follows.

Definition 8.1.2. *Let* ${}^AM_B$ *and* ${}^BN_C$ *be two type DA structures, with operations* m_k^i, *and* n_l^j, *respectively. Let* ${}^AM_B \boxtimes_B {}^BN_C$ *denote the type DA structure* ${}^A(M \otimes N)_C$, *with operations*

$$(m \boxtimes n)_k^i = \sum_{j\geq 1} (m_j^i \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes n_k^{j-1}).$$

In the case when A and C are both trivial, this coincides with the standard operation $M_B \boxtimes {}^BN$.

The constructions generalize to mixed multi-modules of type ${}^{A_1,\dots,A_i}_{B_1,\dots,B_j}M^{C_1,\dots,C_k}_{D_1,\dots,D_l}$. Such a module is left, respectively right, type D with respect to A_p, respectively C_p, and left, respectively right $\mathcal{A}_\infty$–module with respect to B_p, respectively D_p. The category of such modules is denoted ${}^{A_1,\dots,A_i}_{B_1,\dots,B_j}\mathrm{Mod}^{C_1,\dots,C_k}_{D_1,\dots,D_l}$. We can apply the tensor $\boxtimes_X$ to any pair of such modules, as long as one of them has X as an upper (lower) right index, and the other has X as a lower (upper) left index.

We will only use a few special cases of this construction. The most important one is to associate to ${}^AM_B$ a canonical A, B $\mathcal{A}_\infty$–bimodule ${}_A(A \boxtimes M)_B = {}_AA_A \boxtimes_A {}^AM_B$. This allows us to bypass type D and type DA structures. In particular,

$${}_A(A \boxtimes M)_B \ \widetilde{\otimes}_B \ {}_B(B \boxtimes N)_C \simeq {}_A(A \boxtimes (M \boxtimes_B N))_C.$$

8.2 $\widehat{BSD}$ and $\widehat{BSA}$ revisited

Recall that the definition of $\widehat{BSD}$ counted a subset of the moduli spaces used to define $\widehat{BSA}$, and interpreted them differently. This operation can in fact be described completely algebraically. For any arc diagram $\mathcal{Z}$, there is a bimodule (or *type DD structure*) ${}^{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}\mathbb{I}$, such that

$$^{\mathcal{A}(-\mathcal{Z})}\widehat{BSD}(\mathcal{H}, J) = \widehat{BSA}(\mathcal{H}, J)_{\mathcal{A}(\mathcal{Z})} \boxtimes_{\mathcal{A}(\mathcal{Z})} {}^{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}\mathbb{I}.$$

In fact, we could use this as the definition of $\widehat{BSD}$, and use the naturallity of $\boxtimes$ to prove that it is well-defined for $\mathcal{H}$ and J, and its homotopy type is an invariant of the underling bordered sutured manifold.

8.3 Bimodule categories

For two differential graded algebras A and B, the notion of a left-left A, B–module is exactly the same as that of a left $A \otimes B$–module. Similarly, a left type D structure over A and B is exactly the same as a left type D structure over $A \otimes B$. In other words, we can interpret a module ${}^{A,B}M$ as ${}^{A\otimes B}M$, and vice versa, and the categories ${}^{A,B}\mathrm{Mod}$ and ${}^{A\otimes B}\mathrm{Mod}$ are canonically identified.

The situation is not as simple for $\mathcal{A}_\infty$–modules. The categories $\mathrm{Mod}_{A,B}$ and $\mathrm{Mod}_{A\otimes B}$ are *not* the same, or even equivalent. Fortunately, there is a canonical functor $\mathcal{F}\colon \mathrm{Mod}_{A\otimes B} \to \mathrm{Mod}_{A,B}$ which induces an equivalence of the derived categories. For this result, and the precise definition of $\mathcal{F}$ see [LOT10a].

8.4 $\widehat{BSDA}$ and $\widehat{BSDA}_M$

We will give two definitions of the bimodules. One is purely algebraic, and allows us to easily deduce that the bimodules are well-defined and invariant, while the other is more analytic, but is more useful in practice.

Definition 8.4.1. *Suppose $(Y, \Gamma, \mathcal{Z}_1 \cup \mathcal{Z}_2, \phi)$ is a bordered sutured manifold—or equivalently, a decorated sutured cobordism from $F(-\mathcal{Z}_1)$ to $F(\mathcal{Z}_2)$. Note that $\mathcal{A}(\mathcal{Z}_1 \cup \mathcal{Z}_2) = \mathcal{A}(\mathcal{Z}_1) \otimes \mathcal{A}(\mathcal{Z}_2)$. Define*

$$^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{BSDA}(Y, \Gamma, \mathfrak{s})_{\mathcal{A}(\mathcal{Z}_2)} = \mathcal{F}(\widehat{BSA}(Y, \Gamma, \mathfrak{s}))_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)} \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)}\mathbb{I}.$$

The invariance follows easily from the corresponding results for $\widehat{BSA}$ and naturallity.

As promised, below we give a more practical construction. Fix a provincially admissible diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}_1 \cup \mathcal{Z}_2, \psi)$, and an admissible almost complex structure J.

Recall that to define both $\widehat{BSD}$ and $\widehat{BSA}$, we looked at moduli spaces $\mathcal{M}^B_{\text{emb}}(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P})$, where $\overrightarrow{P}$ is a partition of the e punctures on the source $S^\triangleright$. In our case, we can distinguish two sets of e punctures—those labeled by Reeb chords in $\mathcal{Z}_1$, and those labeled by Reeb chords in $\mathcal{Z}_2$. We denote the two sets by E_1 and E_2, respectively. Any partition $\overrightarrow{P}$ restricts to partitions $\overrightarrow{P}_i = \overrightarrow{P}|_{E_i}$ on the two sets.

Definition 8.4.2. *Define the moduli space*

$$\mathcal{M}^B_{\text{emb}}(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P}_1, \overrightarrow{P}_2) = \bigcup_{\overrightarrow{P}|_{E_i} = \overrightarrow{P}_i} \mathcal{M}^B_{\text{emb}}(\mathbf{x}, \mathbf{y}, S^\triangleright, \overrightarrow{P}),$$

with index

$$\begin{aligned} \operatorname{ind}(B, \overrightarrow{\boldsymbol{\rho}}_1, \overrightarrow{\boldsymbol{\rho}}_2) = e(B) &+ n_x(B) + n_y(B) \\ &+ \#\overrightarrow{\boldsymbol{\rho}}_1 + \#\overrightarrow{\boldsymbol{\rho}}_2 + \iota(\overrightarrow{\boldsymbol{\rho}}_1) + \iota(\overrightarrow{\boldsymbol{\rho}}_2), \end{aligned}$$

where $\overrightarrow{\boldsymbol{\rho}}_i = \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P}_i)$ is a sequence of sets of Reeb chords in $\mathcal{Z}_i$, for $i = 1, 2$.

This has the effect of forgetting about the relative height of punctures in E_1 to those in E_2. Its algebraic analogue is applying the functor $\mathcal{F}$, which combines the algebra actions $m_3(x, a \otimes 1, 1 \otimes b)$ and $m_3(x, 1 \otimes b, a \otimes 1)$ into $m_{1,1,1}(x, a, b)$.

The general idea is to treat the $\mathcal{Z}_1$ part of the arc diagram as in $\widehat{BSD}$, and the $\mathcal{Z}_2$ part as in $\widehat{BSA}$. First, to a generator $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ we associate idempotents $I_1(\bar{o}(\mathbf{x})) \in \mathcal{I}(-\mathcal{Z}_1)$ and $I_2(o(\mathbf{x})) \in \mathcal{I}(\mathcal{Z}_2)$, corresponding to unoccupied arcs on the $\mathcal{Z}_1$ side, and occupied arcs on the $\mathcal{Z}_2$ side, respectively. Next, we will look at discrete partitions $\overrightarrow{P}_1 = (\{q_1\}, \ldots, \{q_i\})$ on the $\mathcal{Z}_1$ side, while allowing arbitrary partitions $\overrightarrow{P}_2$ on the $\mathcal{Z}_2$ side.

If the punctures in $\overrightarrow{P}_1$ are labeled by the Reeb chords $(\rho_1, \dots, \rho_i)$, set

$$a_1(\mathbf{x}, \mathbf{y}, \overrightarrow{P}_1) = I_1(\overline{o}(\mathbf{x})) \cdot a(-\rho_1) \cdots a(-\rho_i) \cdot I_1(\overline{o}(\mathbf{y})) \in \mathcal{A}(-\mathcal{Z}_1).$$

If the sets of punctures in $\overrightarrow{P}_2$ are labeled by some sequence of sets of Reeb chords $(\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_j)$, set

$$a_2(\mathbf{x}, \mathbf{y}, \overrightarrow{P}_2) = I_2(\mathbf{x}) \cdot a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_j) \cdot I_2(\mathbf{y}) \in \mathcal{A}(\mathcal{Z}_2)^{\otimes j}.$$

Definition 8.4.3. *Fix $\mathcal{H}$, J, and $\mathfrak{s}$. Let $\widehat{BSDA}(\mathcal{H}, J, \mathfrak{s})$ be freely generated over $\mathbb{Z}/2$ by $\mathcal{G}(\mathcal{H}, \mathfrak{s})$, with $\mathcal{I}(-\mathcal{Z}_1)$ and $\mathcal{I}(\mathcal{Z}_2)$ actions*

$$I(s_1) \cdot \mathbf{x} \cdot I(s_2) = \begin{cases} \mathbf{x} & \text{if } s_1 = \overline{o}(\mathbf{x}) \text{ and } s_2 = o(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

It has type DA operations

$$m_k(\mathbf{x}, b_1, \dots, b_{k-1}) = \sum_{\substack{\operatorname{ind}(B, \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P}_1), \overrightarrow{\boldsymbol{\rho}}(\overrightarrow{P}_2)) = 1 \\ a_2(\mathbf{x}, \mathbf{y}, \overrightarrow{P}_2) = b_1 \otimes \cdots \otimes b_{k-1}}} \#\mathcal{M}^B_{\mathrm{emb}}(\mathbf{x}, \mathbf{y}, S^{\triangleright}, \overrightarrow{P}_1, \overrightarrow{P}_2) \cdot a_1(\mathbf{x}, \mathbf{y}, \overrightarrow{P}_1) \otimes \mathbf{y}.$$

It is easy to check that Definitions 8.4.3 and 8.4.1 are equivalent. The operation of passing from a single partition to pairs of partitions corresponds to applying the functor $\mathcal{F}$, while the operation of restricting to discrete partitions on the $\mathcal{Z}_1$ side and multiplying the corresponding Reeb chords corresponds to the functor $\cdot \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} {}^{\mathcal{A}(-\mathcal{Z}_1), \mathcal{A}(\mathcal{Z}_1)}\mathbb{I}$.

We can use either definition to define

$$_{\mathcal{A}(-\mathcal{Z}_1)}\widehat{BSDA}_M(Y, \Gamma, \mathfrak{s})_{\mathcal{A}(\mathcal{Z}_2)} = {}_{\mathcal{A}(-\mathcal{Z}_1)}\mathcal{A}(-\mathcal{Z}_1)_{\mathcal{A}(-\mathcal{Z}_1)} \boxtimes_{\mathcal{A}(-\mathcal{Z}_1)} {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{BSDA}(Y, \Gamma, \mathfrak{s})_{\mathcal{A}(\mathcal{Z}_2)}.$$

As with the one-sided modules, there is a well-defined grading.

Theorem 8.4.4. *The grading $\underline{\mathrm{gr}}$ on $\widehat{BSDA}(Y, \Gamma, \mathfrak{s})$ is well-defined with values in $\underline{\mathrm{Gr}}(Y, s)$, and makes it a graded DA–structure. In particular, whenever $b \otimes \mathbf{y}$ is a summand in $m_k(\mathbf{x}, a_1, \dots, a_{k-1})$, we have*

$$\underline{\mathrm{gr}}(b) \cdot \underline{\mathrm{gr}}(\mathbf{y}) = \lambda^{k-2} \cdot \underline{\mathrm{gr}}(\mathbf{x}) \cdot \underline{\mathrm{gr}}(a_1) \cdots \underline{\mathrm{gr}}(a_{k-1}).$$

Proof. This is a straightforward combination of the arguments for the gradings on $\widehat{BSD}$ and $\widehat{BSA}$. □

8.5 Nice diagrams and pairing

The key results for bimodules allowing us to talk about a functor from the decorated sutured cobordism category $\mathcal{SD}$ to the category $\mathcal{D}$ of differential graded algebras and $\mathcal{A}_\infty$–bimodules are the full version of Theorem 4, and Theorem 5. Below we give a more precise version of Theorem 4, in the vein of Theorem 7.6.1.

Theorem 8.5.1. *Given two bordered sutured manifolds $(Y_1, \Gamma_1, -\mathcal{Z}_1 \cup \mathcal{Z}_2)$ and $(Y_2, \Gamma_2, -\mathcal{Z}_2 \cup \mathcal{Z}_3)$, representing cobordisms from $\mathcal{Z}_1$ to $\mathcal{Z}_2$ and from $\mathcal{Z}_2$ to $\mathcal{Z}_3$, respectively, there are graded homotopy equivalences of bimodules*

$$\bigoplus_{\mathfrak{s}|Y_i=\mathfrak{s}_i} \widehat{BSDA}(Y_1 \cup Y_2, \mathfrak{s}) \simeq \widehat{BSDA}(Y_1, \mathfrak{s}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSDA}(Y_2, \mathfrak{s}_2).$$

$$\bigoplus_{\mathfrak{s}|Y_i=\mathfrak{s}_i} \widehat{BSDA}_M(Y_1 \cup Y_2, \mathfrak{s}) \simeq \widehat{BSDA}_M(Y_1, \mathfrak{s}_1) \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSDA}_M(Y_2, \mathfrak{s}_2).$$

The gradings are identified in the sense of Theorem 6.4.2.

The proof is completely analogous to that of Theorem 7.6.1. It relies on the combinatorial form of $\widehat{BSDA}$ from a nice diagram, and the fact that gluing two such diagrams also gives a nice diagram with direct correspondence of domains. The actual result for nice diagrams is given below.

Theorem 8.5.2. *For any nice diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}_1 \cup \mathcal{Z}_2, \psi)$ and any admissible almost complex structure J the domains that contribute to m_k are of the following types.*

1. *Provincial bigons and rectangles, which contribute terms of the form $I \otimes \mathbf{y}$ to $m_1(\mathbf{x})$.*

2. *Rectangles hitting a Reeb chord at $\mathcal{Z}_1$, which contribute terms of the form $a \otimes \mathbf{y}$ to $m_1(\mathbf{x})$.*

3. *Collections of rectangles hitting Reeb chords at $\mathcal{Z}_2$, at the same height, which contribute terms of the form $I \otimes \mathbf{y}$ to $m_2(\mathbf{x}, \ldots)$.*

Proof. The proof is the same as those for $\widehat{BSD}$ and $\widehat{BSA}$. The only new step is showing that there are no mixed terms, i.e. combinations of (2) and (3). In other words, the actions of $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$ commute for a nice diagram. The reason is that such a combined domain that hits both $\mathcal{Z}_1$ and $\mathcal{Z}_2$ decomposes into two domains that hit only one side each. There is no constraint of the relative heights, so such a domain will have index at least 2, and would not be counted. □

8.6 Bimodule of the identity

In this subsection we sketch the proof of Theorem 5. We prove a version for $\widehat{BSDA}$, which implies the original statement.

Definition 8.6.1. *Given an arc diagram $\mathcal{Z}$, define the bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$, which as an $\mathcal{I}(\mathcal{Z})$–bimodule is isomorphic to $\mathcal{I}(\mathcal{Z})$ itself, and whose nontrivial operations are*

$$m_2(I_i, a) = a \otimes I_f, \tag{8.1}$$

for all algebra elements $a \in \mathcal{A}(\mathcal{Z})$ with initial and final idempotents I_i and I_f, respectively. It is absolutely graded by $\underline{\mathrm{Gr}}(\mathcal{Z})$, as a subset of $\mathcal{A}(\mathcal{Z})$, i.e. all elements are graded 0.

It is easy to see that ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z}')} \cong {}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z}')}$ canonically, and that $\mathcal{A}(\mathcal{Z}) \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})} \simeq \mathcal{A}(\mathcal{Z})$.

Theorem 8.6.2. *The identity decorated sutured cobordism* $\mathrm{id}_{\mathcal{Z}} = (F(\mathcal{Z}) \times [0,1], \Lambda \times [0,1])$ *from $\mathcal{Z}$ to $\mathcal{Z}$ has a graded bimodule invariant*

$${}^{\mathcal{A}(\mathcal{Z})}\widehat{BSDA}(\mathrm{id}_{\mathcal{Z}})_{\mathcal{A}(\mathcal{Z})} \simeq {}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}.$$

Proof (sketch). The proof is essentially the same as that of the corresponding statement for pure bordered identity cobordisms in [LOT10a]. First we look at an appropriate Heegaard diagram $\mathcal{H}$ for $\mathrm{id}_{\mathcal{Z}}$. For any $\mathcal{Z}$ there is a canonical diagram of the form in Figure 10a, only here we interpret the left side as the $-\mathcal{Z}$, or type D, portion of the boundary, while the right side is the $+\mathcal{Z}$, or $\mathcal{A}_\infty$–type, portion. Indeed, choosing which of the right arcs are occupied in a generator determines it uniquely, and $\mathcal{G}(\mathcal{H})$ is a one-to-one correspondence with elementary idempotents. Thus the underlying space for $\widehat{BSDA}(\mathrm{id}_{\mathcal{Z}})$ is $\mathcal{I}(\mathcal{Z})$. For any Reeb chord ρ of length one there is a convex octagonal domain in $\mathcal{H}$ that makes Eq. (8.1) hold for $a = a(\rho, s)$, for any such ρ, and any completion s.

The rest of the proof is algebraic. Any bimodule with underlying module $\mathcal{I}(\mathcal{Z})$ corresponds to some $\mathcal{A}_\infty$–algebra morphism $\phi\colon \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z})$. We compute the homology of $\mathcal{A}(\mathcal{Z})$ and show it is Massey generated by length one Reeb chords as above. Since Eq. (8.1) holds for such elements, ϕ is a quasi-isomorphism. By Theorem 8.5.1, we know $\widehat{BSDA}(\mathrm{id}_{\mathcal{Z}})$ squares to itself, and so does ϕ, i.e. $\phi \circ \phi \simeq \phi$. Since it is a quasi isomorphism, it is homotopic to the identity morphism, and Eq. (8.1) holds for all a, up to homotopy equivalence.

Finally, for the grading, $\underline{\mathrm{Gr}}(-\mathcal{Z} \cup \mathcal{Z})$ has two copies of $H_1(F(\mathcal{Z}))$, with opposite pairings. For all Spin^c–structures, there are obvious periodic domains, such that $\pi_2(\mathbf{x}, \mathbf{x}) = H_1(F)$. Taking the quotient by the stabilizer subgroup identifies the subgroups $\underline{\mathrm{Gr}}(-\mathcal{Z})$ and $\underline{\mathrm{Gr}}(\mathcal{Z})$ by the canonical anti-isomorphism. All domains have vanishing Maslov grading and canceling homological gradings, so in each Spin^c–structure all generators have the same relative grading. Thus, we can identify it with an absolute grading where all gradings are 0. $\square$

Chapter 9

Applications

In this section we describe some applications of the new invariants. First, as a warm-up we describe how both sutured Floer homology and the regular bordered Floer homology appear as special cases of bordered sutured homology. Then we describe how we can recover the sutured Floer homology of a manifold with boundary from its bordered invariants.

Another application is a new proof for the surface decomposition formula [Juh08, Theorem1.3] of Juhász.

9.1 Sutured Floer homology as a special case

We have already seen that for a bordered sutured manifold $(Y, \Gamma, \varnothing)$, the bordered sutured invariants coincide with the sutured ones. However, there are many more cases when this happens. In fact, for any *balanced* bordered sutured manifold, the $\widehat{BSD}$ and $\widehat{BSA}$ invariants still reduce to *SFH*, no matter what the arc diagram is.

Theorem 9.1.1. *Let* (Y, Γ) *be a balanced sutured manifold, and* $\phi\colon G(\mathcal{Z}) \to \partial Y$ *be a parametrization of any part of* (Y, Γ) *by an arc diagram* $\mathcal{Z}$ *with* k *matched pairs. Let* (SFC, ∂) *be the sutured chain complex for* (Y, Γ)*.*

The following statements hold.

1. $(\widehat{BSA}(Y, \Gamma, \mathcal{Z}, \phi), m_1) \simeq (SFC(Y, \Gamma), \partial)$*, where* $\mathcal{A}(\mathcal{Z}, 0) = \{I(\varnothing)\}$ *acts by identity on* $\widehat{BSA}$ *and* $\mathcal{A}(\mathcal{Z}, k)$ *kills it for any* $k > 0$*.*

2. $\widehat{BSD}(Y,\Gamma,\mathcal{Z},\phi)\cong SFC(Y,\Gamma)$ *as a set, with*

$$\delta(x)=I\otimes\partial(x),$$

where $I=I(\{1,\dots,k\})$ *is the unique idempotent in* $\mathcal{A}(-\mathcal{Z},k)$.

3. $\widehat{BSDD}(Y,\Gamma,\mathcal{Z},\phi)\simeq\mathcal{A}(-\mathcal{Z},k)\otimes SFC(Y,\Gamma)$ *as a product of chain complexes, with the standard action of* $\mathcal{A}(-\mathcal{Z})$ *on* $\mathcal{A}(-\mathcal{Z},k)$ *on the left.*

Proof. Let $\mathcal{H}$ be a provincially admissible Heegaard diagram for $(Y,\Gamma,\mathcal{Z},\phi)$. If we erase $\mathbf{Z}$ and $\boldsymbol{\alpha}^a$ from the diagram, we obtain an admissible sutured diagram $\mathcal{H}'$ for (Y,Γ). (Indeed, any periodic domain for $\mathcal{H}'$ is a provincial periodic domain for $\mathcal{H}$.)

Remember that for a balanced, i.e. 0–unbalanced manifold, each generator occupies 0 arcs in $\boldsymbol{\alpha}^a$. In particular $\mathcal{G}(\mathcal{H})=\mathcal{G}(\mathcal{H}')$.

Let $u\in\mathcal{M}^B(\mathbf{x},\mathbf{y},S^{\triangleright},\overrightarrow{P})$ be a strongly boundary monotonic curve. Let $o_t(u)$ denote the set of $\alpha\in\boldsymbol{\alpha}$, for which $u^{-1}(\alpha\times\{1\}\times\{t\})$ is nonempty. Since $\mathbf{x}$ occupies only α circles, $o_t(u)\subset\boldsymbol{\alpha}^c$ for small t. The only changes in $o_t(u)$ can occur at the heights of e punctures. But at an e puncture, the boundary goes over a Reeb chord, so $o_t(u)$ can only change by replacing some arc in $\boldsymbol{\alpha}^a$ with another. Therefore, $o_t(u)\subset\boldsymbol{\alpha}^c$ for all $t\in\mathbb{R}$, and $S^{\triangleright}$ has no e punctures. Thus, u is a curve with no e punctures and doesn't involve $\boldsymbol{\alpha}^a$. But these are exactly the curves from $\mathcal{H}'$ counted in the definition of *SFH*.

Therefore, the curves counted for the definitions of $\widehat{BSD}$ and $\widehat{BSA}$ from $\mathcal{H}$ are in a one-to-one correspondence with curves counted for the definition of *SFH* from $\mathcal{H}'$. Moreover, in $\widehat{BSD}$ and $\widehat{BSA}$ these curves are all provincial.

Algebraically, in $\widehat{BSD}$ a provincial curve from $\mathbf{x}$ to $\mathbf{y}$ contributes $1\otimes\mathbf{y}$ to $\delta(\mathbf{x})$. In $\widehat{BSA}$ it contributes $\mathbf{y}$ to $m_1(\mathbf{x})$. Finally, in *SFH* it contributes $\mathbf{y}$ to $\partial(\mathbf{x})$. The first two statements follow. The last is a trivial consequence of the definition of $\widehat{BSDD}$. □

In particular, the interesting behavior of the bordered sutured invariants occurs when the underlying sutured manifold is unbalanced. In that case sutured Floer homology is not defined, or is trivially set to 0.

9.2 Bordered Floer homology as a special case

The situation in this section is the opposite of that in the previous one. Here we show that if we look at manifolds that are, in a sense, maximally unbalanced, the bordered sutured invariants reduce to purely bordered invariants.

First we recall a basic result from [Juh06].

Proposition 9.2.1. *Let $\mathcal{C}$ denote the collection of (homeomorphism classes of) closed connected 3–manifolds, and $\mathcal{C}'$ denote the collection of (equivalence classes of) sutured 3–manifolds with one boundary component homeomorphic to S^2, and a single suture on it. The following statements hold.*

1. *$\mathcal{C}$ and $\mathcal{C}'$ are in a one-to-one correspondence given by the map*

$$\xi\colon\ \mathcal{C} \to \mathcal{C}',$$

where $\xi(Y)$ is obtained by removing an open 3–ball from Y, and putting a single suture on the boundary.

2. *There is a homotopy equivalence $\widehat{CF}(Y) \simeq SFC(\xi(Y))$.*

The correspondence is most evident on the level of Heegaard diagrams, where a diagram for $\xi(Y)$ is obtained from a diagram for Y by cutting out a small disc around the basepoint.

There is a natural extension of this result to the bordered category.

Theorem 9.2.2. *Let $\mathcal{B}$ denote the collection of (equivalence classes of) bordered manifolds with one boundary component, and let $\mathcal{B}'$ denote the collection of (equivalence classes of) bordered sutured manifolds of the following form. $(Y, \Gamma, \mathcal{Z}, \phi) \in \mathcal{B}'$ if and only if $D = \partial Y \setminus F(\mathcal{Z})$ is a single disc D and $\Gamma \cap D$ is a single arc. The following statements hold.*

1. *$\mathcal{B}$ and $\mathcal{B}'$ are in a one-to-one correspondence given by the map*

$$\zeta\colon\ \mathcal{B} \to \mathcal{B}',$$

which to a bordered manifold Y parametrized by $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ associates a bordered sutured manifold $\zeta(Y) = (Y, Z, \mathcal{Z}', \phi)$, parametrized by $\mathcal{Z}' = (Z \setminus D, \mathbf{a}, M)$, where D is a small neighborhood of z.

2. For any $Y \in \mathcal{B}$, we have

$$\widehat{BSD}(\zeta(Y)) \simeq \widehat{CFD}(Y),$$
$$\widehat{BSA}(\zeta(Y)) \simeq \widehat{CFA}(Y).$$

3. If Y_1 and Y_2 are bordered manifolds that glue to form a closed manifold Y, then $\zeta(Y_1)$ and $\zeta(Y_2)$ glue to form $\xi(Y)$.

Proof. In the bordered setting the parametrization of $F(\mathcal{Z}) = \partial Y$ means that there is a self-indexing Morse function f on F with one index–0 critical point p, one index–2 critical point q, and $2k$ many index–1 critical points $r_1, \ldots, r_{2k}$. The circle Z is the level set $f^{-1}(3/2)$, the basepoint z is the intersection of a gradient flowline from p to q with Z, and the matched points $M^{-1}(i) \in \mathbf{a}$ are the intersections of the flowlines from r_i with Z.

Note that $F' = F \setminus D$ is a surface with boundary, parametrized by the arc diagram $\mathcal{Z}' = (Z', \mathbf{a}, M)$, where $Z' = Z \setminus D$. Indeed, if we take D to be a neighborhood of a flowline from p to q, then $f|_{F'}$ is a self indexing Morse function for F' with only index 1–critical points, and their stable manifolds intersect the level set Z' at the matched points $\mathbf{a}$.

Moreover, the circle Z separates F into two regions—a disc R_+ around the index–2 critical point q, and a genus k surface R_- with one boundary component. Thus, (Y, Z) is indeed sutured, and the arc Z' embeds into the suture Z. Since $D \cap Z$ is an arc, the manifold we get is indeed in $\mathcal{B}'$.

To see that the construction is reversible we need to check that for any $(Y, \Gamma, \mathcal{Z}, \phi) \in \mathcal{B}'$ there is only one suture in Γ, $\mathbf{Z}$ has only one component, and R_+ is a disc. Indeed, $\mathbf{Z} \cap \Gamma$ consists only of properly embedded arcs in $F(\mathcal{Z})$. But $\Gamma \cap \partial F = \Gamma \cap \partial D$ consists of two points, and therefore there is only one arc. Now $\Gamma = (\Gamma \cap F) \cup (\Gamma \cap D)$ is a circle, and $R_+ \cap F$ is half a disc, so R_+ is a disc. This proves (1).

To see (2), we will investigate the correspondence on Heegaard diagrams. Consider a boundary compatible Morse function f on a bordered 3–manifold Y. On the boundary it behaves as described in the first part of the proof. In the interior, there are only index–1 and index–2 critical points. Let B be a neighborhood in Y of the flowline from the index–0 to the index–3 critical point, which are the index–0 and index–2 critical points on the surface.

Then D is precisely $B \cap \partial Y$. Let $Y' = Y \setminus B$. Topologically, passing from Y to Y' has no effect, except for canceling the two critical points. Now $f|_{Y'}$ is a boundary compatible Morse function for the bordered sutured manifold $Y' = \zeta(Y)$. One can verify this is the same construction as above, except we have pushed D slightly into the manifold.

Looking at the Heegaard diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta})$, compatible with f and $f|_{Y'}$, respectively, one can see that the effect of removing B on $\mathcal{H}$ is that of removing a neighborhood of the basepoint $z \in \partial\Sigma$. Now $\mathbf{Z} = \partial\Sigma \setminus \nu(z)$, the Reeb chords correspond, and $\partial\Sigma' \setminus \mathbf{Z}$ is a small arc in the region where z used to be.

Recall that the definitions of $\widehat{CFD}$ and $\widehat{CFA}$ on one side, and $\widehat{BSD}$ and $\widehat{BSA}$ on the other, are the same, except that $\partial\Sigma' \setminus \mathbf{Z}$ in the latter plays the role of z in the former. Therefore the corresponding moduli spaces $\mathcal{M}$ exactly coincide, and for these particular diagrams there is actual equality of the invariants, proving (2).

For (3), it is enough to notice that $Y = Y_1 \cup_F Y_2$, while $\zeta(Y_1) \cup_{F \setminus D} \zeta(Y_2) = Y_1 \cup_{F \setminus D} Y_2$, which is Y minus a ball. $\square$

9.3 From bordered to sutured homology

In the current section we prove Theorem 1, which was the original motivation for developing the theory of bordered sutured manifolds and their invariants. Recall that it states that for any set of sutures on a bordered manifold, the sutured homology can be obtained from the bordered homology in a functorial way. A refined version is given below.

Theorem 9.3.1. *Let F be a closed connected surface parametrized by some pointed matched circle Z. Let Γ be any set of sutures on F, i.e. an oriented multi curve in F that divides it into positive and negative regions R_+ and R_-.*

There is a (non unique) left type D structure $\widehat{CFD}(\Gamma)$ over $\mathcal{A}(\mathcal{Z})$, with the following property. If Y is any 3–manifold, such that ∂Y is identified with F, making (Y, Γ) a sutured manifold, then

$$SFC(Y, \Gamma) \simeq \widehat{CFA}(Y) \boxtimes \widehat{CFD}(\Gamma).$$

Similarly, there is a (non unique) right $\mathcal{A}_\infty$*–module* $\widehat{CFA}(\Gamma)$ *over* $\mathcal{A}(-\mathcal{Z})$*, such that*

$$SFC(Y,\Gamma) \simeq \widehat{CFA}(\Gamma) \boxtimes \widehat{CFD}(Y).$$

Before we begin the proof, we will note that although $\widehat{CFD}(\Gamma)$ and $\widehat{CFA}(\Gamma)$ are not unique (not even up to homotopy equivalence), they can be easily made so by fixing some extra data. The exact details will become clear below.

Proof. Fix the surface F, pointed matched circle $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$, and the sutures Γ. Repeating the discussion in the proof of Theorem 9.2.2, the parametrization of F means that there is a self-indexing Morse function f on F with exactly one index–0 critical point, and exactly one index–2 critical point, where the circle Z is the level set $f^{-1}(3/2)$.

The choice that breaks uniqueness is the following. Isotope Γ along F until one of the sutures γ becomes tangent to Z at the basepoint z, and so that the orientations of Z and γ agree. Let D be a disc neighborhood of z in F. We can further isotope γ until $\gamma \cap D = Z \cap D$. We will refer to this operation as *picking a basepoint, with direction, on* Γ.

Let $F' = F \setminus D$, and let P be the 3–manifold $F' \times [0,1]$. Let Δ be a set of sutures on P, such that

$$(F' \times \{1\}) \cap \Delta = (F' \cap \Gamma) \times \{1\},$$

$$(F' \times \{0\}) \cap \Delta = (F' \cap Z) \times \{0\},$$

$$(\partial D \times [0,1]) \cap \Delta = (\Gamma \cap \partial D) \times [0,1].$$

We orient Δ so that on the "top" surface $F' \times \{1\}$ its orientation agrees with Γ, its orientation on the "bottom" is opposite from Z, and on $\partial D \times [0,1]$ the two segments are oriented opposite from each other.

As in Theorem 9.2.2, F' is parametrized by the arc diagram $\mathcal{Z}' = (Z \setminus D, \mathbf{a}, M)$. Therefore the "bottom" of P, i.e. $F' \times \{0\}$ is parametrized by $-\mathcal{Z}'$. (Indeed $-(Z \setminus D)$ is part of Δ.) This makes (P, Δ) into a bordered sutured manifold, parametrized by $-\mathcal{Z}'$.

Isotopies of Γ outside of D have no effect on P, except for an isotopy of Δ in the non parametrized part of ∂P. Therefore the bordered sutured manifold P is an invariant of F, Γ, and the choice of basepoint on Γ.

Define

$$\widehat{CFD}(\Gamma) = \widehat{BSD}(P, \Delta),$$
$$\widehat{CFA}(\Gamma) = \widehat{BSA}(P, \Delta).$$

It is clear that their homotopy types are invariants of Γ and the choice of basepoint (with direction). Since $\mathcal{A}(\mathcal{Z}') = \mathcal{A}(\mathcal{Z})$, they are indeed modules over $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$, respectively.

To prove the rest of the theorem, consider any manifold Y with boundary $\partial Y = F$. By the construction in Theorem 9.2.2, $\zeta(Y)$ is the sutured manifold (Y, Z), where F' is parametrized by $\mathcal{Z}'$.

If we glue $\zeta(Y)$ and P along F', we get the sutured manifold

$$(Y \cup F' \times [0,1], (Z \setminus F') \cup (\Delta \setminus F' \times \{0\})).$$

The sutures consist of $Z \setminus F' = Z \cap D = \Gamma \cap D$, $\Delta \cap (\partial D \times [0,1]) = (\Gamma \cap \partial D) \times [0,1]$, and $\Delta \cap (F' \times \{1\}) = (\Gamma \setminus D) \times \{1\}$. Up to homeomorphism, $Y \cup F' \times [0,1] = Y$, and under that homeomorphism the sutures get collapsed to $\Gamma \subset F$. Therefore, $\zeta(Y) \cup_{F'} P$ is precisely (Y, Γ).

Using Theorem 9.2.2, $\widehat{BSD}(\zeta(Y)) \simeq \widehat{CFD}(Y)$, and $\widehat{BSA}(\zeta(Y)) \simeq \widehat{CFA}(Y)$. By Theorem 7.6.1,

$$SFC(Y, \Gamma) \simeq \widehat{BSA}(\zeta(Y)) \boxtimes \widehat{BSD}(P) \simeq \widehat{CFA}(Y) \boxtimes \widehat{CFD}(\Gamma),$$
$$SFC(Y, \Gamma) \simeq \widehat{BSA}(P) \boxtimes \widehat{BSD}(\zeta(Y)) \simeq \widehat{CFA}(\Gamma) \boxtimes \widehat{CFD}(Y). \qquad \square$$

9.4 Surface decompositions

The final application we will show is a new proof of the surface decomposition theorem of Juhász proved in [Juh08].

More precisely we prove the following statement.

Theorem 9.4.1. *Let (Y, Γ) be a balanced sutured manifold. Let S be a properly embedded surface in Y with the following properties. S has no closed components, and each component of ∂S intersects both R_- and R_+. (Juhász calls such a surface a* good decomposing surface*.)*

A Spin^c *structure $\mathfrak{s} \in \mathrm{Spin}^c(Y, \Gamma)$ is* outer *with respect to S if it is represented by a vector field v which is nowhere tangent to a normal vector to $-S$ (with respect to some metric).*

Let (Y', Γ') be the sutured manifold, obtained by decomposing Y along S. *More precisely, Y' is Y cut along S, such that $\partial Y' = \partial Y \cup +S \cup -S$, and the sutures Γ' are chosen so that $R_-(\Gamma') = \overline{R_-(\Gamma) \cup -S}$, and $R_+(\Gamma') = \overline{R_+(\Gamma) \cup +S}$. Here $+S$ (respectively $-S$) is the copy of S on $\partial Y'$, whose orientation agrees (respectively disagrees) with S.*

Then the following statement holds.

$$SFH(Y', \Gamma') \cong \bigoplus_{\mathfrak{s} \text{ outward to } S} SFH(Y, \Gamma, \mathfrak{s}).$$

Proof. We will consider three bordered sutured manifolds. Let $T = S \times [-2, 2] \subset Y$ be a neighborhood of S in Y (so the positive normal of S is in the $+$ direction). Let $W = \overline{Y \setminus T}$, and let $P = S \times ([-2, -1] \cup [1, 2]) \subset T$. We can assume that $\Gamma \cap \partial T$ consists of arcs parallel to the $[-2, 2]$ factor.

Put sutures on T, W and P in the following way. First, notice that $R_+(\Gamma) \cap \partial S$ consists of several arcs $a = \{a_1, \dots, a_n\}$. Let $A_+ \subset S$ be a collection of disjoint discs, such that $A_+ \cap \partial S = a$.

On T put sutures Γ_T, such that

$$R_+(\Gamma_T) \cap \partial Y = R_+(\Gamma) \cap \partial T,$$
$$R_+(\Gamma_T) \cap (S \times \{\pm 2\}) = A_+ \times \{\pm 2\}.$$

On W put sutures Γ_W, such that

$$R_+(\Gamma_W) \cap \partial Y = R_+(\Gamma) \cap \partial W,$$
$$R_+(\Gamma_W) \cap (S \times \{\pm 2\}) = A_+ \times \{\pm 2\}.$$

On P put sutures Γ_P, such that

$$R_+(\Gamma_P) \cap \partial Y = R_+(\Gamma) \cap \partial P,$$
$$R_+(\Gamma_P) \cap (S \times \{\pm 2\}) = A_+ \times \{\pm 2\},$$
$$R_+(\Gamma_P) \cap (S \times \{-1\}) = S \times \{-1\},$$
$$R_+(\Gamma_P) \cap (S \times \{1\}) = \varnothing.$$

Fix a parametrization of S by an arc diagram $\mathcal{Z}_S$ with k many arcs, such that the positive region of S is A_+. This is possible, since S has no closed components, and the arcs a hit every boundary component.

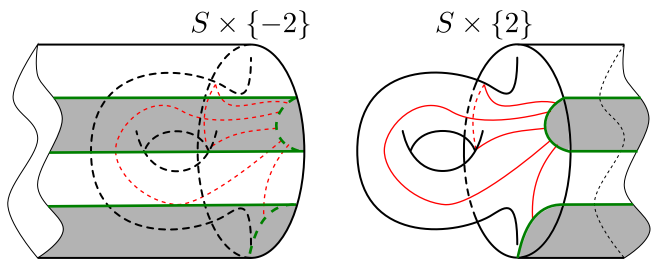
Parametrize the surfaces $S \times \{\pm 2\}$ in each of T, W, and P by $\pm\mathcal{Z}_S$, depending on orientation. If we set $U = S \times \{\pm 2\} \subset W$, with the boundary orientation from W, then U is parametrized by $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$, where $\mathcal{Z}_1 \cong \mathcal{Z}_S$ parametrizes $S \times \{-2\}$, and $\mathcal{Z}_2 \cong -\mathcal{Z}_S$ parametrizes $S \times \{2\}$. Thus, W is a bordered sutured manifold parametrized by $\mathcal{Z}$, while T and P are parametrized by $-\mathcal{Z}$ (see Figure 9). Moreover, gluing along the parametrization,

$$W \cup_U T = (Y, \Gamma),$$
$$W \cup_U P = (Y', \Gamma').$$

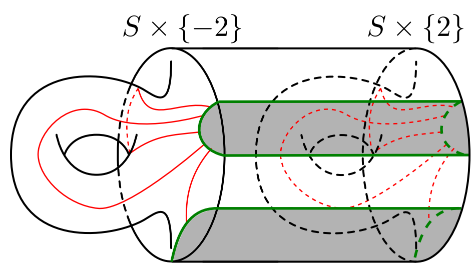
We will look at the relationship between $\widehat{BSD}(T)$ and $\widehat{BSD}(P)$. For simplicity we will assume S is connected. The argument easily generalizes to multiple connected components. Alternatively, it follows by induction. The Heegaard diagrams $\mathcal{H}_T$ and $\mathcal{H}_P$ for T and P are shown in Figure 10. Since all regions D have nonzero $\partial^\partial D$, the diagrams are automatically provincially admissible.

The algebra $\mathcal{A}(\mathcal{Z})$ splits as $\mathcal{A}(\mathcal{Z}_1) \otimes \mathcal{A}(\mathcal{Z}_2)$, and each idempotent $I \in \mathcal{I}(\mathcal{Z})$ splits as the product $I = I_1 \otimes I_2$, where $I_1 \in \mathcal{I}(\mathcal{Z}_1)$ and $I_2 \in \mathcal{I}(\mathcal{Z}_2)$. Moreover, I_1 is in a summand $\mathcal{I}(\mathcal{Z}, l)$ for some $l = 0, \dots, k$. Denote this number by $l(I)$. Intuitively, $l(I)$ means "how many arcs on the $\mathcal{Z}_1$ portion of $\mathcal{Z}$ does I occupy". Similarly, for a generator $\mathbf{x}$ we can define $l(\mathbf{x}) = l(I(\overline{o}(\mathbf{x}))$.

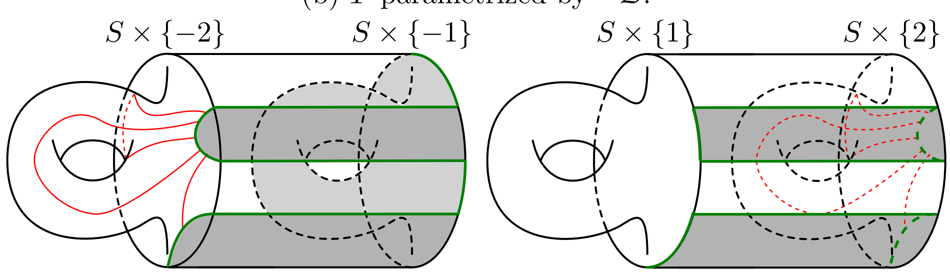
Notice that $\mathcal{H}_P$ has a unique generator $\mathbf{x}_P$, such that $l(\mathbf{x}_P) = k$. Moreover, there are only two regions in the diagram, and both of them are boundary regions. Therefore, no curves contribute to δ. Thus, $\widehat{BSD}(P)$ has a unique generator $\mathbf{x}_P$, with $\delta(\mathbf{x}_P) = 0$.



(a) W parametrized by $\mathcal{Z}$.

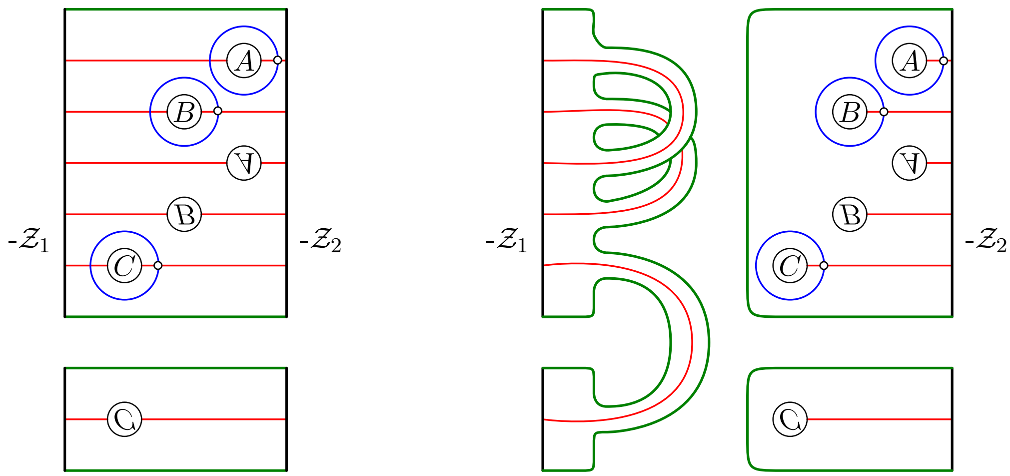


(b) T parametrized by $-\mathcal{Z}$.



(c) P parametrized by $-\mathcal{Z}$.

Figure 9: Bordered sutured decomposition of (Y, Γ) and (Y', Γ').



(a) $\mathcal{H}_T$ with $\mathbf{x}_T$ marked.

(b) $\mathcal{H}_P$ with $\mathbf{x}_P$ marked.

Figure 10: Heegaard diagrams for P and T. Here A, B, and C denote 1–handles.

Now, consider $\mathcal{H}_T$. Every α^a arc intersects a unique β curve, and any β curve intersects a unique pair of α^a arcs, that correspond in $-\mathcal{Z}_1$ and $-\mathcal{Z}_2 \cong \mathcal{Z}_1$. Therefore for any $s \subset \{1,\dots,k\}$ there is a unique generator $\mathbf{x}_s \in \mathcal{G}(\mathcal{H}_t)$, such that $I(\bar{o}(\mathbf{x})) = I_1(s) \otimes I_2(\bar{s})$, and $l(\mathbf{x}_s) = \#s$. These are all the elements of $\mathcal{G}(\mathcal{H}_P)$.

Consider all the Spin^c–structures in $\mathrm{Spin}^c(T, \partial T \setminus S \times \{\pm 2\})$. By Poincaré duality they are an affine space over $H_1(T, S\times\{\pm 2\}) = H_1(S\times[-2,2], S\times\{\pm 2\}) \cong \mathbb{Z}$, generated by an arc $\mu = \{p\}\times[-2,2]$ for any $p \in S$. It is easy to see that $\epsilon(\mathbf{x},\mathbf{y}) = (l(\mathbf{x}) - l(\mathbf{y}))\cdot[\mu]$. Thus, for any $\mathbf{x} \in \mathcal{G}(\mathcal{H}_T)$, its Spin^c–structure $\mathfrak{s}(\mathbf{x})$ depends only on $l(\mathbf{x})$. In particular, there is a unique generator $\mathbf{x}_T$, in the Spin^c–structure $\mathfrak{s}_k = \mathfrak{s}(\mathbf{x}_T)$ which corresponds to $l = k$.

Since $l(\mathbf{x}_T) = k$, any class $B \in \pi_2(\mathbf{x},\mathbf{x})$ that contributes to δ could not hit any Reeb chords on the $\mathcal{Z}_2$ side, and $\partial^\partial B \cap \mathbf{Z}_2$ should be empty. But any elementary region in the diagram hits Reeb chords on both sides. Therefore any such B should be 0, and $\delta(\mathbf{x}_T) = 0$.

Notice that $\mathcal{G}(\mathcal{H}_P) = \{\mathbf{x}_P\} \cong \{\mathbf{x}_T\} = \mathcal{G}(\mathcal{H}_T, \mathfrak{s}_k)$, $I(\bar{o}(\mathbf{x}_P)) = I(\bar{o}(\mathbf{x}_T)) = I_1(\{1,\dots,k\}) \otimes I_2(\varnothing)$, and $\delta(\mathbf{x}_P) = \delta(\mathbf{x}_T) = 0$. Therefore $\widehat{BSD}(\mathcal{H}_P) \cong \widehat{BSD}(\mathcal{H}_T, \mathfrak{s}_k)$, and $\widehat{BSD}(P) \simeq \widehat{BSD}(T, \mathfrak{s}_k)$, as type D structures over $\mathcal{A}(\mathcal{Z})$.

By the pairing theorem,

$$\begin{aligned} SFC(Y',\Gamma') &\simeq \widehat{BSA}(W)\boxtimes\widehat{BSD}(P)\\ &\simeq \widehat{BSA}(W)\boxtimes\widehat{BSD}(T,\mathfrak{s}_k)\simeq\bigoplus_{\mathfrak{s}|_T=\mathfrak{s}_k} SFC(Y,\Gamma,\mathfrak{s}). \end{aligned}$$

To finish the proof, we need to check that $\mathfrak{s}\in\mathrm{Spin}^c(Y,\partial Y)$ is outward to S if and only if $\mathfrak{s}|_T=\mathfrak{s}_k$. This follows from the fact that being outward to S is a local condition. In $T=S\times[-2,2]$ the existence of an outward vector field representing $\mathfrak{s}_l$ is equivalent to $l=k$. □

In fact, using bimodules the proof carries through even when W has another bordered component $\mathcal{Z}'$. Thus we get a somewhat stronger version of the formula.

Theorem 9.4.2. *If $(Y,\Gamma,\mathcal{Z},\phi)$ is a bordered sutured manifold, and S is a nice decomposing surface, where $\partial S\subset\partial Y\setminus F(\mathcal{Z})$, and $(Y',\Gamma',\mathcal{Z},\phi)$ is obtained by decomposing along S, then the following formulas hold.*

$$\begin{aligned} \widehat{BSD}(Y',\Gamma') &\simeq \bigoplus_{\mathfrak{s}\text{ outward to }S}\widehat{BSD}(Y,\Gamma,\mathfrak{s}),\\ \widehat{BSA}(Y',\Gamma') &\simeq \bigoplus_{\mathfrak{s}\text{ outward to }S}\widehat{BSA}(Y,\Gamma,\mathfrak{s}). \end{aligned}$$

Proof. The first statement follows as in Theorem 9.4.1, using $\widehat{BSDA}(W)$. The second follows analogously, replacing the argument for $\widehat{BSD}(T)$ and $\widehat{BSD}(P)$ with one for $\widehat{BSA}(T)$ and $\widehat{BSA}(P)$. □

Chapter 10

Examples

To help the reader understand the definitions we give some simple examples of bordered sutured manifolds and compute their invariants.

10.1 Sutured surfaces and arc diagrams

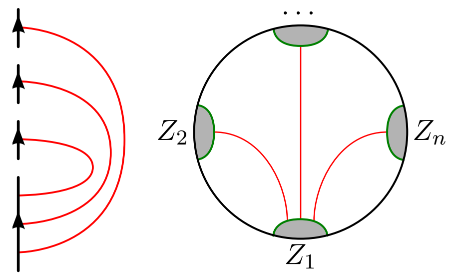
First we discuss some simple arc diagrams and their algebras, that parametrize the same sutured surfaces

Example 10.1.1. One of the simplest classes of examples is the following. Let F_n be the sutured surface (D^2, Λ_n), where Λ_n consists of $2n$ distinct points. That is, F_n is a disc, whose boundary circle is divided into n positive and n negative arcs.

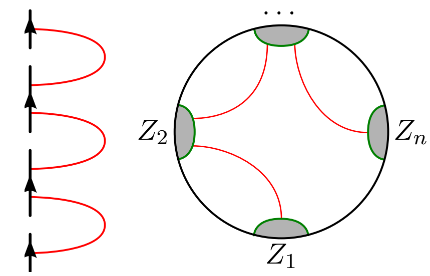
There are many different arc diagrams for F_n, especially for large n, but there are two special cases which we will consider in detail.

Example 10.1.2. Let $\mathbf{Z} = \{Z_1, \dots, Z_n\}$ be a collection of oriented arcs, and $\mathbf{a} = \{a_1, \dots, a_{2n-2}\}$ be a collection of points, such that $a_1, \dots, a_{n-1} \in Z_1$ are in this order, and $a_{n+i-1} \in Z_{i+1}$ for $i = 1, \dots, n-1$. Let M be the matching $M(a_i) = M(a_{2n-i-1}) = i$ for $i = 1, \dots, n-1$. The arc diagram $\mathcal{W}_n = (\mathbf{Z}, \mathbf{a}, M)$ parametrizes F_n, as in Figure 11a.

Proposition 10.1.1. *For the arc diagram $\mathcal{W}_n$ from Example 10.1.2, the algebra $\mathcal{A}(\mathcal{W}_n)$ satisfies $\mathcal{A}(\mathcal{W}_n, k) \cong \mathcal{A}(n-1, k)$ for all $k = 0, \dots, n-1$.*



(a) The arc diagram $\mathcal{W}_n$ for F_n and corresponding parametrization.



(b) The arc diagram $\mathcal{V}_n$ for F_n and corresponding parametrization.

Figure 11: Two parametrizations of F_n.

Proof. The algebra $\mathcal{A}(\mathcal{W}_n)$ is a subalgebra of $\mathcal{A}(n-1,1,1,\dots,1) \cong \mathcal{A}(n-1) \otimes \mathcal{A}(1)^{\otimes(n-1)}$. But $\mathcal{A}(1) = \mathcal{A}(1,0) \oplus \mathcal{A}(1,1)$, where both summands are trivial. The projection π to $\mathcal{A}(n-1) \otimes \mathcal{A}(1,0)^{\otimes(n-1)} \cong \mathcal{A}(n-1)$ respects the algebra structure. For each $\boldsymbol{\rho}$ and completion s, the projection π kills all summands in $a(\boldsymbol{\rho}, s)$, except the one corresponding to the unique section S of s, where $S \subset \{1,\dots,n-1\}$. Therefore $\pi|_{\mathcal{A}(\mathcal{W}_n)}$ is an isomorphism. □

Example 10.1.3. Let $\mathbf{Z} = \{Z_1,\dots,Z_n\}$ and $\mathbf{a} = \{a_1,\dots,a_{2n-2}\}$, again but set $a_1 \in Z_1$, $a_{2n-2} \in Z_n$, while $a_{2i}, a_{2i+1} \in Z_{i+1}$ for $i = 1,\dots,n-2$. Set the matching M to be $M(a_{2i-1}) = M(a_{2i}) = i$ for $i = 1,\dots,n-1$. The arc diagram $\mathcal{V}_n = (\mathbf{Z}, \mathbf{a}, M)$ also parametrizes F_n, as in Figure 11b

Proposition 10.1.2. *For the arc diagram $\mathcal{V}_n$ from Example 10.1.3, its associated algebra $\mathcal{A}(\mathcal{V}_n)$ has no differential.*

Proof. By definition $\mathcal{A}(\mathcal{V}_n)$ is a subalgebra of $\mathcal{A}(1) \otimes \mathcal{A}(2)^{\otimes(n-2)} \otimes \mathcal{A}(1)$. It is trivial to check that neither $\mathcal{A}(1)$, nor $\mathcal{A}(2)$ have differentials. The differential on their product is defined by the Leibniz rule, so it also vanishes. □

It will be useful for next section to compute the two algebras $\mathcal{A}(\mathcal{W}_4)$ and $\mathcal{A}(\mathcal{V}_4)$ explicitly. Recall Definition 2.3.2, which assigns to a collection $\boldsymbol{\rho}$ of Reeb chords, corresponding to moving strands, and a completion s, corresponding to stationary strands, an algebra element $a(\boldsymbol{\rho}, s)$. Abusing notation, we will denote the idempotent $I(\{1,2,4\})$ by I_{124}, etc.

In $\mathcal{W}_4$ there are three Reeb chords—ρ_1 from a_1 to a_2, ρ_2 from a_2 to a_3, and their concatenation ρ_{12} from a_1 to a_3. The algebra splits into 4 summands. The 0– and 3–summands $\mathcal{A}(\mathcal{W}_4,0) = \langle I_\varnothing \rangle$ and $\mathcal{A}(\mathcal{W}_4,3) = \langle I_{123} \rangle$ are trivial.

The 1–summand is $\mathcal{A}(\mathcal{W}_4,1) = \langle I_1, I_2, I_3, \rho'_1, \rho'_2, \rho'_{12} \rangle$. It has three idempotents and three other generators $\rho'_i = a(\{\rho_i\}, \varnothing)$. It has no differential, and the only nontrivial product is $\rho'_1 \cdot \rho'_2 = \rho'_{12}$. The 2–summand $\mathcal{A}(\mathcal{W}_4,2) = \langle I_{12}, I_{13}, I_{23}, \rho''_1, \rho''_2, \rho''_{12}, \rho''_2 \cdot \rho''_1 \rangle$ is the most interesting. Here $\rho''_1 = a(\{\rho_1\},\{3\})$, $\rho''_2 = a(\{\rho_2\},\{1\})$, $\rho''_{12} = a(\{\rho_{12}\},\{2\})$, and $\rho''_2 \cdot \rho''_1 = a(\{\rho_1,\rho_2\},\varnothing)$. There is a nontrivial differential $\partial \rho''_{12} = \rho''_2 \cdot \rho''_1$, and one nontrivial product, which is clear from our notation.

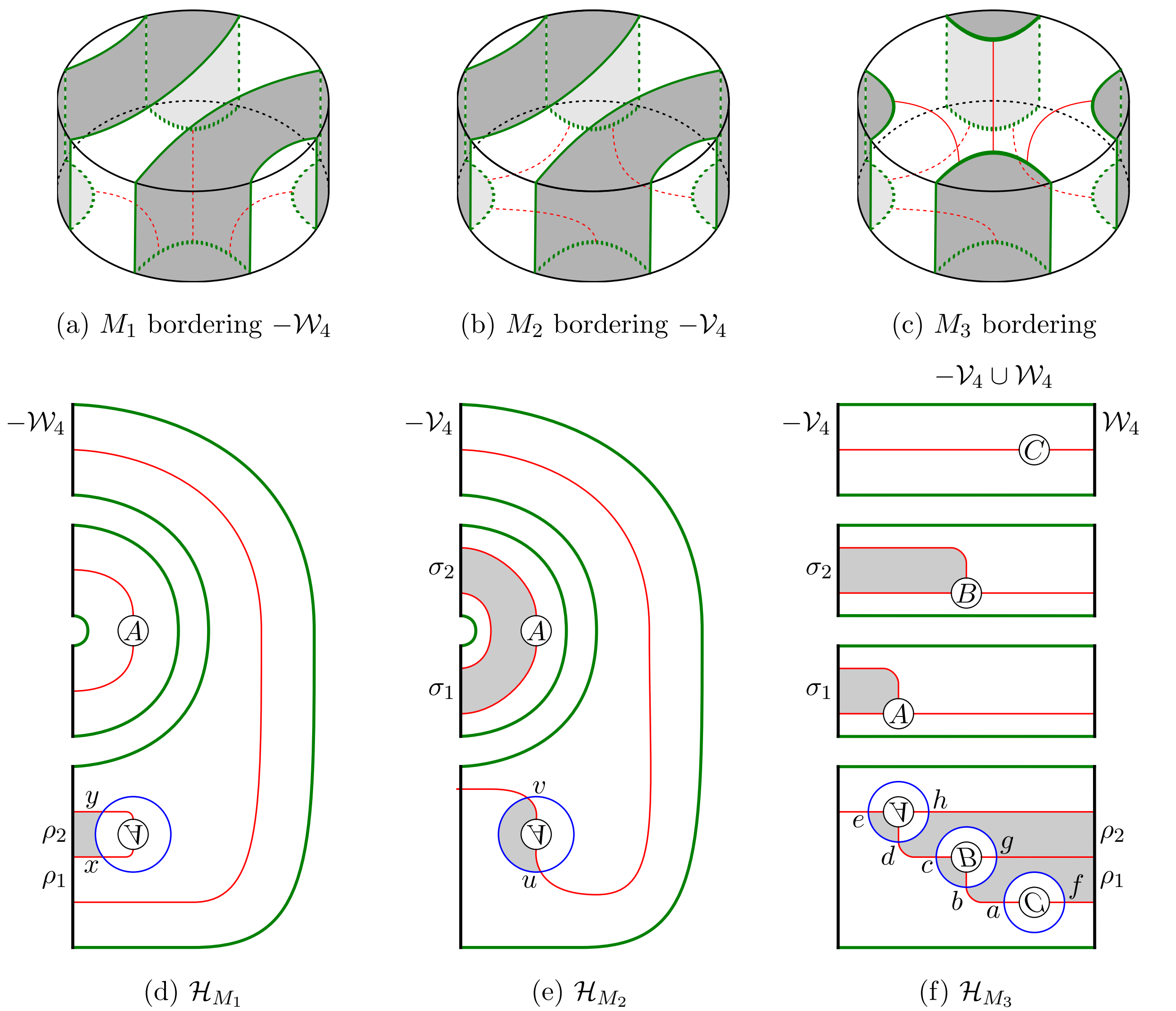
In $\mathcal{V}_4$ there are two Reeb chords—σ_1 from a_2 to a_3, and σ_2 from a_4 to a_5. Again, the summands $\mathcal{A}(\mathcal{V}_4,0) = \langle I_\varnothing \rangle$ and $\mathcal{A}(\mathcal{V}_4,3) = \langle I_{123} \rangle$ are trivial. The 1–summand is $\mathcal{A}(\mathcal{V}_4,1) = \langle I_1, I_2, I_3, \sigma'_1, \sigma'_2 \rangle$, where $\sigma'_i = a(\{\sigma_i\},\varnothing)$. It has no nontrivial differentials or products. The 2–summand is $\mathcal{A}(\mathcal{V}_4,2) = \langle I_{12}, I_{13}, I_{13}, \sigma''_1, \sigma''_2, \sigma''_2 \cdot \sigma''_1 \rangle$, where $\sigma''_1 = a(\{\sigma_1\},\{3\})$, $\sigma''_2 = a(\{\sigma_2\},\{1\})$, and $\sigma''_2 \cdot \sigma''_1 = a(\{\sigma_1,\sigma_2\},\varnothing)$. There are no differentials and there is one nontrivial product.

10.2 Bordered sutured manifolds

We give three examples of bordered sutured manifolds. Topologically they are all very simple—in fact they are all $D^2 \times [0,1]$. They are, nonetheless, interesting and have nontrivial invariants. Bordered sutured manifolds of this type are essential for the study of what happens when we fill in a sutured surface with a chord diagram.

Example 10.2.1. The first example is $M_1 = (D^2 \times [0,1], \Gamma_1, -\mathcal{W}_4)$, where $D^2 \times \{0\}$ is parametrized by $-\mathcal{W}_4$, and the rest of the boundary is divided into two positive and three negative regions (see Figure 12a). An admissible—and in fact nice—Heegaard diagram for M_1 is given in Figure 12d. We will compute ${}^{\mathcal{A}(\mathcal{W}_4)}\widehat{BSD}(M_1)$.

First, notice that the relative Spin^c–structures are in one-to-one correspondence with $H_1(D^2 \times [0,1], D^2 \times \{0\}) = 0$, so there is a unique Spin^c–structure. There are two generators (x) and (y), with idempotents $I_{13} \cdot (x) = (x)$, and $I_{12} \cdot (y) = (y)$ (both in $\mathcal{I}(\mathcal{W}_4, 2)$). There



(a) M_1 bordering $-\mathcal{W}_4$

(b) M_2 bordering $-\mathcal{V}_4$

(c) M_3 bordering $-\mathcal{V}_4 \cup \mathcal{W}_4$

(d) $\mathcal{H}_{M_1}$

(e) $\mathcal{H}_{M_2}$

(f) $\mathcal{H}_{M_3}$

Figure 12: Three examples of bordered sutured manifolds (top row), and their diagrams (bottom row). Capital roman letters denote 1–handles, lower case roman letters denote intersection points, and Greek letters denote Reeb chords (always oriented upward). All non-boundary elementary regions have been shaded.

is a single region contributing to δ. It corresponds to a source $S^{\triangleright}$ which is a bigon from (y) to (x), with one e puncture labeled $-\rho_2$. It contributes $a_2(\rho_2) \otimes (x) = \rho_2'' \otimes (x)$ to $\delta(y)$. Therefore, the only nontrivial term in δ is

$$\delta((y)) = \rho_2'' \otimes (x).$$

If we want to compute $\widehat{BSA}(M_1)_{\mathcal{A}(-\mathcal{W}_4)}$, the same generators have idempotents $(x) \cdot I_2 = (x)$ and $(y) \cdot I_3 = (y)$, and the same region contributes (x) to $m_2((y), a(-\rho_2))$, instead. The only nontrivial term is

$$m_2((y), -\rho_2') = (x).$$

Example 10.2.2. The second example is $M_2 = (D^2 \times [0,1], \Gamma_1, -\mathcal{V}_4)$, which is the same as M_1, except for the different parametrization of $D^2 \times \{0\}$ (see Figure 12b). An admissible diagram for M_2 is given in Figure 12e.

First, we compute ${}^{\mathcal{A}(\mathcal{V}_4)}\widehat{BSD}(M_2)$. It has two generators, with idempotents $I_{12} \cdot (u) = (u)$ and $I_{23} \cdot (v) = (v)$. There is one region which is a bigon with two e punctures labeled $-\sigma_2$ and $-\sigma_1$, at different heights. It contributes $a_2(\sigma_2)a_2(\sigma_1) \otimes (v) = \sigma_2'' \cdot \sigma_1'' \otimes (v)$ to $\delta((u))$. Therefore the differential is

$$\delta((u)) = \sigma_2'' \cdot \sigma_1'' \otimes (v).$$

For $\widehat{BSA}(M_2)_{\mathcal{A}(-\mathcal{V}_4)}$, the idempotents are $(u) \cdot I_3 = (u)$ and $(v) \cdot I_1 = (v)$. The region contributes to m_3, yielding

$$m_3((u), -\sigma_2', -\sigma_1') = (v).$$

Example 10.2.3. Our last—and richest—example is $M_3 = (D^2 \times [0,1], \Gamma_2, -\mathcal{V}_4 \cup \mathcal{W}_4)$, where $-\mathcal{V}_4$ parametrizes $D^2 \times \{0\}$, and $\mathcal{W}_4$ parametrizes $D^2 \times \{1\}$ (see Figure 12c). This is a decorated sutured cobordism from $\mathcal{V}_4$ to $\mathcal{W}_4$, which is an isomorphism in the decorated category $\mathcal{SD}$. An admissible diagram for M_3 is given in Figure 12f.

We will compute (part of) ${}^{\mathcal{A}(\mathcal{V}_4)}\widehat{BSDA}(M_3)_{\mathcal{A}(\mathcal{W}_4)}$. In this case, since $H_1(D^2 \times [0,1], D^2 \times \{0,1\}) = \mathbb{Z}$, there are multiple Spin^c–structures. As in the proof of Theorem 9.4.1, the Spin^c–structures correspond to how many α^a arcs are occupied on the $\mathcal{W}_4$ side of $-\mathcal{V}_4 \cup \mathcal{W}_4$. Let $\mathfrak{s}_k$ be the Spin^c–structure with k arcs occupied. There are $3-k$ arcs occupied on the

$-\mathcal{V}_4$ side for each such generator, and therefore $\widehat{BSDA}(M_3, \mathfrak{s}_k)$ is a bimodule over $\mathcal{A}(\mathcal{V}_4, k)$ and $\mathcal{A}(\mathcal{W}_4, k)$. Moreover, only $k = 0, 1, 2, 3$ give nonzero invariants.

It is easy to check that $\widehat{BSDA}(M_3, \mathfrak{s}_0)$ and $\widehat{BSDA}(M_3, \mathfrak{s}_3)$ have unique generators, (ace) and (fgh), respectively, with no nontrivial actions m_k. We will leave $\widehat{BSDA}(M_3, \mathfrak{s}_1)$ as an exercise and compute $\widehat{BSDA}(M_3, \mathfrak{s}_2)$. There are five generators, with idempotents as follows.

$$I_{12} \cdot (agh) \cdot I_{23} = (agh) \qquad I_{12} \cdot (fbh) \cdot I_{13} = (fbh)$$
$$I_{13} \cdot (fch) \cdot I_{13} = (fch) \qquad I_{13} \cdot (fgd) \cdot I_{12} = (fgd)$$
$$I_{23} \cdot (fge) \cdot I_{12} = (fge)$$

There are four elementary domains, each of which contributes one term to m_1 or m_2. Some of them also contribute to m_1 or m_2 for $\widehat{BSDA}(M_3, \mathfrak{s}_1)$, and there is a composite domain that also contributes in that case. The nontrivial operations for $\widehat{BSDA}(M_3, \mathfrak{s}_2)$ are listed below.

$$m_1((fgd)) = \sigma_1'' \otimes (fge) \qquad m_2((fgd), \rho_2'') = I_{13} \otimes (fch)$$
$$m_1((fbh)) = \sigma_2'' \otimes (fch) \qquad m_2((fbh), \rho_1'') = I_{12} \otimes (agh)$$

10.3 Gluing

Our final example is of gluing of bordered sutured manifolds and the corresponding operation on their invariants.

Example 10.3.1. We will use the manifolds from Examples 10.2.1–10.2.3. If we glue M_1 and M_3 along $F(\mathcal{W}_4)$ we obtain exactly M_2. Treating ${}^{\mathcal{A}(\mathcal{W}_4)}\widehat{BSD}(M_1)$ as ${}^{\mathcal{A}(\mathcal{W}_4)}\widehat{BSDA}(M_1)_{\mathcal{A}(\varnothing)}$, we can compute the product

$$\widehat{BSDA}(M_3) \boxtimes_{\mathcal{A}(\mathcal{W}_4)} \widehat{BSD}(M_1),$$

which is a type D structure over $\mathcal{A}(\mathcal{V}_4)$. Since the only relative Spin^c–structure on M_3 which extends over M_1 is $\mathfrak{s}_2$, the product is equal to $\widehat{BSDA}(M_3, \mathfrak{s}_2) \boxtimes \widehat{BSD}(M_1)$. Another way to see this is to notice that if we decompose the product over $\boxtimes_{\mathcal{A}(\mathcal{W}_4, k)}$, only the $k = 2$ term is nonzero.

After taking the tensor product $\otimes_{\mathcal{I}(\mathcal{W}_{4,2})}$ of the underlying modules, the generators and idempotents are

$$I_{13} \cdot (fch) \boxtimes (x) = (fch) \boxtimes (x) \qquad I_{12} \cdot (fbh) \boxtimes (x) = (fbh) \boxtimes (x)$$
$$I_{23} \cdot (fge) \boxtimes (y) = (fge) \boxtimes (y) \qquad I_{13} \cdot (fgd) \boxtimes (y) = (fgd) \boxtimes (y)$$

The induced operations are

$$\delta((fgd) \boxtimes (y)) = \sigma_1'' \otimes ((fge) \boxtimes (y)) + I_{13} \otimes ((fch) \boxtimes (x))$$
$$\delta((fbh) \boxtimes (x)) = \sigma_2'' \otimes ((fch) \boxtimes (x))$$

There is one pure differential, from $(fgd) \boxtimes (y)$ to $(fch) \boxtimes (x)$. We can cancel it, and see that the complex is homotopy equivalent to $\widehat{BSD}(M_2)$, as expected from the pairing theorem.

Part II

Gluing map for sutured Floer homology

Chapter 11

Topological preliminaries

We recall the definition of a sutured manifold and some auxiliary notions, and define what we mean by gluing and surgery.

Remark. Throughout the thesis all manifolds are oriented. We use $-M$ to denote the manifold M with its orientation reversed.

11.1 Sutured manifolds and surfaces

Definition 11.1.1. *As defined in [Juh06], a* balanced sutured manifold *is a pair* $\mathcal{Y} = (Y, \Gamma)$ *consisting of the following:*

- *An oriented 3–manifold* Y *with boundary.*
- *A collection* Γ *of disjoint oriented simple closed curves in* ∂Y*, called* sutures.

They are required to satisfy the following conditions:

- Y *can be disconnected but cannot have any closed components.*
- ∂Y *is divided by* Γ *into two complementary regions* $R_+(\Gamma)$ *and* $R_-(\Gamma)$*, such that* $\partial R_\pm(Y) = \pm\Gamma$*. (*$R_+$ *and* R_- *may be disconnected.)*
- *Each component of* ∂Y *contains a suture. Equivalently,* R_+ *and* R_- *have no closed components.*

- $\chi(R_+) = \chi(R_-)$.

In Part I we introduced the notion of a sutured surface.

Definition 11.1.2. *A* sutured surface *is a pair* $\mathcal{F} = (F, \Lambda)$ *consisting of the following:*

- *A compact oriented surface* F*.*
- *A finite collection* $\Lambda \subset \partial F$ *of points with sign, called* sutures*.*

They are required to satisfy the following conditions:

- F *can be disconnected but cannot have any closed components.*
- ∂F *is divided by* Λ *into two complementary regions* $S_+(\Gamma)$ *and* $S_-(\Gamma)$*, where* $\partial S_\pm(Y) = \pm\Lambda$*. (*$S_+$ *and* S_- *may be disconnected.)*
- *Each component of* ∂F *contains a suture. Equivalently,* S_+ *and* S_- *have no closed components.*

A sutured surface is precisely the 2–dimensional equivalent of a balanced sutured manifold. The requirement $\chi(S_+) = \chi(S_-)$ follows automatically from the other conditions.

From $\mathcal{F} = (F, \Lambda)$ we can construct two other sutured surfaces: $-\mathcal{F} = (-F, -\Lambda)$, and $\overline{\mathcal{F}} = (-F, \Lambda)$. In both of $-\mathcal{F}$ and $\overline{\mathcal{F}}$, the orientation of the underlying surface F is reversed. The difference between the two is that in $-\mathcal{F}$ the roles of S_+ and S_- are preserved, while in $\overline{\mathcal{F}}$ they are reversed.

Definition 11.1.3. *Suppose* $\mathcal{F} = (F, \Lambda)$ *is a sutured surface. A* dividing set Γ *for* $\mathcal{F}$ *is a finite collection* Γ *of disjoint embedded oriented arcs and simple closed curves in* F*, with the following properties:*

- $\partial\Gamma = -\Lambda$*, as an oriented boundary.*
- Γ *divides* F *into (possibly disconnected) regions* R_+ *and* R_- *with* $\partial R_\pm = (\pm\Gamma) \cup S_\pm$*.*

We can extend the definition of a dividing set to pairs (F, Λ) which do not quite satisfy the conditions for a sutured surface. We can allow some or all of the components F to be closed. We call such a pair *degenerate*. In that case we impose the extra condition that each closed component contains a component of Γ.

Note that the sutures Γ of a sutured manifold (Y, Γ) can be regarded as a dividing set for the (degenerate) sutured surface $(\partial Y, \varnothing)$.

Definition 11.1.4. *A* partially sutured manifold *is a triple* $\mathcal{Y} = (Y, \Gamma, \mathcal{F})$ *consisting of the following:*

- *A 3–manifold* Y *with boundary and 1–dimensional corners.*
- *A sutured surface* $\mathcal{F} = (F, \Lambda)$, *such that* $F \subset \partial Y$, *and such that the 1–dimensional corner of* Y *is* ∂F.
- *A dividing set* Γ *for* $(\partial Y \setminus F, -\Lambda)$ *(which might be degenerate).*

Note that a partially sutured manifold $\mathcal{Y} = (Y, \Gamma, \mathcal{F}_1 \sqcup \mathcal{F}_2)$ can be thought of as a cobordism between $-\mathcal{F}_1$ and $\mathcal{F}_2$. On the other hand, the partially sutured manifold $\mathcal{Y} = (Y, \Gamma, \varnothing)$ is just a sutured manifold, although it may not be balanced. We can *concatenate* $\mathcal{Y} = (Y, \Gamma, \mathcal{F}_1 \sqcup \mathcal{F}_2)$ and $\mathcal{Y}' = (Y', \Gamma', -\mathcal{F}_2 \sqcup \mathcal{F}_3)$ along $\mathcal{F}_2 = (F_2, \Lambda_2)$ and $-\mathcal{F}_2 = (-F_2, -\Lambda_2)$ to obtain

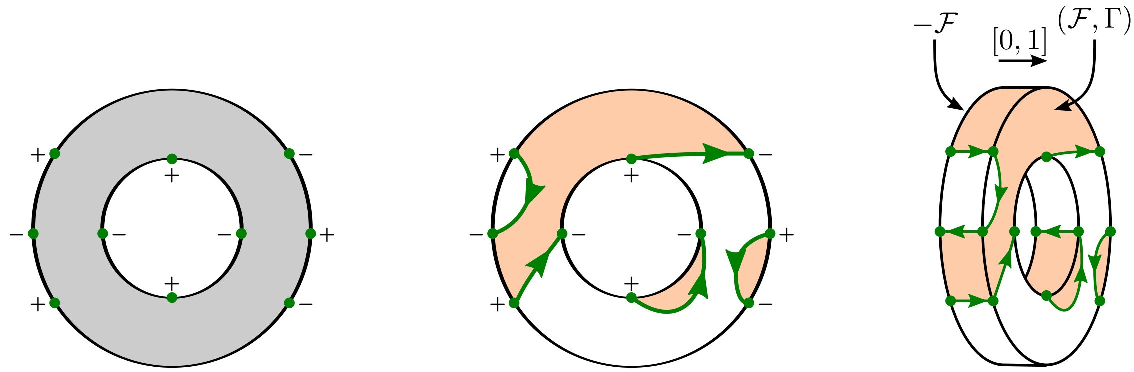
$$\mathcal{Y} \cup_{\mathcal{F}_2} \mathcal{Y}' = (Y \cup_{F_2} Y', \Gamma \cup_{\Lambda_2} \Gamma', \mathcal{F}_1 \sqcup \mathcal{F}_3).$$

We use the term concatenate to distinguish from the operation of *gluing* of two sutured manifolds described in Definition 11.2.4.

A partially sutured manifold whose sutured surface is parametrized by an arc diagram is a *bordered sutured manifold*, as defined in Part I. We will return to this point in section 12, where we give the precise definitions.

An important special case is when Y is a thickening of F.

Definition 11.1.5. *Suppose* Γ *is a dividing set for the sutured surface* $\mathcal{F} = (F, \Lambda)$. *Let* $W = F \times [0,1]$, *and* $W' = F \times [0,1]/\sim$, *where* $(p,t) \sim (p,t')$ *whenever* $p \in \partial F$, *and*



(a) The sutured surface $\mathcal{F}$. (b) A dividing set Γ of $\mathcal{F}$. (c) The cap for Γ.

Figure 13: A sutured annulus $\mathcal{F}$, with a cap associated to a dividing set.

$t, t' \in [0,1]$. *We will refer to the partially sutured manifolds*

$$\mathcal{W}_\Gamma = (W, \Gamma \times \{1\} \cup \Lambda \times [0,1], (-F \times \{0\}, -\Lambda \times \{0\})),$$
$$\mathcal{W}'_\Gamma = (W', \Gamma \times \{1\}, (-F \times \{0\}, -\Lambda \times \{0\}))$$

as the caps *for* $\mathcal{F}$ *associated to* Γ.

Since $\mathcal{W}'_\Gamma$ is just a smoothing of $\mathcal{W}_\Gamma$ along the corner $\partial F \times \{1\}$, we will not distinguish between them. An illustration of a dividing set and a cap is shown in Figure 13. In this and in all other figures we use the convention that the dividing set is colored in green, to avoid confusion with Heegaard diagrams later. We also shade the R_+ regions.

Notice that the sutured surface for $\mathcal{W}_\Gamma$ is $-\mathcal{F}$. This means that if $\mathcal{Y} = (Y, \Gamma', \mathcal{F})$ is a partially sutured manifold, we can concatenate $\mathcal{Y}$ and $\mathcal{W}$ to obtain $(Y, \Gamma' \cup \Gamma)$. That is, the effect is that of "filling in" $F \subset \partial Y$ by Γ.

Definition 11.1.6. *Suppose* $\mathcal{F} = (F, \Lambda)$ *is a sutured surface. An* embedding $\mathcal{W} \hookrightarrow \mathcal{Y}$ *of the partially sutured* $\mathcal{W} = (W, \Gamma_W, \mathcal{F})$ *into the sutured* $\mathcal{Y} = (Y, \Gamma_Y)$ *is an embedding* $W \hookrightarrow Y$ *with the following properties:*

- $F \subset \partial W$ *is properly embedded in* Y *as a separating surface.*
- $\partial W \setminus F = \partial Y \cap W$.
- $\Gamma_W = \Gamma_Y \cap \partial W$.

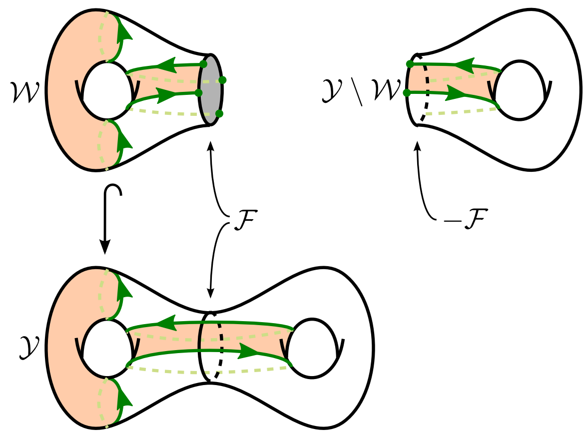


Figure 14: Examples of a partially sutured manifold $\mathcal{W}$ embedding into the sutured manifold $\mathcal{Y}$, and the complement $\mathcal{Y} \setminus \mathcal{W}$, which is also partially sutured.

The complement $Y \setminus W$ also inherits a partially sutured structure. We define

$$\mathcal{Y} \setminus \mathcal{W} = (Y \setminus W, \Gamma_Y \setminus \Gamma_W, -\mathcal{F}).$$

The definition of embeddings easily extends to $\mathcal{W} \hookrightarrow \mathcal{Y}$ where both $\mathcal{W} = (W, \Gamma_W, \mathcal{F})$ and $\mathcal{Y} = (Y, \Gamma_Y, \mathcal{F}')$ are partially sutured. In this case we require that W is disjoint from a collar neighborhood of F'. Then there is still a complement

$$\mathcal{Y} \setminus \mathcal{W} = (Y \setminus W, \Gamma_Y \setminus \Gamma_W, \mathcal{F}' \cup -\mathcal{F}).$$

In both cases $\mathcal{Y}$ is diffeomorphic to the concatenation $\mathcal{W} \cup_{\mathcal{F}} (\mathcal{Y} \setminus \mathcal{W})$. Examples of a partial sutured manifold and of an embedding are given in Figure 14.

11.2 Mirrors and doubles; joining and gluing

We want to define a gluing operation which takes two sutured manifolds (Y_1, Γ_1) and (Y_2, Γ_2), and surfaces $F \subset \partial Y_1$ and $-F \subset \partial Y_2$, and produces a new sutured manifold $(Y_1 \cup_F Y_2, \Gamma_3)$. To do that we have to decide how to match up the dividing sets on and around F and $-F$. One solution is to require that we glue $F \cap R_+(\Gamma_1)$ to $-F \cap R_+(\Gamma_2)$, and $F \cap R_-(\Gamma_1)$ to $-F \cap R_-(\Gamma_2)$. Then $(\Gamma_1 \setminus F) \cup (\Gamma_2 \setminus -F)$ is a valid dividing set, and candidate for Γ_3. The problem with this approach is that even if we glue two balanced sutured manifolds, the result is not guaranteed to be balanced.

Another approach, suggested by contact topology is the following. We glue $F \cap R_+$ to $-F \cap R_-$, and vice versa. To compensate for the fact that the dividing sets $\Gamma_1 \setminus F$ and $\Gamma_2 \setminus -F$ do not match up anymore, we introduce a slight twist along ∂F. In contact topology this twist appears when we smooth the corner between two convex surfaces meeting at an angle.

It turns out that the same approach is the correct one, from the bordered sutured point of view. To be able to define a gluing map on *SFH* with nice formal properties, the underlying topological operation should employ the same kind of twist. However, its direction is opposite from the one in the contact world. This is not unexpected, as orientation reversal is the norm when defining any contact invariant in Heegaard Floer homology.

As we briefly explained in Section 1.3, we will also define a surgery procedure which we call *joining*, and which generalizes this gluing operation. We will associate a map on sutured Floer homology to such a surgery in Chapter 13.2.

First we define some preliminary notions.

Definition 11.2.1. *The* mirror *of a partially sutured manifold $\mathcal{W} = (W, \Gamma, \mathcal{F})$, where $\mathcal{F} = (F, \Lambda)$ is $-\mathcal{W} = (-W, \Gamma, \overline{\mathcal{F}})$. Alternatively, it is a partially sutured manifold $(W', \Gamma', \mathcal{F}')$, with an orientation reversing diffeomorphism $\varphi\colon W \to W'$, such that:*

- *F is sent to $-F'$ (orientation is reversed).*
- *Γ is sent to Γ' (orientation is preserved).*
- *$R_+(\Gamma)$ is sent to $R_-(\Gamma')$, and vice versa.*
- *$S_+(\Lambda)$ is sent to $S_-(\Lambda')$, and vice versa.*

Whenever we talk about a pair of mirrors, we will implicitly assume that a specific diffeomorphism between them has been chosen. An example is shown in Figure 15.

There are two partially sutured manifolds, which will play an important role.

Definition 11.2.2. *A* positive *(respectively* negative*)* twisting slice along the sutured surface *$\mathcal{F} = (F, \Lambda)$ is the partially sutured manifold $\mathcal{TW}_{\mathcal{F},\pm} = (F \times [0,1], \Gamma, -\mathcal{F} \cup -\overline{\mathcal{F}})$ where we*

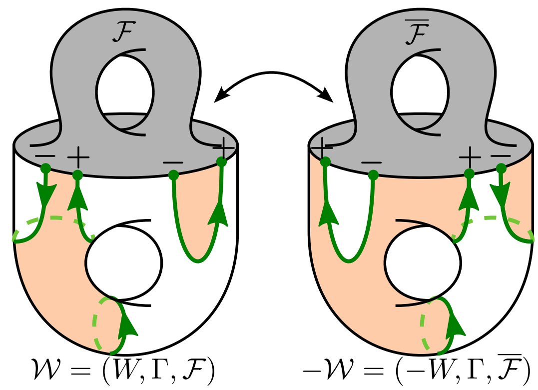


Figure 15: A partially sutured manifold $\mathcal{W}$ and its mirror $-\mathcal{W}$.

Figure 16: Positive and negative twisting slices. The dividing sets are $\Lambda\times[0,1]$, after a fractional Dehn twist has been applied. The R_+ regions have been shaded.

identify $-\mathcal{F}$ *with* $F\times\{0\}$*, and* $-\overline{\mathcal{F}}$ *with* $F\times\{1\}$*. The dividing set* Γ *is obtained from* $\Lambda\times[0,1]$ *by applying* $\frac{1}{n}$*–th of a positive (respectively negative) Dehn twist along each component of* $\partial F\times\{\frac{1}{2}\}$*, containing* n *points of* Λ*. (The twists might be different for different components.)*

Examples of twisting slices are shown in Figure 16.

Definition 11.2.3. *Let* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *be sutured manifolds, and* $\mathcal{W} = (W, \Gamma, -\mathcal{F})$ *be partially sutured. Suppose there are embeddings* $\mathcal{W} \hookrightarrow \mathcal{Y}_1$ *and* $-\mathcal{W} \hookrightarrow \mathcal{Y}_2$*. We will call the new sutured manifold*

$$\mathcal{Y}_1 \ \Cup_{\mathcal{W}} \ \mathcal{Y}_2 = (\mathcal{Y}_1 \setminus \mathcal{W}) \ \cup_{\mathcal{F}} \ \mathcal{TW}_{\mathcal{F},+} \ \cup_{-\overline{\mathcal{F}}} \ (\mathcal{Y}_2 \setminus -\mathcal{W})$$

the join of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ along $\mathcal{W}$*.*

Intuitively, this means that we cut out $\mathcal{W}$ and $-\mathcal{W}$ and concatenate the complements together. There is a mismatch of R_+ with R_- along the boundary, so we introduce a positive twist to fix it. An example of gluing was shown in Figure 4.

Another important operation is gluing.

Definition 11.2.4. *Suppose that $\mathcal{Y}_1 = (Y_1, \Gamma_1, \mathcal{F})$ and $\mathcal{Y}_2 = (Y_2, \Gamma_2, \overline{\mathcal{F}})$ are two partially sutured manifolds, and Γ_0 is a dividing set for $\mathcal{F} = (F, \Lambda)$. We define the* gluing *of the sutured manifolds $(Y_1, \Gamma_1 \cup_\Lambda \Gamma_0)$ and $(Y_2, \Gamma_2 \cup_\Lambda \Gamma_0)$ along (F, Γ_0) to be the concatenation*

$$\mathcal{Y}_1 \cup_{-\mathcal{F}} \mathcal{TW}_{\mathcal{F},+} \cup_{\overline{\mathcal{F}}} \mathcal{Y}_2,$$

and denote it by

$$(Y_1, \Gamma_1 \cup \Gamma_0) \cup_{(F,\Gamma_0)} (Y_2, \Gamma_2 \cup \Gamma_0).$$

An example of gluing was shown in Figure 3. It is easy to see that gluing is a special case of the join. Recall that the concatenation $(Y, \Gamma', \mathcal{F}) \cup_{\mathcal{F}} \mathcal{W}_\Gamma$ is the sutured manifold $(Y, \Gamma' \cup \Gamma)$. Thus we can identify gluing along (F, Γ_0) with join along $\mathcal{W}_{\Gamma_0}$.

Another useful object is the double of a partially sutured manifold.

Definition 11.2.5. *Given a partially sutured manifold $\mathcal{W} = (W, \Gamma, \mathcal{F})$, where $\mathcal{F} = (F, \Lambda)$, define the* double *of $\mathcal{W}$ to be the be sutured manifold obtained by concatenation as follows:*

$$\mathrm{D}(\mathcal{W}) = -\mathcal{W} \ \cup_{-\overline{\mathcal{F}}} \ \mathcal{TW}_{-\overline{\mathcal{F}},-} \ \cup_{\mathcal{F}} \ \mathcal{W}.$$

All the operations we have defined so far keep us in the realm of balanced sutured manifolds.

Proposition 11.2.6. *If we join or glue two balanced sutured manifolds together, the result is balanced. The double of any partially sutured manifold $\mathcal{W}$ is balanced.*

Proof. There are three key observations. The first one is that $\chi(R_+) - \chi(R_-)$ is additive under concatenation. The second is that when passing from $\mathcal{W}$ to its mirror $-\mathcal{W}$, the values of $\chi(R_+)$ and $\chi(R_-)$ are interchanged. Finally, for positive and negative twisting slices $\chi(R_+) = \chi(R_-)$. $\square$

The operations of joining and gluing sutured manifolds have good formal properties described in the following proposition.

Proposition 11.2.7. *The join satisfies the following:*

1. *Commutativity:* $\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2$ *is canonically diffeomorphic to* $\mathcal{Y}_2 \Cup_{-\mathcal{W}} \mathcal{Y}_1$.

2. *Associativity: If there are embeddings* $\mathcal{W} \hookrightarrow \mathcal{Y}_1$, $(-\mathcal{W} \sqcup \mathcal{W}') \hookrightarrow \mathcal{Y}_2$, *and* $-\mathcal{W}' \hookrightarrow \mathcal{W}_3$ *then there are canonical diffeomorphisms*

$$\begin{aligned}(\mathcal{Y}_1 \ \Cup_{\mathcal{W}} \ \mathcal{Y}_2) \ \Cup_{\mathcal{W}'} \ \mathcal{Y}_3 &\cong \mathcal{Y}_1 \ \Cup_{\mathcal{W}} \ (\mathcal{Y}_2 \ \Cup_{\mathcal{W}'} \ \mathcal{Y}_3) \\ &\cong (\mathcal{Y}_1 \ \sqcup \ \mathcal{Y}_3) \ \Cup_{\mathcal{W} \cup -\mathcal{W}'} \ \mathcal{Y}_2.\end{aligned}$$

3. *Identity:* $\mathcal{Y} \ \Cup_{\mathcal{W}} \ \mathcal{D}(\mathcal{W}) \cong \mathcal{Y}$.

Gluing satisfies analogous properties.

Proof. These facts follow immediately from the definitions. $\square$

Chapter 12

Bordered sutured Floer homology with β–arcs

We recall the definitions of bordered sutured manifolds and their invariants, as introduced in Part I.

12.1 Arc diagrams and bordered sutured manifolds

Parametrizations by arc diagrams, as described below are a slight generalization of those originally defined in Part I. The latter corresponded to parametrizations using only α–arcs. While this is sufficient to define invariants for all possible situations, it is somewhat restrictive computationally. Indeed, to define the join map Ψ we need to exploit some symmetries that are not apparent unless we also allow parametrizations using β–arcs.

Definition 12.1.1. *An* arc diagram of rank k *is a triple* $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ *consisting of the following:*

- *A finite collection* $\mathbf{Z}$ *of oriented arcs.*
- *A collection of points* $\mathbf{a} = \{a_1, \ldots, a_{2k}\} \subset \mathbf{Z}$.
- *A 2–to–1 matching* $M\colon\ \mathbf{a} \to \{1, \ldots, k\}$ *of the points into pairs.*

- *A type: "α" or "β".*

We require that the 1–manifold obtained by performing surgery on all the 0–spheres $M^{-1}(i)$ in $\mathbf{Z}$ has no closed components.

We represent arc diagrams graphically by a graph $G(\mathcal{Z})$, which consists of the arcs $\mathbf{Z}$, oriented upwards, and an arc e_i attached at the pair $M^{-1}(i) \in \mathbf{Z}$, for $i = 1, \ldots, k$. Depending on whether the diagram is of α or β type, we draw the arcs to the right or to the left, respectively.

Definition 12.1.2. *The* sutured surface $\mathcal{F}(\mathcal{Z}) = (F(\mathcal{Z}), \Lambda(\mathcal{Z}))$ associated to the α–arc diagram $\mathcal{Z}$ *is constructed in the following way. The underlying surface F is produced from the product $\mathbf{Z} \times [0,1]$ by attaching 1–handles along the 0–spheres $M^{-1}(i) \times \{0\}$, for $i = 1, \ldots, k$. The sutures are $\Lambda = \partial\mathbf{Z} \times \{1/2\}$, with the positive region S_+ being "above", i.e. containing $\mathbf{Z} \times \{1\}$.*

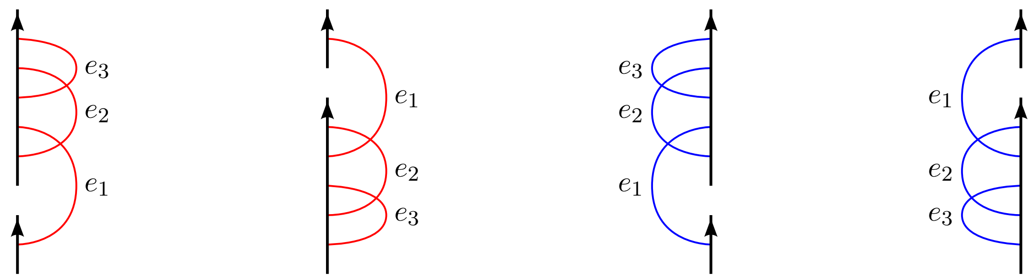
The *sutured surface associated to a β–arc diagram* is constructed in the same fashion, except that the 1–handles are attached "on top", i.e. at $\mathcal{M}^{-1}(i) \times \{1\}$. The positive region S_+ is still above.

Suppose F is a surface with boundary, $G(\mathcal{Z})$ is properly embedded in F, and $\Lambda = \partial G(\mathcal{Z}) \subset \partial F$ are the vertices of valence 1. If F deformation retracts onto $G(\mathcal{Z})$, we can identify (F, Λ) with $\mathcal{F}(\mathcal{Z})$. In fact, the embedding uniquely determines such an identification, up to isotopies fixing the boundary. We say that $\mathcal{Z}$ *parametrizes* (F, Λ).

As mentioned earlier, all arc diagrams considered in Part I are of α–type.

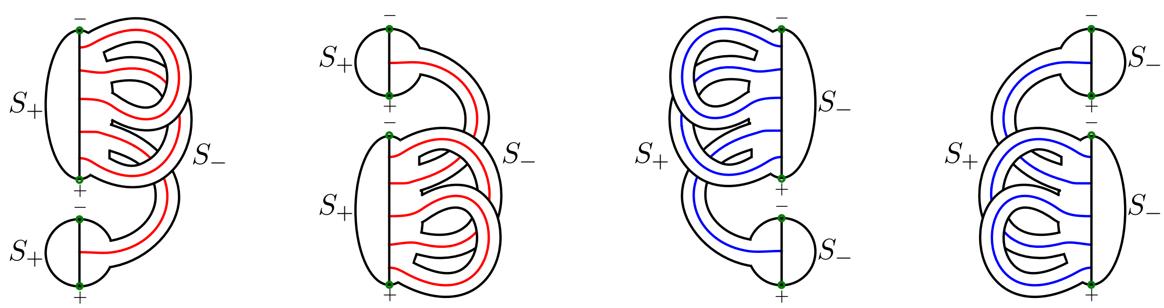
Let $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ be an arc diagram. We will denote by $-\mathcal{Z}$ the diagram obtained by reversing the orientation of $\mathbf{Z}$ (and preserving the type). We will denote by $\overline{\mathcal{Z}}$ the diagram obtained by switching the type—from α to β, or vice versa—and preserving the triple $(\mathbf{Z}, \mathbf{a}, M)$. There are now four related diagrams: $\mathcal{Z}$, $-\mathcal{Z}$, $\overline{\mathcal{Z}}$, and $-\overline{\mathcal{Z}}$. The notation is intentionally similar to the one for the variations on a sutured surface. Indeed, they are related as follows:

$$\mathcal{F}(-\mathcal{Z}) = -\mathcal{F}(\mathcal{Z}), \qquad \mathcal{F}(\overline{\mathcal{Z}}) = \overline{\mathcal{F}(\mathcal{Z})}.$$



(a) $\mathcal{Z}$ of α-type (b) $-\mathcal{Z}$ of α-type (c) $\overline{\mathcal{Z}}$ of β-type (d) $-\overline{\mathcal{Z}}$ of β-type

Figure 17: Four variants of an arc diagram



(a) $\mathcal{F}(\mathcal{Z})$ (b) $\mathcal{F}(-\mathcal{Z})$ (c) $\mathcal{F}(\overline{\mathcal{Z}})$ (d) $\mathcal{F}(-\overline{\mathcal{Z}})$

Figure 18: Parametrizations of surfaces by the arc diagrams in Figure 17

To illustrate that, Figure 17 has four variations of an arc diagram of rank 3. Figure 18 shows the corresponding parametrizations of sutured surfaces, which are all tori with one boundary component and four sutures. Notice the embedding of the graph in each case.

Definition 12.1.3. *A* bordered sutured manifold $\mathcal{Y} = (Y, \Gamma, \mathcal{Z})$ *is a partially sutured manifold* $(Y, \Gamma, \mathcal{F})$*, whose sutured surface* $\mathcal{F}$ *has been parametrized by the arc diagram* $\mathcal{Z}$*.*

As with partially sutured manifolds, $\mathcal{Y} = (Y, \Gamma, \mathcal{Z}_1 \sqcup \mathcal{Z}_2)$ can be thought of as a cobordism from $\mathcal{F}(-\mathcal{Z}_1)$ to $\mathcal{F}(\mathcal{Z}_2)$.

12.2 The bordered algebra

We will briefly recall the definition of the algebra $\mathcal{A}(\mathcal{Z})$ associated to an α–type arc diagram $\mathcal{Z}$. Fix a diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ of rank k. First, we define a larger *strands algebra* $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$, which is independent of the matching M. Then we define $\mathcal{A}(\mathcal{Z})$ as a subalgebra of $\mathcal{A}'(\mathbf{Z},\mathbf{a})$.

Definition 12.2.1. *The* strands algebra associated to $(\mathbf{Z},\mathbf{a})$ *is a* $\mathbb{Z}/2$*–algebra* $\mathcal{A}'(\mathbf{Z},\mathbf{a})$*, which is generated (as a vector space) by diagrams in* $[0,1]\times\mathbf{Z}$ *of the following type. Each diagram consists of several embedded oriented arcs or* strands*, starting in* $\{0\}\times\mathbf{a}$ *and ending in* $\{1\}\times\mathbf{a}$*. All tangent vectors on the strands should project non-negatively on* $\mathbf{Z}$*, i.e. they are "upward-veering". Only transverse intersections are allowed.*

The diagrams are subjects to two relations—any two diagrams related by a Reidemeister III move represent the same element in $\mathcal{A}'(\mathbf{Z},\mathbf{a})$*, and any diagram in which two strands intersect more than once represents zero.*

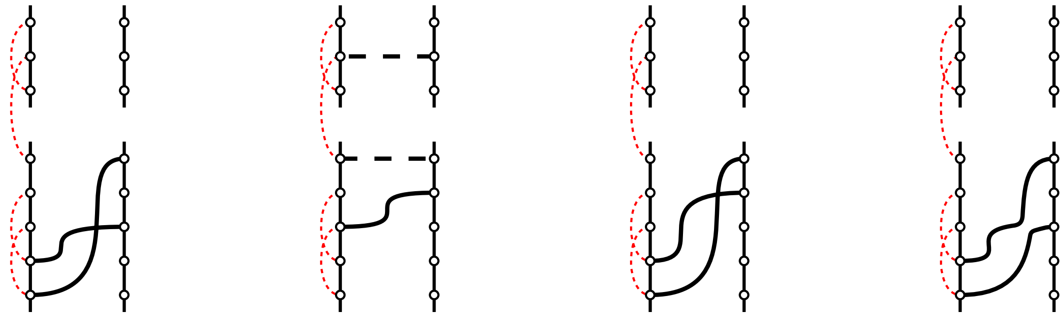
Multiplication is given by concatenation of diagrams in the $[0,1]$*–direction, provided the endpoints of the strands agree. Otherwise the product is zero. The differential of a diagram is the sum of all diagrams obtained from it by taking the oriented resolution of a crossing.*

We refer to a strand connecting $(0,a)$ to $(1,a)$ for some $a\in\mathbf{a}$ as *horizontal*. Notice that the idempotent elements of $\mathcal{A}'(\mathbf{Z},\mathbf{a})$ are precisely those which are sums of diagrams with only horizontal strands. To recover the information carried by the matching M we single out some of these idempotents.

Definition 12.2.2. *The* ground ring $\mathcal{I}(\mathcal{Z})$ associated to $\mathcal{Z}$ *is a ground ring, in the sense of Definition A.1.1, of rank* 2^k *over* $\mathbb{Z}/2$*, with canonical basis* $(\iota_I)_{I\subset\{1,\dots,k\}}$*. It is identified with a subring of the strands algebra* $\mathcal{A}'(\mathbf{Z},\mathbf{a})$*, by setting* $\iota_I=\sum_J D_J$*. The sum is over all* $J\subset\mathbf{a}$ *such that* $M|_J\colon J\to I$ *is a bijection, and* D_J *is the diagram with horizontal strands* $[0,1]\times J$*.*

For all $I\subset\{1,\dots,k\}$, the generator ι_I is a sum of $2^{\#I}$ diagrams.

Definition 12.2.3. *The* bordered algebra $\mathcal{A}(\mathcal{Z})$ associated to $\mathcal{Z}$ *is the subalgebra of* $\mathcal{I}(\mathcal{Z})\cdot\mathcal{A}'(\mathbf{Z},\mathbf{a})\cdot\mathcal{I}(\mathcal{Z})$ *consisting of all elements* α *subject to the following condition. Suppose* $M(a)=M(b)$*, and* D *and* D' *are two diagrams, where* D' *is obtained from* D *by replacing the horizontal arc* $[0,1]\times\{a\}$ *by the horizontal arc* $[0,1]\times\{b\}$*. Then* α *contains* D *as a summand iff it contains* D' *as a summand.*



(a) a_1 (b) a_2 (c) a_3 (d) a_4

Figure 19: Four generators of $\mathcal{A}(\mathcal{Z})$.

We use $\mathcal{I}(\mathcal{Z})$ as the ground ring for $\mathcal{A}(\mathcal{Z})$, in the sense of Definition B.2.1. The condition in Definition 12.2.3 ensures that the canonical basis elements of $\mathcal{I}(\mathcal{Z})$ are indecomposable in $\mathcal{A}(\mathcal{Z})$.

It is straightforward to check that Definition 12.2.3 is equivalent to the definition of $\mathcal{A}(\mathcal{Z})$ in Part I.

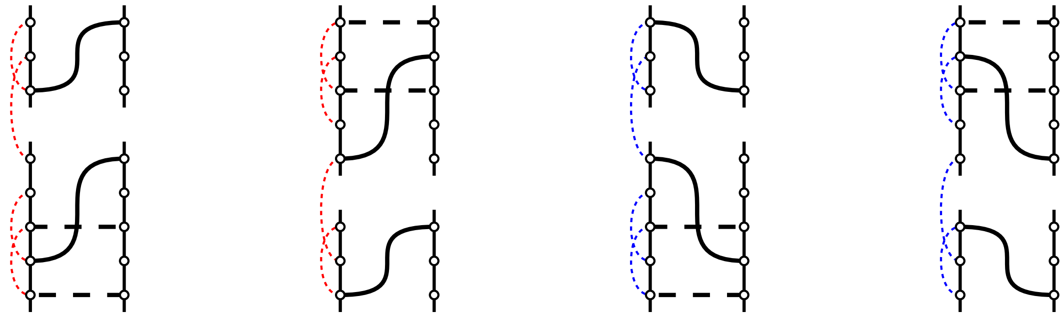
Examples of several algebra elements are given in Figure 19. The dotted lines on the side are given to remind us of the matching in the arc diagram $\mathcal{Z}$. All strands are oriented left to right, so we avoid drawing them with arrows. The horizontal lines in Figure 19b are dotted, as a shorthand for the sum of two diagrams, with a single horizontal line each. For the elements in this example, we have $a_1 \cdot a_2 = a_3$, and $\partial a_1 = a_4$.

The situation for arc diagrams of β–type is completely analogous, with one important difference.

Definition 12.2.4. *The* bordered algebra $\mathcal{A}(\mathcal{Z})$ associated to a β–arc diagram $\mathcal{Z}$*, is defined in the exact same way as in Definitions 12.2.3, except that moving strands are downward veering, instead of upward.*

The relationship between the different types of algebras is summarized in the following proposition.

Proposition 12.2.5. *Suppose $\mathcal{Z}$ is an arc diagram of either α or β–type. The algebras*



(a) $\mathcal{A}(\mathcal{Z})$ (b) $\mathcal{A}(-\mathcal{Z})$ (c) $\mathcal{A}(\overline{\mathcal{Z}})$ (d) $\mathcal{A}(-\overline{\mathcal{Z}})$

Figure 20: Four elements in the algebras for $\mathcal{Z}$, $-\mathcal{Z}$, $\overline{\mathcal{Z}}$, and $-\overline{\mathcal{Z}}$, which correspond to each other.

associated to $\mathcal{Z}$, $-\mathcal{Z}$, $\overline{\mathcal{Z}}$, and $-\overline{\mathcal{Z}}$ are related as follows:

$$\mathcal{A}(-\mathcal{Z}) \cong \mathcal{A}(\overline{\mathcal{Z}}) \cong \mathcal{A}(\mathcal{Z})^{\mathrm{op}},$$
$$\mathcal{A}(-\overline{\mathcal{Z}}) \cong \mathcal{A}(\mathcal{Z}).$$

Here A^{op} denotes the opposite algebra of A. That is, an algebra with the same additive structure and differential, but the order of multiplication reversed.

Proof. This is easily seen by reflecting and rotating diagrams. To get from $\mathcal{A}(\mathcal{Z})$ to $\mathcal{A}(-\mathcal{Z})$ we have to rotate all diagrams by 180 degrees. This means that multiplication switches order, so we get the opposite algebra.

To get from $\mathcal{A}(\mathcal{Z})$ to $\mathcal{A}(\overline{\mathcal{Z}})$ we have to reflect all diagrams along the vertical axis. This again means that multiplication switches order.

An example of the correspondence is shown in Figure 20. □

12.3 The bordered invariants

We will give a brief sketch of the definitions of the bordered invariants from Part I, which apply for the case of α–arc diagrams. Then we discuss the necessary modifications when β–arcs are involved.

For now assume $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is an α–arc diagram.

Definition 12.3.1. *A* bordered sutured Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z})$ *consists of the following:*

- *A compact surface Σ with no closed components.*
- *A collection of circles $\boldsymbol{\alpha}^c$ and a collection of arcs $\boldsymbol{\alpha}^a$, which are pairwise disjoint and properly embedded in Σ. We set $\boldsymbol{\alpha} = \boldsymbol{\alpha}^a \cup \boldsymbol{\alpha}^c$.*
- *A collection of disjoint circles $\boldsymbol{\beta}$, properly embedded in Σ.*
- *An embedding $G(\mathcal{Z}) \hookrightarrow \Sigma$, such that $\mathbf{Z}$ is sent into $\partial\Sigma$, preserving orientation, while $\boldsymbol{\alpha}^a$ is the image of the arcs e_i in $G(\mathcal{Z})$.*

We require that $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \to \pi_0(\Sigma \setminus (\boldsymbol{\alpha}^c \cup \boldsymbol{\alpha}^a))$ and $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \to \pi_0(\Sigma \setminus \boldsymbol{\beta})$ be surjective.

To such a diagram we can associate a bordered sutured manifold $(Y, \Gamma, \mathcal{Z})$ as follows. We obtain Y from $\Sigma \times [0,1]$ by gluing 2–handles to $\boldsymbol{\beta} \times \{1\}$ and $\boldsymbol{\alpha}^c \times \{0\}$. The dividing set is $\Gamma = (\partial\Sigma \setminus \mathbf{Z}) \times \{1/2\}$, and $F(\mathcal{Z})$ is a neighborhood of $\mathbf{Z} \times [0,1] \cup \boldsymbol{\alpha}^a \times \{0\}$.

As proved in Part I, for every bordered sutured manifold there is a unique Heegaard diagram, up to isotopy and some moves.

The bordered invariants are certain homotopy-equivalence classes of $\mathcal{A}_\infty$–modules (see Appendix B). For a given Heegaard diagram $\mathcal{H}$, we can form the set of *generators* $\mathcal{G}(\mathcal{H})$ consisting of collections of intersection points of $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$.

The invariant $\widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ is a right type–A $\mathcal{A}_\infty$–module over $\mathcal{A}(\mathcal{Z})$, with $\mathbb{Z}/2$–basis $\mathcal{G}(\mathcal{H})$. The ground ring $\mathcal{I}(\mathcal{Z})$ acts as follows. The only idempotent in $\mathcal{I}(\mathcal{Z})$ which acts nontrivially on $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ is $\iota_{I(\mathbf{x})}$ where $I(\mathbf{x}) \subset \{1, \ldots, k\}$ records the α–arcs which contain a point of $\mathbf{x}$.

The structure map m of $\widehat{BSA}(\mathcal{H})$ counts certain holomorphic curves in $\operatorname{Int}\Sigma \times [0,1] \times \mathbb{R}$, with boundary on $(\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})$. Each such curve has two types of asymptotics—ends at $(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \times [0,1] \times \pm\infty$, and ends at $\partial\Sigma \times \{0\} \times \{h\}$ where $h \in \mathbb{R}$ is finite. The possible ends at $\partial\Sigma$ are in 1-to-1 correspondence with elements of $\mathcal{A}(\mathcal{Z})$.

The expression $\langle m(\mathbf{x}, a_1, \ldots, a_n), \mathbf{y}^\vee \rangle$ counts curves as above, which have asymptotics $\mathbf{x} \times [0,1]$ at $-\infty$, $\mathbf{y} \times [0,1]$ at $+\infty$, and $a_1, a_2, \ldots, a_n$ at some finite values $h_1 < h_2 < \ldots < h_n$.

We write $\widehat{BSA}(\mathcal{Y})$ for the homotopy equivalence class of $\widehat{BSA}(\mathcal{H})$. (Invariance was proven in Part I.)

The invariant ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{BSD}(\mathcal{H})$ is a left type–D $\mathcal{A}_\infty$–module over $\mathcal{A}(-\mathcal{Z}) = \mathcal{A}(\mathcal{Z})^{\mathrm{op}}$, with $\mathbb{Z}/2$–basis $\mathcal{G}(\mathcal{H})$. (See Appendix B.2 for type–D modules, and the meaning of upper and lower indices). The ground ring $\mathcal{I}(-\mathcal{Z})$ acts as follows. The only idempotent in $\mathcal{I}(-\mathcal{Z})$ which acts nontrivially on $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ is $\iota_{I^c(\mathbf{x})}$ where $I^c(\mathbf{x}) \subset \{1, \ldots, k\}$ records the α–arcs which *do not* contain a point of $\mathbf{x}$.

The structure map δ of $\widehat{BSD}(\mathcal{H})$ counts a subset of the same holomorphic curves as for $\widehat{BSA}(\mathcal{H})$. The interpretation is somewhat different, though. Equivalently, ${}^{\mathcal{A}(\mathcal{Z})^{\mathrm{op}}}\widehat{BSD}(\mathcal{H}) = \widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(\mathcal{Z})^{\mathrm{op}}}\mathbb{I}$, where $\mathbb{I}$ is a certain bimodule defined in [LOT10a].

Again, we write $\widehat{BSD}(\mathcal{Y})$ for the homotopy equivalence class of $\widehat{BSD}(\mathcal{H})$. (Invariance was proven in Part I.)

We can also construct invariants ${}_{\mathcal{A}(\mathcal{Z})^{\mathrm{op}}}\widehat{BSA}(\mathcal{Y})$ and $\widehat{BSD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z})}$ purely algebraically from the usual $\widehat{BSA}$ and $\widehat{BSD}$. Indeed, as discussed in Appendix B.6, any right A–module is a left–A^{op} module and vice versa.

If $\mathcal{Y}$ is bordered by $\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, we can similarly define several bimodules invariants for $\mathcal{Y}$:

$$
\begin{array}{ll}
{}_{\mathcal{A}(\mathcal{Z}_1)^{\mathrm{op}}}\widehat{BSAA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)^{\mathrm{op}}}\widehat{BSDA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} \\
{}_{\mathcal{A}(\mathcal{Z}_1)^{\mathrm{op}}}\widehat{BSAD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)^{\mathrm{op}}}\widehat{BSDD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)}
\end{array}
$$

For the invariants of β–diagrams little changes. Suppose $\mathcal{Z}$ is a β–type arc diagram. Heegaard diagrams will now involve β–arcs as the images of $e_i \subset G(\mathcal{Z})$, instead of α–arcs. We still count holomorphic curves in $\operatorname{Int}\Sigma \times [0,1] \times \mathbb{R}$. However, since there are β–curves hitting $\partial\Sigma$ instead of α, the asymptotic ends at $\partial\Sigma \times \{1\} \times \{h\}$ are replaced by ends at $\partial\Sigma \times \{0\} \times \{h\}$, which again correspond to elements of $\mathcal{A}(\mathcal{Z})$. The rest of the definition is essentially unchanged.

The last case is when $\mathcal{Y}$ is bordered by $\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, where $\mathcal{Z}_1$ is a diagram of α–type and $\mathcal{Z}_2$ is of β–type. We can extend the definition of $\widehat{BSAA}(\mathcal{Y})$ as before. There are now four types of asymptotic ends:

- The ones at $\pm\infty$ which correspond to generators $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$.
- $\partial\Sigma \times \{1\} \times \{h\}$ (or α–*ends*) which correspond to $\mathcal{A}(\mathcal{Z}_1)$.
- $\partial\Sigma \times \{0\} \times \{h\}$ (or β–*ends*) which correspond to $\mathcal{A}(\mathcal{Z}_2)$.

Each holomorphic curve will have some number $k \geq 0$ of α–ends, and some number $l \geq 0$ of β–ends. Such a curve contributes to the structure map $m_{k|1|l}$ which takes k elements of $\mathcal{A}(\mathcal{Z}_1)$ and l elements of $\mathcal{A}(\mathcal{Z}_2)$.

To summarize we have the following theorem.

Theorem 12.3.2. *Let $\mathcal{Y}$ be a bordered sutured manifold, bordered by $-\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ can be any combination of α and β types. Then there are bimodules, well defined up to homotopy equivalence:*

$$
\begin{array}{ll}
{}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{BSAA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{BSDA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} \\
{}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{BSAD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{BSDD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)}
\end{array}
$$

If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two such manifolds, bordered by $-\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$ and $-\mathcal{F}(\mathcal{Z}_2) \sqcup \mathcal{F}(\mathcal{Z}_3)$, respectively, then there are homotopy equivalences

$$
\begin{aligned}
\widehat{BSAA}(\mathcal{Y}_1 \cup \mathcal{Y}_2) &\simeq \widehat{BSAA}(\mathcal{Y}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSDA}(\mathcal{Y}_2), \\
\widehat{BSDA}(\mathcal{Y}_1 \cup \mathcal{Y}_2) &\simeq \widehat{BSDD}(\mathcal{Y}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSAA}(\mathcal{Y}_2),
\end{aligned}
$$

etc. Any combination of bimodules for $\mathcal{Y}_1$ and $\mathcal{Y}_2$ can be used, where one is type–A for $\mathcal{A}(\mathcal{Z}_2)$, and the other is type–D for $\mathcal{A}(\mathcal{Z}_2)$.

The latter statement is referred to as the *pairing theorem*. The proof of Proposition 12.3.2 is a straightforward adaptation of the corresponding proofs when dealing with only type–α diagrams. An analogous construction involving both α and β arcs in the purely bordered setting is given in [LOT10b].

12.4 Mirrors and twisting slices

In this section we give two computations of bordered invariants. One of them relates the invariants for a bordered sutured manifold $\mathcal{W}$ and its mirror $-\mathcal{W}$. The other gives the invariants for a positive and negative twisting slice.

Recall that if $\mathcal{W} = (W, \Gamma, \mathcal{F}(\mathcal{Z}))$, its mirror is $-\mathcal{W} = (-W, \Gamma, \overline{\mathcal{F}(\mathcal{Z})}) = (-W, \Gamma, \mathcal{F}(\overline{\mathcal{Z}}))$.

Proposition 12.4.1. *Let $\mathcal{W}$ and $-\mathcal{W}$ be as above. Let $M_{\mathcal{A}(\mathcal{Z})}$ be a representative for the homotopy equivalence class $\widehat{BSA}(\mathcal{W})_{\mathcal{A}(\mathcal{Z})}$. Then its dual ${}_{\mathcal{A}(\mathcal{Z})}M^\vee$ is a representative for ${}_{\mathcal{A}(\mathcal{Z})}\widehat{BSA}(-\mathcal{W})$. Similarly, there are homotopy equivalences*

$$\left(\widehat{BSD}(\mathcal{W})^{\mathcal{A}(\mathcal{Z})}\right)^\vee \simeq {}^{\mathcal{A}(\mathcal{Z})}\widehat{BSD}(-\mathcal{W}),$$
$$\left({}_{\mathcal{A}(\mathcal{Z})^{\mathrm{op}}}\widehat{BSA}(\mathcal{W})\right)^\vee \simeq \widehat{BSA}(-\mathcal{W})_{\mathcal{A}(\mathcal{Z})^{\mathrm{op}}},$$
$$\left({}^{\mathcal{A}(\mathcal{Z})^{\mathrm{op}}}\widehat{BSD}(\mathcal{W})\right)^\vee \simeq \widehat{BSD}(-\mathcal{W})^{\mathcal{A}(\mathcal{Z})^{\mathrm{op}}}.$$

A similar statement holds for bimodules—if $\mathcal{W}$ is bordered by $\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, then the corresponding bimodule invariants of $\mathcal{W}$ and $-\mathcal{W}$ are duals of each other.

Proof. We prove one case. All others follow by analogy. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z})$ be a Heegaard diagram for $\mathcal{W}$. Let $\mathcal{H}' = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \overline{\mathcal{Z}})$ be the diagram obtained by switching all α and β curves. (Note that if $\mathcal{Z}$ was an α–type diagram, this turns it into the β–type diagram $\overline{\mathcal{Z}}$, and vice versa.)

The bordered sutured manifold described by $\mathcal{H}'$ is precisely $-\mathcal{W}$. Indeed, it is obtained from the same manifold $\Sigma \times [0,1]$ by attaching all 2–handles on the opposite side, and taking the sutured surface $\mathcal{F}$ also on the opposite side. This is equivalent to reversing the orientation of W, while keeping the orientations of $\Gamma \subset \partial\Sigma$ and $\mathbf{Z} \subset \partial\Sigma$ the same. (Compare to [HKM09], where the EH–invariant for contact structures on (Y, Γ) is defined in $SFH(-Y, +\Gamma)$.)

The generators $\mathcal{G}(\mathcal{H})$ and $\mathcal{G}(\mathcal{H}')$ of the two diagrams are the same. There is also a 1–to–1 correspondence between the holomorphic curves u in the definition of $\widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ and the curves u' in the definition of $\widehat{BSA}(\mathcal{H}')_{\mathcal{A}(\overline{\mathcal{Z}})}$. This is given by reflecting both the $[0,1]$–factor and the $\mathbb{R}$–factor in the domain $\operatorname{Int}\Sigma \times [0,1] \times \mathbb{R}$. The $\pm\infty$ asymptotic ends are reversed.

The α–ends of u are sent to the β–ends of u', and vice versa, while their heights h on the $\mathbb{R}$–scale are reversed. When turning α–ends to β–ends, the corresponding elements of $\mathcal{A}(\mathcal{Z})$ are reflected (as in the correspondence $\mathcal{A}(\overline{\mathcal{Z}}) \cong \mathcal{A}(\mathcal{Z})^{\mathrm{op}}$ from Proposition 12.2.5).

This implies the following relation between the structure maps m of $\widehat{BSA}(\mathcal{H})$ and m' of $\widehat{BSA}(\mathcal{H}')$:

$$\langle m(\mathbf{x}, a_1, \ldots, a_n), \mathbf{y}^\vee \rangle = \langle m'(\mathbf{y}', a_n^{\mathrm{op}}, \ldots, a_1^{op}), \mathbf{x}'^\vee \rangle .$$

Turning $\widehat{BSA}(\mathcal{H}')$ into a left module over $(\mathcal{A}(\mathcal{Z})^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}(\mathcal{Z})$, we get the relation

$$\langle m(\mathbf{x}, a_1, \ldots, a_n), \mathbf{y}^\vee \rangle = \langle m'(a_1, \ldots, a_n, \mathbf{y}'), \mathbf{x}'^\vee \rangle .$$

This is precisely the statement that $\widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ and ${}_{\mathcal{A}(\mathcal{Z})}\widehat{BSA}(\mathcal{H}')$ are duals, with $\mathcal{G}(\mathcal{H})$ and $\mathcal{G}(\mathcal{H}')$ as dual bases. $\square$

A similar statement for purely bordered invariants is proven in [LOT10b].

Proposition 12.4.2. *Let $\mathcal{Z}$ be any arc diagram, and let $A = \mathcal{A}(\mathcal{Z})$. The twisting slices $\mathcal{TW}_{\mathcal{F}(\mathcal{Z}),\pm}$ are bordered by $-\mathcal{F}(\mathcal{Z}) \sqcup -\overline{\mathcal{F}(\mathcal{Z})}$. They have bimodule invariants*

$${}_A\widehat{BSAA}(\mathcal{TW}_{\mathcal{F}(\mathcal{Z}),-})_A \simeq {}_A A_A, \qquad {}_A\widehat{BSAA}(\mathcal{TW}_{\mathcal{F}(\mathcal{Z}),+})_A \simeq {}_A A^\vee{}_A.$$

Proof. Since $\mathcal{TW}_{\mathcal{F}(\mathcal{Z}),\pm}$ are mirrors of each other, by Proposition 12.4.1, it is enough to prove the first equivalence. The key ingredient is a very convenient *nice diagram* $\mathcal{H}$ for $\mathcal{TW}_{\mathcal{F}(\mathcal{Z}),-}$. This diagram was discovered by the author, and independently by Auroux in [Aur10], where it appears in a rather different setting.

Recall from Part I that a nice diagram is a diagram, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z})$ where each region of $\Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ is either a boundary region, a rectangle, or a bigon. The definition trivially extends to the current more general setting. Nice diagrams can still be used to combinatorially compute bordered sutured invariants.

The diagram is obtained as follows. For concreteness assume that $\mathcal{Z}$ is of α–type. To construct the Heegaard surface Σ, start with several squares $[0,1] \times [0,1]$, one for each component $Z \in \mathbf{Z}$. There are three identifications of $\mathbf{Z}$ with sides of the squares:

- φ sending $\mathbf{Z}$ to the "left sides" $\{0\} \times [0,1]$, oriented from 0 to 1.

- φ' sending $\mathbf{Z}$ to the "right sides" $\{1\} \times [0,1]$, oriented from 1 to 0.

- ψ sending $\mathbf{Z}$ to the "top sides" $[0,1] \times \{1\}$, oriented from 1 to 0.

For each matched pair $\{a,b\} = M^{-1}(i) \subset \mathbf{a} \subset \mathbf{Z}$, attach a 1–handle at $\psi(\{a,b\})$. Add an α–arc α_i^a from $\varphi(a)$ to $\varphi(b)$, and a β–arc β_i^a from $\varphi'(a)$ to $\varphi'(b)$, both running through the handle corresponding to a,b. To see that this gives the correct manifold, notice that there are no α or β–circles, so the manifold is topologically $\Sigma \times [0,1]$. The pattern of attachment of the 1–handles shows that $\Sigma = F(\mathcal{Z})$. It is easy to check that Γ and the arcs are in the correct positions.

This construction is demonstrated in Figure 21. The figure corresponds to the arc diagram $\mathcal{Z}$ from Figure 17c.

Calculations with the same diagram in [Aur10] and [LOT10b] show that the bimodule $\widehat{BSAA}(\mathcal{H})$ is indeed the algebra A as a bimodule over itself. While the statements in those cases are not about bordered sutured Floer homology, the argument is purely combinatorial and caries over completely.

We give a brief summary of this argument. Intersection points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ are of two types:

- $x_i \in \alpha_i^a \cap \beta_i^a$ inside the 1–handle corresponding to $M^{-1}(i)$, for $i \in \{1,\ldots,k\}$. The point x_i corresponds to the two horizontal strands $[0,1] \times M^{-1}(i)$ in $\mathcal{A}(\mathcal{Z})$.

- $y_{ab} \in \alpha_{M(a)}^a \cap \beta_{M(b)}^a$, inside the square regions of $\mathcal{H}$. The point y_{ab} corresponds to a strand $(0,a) \to (1,b)$ (or $a \to b$ for short) in $\mathcal{A}(\mathcal{Z})$.

The allowed combinations of intersection points correspond to the allowed diagrams in $\mathcal{A}(\mathcal{Z})$, so $\widehat{BSA}(\mathcal{H}) \cong \mathcal{A}(\mathcal{Z})$ as a $\mathbb{Z}/2$–vector space.

Since $\mathcal{H}$ is a nice diagram the differential counts embedded rectangles in $\mathcal{H}$, with sides on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The rectangle with corners $(y_{ad}, y_{bc}, y_{ac}, y_{ad})$ corresponds to resolving the crossing between the strands $a \to d$ and $b \to c$ (getting $a \to c$ and $b \to d$).

The left action $m_{1|1|0}$ of A counts rectangles hitting the $-\mathcal{Z}$–part of the boundary. The rectangle with corners $(\varphi(a), y_{ac}, y_{bc}, \varphi(b))$ corresponds to concatenating the strands $a \to b$ and $b \to c$ (getting $a \to c$). The right action is similar, with rectangles hitting the $-\overline{\mathcal{Z}}$–part of the boundary.

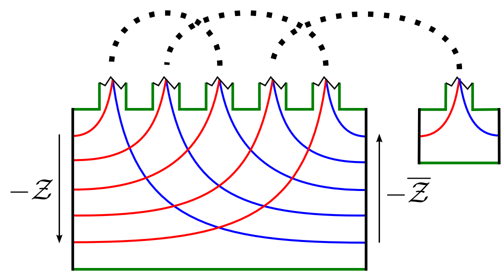
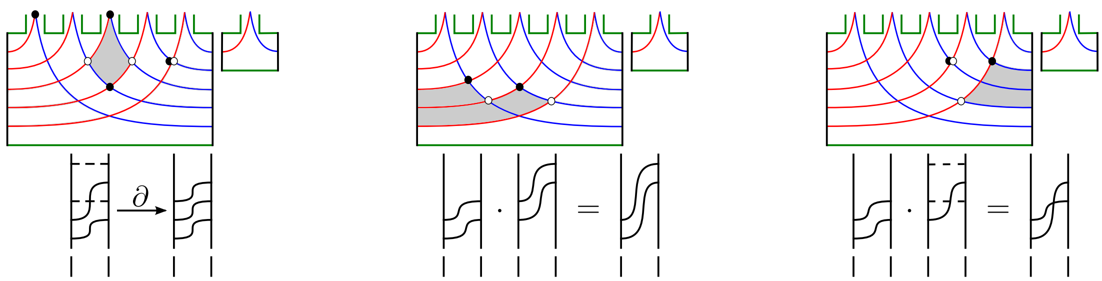


Figure 21: Heegaard diagram for a negative twisted slice $\mathcal{TW}_{\mathcal{F},-}$.



(a) Differential.

(b) Left action.

(c) Right action.

Figure 22: Examples of domains counted in the diagram for $\mathcal{TW}_{\mathcal{F},-}$. In each case the domain goes from the black dots to the white dots. Below them we show the corresponding operations on the algebra.

Some examples of domains in $\mathcal{H}$ contributing to $m_{0|1|0}$, $m_{1|1|0}$, and $m_{0|1|1}$ are shown in Figure 22. They are for the diagram $\mathcal{H}$ from Figure 21. $\square$

Chapter 13

The join map

In this section we will define the join and gluing maps, and prove some basic properties. Recall that the gluing operation is defined as a special case of the join operation. The gluing map is similarly a special case of the join map. Thus for the most part we will only talk about the general case, i.e. the join map.

13.1 The algebraic map

We will first define an abstract algebraic map, on the level of $\mathcal{A}_\infty$–modules.

Let A be a differential graded algebra, and ${}_AM$ be a left $\mathcal{A}_\infty$–module over it. As discussed in Appendix B.6, the dual $M^\vee{}_A$ is a right $\mathcal{A}_\infty$–module over A. Thus ${}_A(M \otimes M^\vee)_A$ is an $\mathcal{A}_\infty$–bimodule. On the other hand, since A is a bimodule over itself, so is its dual ${}_AA^\vee{}_A$. We define a map $M \otimes M^\vee \to A^\vee$ which is an $\mathcal{A}_\infty$–analog of the natural pairing of a module and its dual.

Definition 13.1.1. *The* algebraic join map $\nabla_M\colon {}_A(M \otimes M^\vee)_A \to {}_AA^\vee{}_A$*—or just* ∇ *when unambiguous—is an* $\mathcal{A}_\infty$*–bimodule morphism, defined as follows. It is the unique morphism satisfying*

$$\left\langle \nabla_{i|1|j}(a_1, \ldots, a_i,\ p,\ q^\vee,\ , a'_1, \ldots, a'_j), a'' \right\rangle$$
$$= \left\langle m_{i+j+1|1}(a'_1, \ldots, a'_j, a'', a_1, \ldots, a_i,\ p), q^\vee \right\rangle, \quad (13.1)$$

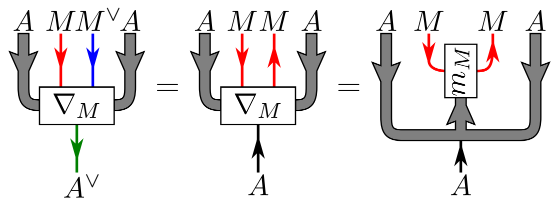


Figure 23: Definition of the join map ∇.

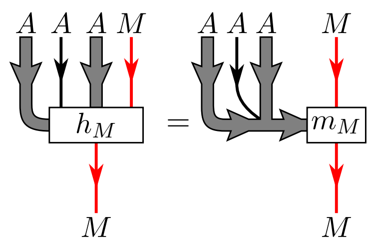


Figure 24: The homotopy equivalence $h_M\colon A \widetilde{\otimes} M \to M$.

for any $i, j \geq 0$, $p \in M$, $q^\vee \in M^\vee$, *and* $a_*^* \in A$.

Eq. (13.1) is best represented diagrammatically, as in Figure 23. Note that ∇_M is a bounded morphism if and only if M is a bounded module.

As discussed in Appendix B.4, morphisms of $\mathcal{A}_\infty$–modules form chain complexes, where cycles are *homomorphisms*. Only homomorphisms descend to maps on homology.

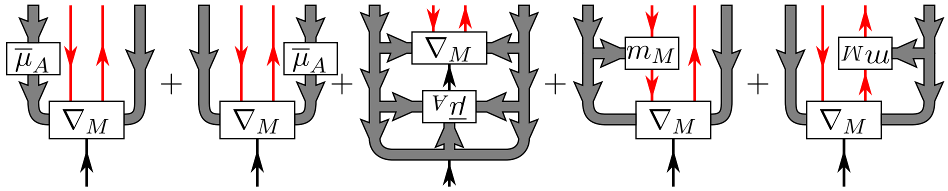
Proposition 13.1.2. *For any* ${}_AM$*, the join map* ∇_M *is a homomorphism.*

Proof. It is a straightforward but tedious computation to see that $\partial\nabla_M = 0$ is equivalent to the structure equation for m_M.

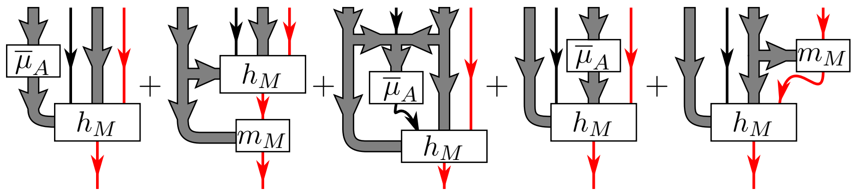
A more enlightening way to see this is to notice that by turning the diagram in Figure 23 partly sideways, we get a diagram for the homotopy equivalence $h_M\colon A \widetilde{\otimes} M \to M$, shown in Figure 24. Taking the differential $\partial\nabla_M$ and turning the resulting diagrams sideways, we get precisely ∂h_M. We know that h_M is a homomorphism and, so $\partial h_M = 0$.

The equivalences are presented in Figure 25. $\square$

We will prove two naturallity statements about ∇ that together imply that ∇ descends to a well defined map on the derived category. The first shows that ∇ is natural with respect to isomorphisms in the derived category of the DG-algebra A, i.e. homotopy equivalences



(a) The differential $\partial\nabla_M$ which needs to vanish to show that ∇_M is an $\mathcal{A}_\infty$–bimodule homomorphism.



(b) The differential ∂h_M of the homotopy equivalence h_M.

Figure 25: Proof that ∇ is a homomorphism, by rotating diagrams.

of modules. The second shows that ∇ is natural with respect to equivalences of derived categories. (Recall from Part I that different algebras corresponding to the same sutured surface are derived-equivalent.)

Proposition 13.1.3. *Suppose ${}_AM$ and ${}_AN$ are two $\mathcal{A}_\infty$–modules over A, such that there are inverse homotopy equivalences $\varphi\colon M \to N$ and $\psi\colon N \to M$. Then there is an $\mathcal{A}_\infty$–homotopy equivalence of A, A–bimodules*

$$\varphi \otimes \psi^\vee\colon M \otimes M^\vee \to N \otimes N^\vee,$$

and the following diagram commutes up to $\mathcal{A}_\infty$–homotopy:

$$\begin{tikzcd}
M \otimes M^\vee \arrow[d, "\varphi\otimes\psi^\vee"'] \arrow[dr, "\nabla_M"] & \\
N \otimes N^\vee \arrow[r, "\nabla_N"] & A^\vee.
\end{tikzcd}$$

Proposition 13.1.4. *Suppose A and B are differential graded algebras, and ${}_BX^A$ and ${}_AY^B$ are two type–DA bimodules, which are quasi-inverses. That is, there are $\mathcal{A}_\infty$–homotopy equivalences*

$${}_A(Y \boxtimes X)^A \simeq {}_A\mathbb{I}^A, \qquad {}_B(X \boxtimes Y)^B \simeq {}_B\mathbb{I}^B.$$

Moreover, suppose $H_*(B^\vee)$ *and* $H_*(X \boxtimes A^\vee \boxtimes X^\vee)$ *have the same rank (over* $\mathbb{Z}/2$*).*

Then there is a B, B*–bimodule homotopy equivalence*

$$\varphi_X \colon X \boxtimes A^\vee \boxtimes X^\vee \to B^\vee.$$

Moreover, for any $\mathcal{A}_\infty$*–module* ${}_A M$*, such that* $X \boxtimes M$ *is well defined, the following diagram commutes up to* $\mathcal{A}_\infty$*–homotopy:*

$$\begin{tikzcd}
X \boxtimes M \otimes M^\vee \boxtimes X^\vee \arrow[d, "\mathrm{id}_X \boxtimes \nabla_M \boxtimes \mathrm{id}_{X^\vee}"'] \arrow[dr, "\nabla_{X\boxtimes M}"] & \\
X \boxtimes A^\vee \boxtimes X^\vee \arrow[r, "\varphi_X"] & B^\vee.
\end{tikzcd}$$

Notice the condition that $X \boxtimes M$ be well defined. This can be satisfied for example if M is a bounded module, or if X is reletively bounded in A with respect to B. Before proving Propositions 13.1.3 and 13.1.4 in Chapter 13.3, we will use them to define the join Ψ.

13.2 The geometric map

Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two sutured manifolds, and $\mathcal{W} = (W, \Gamma, -\mathcal{F})$ is a partially sutured manifold, with embeddings $\mathcal{W} \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W} \hookrightarrow \mathcal{Y}_2$. Let $\mathcal{Z}$ be any arc diagram parametrizing the surface $\mathcal{F}$. Recall that $-\mathcal{W} = (-W, \Gamma, -\overline{\mathcal{F}})$. Also recall the twisting slice $\mathcal{TW}_{\mathcal{F},+}$, from Definition 11.2.2. The join $\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2$ of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ along $\mathcal{W}$ was defined as

$$\mathcal{Y}_1 \ \Cup_{\mathcal{W}} \ \mathcal{Y}_2 = (\mathcal{Y}_1 \setminus \mathcal{W}) \ \cup_{\mathcal{F}} \ \mathcal{TW}_{\mathcal{F},+} \ \cup_{-\overline{\mathcal{F}}} \ (\mathcal{Y}_2 \setminus -\mathcal{W}).$$

Let $A = \mathcal{A}(\mathcal{Z})$ be the algebra associated to $\mathcal{Z}$. Let ${}_A M$, U^A, and ${}^A V$ be representatives for the bordered sutured modules ${}_A\widehat{BSA}(\mathcal{W})$, $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W})^A$, and ${}^A\widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})$, respectively such that $U \boxtimes M$ and $M^\vee \boxtimes V$ are well-defined. (Recall that the modules are only defined up to homotopy equivalence, and that the $\boxtimes$ product is only defined under some boundedness conditions.) We proved in Proposition 12.4.1 that $M^\vee{}_A$ is a representative for $\widehat{BSA}(-\mathcal{W})_A$, and in Proposition 12.4.2 that ${}_A A^\vee{}_A$ is a representative for $\widehat{BSAA}(\mathcal{TW}_{\mathcal{F},+})$.

From the Künneth formula for *SFH* of a disjoint union, and from Theorem 12.3.2, we have the following homotopy equivalences of chain complexes.

$$\begin{aligned}
SFC(\mathcal{Y}_1 \cup \mathcal{Y}_2) &\cong SFC(\mathcal{Y}_1) \otimes SFC(\mathcal{Y}_2) \\
&\simeq \left(\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}) \boxtimes_A \widehat{BSA}(\mathcal{W})\right) \otimes \left(\widehat{BSA}(-\mathcal{W}) \boxtimes_A \widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})\right) \\
&\simeq U^A \ \boxtimes \ {}_A(M \otimes M^\vee)_A \ \boxtimes \ {}^A V.
\end{aligned}$$

$$\begin{aligned}
&SFC(\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2) \\
&\simeq \widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}) \boxtimes_A \widehat{BSAA}(\mathcal{TW}_{\mathcal{F},+}) \boxtimes_A \widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W}) \\
&\simeq U^A \ \boxtimes \ {}_A A^\vee{}_A \ \boxtimes \ {}^A V.
\end{aligned}$$

Definition 13.2.1. *Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{W}$ be as described above. Define the* geometric join map

$$\Psi_M \colon\ SFC(\mathcal{Y}_1) \otimes SFC(\mathcal{Y}_2) \to SFC(\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2)$$

by the formula

$$\Psi_M = \mathrm{id}_U \boxtimes \nabla_M \boxtimes \mathrm{id}_V \colon\ U \boxtimes M \otimes M^\vee \boxtimes V \to U \boxtimes A^\vee \boxtimes V. \tag{13.2}$$

Note that such an induced map is not generally well defined (it might involve an infinite sum). In this case, however, we have made some boundedness assumptions. Since $U \boxtimes M$ and $M^\vee \boxtimes V$ are defined, either M must be bounded, or both of U and V must be bounded. In the former case, ∇_M is also bounded. Either of these situations guarantees that the sum defining Ψ_M is finite.

Theorem 13.2.2. *The map Ψ_M from Definition 13.2.1 is, up to homotopy, independent on the choice of parametrization $\mathcal{Z}$, and on the choices of representatives M, U, and V.*

Proof. First, we will give a more precise version of the statement. Let $\mathcal{Z}'$ be any other parametrization of $\mathcal{F}$, with $B = \mathcal{A}(-\mathcal{Z}')$, and let ${}_B M'$, U'^B and ${}^B V'$, be representatives for the respective bordered sutured modules. Then there are homotopy equivalences φ and ψ making the following diagram commute up to $\mathcal{A}_\infty$–homotopy:

$$\begin{tikzcd}
U \boxtimes M \otimes M^\vee \boxtimes V \arrow[r, "\varphi"] \arrow[d, "\Psi_M"'] & U' \boxtimes M' \otimes M'^\vee \boxtimes V' \arrow[d, "\Psi_{M'}"] \\
U \boxtimes A^\vee \boxtimes V \arrow[r, "\psi"] & U' \boxtimes B^\vee \boxtimes V'.
\end{tikzcd}$$

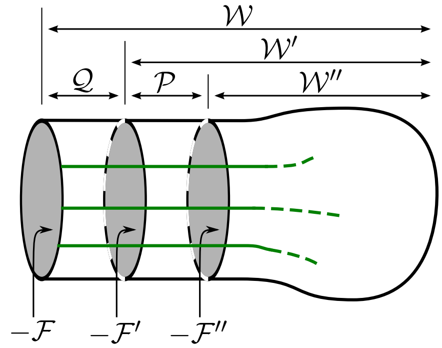


Figure 26: The various pieces produced by slicing $\mathcal{W}$ at two surfaces parallel to $\mathcal{F}$.

The proof can be broken up into several steps. The first step is independence from the choice of U and V, given a fixed choice for A and M. This follows directly from the fact $\mathrm{id}\boxtimes\cdot$ and $\cdot\boxtimes\mathrm{id}$ are DG-functors.

The second step is to show independence from the choice of M, for fixed A, U, and V. This follows from Proposition 13.1.3. Indeed, suppose $\varphi\colon M\to M'$ is a homotopy equivalence with homotopy inverse $\psi\colon M'\to M$. Then $\psi^\vee\colon M^\vee\to M'^\vee$ is also a homotopy equivalence inducing the homotopy equivalence

$$\mathrm{id}_U\boxtimes\varphi\otimes\psi^\vee\boxtimes\mathrm{id}_V\colon U\boxtimes M\otimes M^\vee\boxtimes V\to U\boxtimes M'\otimes M'^\vee\boxtimes V.$$

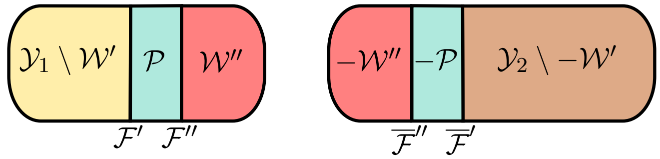
By Proposition 13.1.3, $\nabla_M\simeq\nabla_{M'}\circ(\varphi\otimes\psi^\vee)$, which implies

$$\mathrm{id}_U\boxtimes\nabla_M\boxtimes\mathrm{id}_V\simeq(\mathrm{id}_U\boxtimes\nabla_{M'}\boxtimes\mathrm{id}_V)\circ(\mathrm{id}_U\boxtimes\varphi\otimes\psi^\vee\boxtimes\mathrm{id}_V).$$

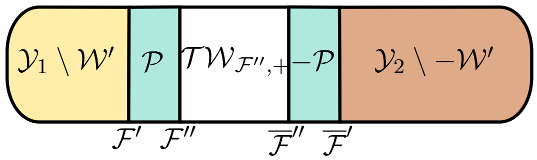
The final step is to show independence from the choice of algebra A. We will cut $\mathcal{Y}_1$ and $\mathcal{Y}_2$ into several pieces, so we can evaluate the two different versions of Ψ from the same geometric picture.

Let $-\mathcal{F}'$ and $-\mathcal{F}''$ be two parallel copies of $-\mathcal{F}$ in $\mathcal{W}$, which cut out $\mathcal{W}'=(W',\Gamma',-\mathcal{F}')$ and $\mathcal{W}''=(W'',\Gamma'',-\mathcal{F}'')$, where $\mathcal{W}''\subset\mathcal{W}'\subset\mathcal{W}$. Let $\mathcal{P}=\mathcal{W}'\setminus\mathcal{W}''$ and $\mathcal{Q}=\mathcal{W}\setminus\mathcal{W}'$ (see Figure 26). Both $\mathcal{P}$ and $\mathcal{Q}$ are topologically $F\times[0,1]$.

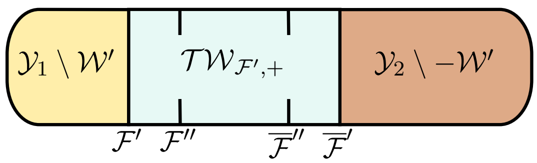
Parametrize $\mathcal{F}$ and $\mathcal{F}''$ by $\mathcal{Z}$, and $\mathcal{F}'$ by $\mathcal{Z}'$, where $\mathcal{A}(\mathcal{Z})=A$, and $\mathcal{A}(\mathcal{Z}')=B$. Let ${}_BX^A$ and ${}_AY^B$ be representatives for ${}_B\widehat{BSAD}(\mathcal{P})^A$ and ${}_A\widehat{BSAD}(\mathcal{Q})^B$, respectively. Note that $\mathcal{Q}\cup_{\mathcal{F}'}\mathcal{P}$ is a product bordered sutured manifold, and thus has trivial invariant ${}_A\widehat{BSAD}(\mathcal{Q}\cup$



(a) Cutting $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in two different places.



(b) The join by $\mathcal{W}''$.



(c) The join by $\mathcal{W}'$.

Figure 27: Two ways of cutting and pasting to get the join of $\mathcal{Y}_1$ and $\mathcal{Y}_2$.

$\mathcal{P})^A \simeq {}_A\mathbb{I}^A$. By the pairing theorem, this implies $Y \boxtimes X \simeq {}_A\mathbb{I}^A$. Similarly, by stacking $\mathcal{P}$ and $\mathcal{Q}$ in the opposite order we get $X \boxtimes Y \simeq {}_B\mathbb{I}^B$.

There are embeddings $\mathcal{W}', \mathcal{W}'' \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W}', -\mathcal{W}'' \hookrightarrow \mathcal{Y}_2$ and two distinct ways to cut and glue them together, getting $\mathcal{Y}_1 \Cup_{\mathcal{W}'} \mathcal{Y}_2 \cong \mathcal{Y}_1 \Cup_{\mathcal{W}''} \mathcal{Y}_2$. This is illustrated schematically in Figure 27.

Let ${}_AM$ be a representative for ${}_A\widehat{BSA}(\mathcal{W}'')$. By the pairing theorem, ${}_B(X \boxtimes M)$ is a representative for ${}_B\widehat{BSA}(\mathcal{W}')$. Notice that $\mathcal{TW}_{\mathcal{F}',+} \cong \mathcal{P} \cup \mathcal{TW}_{\mathcal{F}'',+} \cup -\mathcal{P}$ and ${}_BB^\vee{}_B$ and ${}_B(X \boxtimes A^\vee \boxtimes X^\vee)_B$ are both representatives for its $\widehat{BSAA}$ invariant. In particular, they have the same homology. Finally, let U^B and ${}^BV$ be representatives for $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}')^B$ and ${}^B\widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W}')$, respectively.

The two join maps Ψ_M and $\Psi_{X\boxtimes M}$ are described by the following equations.

$$\Psi_M = \mathrm{id}_{U\boxtimes X} \boxtimes \nabla_M \boxtimes \mathrm{id}_{X^\vee \boxtimes V} \colon$$
$$(U \boxtimes X) \boxtimes M \otimes M^\vee \boxtimes (X^\vee \boxtimes V) \to (U \boxtimes X) \boxtimes A^\vee \boxtimes (X^\vee \boxtimes V),$$

$$\Psi_{X\boxtimes M} = \mathrm{id}_U \boxtimes \nabla_{X\boxtimes M} \boxtimes \mathrm{id}_V \colon$$
$$U \boxtimes (X \boxtimes M) \otimes (M^\vee \boxtimes X^\vee) \boxtimes V \to U \boxtimes B^\vee \boxtimes V.$$

We can apply Proposition 13.1.4. The boundedness condition can be satisfied by requiring that X and Y are bounded modules. There is a homotopy equivalence $\varphi_X \colon X \boxtimes A^\vee \boxtimes X^\vee \to B$, and a homotopy $\nabla_{X\boxtimes M} \sim \varphi_X \circ (\mathrm{id}_X \boxtimes \nabla_M \boxtimes \mathrm{id}_{X^\vee})$. These induce a homotopy

$$(\mathrm{id}_U \boxtimes \varphi_X \boxtimes \mathrm{id}_V) \circ \Psi_M = \mathrm{id}_U \boxtimes (\varphi_X \circ (\mathrm{id}_X \boxtimes \nabla_M \boxtimes \mathrm{id}_{X^\vee})) \boxtimes \mathrm{id}_V$$
$$\sim \mathrm{id}_U \boxtimes \nabla_{X\boxtimes M} \boxtimes \mathrm{id}_V = \Psi_{X\boxtimes M}.$$

This finishes the last step. Combining all three gives complete invariance. Thus we can refer to $\Psi_{\mathcal{W}}$ from now on. □

13.3 Proof of algebraic invariance

In this section we prove Propositions 13.1.3 and 13.1.4.

Proof of Proposition 13.1.3. The proof will be mostly diagrammatic. There are two modules ${}_AM$ and ${}_AN$, and two inverse homotopy equivalences, $\varphi \colon M \to N$ and $\psi \colon N \to M$. The dualizing functor ${}_A\mathrm{Mod} \to \mathrm{Mod}_A$ is a DG-functor. Thus it is easy to see that

$$\varphi \otimes \psi^\vee = (\varphi \otimes \mathrm{id}_{N^\vee}) \circ (\mathrm{id}_M \otimes \psi^\vee)$$

is also a homotopy equivalence. Let $H \colon M \to M$ be the homotopy between id_M and $\psi \circ \varphi$.

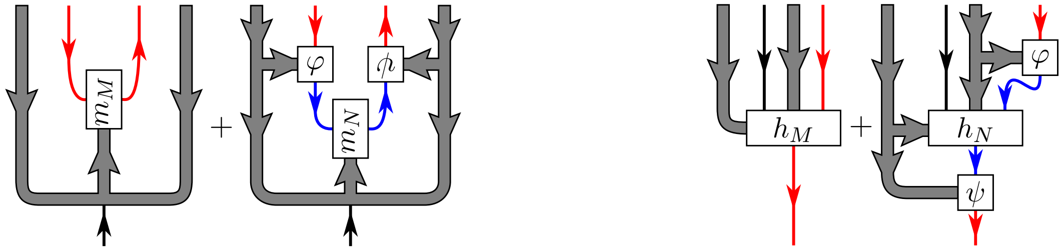
We have to show that the homomorphism

$$\nabla_M + \nabla_N \circ (\varphi \otimes \psi^\vee) \tag{13.3}$$

is null-homotopic (see Figure 28a). Again, it helps if we turn the diagram sideways, where bar resolutions come into play. Let $h_M \colon A \widetilde{\otimes} M \to M$ and $h_N \colon A \widetilde{\otimes} N \to N$ be the natural homotopy equivalences.

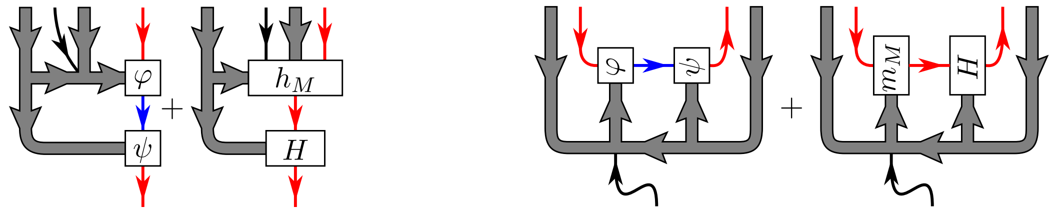
Turning the first term in Eq. (13.3) sideways, we get h_M. Turning the second term sideways we get $\psi \circ h_N \circ (\mathrm{id}_A \widetilde{\otimes} \varphi)$. Thus we need to show that

$$h_M + \psi \circ h_N \circ (\mathrm{id}_A \widetilde{\otimes} \varphi) \tag{13.4}$$



(a) Representation of Eq. (13.3).

(b) Representation of Eq. (13.4).



(c) Null-homotopy of (13.4).

(d) Null-homotopy of (13.3).

Figure 28: Diagrams from the proof of Proposition 13.1.3.

is null-homotopic (see Figure 28b).

There is a canonical homotopy $h_\varphi\colon A \mathbin{\widetilde{\otimes}} M \to N$ between $\varphi \circ h_M$ and $h_N \circ (\mathrm{id}_A \mathbin{\widetilde{\otimes}} \varphi)$, given by

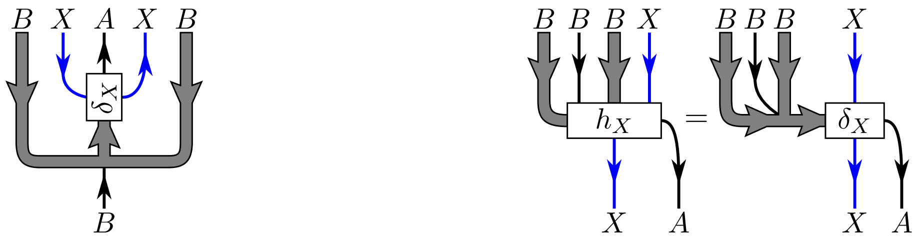
$$h_\varphi(a_1, \dots, a_i, \quad (a', \quad a''_1, \dots, a''_j, \quad m)) \quad = \quad \varphi(a_1, \dots, a_i, a', a''_1, \dots, a''_j, \quad m).$$

Thus we can build the null-homotopy $\psi \circ h_\varphi + H \circ h_M$ (see Figure 28c). Indeed,

$$\partial(\psi \circ h_\varphi) = \psi \circ \varphi \circ h_M + \psi \circ h_N \circ (\mathrm{id}_A \mathbin{\widetilde{\otimes}} \varphi),$$
$$\partial(H \circ h_M) = \mathrm{id}_M \circ h_M + \psi \circ \varphi \circ h_M.$$

Alternatively, we can express the null-homotopy of the expression (13.3) directly as in Figure 28d. □

Proof of Proposition 13.1.4. Recall the statement of Proposition 13.1.4. We are given two differential graded algebras A and B, and three modules—${}_BX^A$, ${}_AY^B$, and ${}_AM$. We assume that there are homotopy equivalences $X \boxtimes Y \simeq {}_B\mathbb{I}^B$ and $Y \boxtimes X \simeq {}_A\mathbb{I}^A$, and that $X \boxtimes A^\vee \boxtimes X^\vee$ and $B^\vee$ have homologies of the same rank.



(a) Definition of φ. (b) View as a map $B \mathbin{\widetilde{\otimes}} X \to X$.

Figure 29: Two views of the homotopy equivalence φ from Eq. (13.5).

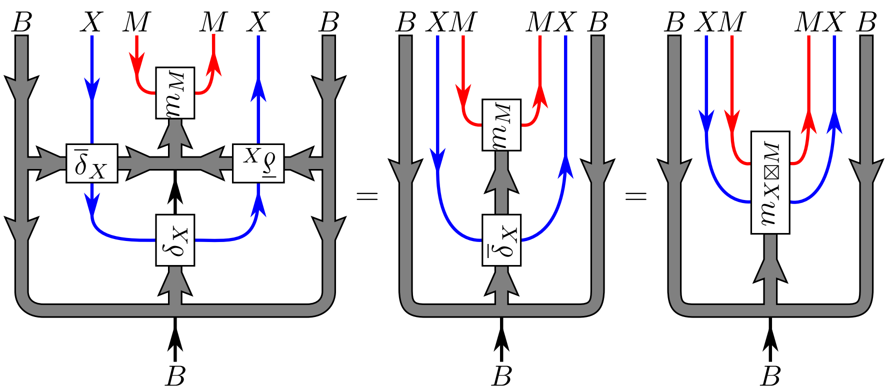


Figure 30: Equality of the direct and induced ∇ maps for $X \boxtimes M$.

We have to construct a homotopy equivalence $\varphi_X \colon X \boxtimes A^\vee \boxtimes X^\vee \to B^\vee$, and a homotopy $\nabla_{X \boxtimes M} \simeq \varphi_X \circ (\mathrm{id}_X \boxtimes \nabla_M \boxtimes \mathrm{id}_{X^\vee})$.

We start by constructing the morphism φ. We can define it by the following equation:

$$\begin{aligned} &\left\langle (\varphi_X)_{i|1|j}(b_1, \dots, b_i,\ (x, a^\vee, x'^\vee),\ b'_1, \dots, b'_j), b'' \right\rangle \\ &\qquad = \left\langle \delta_{i+j+1|1|1}(b'_1, \dots, b'_j, b'', b_1, \dots, b_i,\ x), (x', a)^\vee \right\rangle . \qquad (13.5) \end{aligned}$$

Again, it is useful to "turn it sideways". We can reinterpret φ_X as a morphism of type–AD modules $B \mathbin{\widetilde{\otimes}} X \to X$. In fact, it is precisely the canonical homotopy equivalence h_X between the two. Diagrams for φ_X and h_X are shown in Figure 29. Since the h_X is a homomorphism, it follows that φ_X is one as well.

Next we show that $\nabla_{X \boxtimes M}$ is homotopic to $\varphi_X \circ (\mathrm{id}_X \boxtimes \nabla_M \boxtimes \mathrm{id}_{X^\vee})$. They are in fact equal. This is best seen in Figure 30. We use the fact that $\overline{\delta}_X$ and δ_X commute with merges and splits.

Finally, we need to show that φ_X is a homotopy equivalence. We will do that by constructing a right homotopy inverse for it. Combined with the fact that the homologies of the two sides have equal rank, this is enough to ascertain that it is indeed a homotopy equivalence.

Recall that $X \boxtimes Y \simeq \mathbb{I}$. Thus there exist morphisms of type–AD B, B–bimodules $f \colon \mathbb{I} \to X \boxtimes Y$, and $g \colon X \boxtimes Y \to \mathbb{I}$, and a null-homotopy $H \colon \mathbb{I} \to \mathbb{I}$ of $\mathrm{id}_{\mathbb{I}} - g \circ f$. Note that $g^{\vee} \colon \mathbb{I}^{\vee} \to Y^{\vee} \boxtimes X^{\vee}$ is a map of type–DA–modules, and $({}_B\mathbb{I}^B)^{\vee} = {}^B\mathbb{I}_B$.

Let $\varphi_Y \colon Y \boxtimes B^{\vee} \boxtimes Y^{\vee} \to A$ be defined analogous to φ_X. Construct the homomorphism

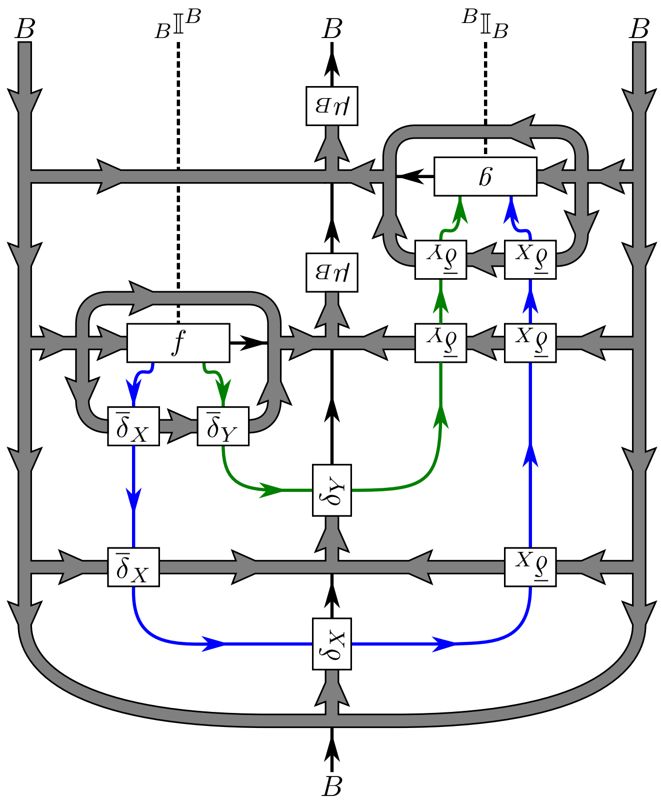
$$\psi = (\mathrm{id}_X \boxtimes \varphi_Y \boxtimes \mathrm{id}_{X^{\vee}}) \circ (f \boxtimes \mathrm{id}_{B^{\vee}} \boxtimes \mathrm{id}_{Y^{\vee}} \boxtimes \mathrm{id}_{X^{\vee}}) \circ (\mathrm{id}_{\mathbb{I}} \boxtimes \mathrm{id}_B \boxtimes g^{\vee}) \colon$$
$$\mathbb{I} \boxtimes B^{\vee} \boxtimes \mathbb{I} \to X \boxtimes A^{\vee} \boxtimes X^{\vee}.$$

We need to show that $\varphi_X \circ \psi$ is homotopic to $\mathrm{id}_{B^{\vee}}$, or more precisely to the canonical isomorphism $\iota \colon \mathbb{I} \boxtimes B^{\vee} \boxtimes \mathbb{I} \to B^{\vee}$. A graphical representation of $\varphi_X \circ \psi$ is shown in Figure 31a. It simplifies significantly, due to the fact that B is a DG-algebra, and μ_B only has two nonzero terms. The simplified version of $\varphi_X \circ \psi$ is shown in Figure 31b. As usual, it helps to turn the diagram sideways. We can view it as a homomorphism $B \widetilde{\otimes} \mathbb{I} \to \mathbb{I}$ of type–AD B, B–bimodules. As can be seen from Figure 31c, we get the composition

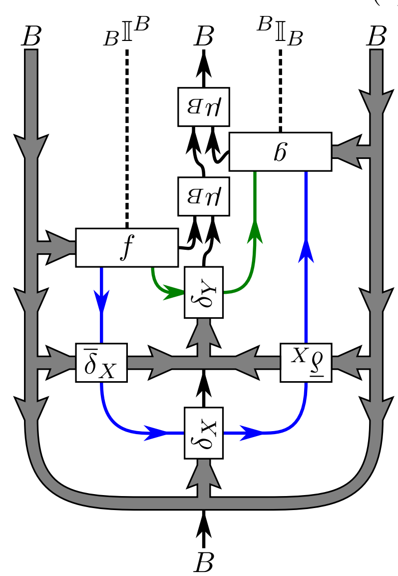
$$g \circ (h_X \boxtimes \mathrm{id}_Y) \circ (\mathrm{id}_B \widetilde{\otimes} f) = g \circ h_{X \boxtimes Y} \circ (\mathrm{id}_B \widetilde{\otimes} f) \colon B \widetilde{\otimes} \mathbb{I} \to \mathbb{I}. \tag{13.6}$$

On the other hand, the homomorphism $\iota \colon \mathbb{I} \boxtimes B^{\vee} \boxtimes \mathbb{I} \to B^{\vee}$, if written sideways, becomes the homotopy equivalence $h_{\mathbb{I}} \colon B \widetilde{\otimes} \mathbb{I} \to \mathbb{I}$. See Figure 32 for the calculation. In the second step we use some new notation. The caps on the thick strands denote a map $\mathrm{Bar}\, B \to K$ to the ground ring, which is the identity on $B^{\otimes 0}$, and zero on $B^{\otimes i}$ for any $i > 0$. The dots on the $\mathbb{I}$ strands denote the canonical isomorphism of $\mathbb{I} \boxtimes B^{\vee} \boxtimes \mathbb{I}$ and $B^{\vee}$ as modules over the ground ring.

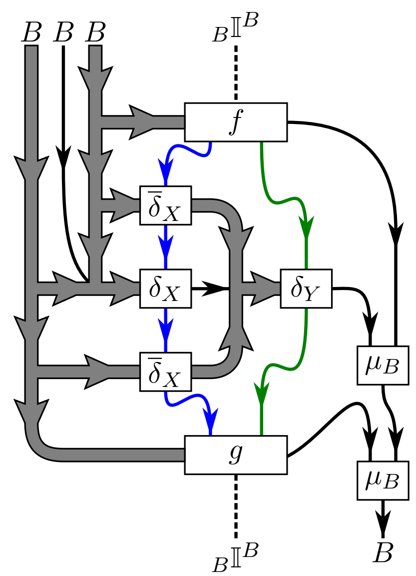
Finding a null-homotopy for $\iota + \varphi_X \circ \psi$ is equivalent to finding a null-homotopy $B \widetilde{\otimes} \mathbb{I} \to \mathbb{I}$ of $h_{\mathbb{I}} + g \circ h_{X \boxtimes Y} \circ (\mathrm{id}_B \widetilde{\otimes} f)$. There is a null-homotopy $\zeta_f \colon B \widetilde{\otimes} \mathbb{I} \to B \widetilde{\otimes} \boxtimes X \boxtimes Y$ of



(a) Before simplification.



(b) After simplification.



(c) Written sideways.

Figure 31: Three views of $\varphi_X \circ \psi \colon \mathbb{I} \boxtimes B^\vee \boxtimes \mathbb{I} \to B^\vee$.

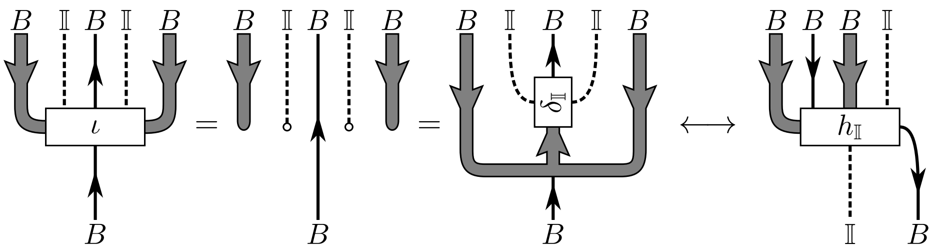


Figure 32: The equivalence of the morphism ι and $h_{\mathbb{I}}$.

$f \circ h_{\mathbb{I}} + h_{X\boxtimes Y} \circ (\mathrm{id}_B \widetilde{\otimes} f)$. Recall that H was a null-homotopy of $\mathrm{id}_{\mathbb{I}} + g \circ f$. Thus we have

$$\begin{aligned}\partial(H \circ h_{\mathbb{I}} + g \circ \zeta_f) &= (\mathrm{id}_{\mathbb{I}} \circ h_{\mathbb{I}} + g \circ f \circ h_{\mathbb{I}}) \\ &\quad + (g \circ f \circ h_{\mathbb{I}} + g \circ h_{X\boxtimes Y} \circ (\mathrm{id}_B \widetilde{\otimes} F) \\ &= h_{\mathbb{I}} + g \circ h_{X\boxtimes Y} \circ (\mathrm{id}_B \widetilde{\otimes} F),\end{aligned}$$

giving us the required null-homotopy.

To finish the proof, notice that if $\varphi_X \circ \psi$ is homotopic to id_B, then it is a quasi-isomorphism, i.e. a homomorphism whose scalar component is a quasi-isomorphism of chain complexes. Moreover, when working with $\mathbb{Z}/2$–coefficients, as we do, quasi-isomorphisms of $\mathcal{A}_\infty$–modules and bimodules coincide with homotopy equivalences.

In particular we have that $(\varphi_X \circ \psi)_{0|1|0} = (\varphi_X)_{0|1|0} \circ \psi_{0|1|0}$ induces an isomorphism on homology (in this case the identity map on homology). In particular ψ induces an injection, while φ_X induces a surjection. Combined with the initial assumption that $B^\vee$ and $X \boxtimes A^\vee \boxtimes X^\vee$ have homologies of equal rank, this implies that $(\varphi_X)_{0|1|0}$ and $\psi_{0|1|0}$ induce isomorphisms on homology. That is, φ_X and ψ are quasi-isomorphisms, and so homotopy equivalences. This concludes the proof of Proposition 13.1.4, and with it, of Theorem 13.2.2. $\square$

Chapter 14

Properties of the join map

In this section we give some formulas for the join and gluing maps, and prove their formal properties.

14.1 Explicit formulas

We have abstractly defined the join map $\Psi_{\mathcal{W}}$ in terms of $\nabla_{\widehat{BSA}(\mathcal{W})}$ but so far have not given any explicit formula for it. Here we give the general formula, as well as some special cases which are somewhat simpler.

If we want to compute $\Psi_{\mathcal{W}}$ for the join $\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2$, we need to pick a parametrization by an arc diagram $\mathcal{Z}$, with associated algebra A, and representatives U for $\widehat{BSD}(\mathcal{Y}_1)^A$, V for ${}^A\widehat{BSD}(\mathcal{Y}_2)$, and M for ${}_A\widehat{BSA}(\mathcal{W})$. Then we know $SFC(\mathcal{Y}_1) = U \boxtimes M$, $SFC(\mathcal{Y}_2) = M^\vee \boxtimes V$, and $SFC(\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2) = U \boxtimes A^\vee \boxtimes V$. As given in Definition 13.2.1, the join map $\Psi_{\mathcal{W}}$ is

$$\Psi_{\mathcal{W}} = \mathrm{id}_U \boxtimes \nabla_M \boxtimes \mathrm{id}_V \colon U \boxtimes M \otimes M^\vee \boxtimes V \to U \boxtimes A^\vee \boxtimes V.$$

In graphic form this can be seen in Figure 33a.

This general form is not good for computations, especially if we try to write it algebraically. However $\Psi_{\mathcal{W}}$ has a much simpler form when M is a *DG-type* module.

Definition 14.1.1. *An $\mathcal{A}_\infty$–module M_A is* of DG-type *if it is a DG-module, i.e., if its structure maps $m_{1|i}$ vanish for $i \geq 2$. A bimodule ${}_AM_B$ is* of DG-type *if $m_{i|1|j}$ vanish, unless*

(i,j) is one of $(0,0)$, $(1,0)$ or $(0,1)$ (i.e. it is a DG-module over $A \otimes B$).

A type–DA bimodule ${}^{A}M_{B}$ is of DG-type *if $\delta_{1|1|j}$ vanish for all $j \geq 2$. A type–DD bimodule ${}^{A}M^{B}$ is* of DG-type *if $\delta_{1|1|1}(x)$ is always in $A \otimes X \otimes 1 + 1 \otimes X \otimes B$ (i.e. it is separated). All type D–modules M^{A} are DG-type.*

The $\boxtimes$–product of any combination of DG-type modules is also DG-type. All modules $\widehat{BSA}$, $\widehat{BSD}$, $\widehat{BSAA}$, etc., computed from a nice diagram are of DG-type.

Proposition 14.1.2. *Let the manifolds $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{W}$, and the modules U, V, and M be as in the above discussion. If M is DG-type, the formula for the join map $\Psi_{\mathcal{W}}$ simplifies to:*

$$\Psi_{\mathcal{W}}(u \boxtimes m \otimes n^{\vee} \boxtimes v) = \sum_{a} \langle m_M(a,m), n^{\vee} \rangle \cdot u \boxtimes a^{\vee} \boxtimes v, \tag{14.1}$$

where the sum is over a $\mathbb{Z}/2$–basis for A. A graphical representation is given in Figure 33b.

Finally, an even simpler case is that of elementary modules. We will see later that elementary modules play an important role for gluing, and for the relationship between the bordered and sutured theories.

Definition 14.1.3. *A type–A module ${}_{A}M$ (or similarly M_{A}) is called* elementary *if the following conditions hold:*

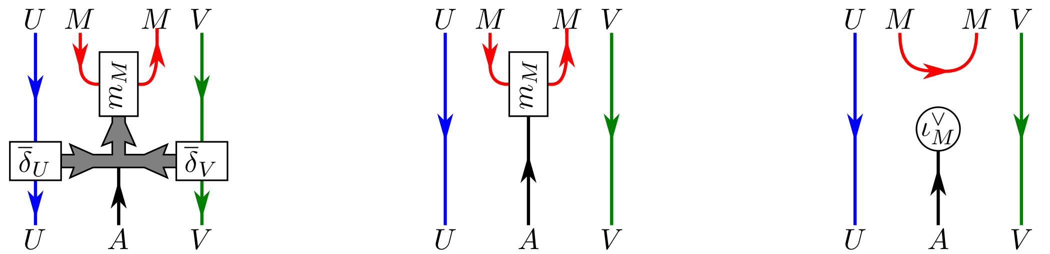
1. *M is generated by a single element m over $\mathbb{Z}/2$.*
2. *All structural operations on M vanish (except for multiplication by an idempotent, which might be identity).*

A type–D module ${}^{A}M$ (or M^{A}), is called elementary *if the following conditions hold:*

1. *M is generated by a single element m over $\mathbb{Z}/2$.*
2. *$\delta(m) = 0$.*

Notice that for an elementary module $M = \{0, m\}$ we can decompose m as a sum $m = \iota_1 m + \cdots + \iota_k m$, where (ι_i) is the canonical basis of the ground ring. Thus we must have $\iota_i m = m$ for some i, and $\iota_j m = 0$ for all $i \neq j$. Therefore, elementary (left) modules over A are in a 1–to–1 correspondence with the canonical basis for its ground ring.

We only use elementary type–A modules in this section but we will need both types later.



(a) The general case. (b) M of DG-type. (c) M elementary.

Figure 33: Full expression for join map in three cases.

Remark. For the algebras we discuss, the elementary type–A modules are precisely the simple modules. The elementary type–D modules are the those ${}^{A}M$ for which $A \boxtimes M \in {}_{A}\mathrm{Mod}$ is an elementary projective module.

Proposition 14.1.4. *If ${}_{A}M = \{m, 0\}$ is an elementary module corresponding to the basis idempotent ι_M, then the join map $\Psi_{\mathcal{W}}$ reduces to*

$$\Psi_{\mathcal{W}}(u \boxtimes m \otimes m^\vee \boxtimes v) = u \boxtimes {\iota_M}^\vee \boxtimes v. \tag{14.2}$$

Graphically, this is given in Figure 33c.

Moreover, in this case, SFC($\mathcal{Y}_1$) $= U \boxtimes M \cong U \cdot \iota_M \subset U$ and SFC($\mathcal{Y}_2$) $= M \boxtimes V \cong \iota_M \cdot V \subset V$ as chain complexes.

Proposition 14.1.2 and Proposition 14.1.4 follow directly from the definitions of DG-type and elementary modules.

14.2 Formal properties

In this section we will show that the join map has the formal properties stated in Theorem 7. A more precise statement of the properties is given below.

Theorem 14.2.1. *The following properties hold:*

1. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be sutured and $\mathcal{W}$ be partially sutured, with embeddings $\mathcal{W} \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W} \hookrightarrow \mathcal{Y}_2$. There are natural identifications of the disjoint unions $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$ and $\mathcal{Y}_2 \sqcup \mathcal{Y}_1$,*

and of of the joins $\mathcal{Y}_1 \Cup_{\mathcal{W}} \mathcal{Y}_2$ and $\mathcal{Y}_2 \Cup_{-\mathcal{W}} \mathcal{Y}_1$. Under this identification, there is a homotopy

$$\Psi_{\mathcal{W}} \simeq \Psi_{-\mathcal{W}}.$$

2. *Let $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{Y}_3$ be sutured, and $\mathcal{W}_1$ and $\mathcal{W}_2$ be partially sutured, such that there are embeddings $\mathcal{W}_1 \hookrightarrow \mathcal{Y}_1$, $(-\mathcal{W}_1 \sqcup \mathcal{W}_2) \hookrightarrow \mathcal{Y}_2$, and $-\mathcal{W}_2 \hookrightarrow \mathcal{Y}_3$. The following diagram commutes up to homotopy:*

$$\begin{tikzcd}
SFC(\mathcal{Y}_1 \sqcup \mathcal{Y}_2 \sqcup \mathcal{Y}_3) \arrow[r, "\Psi_{\mathcal{W}_1}"] \arrow[d, "\Psi_{\mathcal{W}_2}"'] \arrow[dr, "\Psi_{\mathcal{W}_1 \sqcup -\mathcal{W}_2}"] & SFC(\mathcal{Y}_1 \Cup \mathcal{Y}_2 \sqcup \mathcal{Y}_3) \arrow[d, "\Psi_{\mathcal{W}_2}"] \\
SFC(\mathcal{Y}_1 \sqcup \mathcal{Y}_2 \Cup \mathcal{Y}_3) \arrow[r, "\Psi_{\mathcal{W}_1}"'] & SFC(\mathcal{Y}_1 \Cup \mathcal{Y}_2 \Cup \mathcal{Y}_3)
\end{tikzcd}$$

3. *Let $\mathcal{W}$ be partially sutured. There is a canonical element $[\Delta_{\mathcal{W}}]$ in the sutured Floer homology $SFH(\mathcal{D}(\mathcal{W}))$ of the double of $\mathcal{W}$. If Δ is any representative for $[\Delta_{\mathcal{W}}]$, and there is an embedding $\mathcal{W} \hookrightarrow \mathcal{Y}$, then*

$$\Psi_{\mathcal{W}}(\cdot, \Delta) \simeq \mathrm{id}_{SFC(\mathcal{Y})} \colon SFC(\mathcal{Y}) \to SFC(\mathcal{Y}). \tag{14.3}$$

Proof. We will prove the three parts in order.

For part (1), take representatives U^A for $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W})$, ${}^AV$ for $\widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})$, and ${}_AM$ for $\widehat{BSA}(\mathcal{W})$. The main observation here is that we can turn left modules into right modules and vice versa, by reflecting all diagrams along the vertical axis (see Appendix B.6). If we reflect the entire diagram for $\Psi_{\mathcal{M}}$, domain and target chain complexes are turned into isomorphic ones and we get a new map that is equivalent.

The domain $U^A \boxtimes {}_AM \otimes M^\vee{}_A \boxtimes {}^AV$ becomes $V^{A^{\mathrm{op}}} \boxtimes {}_{A^{\mathrm{op}}}M^\vee \otimes M_{A^{\mathrm{op}}} \boxtimes {}^{A^{\mathrm{op}}}U$, and the target $U^A \boxtimes {}_AA^\vee{}_A \boxtimes {}^AV$ becomes $V^{A^{\mathrm{op}}} \boxtimes {}_{A^{\mathrm{op}}}(A^\vee)^{\mathrm{op}}{}_{A^{\mathrm{op}}} \boxtimes {}^{A^{\mathrm{op}}}U$.

Notice that $V^{A^{\mathrm{op}}}$ is $\widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})$, ${}_{A^{\mathrm{op}}}U$ is $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W})$, and ${}_{A^{\mathrm{op}}}M^\vee$ is $\widehat{BSA}(-\mathcal{W})$. In addition $(A^\vee)^{\mathrm{op}} = (A^{\mathrm{op}})^\vee$. Since the map ∇_M is completely symmetric, when we reflect it, we get $\nabla_{M^\vee}$. Everything else is preserved, so reflecting $\Psi_{\mathcal{W}}$ gives precisely $\Psi_{-\mathcal{W}}$. This finishes part (1).

For part (2), the equivalence is best seen by working with convenient representatives. Pick the following modules as representatives: U^A for $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}_1)$, ${}^AX^B$ for $\widehat{BSDD}(\mathcal{Y}_2 \setminus$

$(-\mathcal{W}_1 \cup \mathcal{W}_2))$, ${}^B V$ for $\widehat{BSD}(\mathcal{Y}_1)$, ${}_A M$ for $\widehat{BSA}(\mathcal{W}_1)$ and ${}_B N$ for $\widehat{BSD}(\mathcal{W}_2)$. We can always choose M, N, and X to be of DG-type in the sense of Definition 14.1.1. Since X is of DG-type, taking the $\boxtimes$–product with it is associative. (This is only true up to homotopy in general). Since M and N are DG-type, we can apply Proposition 14.1.2 to get formulas for $\Psi_{\mathcal{W}_1}$ and $\Psi_{\mathcal{W}_2}$. The two possible compositions are shown in Figures 34a and 34b.

To compute $\Psi_{\mathcal{W}_1 \cup -\mathcal{W}_2}$, notice that $(U \otimes V)^{A,B^{\mathrm{op}}}$ represents $\widehat{BSDD}((\mathcal{Y}_1 \cup \mathcal{Y}_3) \setminus (\mathcal{W}_1 \cup -\mathcal{W}_3))$, ${}^{A,B^{\mathrm{op}}}X$ represents $\widehat{BSDD}(\mathcal{Y}_2 \setminus (-\mathcal{W}_1 \cup \mathcal{W}_2))$, and ${}_{A,B^{\mathrm{op}}}(M \otimes N^\vee)$ is a DG-type module representing $\widehat{BSAA}(\mathcal{W}_1 \cup -\mathcal{W}_2)$. To compute the join map, we need to convert them to single modules. For type–DD modules, this is trivial (any A, B–bimodule is automatically an $A \otimes B$–module and vice versa). For type–AA modules, this could be complicated in general. Luckily, it is easy for DG-type modules. Indeed, if $P_{A,B}$ is DG-type, the corresponding $A \otimes B$–module $P_{A\otimes B}$ is also DG-type, with algebra action

$$m_{1|1}(\cdot, a \otimes b) = m_{1|1|0}(\cdot, a) \circ m_{1|0|1}(\cdot, b) = m_{1|0|1}(\cdot, b) \circ m_{1|1|0}(\cdot, a).$$

In the definition of bimodule invariants in Part I, the procedure used to get $\widehat{BSAA}$ from $\widehat{BSA}$, and $\widehat{BSDD}$ from $\widehat{BSD}$ is exactly the reverse of this construction.

Thus, we can see that $(U \otimes V)^{A\otimes B^{\mathrm{op}}}$ represents $\widehat{BSD}((\mathcal{Y}_1 \cup \mathcal{Y}_3) \setminus (\mathcal{W}_1 \cup -\mathcal{W}_3))$, ${}^{A\otimes B^{\mathrm{op}}}X$ represents $\widehat{BSD}(\mathcal{Y}_2 \setminus (-\mathcal{W}_1 \cup \mathcal{W}_2))$, and ${}_{A\otimes B^{\mathrm{op}}}(M \otimes N^\vee)$ represents $\widehat{BSA}(\mathcal{W}_1 \cup -\mathcal{W}_2)$. It is also easy to check that

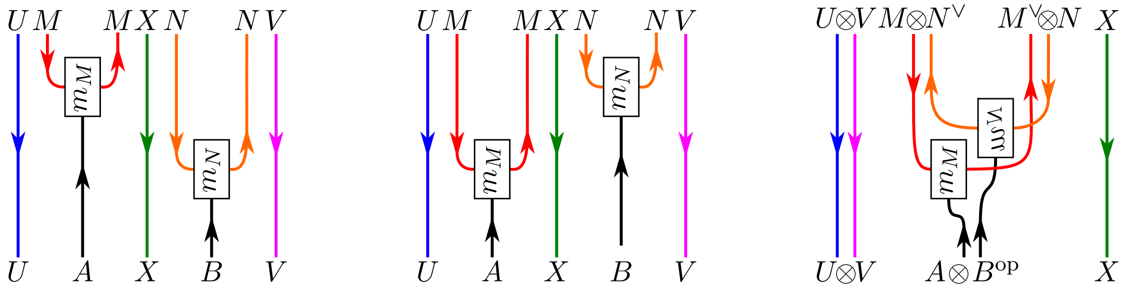
$${}_A A^\vee{}_A \otimes {}_{B^{\mathrm{op}}}(B^{\mathrm{op}})^\vee{}_{B^{\mathrm{op}}} \cong {}_{A\otimes B^{\mathrm{op}}}(A \otimes B^{\mathrm{op}})^\vee{}_{A\otimes B^{\mathrm{op}}}.$$

We can see a diagram for $\Psi_{\mathcal{W}_1 \cup -\mathcal{W}_2}$ in Figure 34c. By examining the diagrams, we see that the three maps are the same, which finishes part (2).

Part (3) requires some more work, so we will split it in several steps. We will define Δ_M for a fixed representative M of $\widehat{BSD}(\mathcal{W})$. We will prove that $[\Delta_M]$ does no depend on the choice of M. Finally, we will use a computational lemma to show that Eq. (14.3) holds for Δ_M.

First we will introduce some notation. Given an $\mathcal{A}_\infty$–module ${}_A M$ over $A = \mathcal{A}(\mathcal{Z})$, define the *double* of M to be

$$\mathcal{D}(M) = M^\vee \boxtimes ({}^A\mathbb{I}^A \boxtimes A \boxtimes {}^A\mathbb{I}^A) \boxtimes M. \tag{14.4}$$



(a) $\Psi_{\mathcal{W}_2} \circ \Psi_{\mathcal{W}_1}$ (b) $\Psi_{\mathcal{W}_1} \circ \Psi_{\mathcal{W}_2}$ (c) $\Psi_{\mathcal{W}_1 \cup -\mathcal{W}_2}$

Figure 34: Three ways to join $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{Y}_3$.

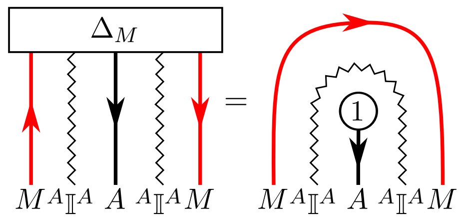


Figure 35: The diagonal element Δ_M.

Note that if $M = \widehat{BSA}(\mathcal{W})$, then $\mathcal{D}(M) = \widehat{BSA}(-\mathcal{W}) \boxtimes \widehat{BSDD}(\mathcal{TW}_{\mathcal{F},-}) \boxtimes \widehat{BSA}(\mathcal{W}) \simeq SFC(\mathcal{D}(\mathcal{W}))$. Next we define the *diagonal element* $\Delta_M \in \mathcal{D}(M)$ as follows. Pick a basis $(m_1, \ldots, m_k)$ of M over $\mathbb{Z}/2$. Define

$$\Delta_M = \sum_{i=1}^{k} m_i \boxtimes (* \boxtimes 1 \boxtimes *) \boxtimes m_i{}^{\vee}. \tag{14.5}$$

It is easy to check that this definition does not depend on the choice of basis. Indeed there is a simple diagrammatic representation of ΔM, given in Figure 35. We think of it as a linear map from $\mathbb{Z}/2$ to $\mathcal{D}(M)$. It is also easy to check that $\partial \Delta_M = 0$. Indeed, writing out the definition of $\partial \Delta_M$, there are are only two nonzero terms which cancel.

The proof that $[\Delta_M]$ does not depend on the choices of A and M is very similar to the proof of Theorem 13.2.2, so will omit it. (It involves showing independence from M, as well as from A via a quasi-invertible bimodule ${}^A X_B$.)

Lemma 14.2.2. *Let A be a differential graded algebra, coming from an arc diagram $\mathcal{Z}$. There is a homotopy equivalence*

$$c_A \colon {}^A\mathbb{I}^A \boxtimes A^{\vee} \boxtimes {}^A\mathbb{I}^A \boxtimes {}_AA_A \to {}^A\mathbb{I}_A,$$

given by

$$(c_A)_{1|1|0}\,(*\boxtimes a^\vee\boxtimes *\boxtimes b)=\begin{cases} b\otimes * & \text{if } a \text{ is an idempotent,}\\ 0 & \text{otherwise.}\end{cases}$$

Here we use $*$ to denote the unique element with compatible idempotents in the two versions of $\mathbb{I}$. (Both versions have generators in 1–to–1 correspondence with the basis idempotents.)

Remark. As we mentioned earlier, one has to be careful when working with type–DD modules. While $\boxtimes$ and $\widetilde{\otimes}$ are usually associative by themselves, and with each other, this might fail when a DD–module is involved, in which case we only have associativity up to homotopy equivalence. However, this could be mitigated in two situations. If the DD–module is DG-type (which fails for ${}^A\mathbb{I}^A$), or if the type–A modules on both sides are DG-type, then true associativity still holds. This is true for A and $A^\vee$, so the statement of the lemma makes sense.

Proof of Lemma 14.2.2. Note that we can easily see that there is *some* homotopy equivalence $(\mathbb{I}\boxtimes A^\vee\boxtimes\mathbb{I})\boxtimes A\simeq\mathbb{I}$, since the left-hand side is

$$\widehat{BSDD}(\mathcal{TW}_{\mathcal{F},+})\boxtimes\widehat{BSAA}(\mathcal{TW}_{-\overline{\mathcal{F}},-})\simeq\widehat{BSDA}(\mathcal{TW}_{\mathcal{F},+}\cup\mathcal{TW}_{-\overline{\mathcal{F}},-}),$$

while the right side is $\widehat{BSDA}(\mathcal{F}\times[0,1])$, and those bordered sutured manifolds are the same. The difficulty is in finding the precise homotopy equivalence, which we need for computations, in order to "cancel" $A^\vee$ and A.

First, we need to show that c_A is a homomorphism. This is best done graphically. The definition of c_A is represented in Figure 36. The notation we use there is that ${}^A\mathbb{I}^A$ is a jagged line, without a direction, since $\mathbb{I}$ is its own dual. ${}^A\mathbb{I}_A$ is represented by a dashed line. As before the line can start or end with a dot, signifying the canonical isomorphism given by $\cdot\boxtimes *$.

We need to show that $\partial c_A=0$. Note that by definition c_A only has a $1|1|0$–term. On the other hand δ on $\mathbb{I}\boxtimes A^\vee\boxtimes\mathbb{I}\boxtimes A$ has only $1|1|0$– and $1|1|1$–terms, while δ on $\mathbb{I}$ has only a $1|1|1$–term.

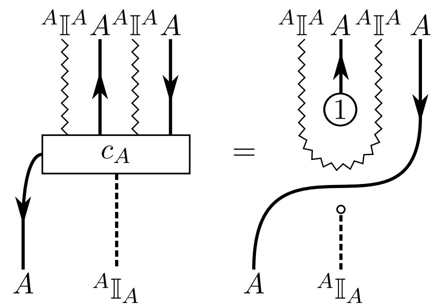
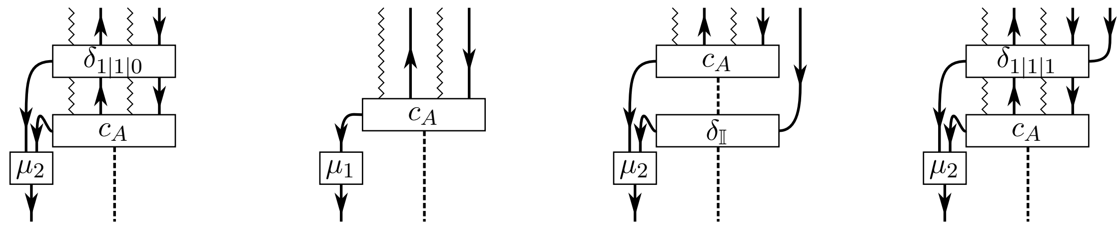


Figure 36: The cancellation homotopy equivalence $c_A : \mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A \to \mathbb{I}$.



(a) (b) (c) (d)

Figure 37: Nontrivial terms of ∂c_A.

Thus only four terms from the definition of ∂c_A survive. These are shown in Figure 37. Expanding the definition of δ on $\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A$ in terms of the operations of $\mathbb{I}$, A, and $A^\vee$, we get seven terms. We can see them in Figure 38. The terms in Figures 38a—38d correspond to Figure 37a, while those in Figures 38e—38g correspond to Figures 37b—37d, respectively. Six of the terms cancel in pairs, while the one in 38b equals 0.

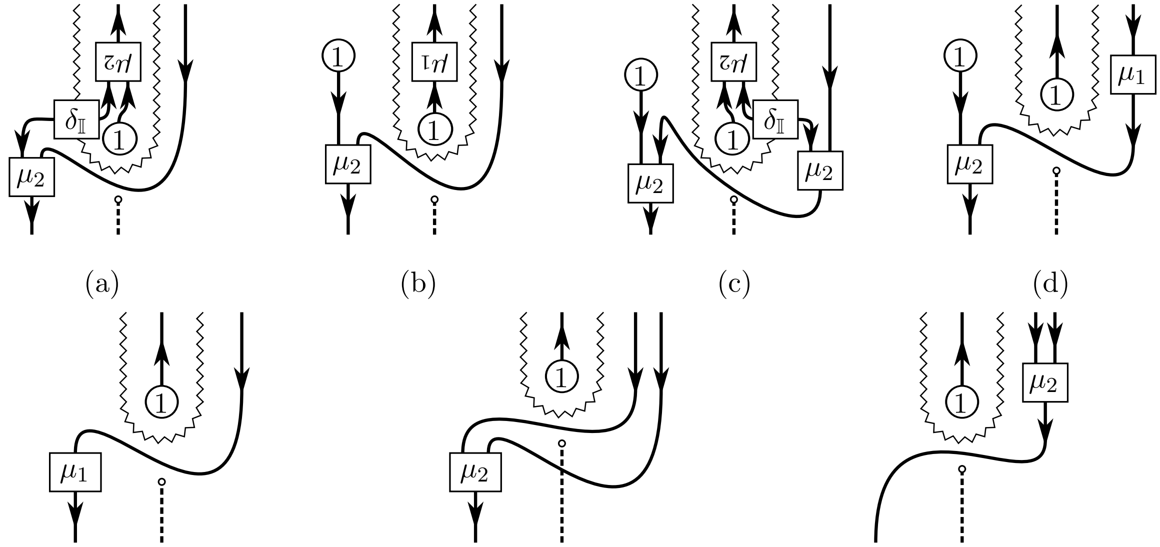
Showing that c_A is a homotopy equivalence is somewhat roundabout. First we will show that the induced map

$$\mathrm{id}_A \boxtimes c_A \colon\ A \boxtimes ({}^A\mathbb{I}^A \boxtimes A^\vee \boxtimes {}^A\mathbb{I}^A \boxtimes A) \to A \boxtimes {}^A\mathbb{I}_A \cong A$$

is a homotopy equivalence. It is easy to see that the map is

$$(\mathrm{id}_A \boxtimes c_A)_{0|1|0}(a \boxtimes * \boxtimes b^\vee \boxtimes * \boxtimes c) = \begin{cases} a \cdot c & \text{if } b \text{ is an idempotent,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it is surjective. Indeed, $\mathrm{id}_A \boxtimes c_A(a \boxtimes * \boxtimes 1^\vee \boxtimes * \boxtimes 1) = a$ for all $a \in A$. Thus the induced map on homology is surjective. But the source and domain are homotopy equivalent



(a) (b) (c) (d)

(e) (f) (g)

Figure 38: Elementary terms of ∂h.

for topological reasons (both represent $\widehat{BSAA}(\mathcal{TW}_{\overline{\mathcal{F}},-})$). This implies that $\mathrm{id}_A \boxtimes c_A$ is a quasi-isomorphism, and a homotopy equivalence. But $(\mathbb{I}\boxtimes A^\vee \boxtimes \mathbb{I})\boxtimes A \simeq \mathbb{I}$ and $A\boxtimes(\mathbb{I}\boxtimes A^\vee\boxtimes\mathbb{I}) \simeq \mathbb{I}$ for topological reasons, so $A\boxtimes\cdot$ is an equivalence of derived categories. Thus, c_A itself must have been a homotopy equivalence, which finishes the proof of the lemma. $\square$

We will now use Lemma 14.2.2, to show that for any $\mathcal{Y}$ there is a homotopy $\Psi_{\mathcal{W}}(\cdot,\Delta_M) \simeq \mathrm{id}_{SFC(\mathcal{Y})}$. Let c_A be the homotopy equivalence from the lemma. There is a sequence of homomorphisms as follows.

$$\begin{tikzcd}
\mathbb{I}\boxtimes M \arrow[d, "\mathrm{id}_{\mathbb{I}\boxtimes M}\otimes\Delta_M"] \\
\mathbb{I}\boxtimes M\otimes\mathcal{D}(M) \cong \mathbb{I}\boxtimes M\otimes M^\vee\boxtimes\mathbb{I}\boxtimes A\boxtimes\mathbb{I}\boxtimes M \arrow[d, "\mathrm{id}_{\mathbb{I}}\boxtimes\nabla_M\boxtimes\mathrm{id}_{\mathbb{I}\boxtimes A\boxtimes\mathbb{I}\boxtimes M}"] \\
\mathbb{I}\boxtimes A^\vee\boxtimes\mathbb{I}\boxtimes A\boxtimes\mathbb{I}\boxtimes M \arrow[d, "c_A\boxtimes\mathrm{id}_{\mathbb{I}\boxtimes M}"] \\
\mathbb{I}\boxtimes M
\end{tikzcd}$$

The composition of these maps is shown in Figure 39. As we can see from the diagram, it is equal to $\mathrm{id}_{\mathbb{I}}\boxtimes\mathrm{id}_M$. If $U=\widehat{BSD}(\mathcal{Y}\setminus\mathcal{W})$, then by applying the functor $\mathrm{id}_U\boxtimes\cdot$ to both

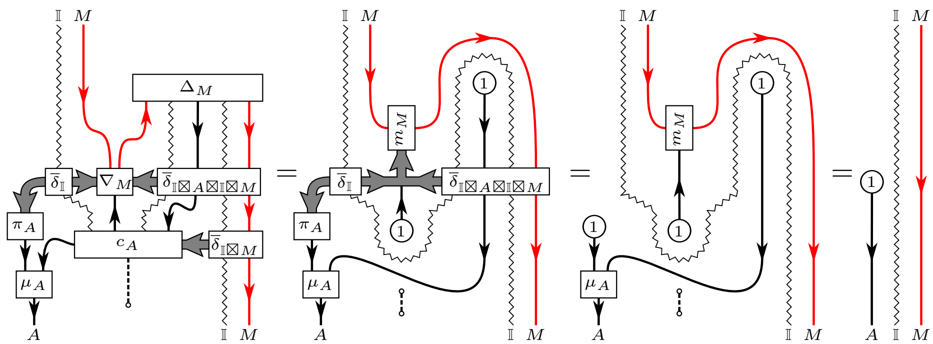


Figure 39: Proof that $\Psi_M(\cdot, \Delta_M) \simeq \mathrm{id}$.

homomorphisms, we see that

$$(\mathrm{id}_U \boxtimes c_A \boxtimes \mathrm{id}_{\mathbb{I}\boxtimes M}) \circ \Psi_M \circ (\mathrm{id}_{SFC(\mathcal{Y})} \boxtimes \Delta_M) = \mathrm{id}_{SFC(\mathcal{Y})},$$

which is equivalent to Eq. (14.3). □

14.3 Self-join and self-gluing

So far we have been talking about the join or gluing of two disjoint sutured manifolds. However, we can extend these notions to a self-join or self-gluing of a single manifold. For example if there is an embedding $(\mathcal{W} \sqcup -\mathcal{W}) \hookrightarrow \mathcal{Y}$, then we can define the *self-join* of $\mathcal{Y}$ along $\mathcal{W}$ to be the concatenation

$$\mathcal{Y}\Cup_{\mathcal{W},\circlearrowleft} = (Y \setminus (\mathcal{W} \sqcup -\mathcal{W})) \cup_{\mathcal{F}\cup\overline{\mathcal{F}}} \mathcal{TW}_{\mathcal{F},+} \cong \mathcal{Y} \Cup_{\mathcal{W}\sqcup-\mathcal{W}} \mathcal{D}(\mathcal{W}).$$

It is easy to see that if $\mathcal{W}$ and $-\mathcal{W}$ embed into different components of $\mathcal{Y}$, this is the same as the regular join.

Similarly, we can extend the join map to a *self-join map*

$$\Psi_{\mathcal{W},\circlearrowleft} \colon SFC(\mathcal{Y}) \to SFC(\mathcal{Y} \Cup_{\mathcal{W}\sqcup-\mathcal{W}} \mathcal{D}(\mathcal{W})) \simeq SFC(\mathcal{Y}\Cup_{\mathcal{W},\circlearrowleft}),$$

by setting

$$\Psi_{\mathcal{W},\circlearrowleft} = \Psi_{\mathcal{W}\sqcup-\mathcal{W}}(\cdot, \Delta_{\mathcal{W}}).$$

Again, if $\mathcal{W}$ and $-\mathcal{W}$ embed into disjoint components of $\mathcal{Y}$, $\Psi_{\mathcal{W},\circlearrowleft}$ is, up to homotopy, the same as the regular join map $\Psi_{\mathcal{W}}$. This follows quickly from properties (2) and (3) in Theorem 14.2.1.

Chapter 15

The bordered invariants in terms of *SFH*

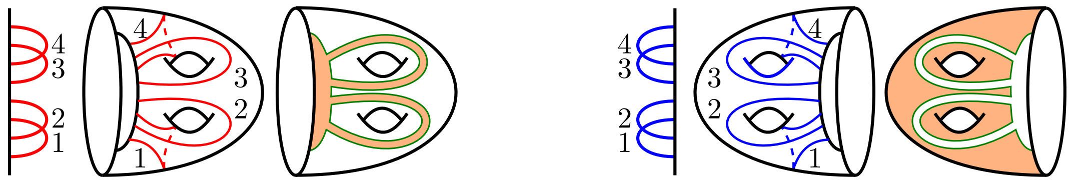
In this section we give a (partial) reinterpretation of bordered and bordered sutured invariants in terms of *SFH* and the gluing map Ψ. This is a more detailed version of Theorem 2.

15.1 Elementary dividing sets

Recall Definition 11.1.3 of a dividing set. Suppose we have a sutured surface $\mathcal{F} = (F, \Lambda)$ parametrized by an arc diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ of rank k. We will define a set of 2^k distinguished dividing sets.

Before we do that, recall the way an arc diagram parametrizes a sutured surface, from Chapter 12.1. There is an embedding of the graph $G(\mathcal{Z})$ into F, such that $\partial \mathbf{Z} = \Lambda$ (Recall Figure 18). We will first define the elementary dividing sets in the cases that $\mathcal{Z}$ is of α–type. In that case the image of $\mathbf{Z}$ is a push-off of S_+ into the interior of F. Denote the regions between S_+ and $\mathbf{Z}$ by R_0. It is a collection of discs, one for each component of S_+. The images of the arcs $e_i \subset G(\mathcal{Z})$ are in the complement $F \setminus R_0$.

Definition 15.1.1. *Let $I \subset \{1, \ldots, k\}$. The* elementary dividing set for $\mathcal{F}$ *associated to I*



(a) α–type diagram. (b) β–type diagram.

Figure 40: Elementary dividing sets for an arc diagram. In each case we show the arc diagram, its embedding into the surface, and the dividing set $\Gamma_{\{2,3\}}$. The shaded regions are R_+.

is the dividing set Γ_I *constructed as follows. Let* R_0 *be the region defined above. Set*

$$R_+ = R_0 \cup \bigcup_{i \in I} \nu(e_i) \subset F.$$

Then $\Gamma_I = (\partial R_+) \setminus S_+$*.*

If $\mathcal{Z}$ is of β–type, repeat the same procedure, substituting R_- for R_+ and S_- for S_+. For example the region R_0 consists of discs bounded by $S_- \cup \mathbf{Z}$. Examples of both cases are given in Figure 40.

We refer to the collection of Γ_I for all 2^k–many subsets of $\{1, \ldots, k\}$ as *elementary dividing sets for* $\mathcal{Z}$. The reason they are important is the following proposition.

Proposition 15.1.2. *Let* $\mathcal{Z}$ *be an arc diagram of rank* k*, and let* $I \subset \{1, \ldots, k\}$ *be any subset. Let* ι_I *be the idempotent for* $A = \mathcal{A}(\mathcal{Z})$ *corresponding to horizontal arcs at all* $i \in I$*, and let* ι_{I^c} *be the idempotent corresponding to the complement of* I*. Let* $\mathcal{W}_I$ *be the cap associated to the elementary dividing set* Γ_I*.*

Then the following hold:

- ${}^A\widehat{BSD}(\mathcal{W})$ *is (represented by) the elementary type–D module for* ι_I*.*
- ${}_A\widehat{BSA}(\mathcal{W})$ *is (represented by) the elementary type–A module for* ι_{I^c}*.*

Proof. The key fact is that there is a particularly simple Heegaard diagram $\mathcal{H}$ for $\mathcal{W}_I$. For concreteness we will assume $\mathcal{Z}$ is a type–α diagram, though the case of a type–β diagram is completely analogous.

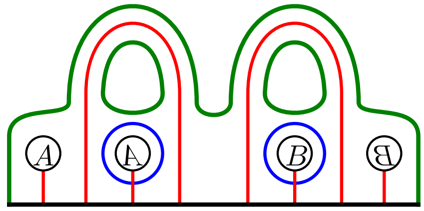


Figure 41: Heegaard diagram $\mathcal{H}$ for the cap $\mathcal{W}_{2,3}$ corresponding to the dividing set from Figure 40a.

The diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z})$ contains no α–circles, exactly one α–arc α_i^a for each matched pair $M^{-1}(i)$, and $k - \#I$ many β–circles. Each β–circle has exactly one intersection point on it, with one of α_i^a, for $i \notin I$. This implies that there is exactly one generator $\mathbf{x} \in \mathcal{G}(\mathcal{H})$, that occupies the arcs for I^c. This implies that $\widehat{BSD}(\mathcal{W}_I)$ and $\widehat{BSA}(\mathcal{W}_I)$ are both $\{\mathbf{x}, 0\}$ as $\mathbb{Z}/2$–modules. The actions of the ground ring are $\iota_I \cdot \mathbf{x} = \mathbf{x}$ for $\widehat{BSD}(\mathcal{W}_I)$ and $\iota_{I^c} \cdot \mathbf{x} = \mathbf{x}$ for $\widehat{BSA}(\mathcal{W}_I)$. This was one of the two requirements for an elementary module.

The connected components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ are in 1–to–1 correspondence with components of ∂R_+. In fact each such region is adjacent to exactly one component of $\partial\Sigma \setminus \mathbf{Z}$. Therefore, there are only boundary regions and no holomorphic curves are counted for the definitions of $\widehat{BSD}(\mathcal{W}_I)$ and $\widehat{BSA}(\mathcal{W}_I)$. This was the other requirement for an elementary module, so the proof is complete. The diagram $\mathcal{H}$ can be seen in Figure 41. $\square$

We will define one more type of object. Let $\mathcal{F}$ be a sutured surface parametrized by some arc diagram $\mathcal{Z}$. Let I and J be two subsets of $\{1, \dots, k\}$. Consider the sutured manifold $-\mathcal{W}_I \cup \mathcal{TW}_{-\overline{\mathcal{F}},-} \cup \mathcal{W}_J$. Since $-\mathcal{W}_I$ and $\mathcal{W}_J$ are caps, topologically this is $F \times [0,1]$. The dividing set can be described as follows. Along $F \times \{0\}$ it is $\Gamma_I \times \{0\}$, along $F \times \{1\}$ it is $\Gamma_J \times \{1\}$, and along $\partial F \times [0,1]$ it consists of arcs in the $[0,1]$ direction with a partial negative twist.

Definition 15.1.3. *Let $\Gamma_{I \to J}$ denote the dividing set on $\partial(F \times [0,1])$, such that*

$$(F \times [0,1], \Gamma_{I \to J}) = -\mathcal{W}_I \cup \mathcal{TW}_{\overline{\mathcal{F}},-} \cup \mathcal{W}_J.$$

15.2 Main results

The main results of this section are the following two theorems. We will give the proofs in the next subsection.

Theorem 15.2.1. *Let $\mathcal{F}$ be a sutured surface parametrized by an arc diagram $\mathcal{Z}$. The homology of $A = \mathcal{A}(\mathcal{Z})$ decomposes as the sum*

$$H_*(A) = \bigoplus_{I,J \subset \{1,\dots,k\}} \iota_I \cdot H_*(A) \cdot \iota_J = \bigoplus_{I,J \subset \{1,\dots,k\}} H_*(\iota_I \cdot A \cdot \iota_J), \tag{15.1}$$

where

$$\iota_I \cdot H_*(A) \cdot \iota_J \cong SFH(F \times [0,1], \Gamma_{I \to J}). \tag{15.2}$$

Multiplication μ_2 descends to homology as

$$\begin{aligned}\mu_H = \Psi_{(F,\Gamma_J)} \colon\ & SFH(F \times [0,1], \Gamma_{I \to J}) \otimes SFH(F \times [0,1], \Gamma_{J \to K}) \\ & \to SFH(F \times [0,1], \Gamma_{I \to K}),\end{aligned} \tag{15.3}$$

and is 0 on all other summands.

Theorem 15.2.2. *Let $\mathcal{Y} = (Y, \Gamma, \mathcal{F})$ be a bordered sutured manifold where $\mathcal{F}$ parametrized by $\mathcal{Z}$. Then there is a decomposition*

$$\begin{aligned} H_*\left(\widehat{BSA}(Y)_A\right) &= \bigoplus_{I \subset \{1,\dots,k\}} H_*\left(\widehat{BSA}(\mathcal{Y})\right) \cdot \iota_I \\ &= \bigoplus_{I \subset \{1,\dots,k\}} H_*\left(\widehat{BSA}(Y) \cdot \iota_I\right),\end{aligned} \tag{15.4}$$

where

$$H_*\left(\widehat{BSA}(Y)\right) \cdot \iota_I \cong SFH(Y, \Gamma \cup \Gamma_I). \tag{15.5}$$

Moreover, the $m_{1|1}$ action of A on $\widehat{BSA}$ descends to the following action on homology:

$$m_H = \Psi_{(F,\Gamma_I)} \colon\ SFH(Y, \Gamma \cup \Gamma_I) \otimes SFH(F \times I, \Gamma_{I \to J}) \to SFH(Y, \Gamma \cup \Gamma_J), \tag{15.6}$$

and $m_H = 0$ on all other summands.

Similar statements hold for left A–modules ${}_A\widehat{BSA}(\mathcal{Y})$, and for bimodules ${}_A\widehat{BSAA}(\mathcal{Y})_B$.

Theorem 15.2.1 and 15.2.2, give us an alternative way to think about bordered sutured Floer homology, or pure bordered Floer homology. (Recall that as shown in Part I, the bordered invariants $\widehat{CFD}$ and $\widehat{CFA}$ are special cases of $\widehat{BSD}$ and $\widehat{BSA}$.) More remarkably, as we show in [Zar11b], $H_*(A)$, μ_H, and m_H can be expressed in purely contact-geometric terms.

For practical purposes, A and $\widehat{BSA}$ can be replaced by the $\mathcal{A}_\infty$–algebra $H_*(A)$ and the $\mathcal{A}_\infty$–module $H_*(\widehat{BSA})$ over it. For example, the pairing theorem will still hold. This is due to the fact that (using $\mathbb{Z}/2$–coefficients), an $\mathcal{A}_\infty$–algebra or module is always homotopy equivalent to its homology.

We would need, however, the higher multiplication maps of $H_*(A)$, and the higher actions of $H_*(A)$ on $H_*(\widehat{BSA})$. The maps μ_H and m_H that we just computed are only single terms of those higher operations. (Even though A is a DG-algebra, $H_*(A)$ usually has nontrivial higher multiplication.)

15.3 Proofs

In this section we prove Theorems 15.2.1 and 15.2.2. Since there is a lot of overlap of the two results and the arguments, we will actually give a combined proof of a mix of statements from both theorems. The rest follow as corollary.

Combined proof of Theorem 15.2.1 and Theorem 15.2.2. First, note that both Eq. (15.1) and Eq. (15.4) follow directly from the fact that the idempotents generate the ground ring over $\mathbb{Z}/2$.

We will start by proving a generalization of Eq. (15.2) and Eq. (15.5). The statement is as follows. Let $\mathcal{F}$ and $\mathcal{F}'$ be two sutured surfaces parametrized by the arc diagrams $\mathcal{Z}$ and $\mathcal{Z}'$ of rank k and k', respectively. Let $A = \mathcal{A}(\mathcal{Z})$ and $B = \mathcal{A}(\mathcal{Z}')$. Let $\mathcal{Y} = (Y, \Gamma, \overline{\mathcal{F}} \sqcup \mathcal{F}')$ be a bordered sutured manifold, and let $M = {}_A\widehat{BSAA}(\mathcal{Y})_B$.

Fix $I \subset \{1, \ldots, k\}$ and $J \subset \{1, \ldots, k'\}$. Let $\mathcal{W}_I$ and $\mathcal{W}'_J$ be the respective caps associated to the dividing sets Γ_I on $\mathcal{F}$ and Γ'_J on $\mathcal{F}'$. Then the following homotopy equivalence holds.

$$\iota_I \cdot \widehat{BSAA}(\mathcal{Y}) \cdot \iota_J \simeq SFC(Y, \Gamma_I \cup \Gamma \cup \Gamma'_J). \tag{15.7}$$

The proof is easy. Notice that the sutured manifold $(Y, \Gamma_I \cup \Gamma \cup \Gamma'_J)$ is just $-\mathcal{W}_I \cup \mathcal{Y} \cup \mathcal{W}'_J$. By the pairing theorem, $SFC(Y, \Gamma_I \cup \Gamma \cup \Gamma'_J) \simeq \widehat{BSD}(-\mathcal{W}_I) \boxtimes \widehat{BSAA}(\mathcal{Y}) \boxtimes \widehat{BSD}(\mathcal{W}'_J)$. But by Proposition 15.1.2, $\widehat{BSD}(-\mathcal{W}_I) = \{x_I, 0\}$ is the elementary module corresponding to ι_I, while $\widehat{BSD}(\mathcal{W}'_J) = \{y_J, 0\}$ is the elementary idempotent corresponding to ι'_J. Thus we have

$$\begin{aligned}\widehat{BSD}(-\mathcal{W}_I) \boxtimes \widehat{BSAA}(\mathcal{Y}) \boxtimes \widehat{BSD}(\mathcal{W}'_J) &= x_I \boxtimes \widehat{BSAA}(\mathcal{Y}) \boxtimes y_J \\ &\cong \iota_I \cdot \widehat{BSAA}(\mathcal{Y}) \cdot \iota'_J.\end{aligned}$$

Eq. (15.2) follows from Eq. (15.7) by substituting the empty sutured surface $\varnothing = (\varnothing, \varnothing)$ for $\mathcal{F}$. Its algebra is $\mathcal{A}(\varnothing) = \mathbb{Z}/2$, so ${}_{\mathbb{Z}/2}\widehat{BSAA}(\mathcal{Y})_B$ and $\widehat{BSA}(\mathcal{Y})_B$ can be identified.

Eq. (15.5) follows from Eq. (15.7) by substituting $\mathcal{F}(\mathcal{Z})$ for both $\mathcal{F}$ and $\mathcal{F}'$, and $\mathcal{TW}_{-\overline{\mathcal{F}},-}$ for $\mathcal{Y}$. Indeed, $\widehat{BSAA}(-\mathcal{TW}_{\overline{\mathcal{F}},-}) \simeq \mathcal{A}(\mathcal{Z})$, as a bimodule over itself, by Proposition 12.4.2.

Next we prove Eq. (15.6). Let U_A be a DG-type representative for $\widehat{BSA}(\mathcal{Y})_A$, and let M_I be the elementary representative for ${}_A\widehat{BSA}(\mathcal{W}_I)$. Since both are DG-type, we can form the associative product

$$\begin{aligned}U \boxtimes {}^A\mathbb{I}^A \boxtimes M_I &\simeq \widehat{BSA}(Y) \boxtimes \widehat{BSD}(\mathcal{W}_I) \\ &\simeq SFC(Y, \Gamma \cup \Gamma_I).\end{aligned}$$

Similarly, pick M_J to be the elementary representative for ${}_A\widehat{BSA}(\mathcal{W}_J)$. We also know that ${}_AA_A$ is a DG-type representative for ${}_A\widehat{BSAA}(\mathcal{TW}_{-\overline{\mathcal{F}},-})_A$. We have the associative product

$$\begin{aligned}{M_I}^\vee \boxtimes {}^A\mathbb{I}^A \boxtimes A \boxtimes {}^A\mathbb{I}^A \boxtimes M_J &\simeq \widehat{BSD}(-\mathcal{W}_I) \boxtimes A \boxtimes \widehat{BSD}(\mathcal{W}_J) \\ &\simeq SFC(F \times [0,1], \Gamma_{I \to J}).\end{aligned}$$

Gluing the two sutured manifolds along (F, Γ_I) results in

$$\mathcal{Y} \cup \mathcal{TW}_{\mathcal{F},+} \cup \mathcal{TW}_{-\overline{\mathcal{F}},-} \cup \mathcal{W}_J \cong \mathcal{Y} \cup \mathcal{W}_J = (Y, \Gamma \cup \Gamma_J),$$

so we get the correct manifold.

The gluing map can be written as the composition of

$$\begin{aligned}\Psi_{M_I}\colon\ (U \boxtimes \mathbb{I}) \boxtimes M_I \otimes {M_I}^\vee \boxtimes (\mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M_J) \\ \to (U \boxtimes \mathbb{I}) \boxtimes A^\vee \boxtimes (\mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M_J), \\ \mathrm{id}_U \boxtimes c_A \boxtimes \mathrm{id}_{\mathbb{I} \boxtimes M_J}\colon\ U \boxtimes (\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A) \boxtimes \mathbb{I} \boxtimes M_J \to U \boxtimes \mathbb{I} \boxtimes M_J,\end{aligned}$$

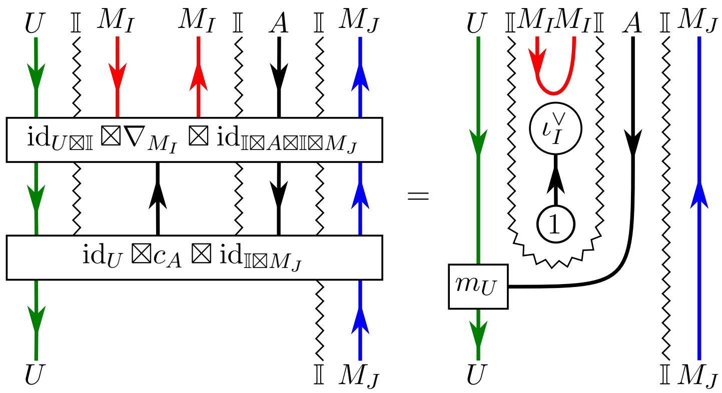


Figure 42: The gluing map Ψ_{M_I} on $SFC(Y,\Gamma_I)\otimes SFC(F\times[0,1],\Gamma_{I\to J})$, followed by the chain homotopy equivalence $\mathrm{id}\boxtimes c_A\boxtimes\mathrm{id}$.

where c_A is the homotopy equivalence from Lemma 14.2.2.

Luckily, since M_I is elementary, Ψ_{M_I} takes the simple form from Proposition 14.1.4. In addition, since U and M_J are DG-type, $\mathrm{id}\boxtimes h\boxtimes\mathrm{id}$ is also very simple. As can be seen in Figure 42, the composition is in fact

$$u\boxtimes *\boxtimes x_{I^c}\otimes {x_{I^c}}^\vee\boxtimes *\boxtimes a\boxtimes *\boxtimes x_{J^c}\mapsto m_{1|1}(u,a)\boxtimes *\boxtimes x_{J^c}.$$

Since $\cdot\boxtimes *\boxtimes x_{I^c}$ corresponds to $\cdot\iota_I$, this translates to the map

$$\begin{aligned}\Psi_{(F,\Gamma_I)}\colon\ &(U\cdot\iota_I)\otimes(\iota_I\cdot A\cdot\iota_J)\to U\cdot\iota_J,\\ &(u\cdot\iota_I)\otimes(\iota_I\cdot a\cdot\iota_J).\mapsto m(u,a)\cdot\iota_J\end{aligned}$$

Note that even though we picked a specific representative for $\widehat{BSA}(\mathcal{Y})_A$, the group $H_*(\widehat{BSA}(\mathcal{Y}))$ and the induced action m_H of $H_*(A)$ do not depend on this choice. Finally, Eq. (15.3) follows by treating A as a right module over itself. $\square$

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Appendices

Appendix A

Calculus of diagrams

This appendix summarizes the principles of the diagrammatic calculus we have used throughout the thesis. First we describe the algebraic objects we work with, and the necessary assumptions on them. Then we describe the diagrams representing these objects.

A.1 Ground rings

The two basic objects we work with are a special class of rings, and bimodules over them. We call these rings *ground rings*.

Definition A.1.1. *A* ground ring K *is a finite dimensional* $\mathbb{Z}/2$*–algebra with a distinguished basis* $(e_1, \ldots, e_k)$ *such that multiplication is given by the formula*

$$e_i \cdot e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Such a basis for K *is called a* canonical basis.

The canonical basis elements are uniquely determined by the property that e_i cannot be written as a sum $u + v$, where u and v are nonzero and $u \cdot v = 0$. Each element of K is an idempotent, while $1_K = e_1 + \cdots + e_k$ is an identity element.

We consider only finite dimensional bimodules ${}_K M_L$ over ground rings K and L, and collections $({}_K M_{iL})_{i \in I}$ where I is a countable index set (usually $I = \{0, 1, 2, \ldots\}$, or some

Cartesian power of the same), and each M_i is a finite-dimensional K, L–bimodule. It is often useful to think of the collection (M_i) as the direct sum $\bigoplus_{i\in I} M_i$, but that sometimes leads to problems, so we will not make this identification.

There are some basic properties of bimodules over ground rings as defined above.

Proposition A.1.2. *Suppose K, L, and R are ground rings with canonical bases $(e_1, \ldots, e_k)$, $(e'_1, \ldots, e'_l)$, and $(e''_1, \ldots, e''_r)$, respectively.*

- *A bimodule ${}_KM_L$ is uniquely determined by the collection of $\mathbb{Z}/2$–vector spaces*
$$e_i \cdot M \cdot e'_j, \qquad\qquad i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\},$$
which we will call the components *of M.*

- *A K, L–bilinear map $f\colon M \to N$ is determined by the collection of $\mathbb{Z}/2$–linear maps*
$$f|_{e_i \cdot M \cdot e'_j}\colon e_i \cdot M \cdot e'_j \to e_i \cdot N \cdot e'_j.$$

- *The tensor product $({}_KM_L) \otimes_L ({}_LN_R)$ has components*
$$e_i \cdot (M \otimes_L N) \cdot e''_j = \bigoplus_{p=1}^{l} (e_i \cdot M \cdot e'_p) \otimes_{\mathbb{Z}/2} (e'_p \cdot N \cdot e''_j).$$

- *The dual ${}_LM^\vee{}_K$ of ${}_KM_L$ has components*
$$e_i \cdot M^\vee \cdot e'_j \cong (e'_j \cdot M \cdot e_i)^\vee,$$
and the double dual $(M^\vee)^\vee$ is canonically isomorphic to M.

Proof. These follow immediately. The fact that $M^{\vee\vee} \cong M$ is due to the fact each component is a finite dimensional vector space. $\square$

Finally, when dealing with countable collections we introduce the following conventions. For consistency we can think of a single module M as a collection (M_i) indexed by the set $I = \{1\}$.

Definition A.1.3. *Let K, L, and M be as in Proposition A.1.2.*

- *An* element *of* $(M_i)_{i\in I}$ *is a collection* $(m_i)_{i\in I}$ *where* $m_i \in M_i$*.*

- *A* bilinear map $f\colon\ ({}_K M_{i L})_{i\in I} \to ({}_K N_{j L})_{j\in J}$ *is a collection*

$$f_{(i,j)}\colon\ M_i \to N_j \qquad\qquad (i,j) \in I \times J.$$

Equivalently, a map f is an element of the collection

$$\mathrm{Hom}_{K,L}((M_i)_{i\in I}, (N_j)_{j\in J}) = (\mathrm{Hom}(M_i, N_j))_{(i,j)\in I\times J}.$$

- *The* tensor ${}_K(M_i)_L \otimes {}_L(N_j)_R$ *is the collection*

$$((M \otimes N)_{(i,j)})_{(i,j)\in I\times J} = (M_i \otimes N_j)_{(i,j)\in I\times J}.$$

- *The* dual $((M_i)_{i\in I})^\vee$ *is the collection* $({M_i}^\vee)_{i\in I}$*.*

- *Given bilinear maps* $f\colon\ (M_i) \to (N_j)$ *and* $g\colon\ (N_j) \to (P_p)$*, their composition* $g \circ f\colon\ (M_i) \to (P_p)$ *is the collection*

$$(g \circ f)_{(i,p)} = \sum_{j\in J} g_{(j,p)} \circ f_{(i,j)}.$$

Note that the composition of maps on collections may not always be defined due to a potentially infinite sum. On the other hand, the double dual $(M_i)^{\vee\vee}$ is still canonically isomorphic to (M_i).

A.2 Diagrams for maps

We will use the following convention for our diagram calculus. There is a TQFT-like structure, where to decorated planar graphs we assign bimodule maps.

Proposition A.2.1. *Suppose $K_0, K_1, \dots, K_n = K_0$ are ground rings, $n \geq 0$, and ${}_{K_{i-1}}M_{i K_i}$ are bimodules, or collections of bimodules. Then the following $\mathbb{Z}/2$–spaces are canonically isomorphic.*

$$\begin{aligned}
A_i &= M_i \otimes M_{i+1} \otimes \cdots \otimes M_n \otimes M_1 \otimes \cdots \otimes M_{i-1} / \sim,\\
B_{i,j} &= \mathrm{Hom}_{K_i,K_j}(M_i^\vee \otimes \cdots \otimes M_1^\vee \otimes M_n^\vee \otimes \cdots \otimes M_{j+1}^\vee,\ M_{i+1} \otimes \cdots \otimes M_j),\\
C_{i,j} &= \mathrm{Hom}_{K_j,K_i}(M_j^\vee \otimes \cdots \otimes M_{i+1}^\vee,\ M_{j+1} \otimes \cdots \otimes M_n \otimes M_1 \otimes \cdots \otimes M_i),
\end{aligned}$$

for $0 \leq i \leq j \leq n$, where the relation $\sim$ in the definition of A_i is $k \cdot x \sim x \cdot k$, for $k \in K_{i-1}$.

Proof. The proof is straightforward. If all M_i are single modules, then we are only dealing with finite-dimensional $\mathbb{Z}/2$–vector spaces. If some of them are collections, then the index sets for A_i, $B_{i,j}$ and $C_{i,j}$ are all the same, and any individual component still consists of finite dimensional vector spaces. $\square$

This property is usually referred to as *Frobenius duality*. Our bimodules behave similar to a pivotal tensor category. Of course we do not have a real category, as even compositions are not always defined.

Definition A.2.2. *A* diagram *is a planar oriented graph, embedded in a disc, with some degree–1 vertices on the boundary of the disc There are labels as follows.*

- *Each planar region (and thus each arc of the boundary) is labeled by a ground ring K.*
- *Each edge is labeled by a bimodule ${}_KM_L$, such that when traversing the edge in its direction, the region on the left is labeled by K, while the one on the right is labeled by L.*
- *An internal vertex with all outgoing edges labeled by $M_1, \ldots, M_n$, in cyclic counter-clockwise order, is labeled by an element of one of the isomorphic spaces in Proposition A.2.1.*
- *If any of the edges adjacent to a vertex are incoming, we replace the corresponding modules by their duals.*

When drawing diagrams we will omit the bounding disc, and the boundary vertices. We will usually interpret diagrams consisting of a single internal vertex having several incoming edges $M_1, \ldots, M_m$ "on top", and several outgoing edges $N_1, \ldots, N_n$ "on the bottom", as a bilinear map in $\mathrm{Hom}(M_1 \otimes \cdots \otimes M_m,\ N_1 \otimes \cdots \otimes N_n)$. See Figure 43 for an example.

Under some extra assumptions, discussed in Chapter A.3, a diagram with more vertices can also be evaluated, or interpreted as an element of some set, corresponding to all outgoing

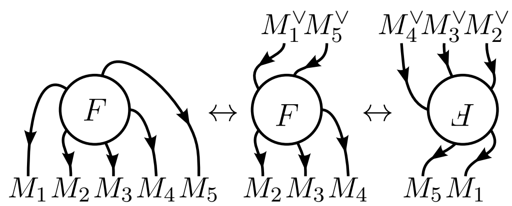


Figure 43: Three equivalent diagrams with a single vertex. The label F is interpreted as an element of $A_1 = M_1 \otimes \cdots \otimes M_5 / \sim$, $B_{1,4} = \operatorname{Hom}(M_1^\vee \otimes M_5^\vee, M_2 \otimes M_3 \otimes M_4)$, and $C_{1,4}\operatorname{Hom}(M_4^\vee \otimes M_3^\vee \otimes M_2^\vee, M_5 \otimes M_1)$, respectively.

edges. The most common example is having two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ representing linear maps

$$M \xrightarrow{f_1} N \xrightarrow{f_2} P.$$

Stacking the two diagrams together, feeding the outgoing edges of $\mathcal{D}_1$ into the incoming edges of $\mathcal{D}_2$, we get a new diagram $\mathcal{D}$, corresponding to the map $f_2 \circ f_1 \colon M \to P$. More generally, we can "contract" along all internal edges, pairing the elements assigned to the two ends of an edge. As an example we will compute the diagram D in Figure 44. Suppose the values of the vertices F, G, and H are as follows:

$$F = \sum_i m_i \otimes q_i \otimes s_i \in M \otimes Q \otimes S,$$

$$G = \sum_j s'_j \otimes r'_j \otimes p'_j \in S^\vee \otimes R^\vee \otimes P^\vee,$$

$$H = \sum_k q'_k \otimes n_k \otimes r_k \in Q^\vee \otimes N \otimes R.$$

Then the value of D is given by

$$D = \sum_{i,j,k} \langle q_i, q'_k \rangle_Q \cdot \left\langle s_i, s'_j \right\rangle_S \cdot \left\langle r_k, r'_j \right\rangle_R \cdot m_i \otimes n_k \otimes p'_j \in M \otimes N \otimes P^\vee.$$

Edges that go from boundary to boundary and closed loops can be interpreted as having an identity vertex in the middle. As with individual vertices, we can rotate a diagram to interpret it as an element of different spaces, or different linear maps.

Note that the above construction might fail if any of the internal edges corresponds to a collection, since there might be an infinite sum involved. The next section discusses how to deal with this problem.

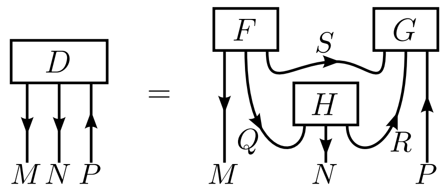


Figure 44: Evaluation of a complex diagram.

A.3 Boundedness

When using collections of modules we have to make additional assumptions to avoid infinite sums. We use the concept of boundedness of maps and diagrams.

Definition A.3.1. *An element* $(m_i)_{i\in I}$ *of the collection* $(M_i)_{i\in I}$ *is called* bounded *if only finitely many of its components* m_i *are nonzero. Equivalently, the bounded elements of* (M_i) *can be identified with the elements of* $\bigoplus_i M_i$*.*

For a collection $(M_{i,j})_{i\in I, j\in J}$ *there are several different concepts of boundedness. An element* $(m_{i,j})$ *is* totally bounded *if it is bounded in the above sense, considering* $I \times J$ *as a single index-set. A weaker condition is that* $(m_{i,j})$ *is* bounded in J relative to I*. This means that for each* $i \in I$*, there are only finitely many* $j \in J$*, such that* $m_{i,j}$ *is nonzero. Similarly, an element can be bounded in* I *relative to* J*.*

Note that $f\colon (M_i) \to (N_j)$ is bounded in J relative to I exactly when f represents a map from $\bigoplus_i M_i$ to $\bigoplus_j N_j$. In computations relatively bounded maps are more common than totally bounded ones. For instance the identity map $\mathrm{id}\colon (M_i) \to (M_i)$ and the natural pairing $\langle \cdot, \cdot \rangle : (M_i)^\vee \otimes (M_i) \to K$ are not totally bounded, but are bounded in each index relative to the other.

To be able to collapse an edge labeled by a collection $(M_i)_{i\in I}$ in a diagram, at least one of the two adjacent vertices needs to be labeled by an element relatively bounded in the I–index. For a given diagram D we can ensure that it has a well-defined evaluation by imposing enough boundedness conditions on individual vertices. (There is usually no unique minimal set of conditions.) Total or relative boundedness of D can also be achieved by a stronger set of conditions. For example, if all vertices are totally bounded, the entire diagram is also totally bounded.

Appendix B

$\mathcal{A}_\infty$–algebras and modules

In this section we will present some of the background on $\mathcal{A}_\infty$–algebras and modules, and the way they are used in the bordered setting. A more thorough treatment is given in [LOT10a].

As in Appendix A, we always work with $\mathbb{Z}/2$–coefficients which avoids dealing with signs. Everything is expressed in terms of the diagram calculus of Appendix A. As described there, all modules are finite dimensional, although we also deal with countable collections of such modules. There is essentially only one example of collections that we use, which is presented below.

B.1 The bar construction

Suppose K is a ground ring and ${}_K M_K$ is a bimodule over it.

Definition B.1.1. *The* bar of M *is the collection*

$$\operatorname{Bar} M = (M^{\otimes i})_{i=0,\dots,\infty},$$

of tensor powers of M.

There are two important maps on the bar of M.

(a) Split maps. (b) Merge maps.

Figure 45

Definition B.1.2. *The* split *on* Bar M *is the map* $s\colon \operatorname{Bar} M \to \operatorname{Bar} M \otimes \operatorname{Bar} M$ *with components*

$$s_{(i,j,k)} = \begin{cases} \mathrm{id}\colon M^{\otimes i} \to (M^{\otimes j}) \otimes (M^{\otimes k}) & \text{if } i = j+k, \\ 0 & \text{otherwise.} \end{cases}$$

The merge *map* $\operatorname{Bar} M \otimes \operatorname{Bar} M \to \operatorname{Bar} M$ *is similarly defined.*

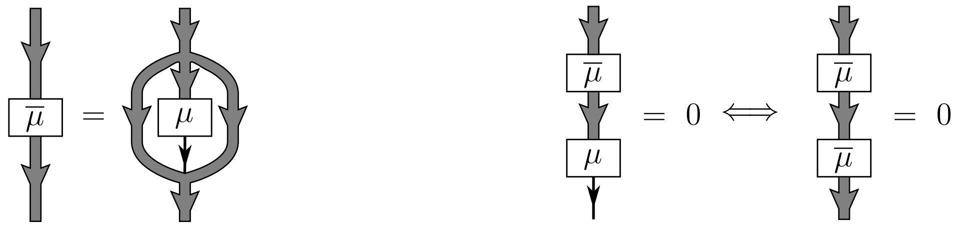
Merges and splits can be extended to more complicated situations where any combination of copies of Bar M and M merge into Bar M, or split from Bar M. All merges are associative, and all splits are coassociative.

Like the identity map, splits and merges are bounded in incoming indices, relative to outgoing, and vice versa. To simplify diagrams, we draw merges and splits as merges ans splits of arrows, respectively, without using a box for the corresponding vertex (see Figure 45).

B.2 Algebras and modules

The notion of an $\mathcal{A}_\infty$–algebra is a generalization of that of a differential graded (or DG) algebra. While the algebras that arise in the context of bordered Floer homology are only DG, we give the general definition for completeness. We will omit grading shifts.

Definition B.2.1. *An* $\mathcal{A}_\infty$–algebra A over the base ring K *consists a* K*–bimodule* ${}_K A_K$*, together with a collection of linear maps* $\mu_i\colon A^{\otimes i} \to A$*,* $i \geq 1$*, satisfying certain compatibility conditions. By adding the trivial map* $\mu_0 = 0\colon K \to A$*, we can regard this as a map* $\mu = (\mu_i)\colon \operatorname{Bar} A \to A$*. This induces a map* $\overline{\mu}\colon \operatorname{Bar} A \to \operatorname{Bar} A$*, given by splitting* Bar M *into three copies of itself, applying* μ *to the middle one, and merging again (see Figure 46a).*



(a) $\bar{\mu}$ in terms of μ.

(b) Compatibility conditions.

Figure 46: Definition of $\mathcal{A}_\infty$–algebras

The compatibility condition is $\overline{\mu} \circ \mu = 0$*, or equivalently* $\overline{\mu} \circ \overline{\mu} = 0$ *(see Figure 46b).*

The algebra is unital *if there is a map* $1\colon\ K \to A$ *(which we draw as a circle labeled "1" with an outgoing arrow labeled "A"), such that* $\mu_2(1, a) = \mu_2(a, 1) = a$*, and* $\mu_i(\ldots, 1, \ldots) = 0$ *if* $i \neq 2$*.*

The algebra A *is* bounded *if* μ *is bounded, or equivalently if* $\overline{\mu}$ *is relatively bounded in both directions.*

Notice that a DG-algebra with multiplication m and differential d is just an $\mathcal{A}_\infty$ algebra with $\mu_1 = d$, $\mu_2 = m$, and $\mu_i = 0$ for $i \geq 3$. Moreover, DG-algebras are always bounded.

Since DG-algebras are associative, there is one more operation that is specific to them.

Definition B.2.2. *The* associative multiplication $\pi\colon$ Bar $A \to A$ *for a DG-algebra* A *is the map with components*

$$\pi_i(a_1 \otimes \cdots \otimes a_i) = \begin{cases} a_1 a_2 \cdots a_i & i > 0, \\ 1 & i = 0. \end{cases}$$

There are two types of modules: type–A, which is the usual notion of an $\mathcal{A}_\infty$–module, and type–D. There are four types of bimodules: type–AA, type–DA, etc. These can be extend to tri-modules and so on. We describe several of the bimodules. Other cases can be easily deduced.

Suppose A and B are unital $\mathcal{A}_\infty$–algebras with ground rings K and L, respectively. We use the following notation. A type–A module over A will have A as a lower index. A type–D module over A will have A as an upper index. Module structures over the ground rings K and L are denoted with the usual lower index notation.

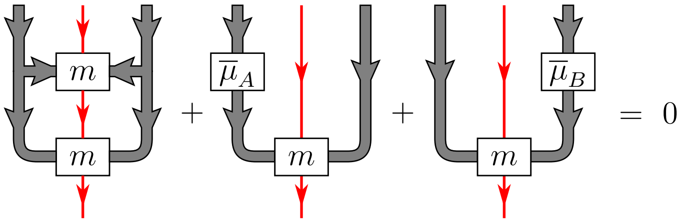
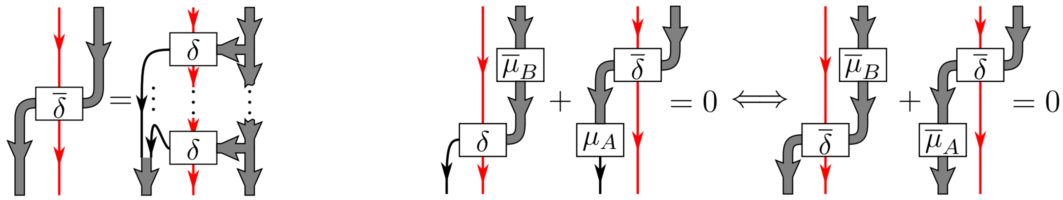


Figure 47: Structure equation for a type–AA module.



(a) $\overline{\delta}$ in terms of δ.

(b) Structure equation for a type–DA module.

Figure 48

Definition B.2.3. *A type–AA bimodule ${}_AM_B$ consists of a bimodule ${}_KM_L$ over the ground rings, together with a map $m = (m_{i|1|j})\colon \operatorname{Bar} A \otimes M \otimes \operatorname{Bar} B \to M$. The compatibility conditions for m are given in Figure 47.*

The bimodule M is *unital* if $m_{1|1|0}(1_A, m) = m_{0|1|1}(m, 1_B) = m$, and $m_{i|1|j}$ vanishes in all other cases where one of the inputs is 1_A or 1_B.

The bimodule can be *bounded*, *bounded only in A*, *relatively bounded in A with respect to B*, etc. These are defined in terms of the index sets of $\operatorname{Bar} A$ and $\operatorname{Bar} B$.

Definition B.2.4. *A type–DA bimodule ${}^AM_B$ consists of a bimodule ${}_KM_L$ over the ground rings, together with a map $\delta = (\delta_{1|1|j})\colon M \otimes \operatorname{Bar} B \to A \otimes M$. This induces another map $\overline{\delta} = (\delta_{i|1|j})\colon M \otimes \operatorname{Bar} B \to \operatorname{Bar} A \otimes M$, by splitting* $\operatorname{Bar} B$ *into i copies, and applying δ i–many times (see Figure 48a). The compatibility conditions for δ and $\overline{\delta}$ are given in Figure 48b.*

The bimodule M is *unital* if $\delta_{1|1|1}(m, 1_B) = 1_A \otimes m$, and $\delta_{1|1|i}$ vanishes for $i > 1$ if one of the inputs is 1_B.

Again, there are various boundedness conditions that can be imposed.

Type–DD modules only behave well if the algebras involved are DG, so we only give the definition for that case.

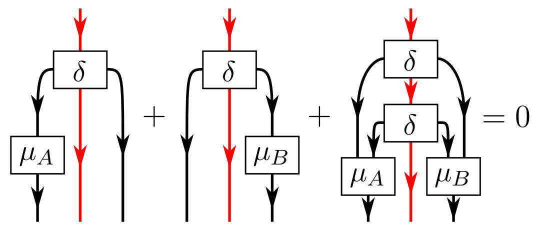


Figure 49: Structure equation for a type–DD module.

Definition B.2.5. *Suppose A and B are DG-algebras. A type DD–module ${}^AM^B$ consists of a bimodule ${}_KM_L$ over the ground rings, together with a map $\delta_{1|1|1}\colon M \to A \otimes M \otimes B$ satisfying the condition in Figure 49.*

We omit the definition of one-sided type–A and type–D modules, as they can be regarded as special cases of bimodules. Type–A modules over A can be interpreted as type–AA bimodules over A and $B = \mathbb{Z}/2$. Similarly, type–D modules are type DA–modules over $\mathbb{Z}/2$.

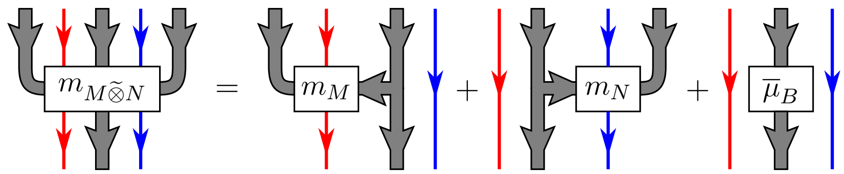
B.3 Tensor products

There are two types of tensor products for $\mathcal{A}_\infty$–modules. One is the more traditional derived tensor product $\widetilde{\otimes}$. It is generally hard to work with, as $M\widetilde{\otimes}N$ is infinite dimensional over $\mathbb{Z}/2$ even when M and N are finite dimensional. This is bad for computational reasons, as well as when using diagrams—it violates some of the assumptions of Appendix A. Nevertheless, we do use it in a few places throughout the thesis.

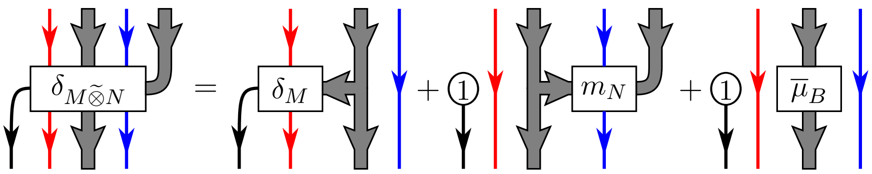
Throughout the rest of this section assume that A, B, and C are DG-algebras over the ground rings K, L, and P, respectively.

Definition B.3.1. *Suppose ${}_AM_B$ and ${}_BN_C$ are two type–AA bimodules. The* derived tensor product $({}_AM_B)\widetilde{\otimes}_B({}_BN_C)$ *is a type–AA bimodule ${}_A(M\widetilde{\otimes}N)_B$ defined as follows. Its underling bimodule over the ground rings is*

$$
\begin{aligned}
{}_K(M \,\widetilde{\otimes}\, N)_P &= ({}_KM_L) \otimes_L \left(\bigoplus_{i=0}^{\infty} {}_LB_L{}^{\otimes i} \right) \otimes_L ({}_LN_P) \\
&= M \otimes_L \operatorname{Bar} B \otimes_L N.
\end{aligned}
$$



(a) ${}_AM_B \mathbin{\widetilde{\otimes}} {}_BN_C$



(b) ${}^AM_B \mathbin{\widetilde{\otimes}} {}_BN_C$

Figure 50: Structure maps for two types of $\widetilde{\otimes}$ products.

Here we're slightly abusing notation in identifying $\operatorname{Bar} B$ *with a direct sum. The structure map as an $\mathcal{A}_\infty$–bimodule over A and C is $m_{M\widetilde{\otimes}N}$, as shown in Figure 50a.*

Similarly, we can take the derived tensor product of a DA module and an AA module, or a DA module and an AD module. The former is demonstrated in Figure 50b.

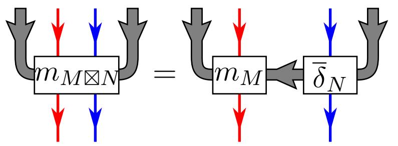
The other type of tensor product is the square tensor product $\boxtimes$. It is asymmetric, as it requires one side to be a type–D module, and the other to be a type–A module. The main advantage of $\boxtimes$ over $\widetilde{\otimes}$ is that $M \boxtimes N$ is finite dimensional over $\mathbb{Z}/2$ whenever M and N are. Its main disadvantage is that $M \boxtimes N$ is only defined subject to some boundedness conditions on M and N.

Definition B.3.2. *Suppose ${}_AM_B$ is a type–AA bimodule and ${}^BN_C$ is a type–DA bimodule, such that at least one of M and N is relatively bounded in B. The* square tensor product $({}_AM_B) \boxtimes_B ({}^BN_C)$ *is a type–AA bimodule ${}_A(M \boxtimes N)_C$ defined as follows. Its underlying bimodule over the ground rings is*

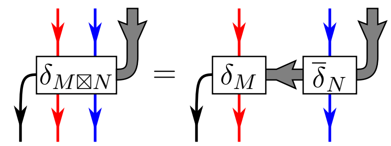
$$ {}_K(M \boxtimes N)_P = ({}_KM_L) \boxtimes_L ({}_LN_P), $$

and its structure map is $m_{M\boxtimes N}$ as shown in Figure 51a.

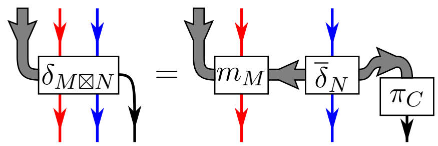
There are three other combinations depending on whether the modules are of type D or A with respect A and C. All combinations are shown in Figure 51.



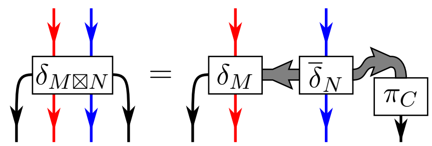
(a) ${}_AM_B \boxtimes {}^BN_C$



(b) ${}^AM_B \boxtimes {}^BN_C$



(c) ${}_AM_B \boxtimes {}^BN^C$



(d) ${}^AM_B \boxtimes {}^BN^C$

Figure 51: Structure maps for the four types of $\boxtimes$ products.

B.4 Morphisms and homomorphisms

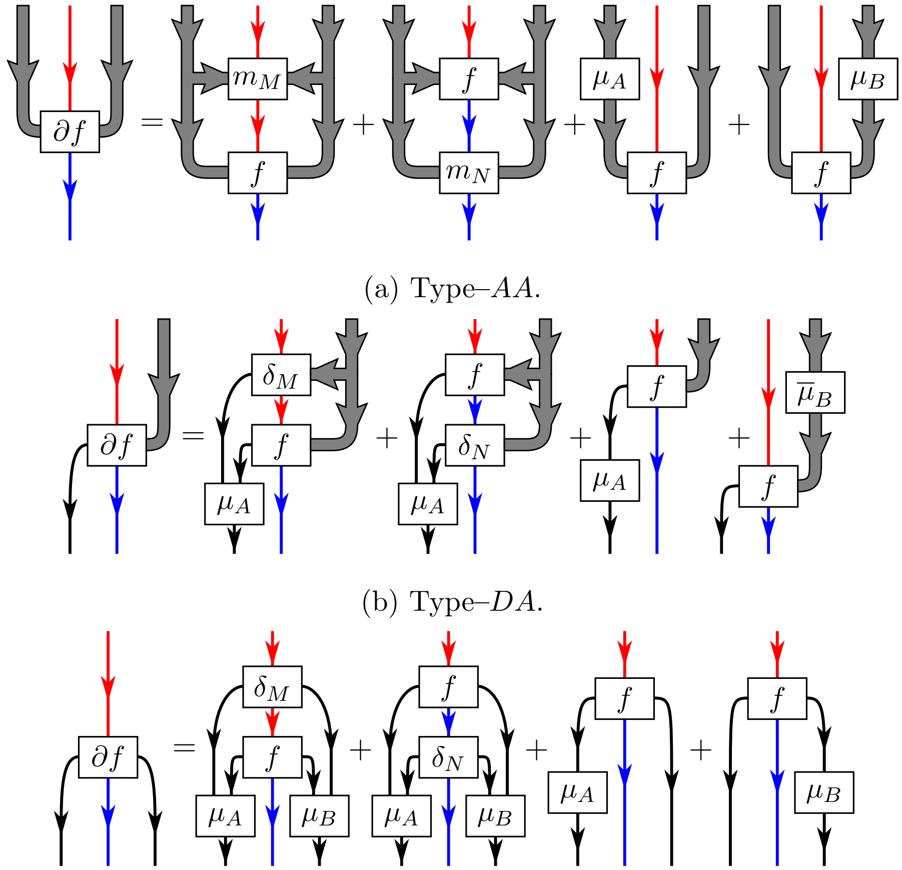
There are two different notions of morphisms when working with $\mathcal{A}_\infty$–modules and bimodules. The more natural one is that of *homomorphisms*, which generalize chain maps. However, if we work only with homomorphisms, too much information is lost. For this reason we also consider the more general *morphisms*. These generalize linear maps of chain complexes, which do not necessarily respect differentials.

Definition B.4.1. *A* morphism $f\colon M \to N$ *between two bimodules M and N of the same type is a collection of maps of the same type as the structure maps for M and N. For example, $f\colon {}_AM_B \to {}_AN_B$ has components $f_{i|1|j}\colon \operatorname{Bar} A \otimes M \otimes \operatorname{Bar} B \to N$. The spaces of morphisms are denoted by ${}_A\operatorname{Mor}_B(M,N)$, etc.*

Suppose A and B are DG-algebras. The bimodules of each type, e.g. ${}_A\operatorname{Mod}_B$, form a DG-category, with morphism spaces ${}_A\operatorname{Mor}_B$, etc. The differentials and composition maps for each type are shown in Figures 52 and 53, respectively.

Definition B.4.2. *A* homomorphism $f\colon M \to N$ *of bimodules is a morphism f which is a cycle, i.e., $\partial f = 0$. A* null-homotopy *of f is a morphism H, such that $\partial H = f$. The space of homomorphisms up to homotopy is denoted by ${}_A\operatorname{Hom}_B$, etc.*

Notice that the homomorphism space ${}_A\operatorname{Hom}_B(M,N)$ is precisely the homology of the morphism space ${}_A\operatorname{Mor}_B(M,N)$. This gives us a new category of bimodules.



(a) Type–AA.

(b) Type–DA.

(c) Type–DD.

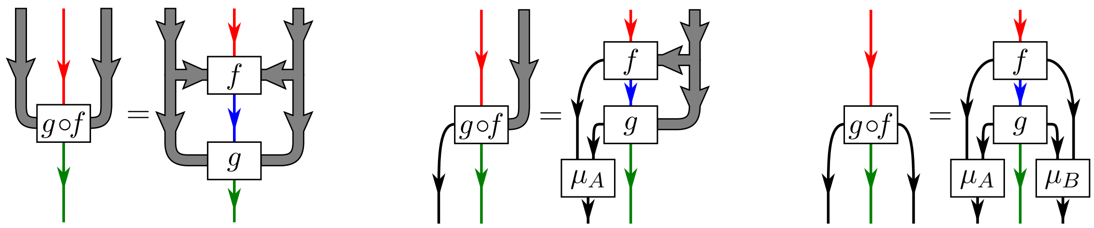
Figure 52: Differentials of the different types of morphisms.

Having homomorphisms and homotopies allows us to talk about homotopy equivalences of modules. For example, if ${}_AM_B$ is a bimodule, then $A \mathbin{\widetilde{\otimes}} M \simeq M \simeq M \mathbin{\widetilde{\otimes}} B$, via canonical homotopy equivalences. For example, there is $h_M\colon A \mathbin{\widetilde{\otimes}} M \to M$, which we used in several places.

B.5 Induced morphisms

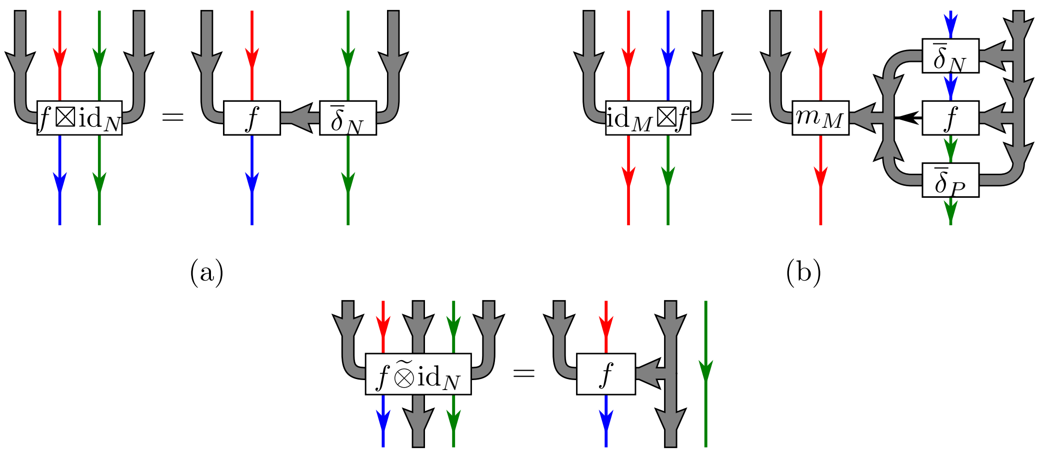
Suppose $f\colon M \to N$ is a bimodule morphism. This induces morphisms

$$f \mathbin{\widetilde{\otimes}} \mathrm{id}\colon M \mathbin{\widetilde{\otimes}} P \to N \mathbin{\widetilde{\otimes}} P \qquad\qquad f \boxtimes \mathrm{id}\colon M \boxtimes P \to N \boxtimes P,$$



(a) Type–AA. (b) Type–DA. (c) Type–DA.

Figure 53: Compositions of the different types of morphisms.



(a) (b) (c)

Figure 54: Three types of induced maps on tensor products.

whenever the tensor products are defined. The main types of induced morphisms are shown in Figure 54. The functors $\cdot\boxtimes\mathrm{id}$ and $\cdot\,\widetilde{\otimes}\,\mathrm{id}$ are DG-functors. That is, they preserve homomorphisms, homotopies, and compositions.

B.6 Duals

There are two operations on modules, which can be neatly expressed by diagrams. One is the operation of turning a bimodule ${}_AM_B$ into a bimodule ${}_{B^{\mathrm{op}}}M_{A^{\mathrm{op}}}$. (Similarly, type–$DA$ bimodules become type–AD bimodules, etc.) Diagrammatically this is achieved by reflecting diagrams along the vertical axis. See Figure 55 for an example.

The other operation is dualizing modules and bimodules. If ${}_AM_B$ has an underlying

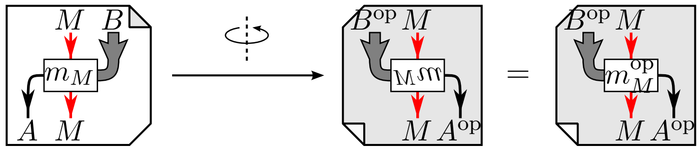


Figure 55: Passing from ${}^{A}\mathrm{Mod}_{B}$ to ${}_{B^{\mathrm{op}}}\mathrm{Mod}^{A^{\mathrm{op}}}$ by reflection.

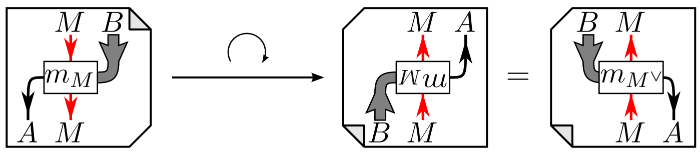


Figure 56: Passing from ${}^{A}\mathrm{Mod}_{B}$ to ${}_{B}\mathrm{Mod}^{A}$ by rotation.

bimodule ${}_{K}M_{L}$ over the ground rings, then its dual ${}_{B}M^{\vee}{}_{A}$ has an underlying bimodule ${}_{L}M^{\vee}{}_{K} = ({}_{K}M_{L})^{\vee}$. Diagrammatically this is achieved by rotating diagrams by 180 degrees. Again, there are variations for type–D modules. See Figure 56 for an example.

Since the structure equations are symmetric, it is immediate that both of these operations send bimodules to bimodules, as long as we restrict to modules finitely generated over $\mathbb{Z}/2$.

This gives equivalences of the DG-categories

$$ {}^{A}\mathrm{Mod}_{B} \cong {}_{B^{\mathrm{op}}}\mathrm{Mod}^{A^{\mathrm{op}}} \cong \left({}_{B}\mathrm{Mod}^{A}\right)^{\mathrm{op}}, $$

etc. One can check that both constructions extend to tensors, induced morphisms, etc.