F-virtual Abelian Varieties of GL_2-type and Rallis Inner Product Formula

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Abstract

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This thesis consists of two topics. First we study F-virtual Abelian varieties of GL_2-type where F is a number field. We show the relation between these Abelian varieties and those defined over F. We compare their ℓ-adic representations and study the modularity of F-virtual Abelian varieties of GL_2-type. Then we construct their moduli spaces and in the case where the moduli space is a surface we give criteria when it is of general type. We also give two examples of surfaces that are rational and one that is neither rational nor of general type.

Second we prove a crucial case of Siegel-Weil formula for orthogonal groups and metaplectic groups. With this we can compute the pairing of theta functions and show in this case that it is related to the central value of Langlands L-function. This new case of Rallis inner product formula enables us to relate nonvanishing of L-value to the nonvanishing of theta lifting.

Contents

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Chapter 1

Introduction

This thesis is made up of two topics. In Chapter 2 we study F-virtual Abelian varieties of GL_2-type where F is a number field. Virtual Abelian variety is a generalization of $\mathbb{Q}$-curve first studied by Gross[4] in the the CM case. The concept of $\mathbb{Q}$-curve was generalized by Ribet[25][26] to include non-CM elliptic curves and Abelian varieties of GL_2-type. Ribet[26] showed that every elliptic curve coming from a quotient of $J_1(N)$ is a $\mathbb{Q}$-curve and that Serre conjecture implies that the converse holds, namely that all $\mathbb{Q}$-curves are modular. Then Ellenberg-Skinner[2] showed that all $\mathbb{Q}$-curves are modular under some local condition at 3. Later Serre conjecture was proved by Khare and Wintenberger[12][13], implying by Ribet's work[26] that all $\mathbb{Q}$-virtual Abelian varieties of GL_2-type are modular. The Heegner points on $J_1(N)$ then can be used to construct rational points on $\mathbb{Q}$-virtual Abelian varieties of GL_2-type and this gives supporting evidence to the BSD conjecture. We would like to investigate if general F-virtual Abelian varieties of GL_2-type are modular. In Chapter 2.2 we will show some properties of the associated ℓ-adic representations.

Elkies studied the F-virtual elliptic curves in [1] and produced concrete examples of moduli space of virtual elliptic curves and Quer[21] computed some equations for $\mathbb{Q}$-virtual elliptic curves (i.e., $\mathbb{Q}$-curves) based on the parametrization by Elkies[1] and González-Lario[3]. However little is known for $F \neq \mathbb{Q}$ and for Abelian varieties of higher dimension. This work attempts to produce concrete examples of F-virtual Abelian varieties of GL_2-type in higher dimensions. We will describe the moduli spaces which turn out to be generalized Atkin-Lehner quotients of Shimura varieties and in the cases where the moduli spaces are surfaces, classify them birationally.

An Abelian variety A is said to be of $\mathrm{GL}_2(E)$-type if its endomorphism algebra contains the number field E of degree over $\mathbb{Q}$ equal to the dimension of A. This leads to Galois representations on Tate modules of rank 2 over $E \otimes \mathbb{Q}_\ell$ and hence the nomenclature GL_2. Let F be a number field and let A be an Abelian variety defined over $\overline{F}$. We say A is an F-virtual Abelian variety if for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ there exists an isogeny ${}^\sigma\! A$ to A over $\overline{F}$.

We focus on non-CM simple Abelian varieties. We can actually attach Galois representations of $\mathrm{Gal}(\overline{F}/F)$ to F-virtual Abelian varieties (Chapter 2.2). Since simple Abelian variety B over F of GL_2-type factors over $\overline{F}$ isotypically into a power of simple F-virtual Abelian variety of GL_2-type, the Galois representation associated to F-virtual Abelian variety of GL_2-type is essentially that associated to Abelian variety over F of GL_2-type (Prop. 2.2.1).

For F-virtual elliptic curves Elkies[1] constructed ℓ-local trees where vertices are isomorphism classes of F-virtual elliptic curves and edges represent cyclic isogenies. Note that the endomorphism algebra for F-virtual elliptic curves is $\mathbb{Q}$. For Abelian varieties of $\mathrm{GL}_2(E)$-type the field E does not necessarily have class number 1. The same construction would produce loops in the λ-local graph for λ a prime of E. We work around this problem by introducing certain equivalence relation on the category of Abelian varieties. Essentially we are modding out $\mathrm{Pic}(\mathcal{O}_E)$ by making the Serre tensor $A \otimes \mathfrak{A}$ equivalent to A where $\mathfrak{A}$ is a (fractional) ideal of E. Then the λ-local graphs are still trees. Applying graph theory we get

Proposition 1.0.1. [Prop. 2.3.14] *For an F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type there exists an isogenous Abelian variety A' and a minimal level structure $\mathfrak{n} \subset \mathcal{O}_E$ such that its Galois orbit is contained in the generalized $\mathfrak{n}$-Atkin-Lehner orbit.*

Remark 1.0.2. Note that the Atkin-Lehner operators are no longer involutions. Because of the equivalence relation we have introduced, we need to extend the group W of Atkin-Lehner operators to incorporate the action of $\mathrm{Pic}(\mathcal{O}_E)$. Thus the generalized group of Atkin-Lehner operators is $W \ltimes \mathrm{Pic}(\mathcal{O}_E)$. Please refer to Chapter 2.3. Here $\mathfrak{n}$ is an isogeny invariant.

Let $S^+_{E,\mathfrak{n}}$ denote the quotient by the group of generalized $\mathfrak{n}$-Atkin-Lehner operators on the Shimura variety parametrizing Abelian varieties with endomorphism algebra E and with level $\mathfrak{n}$. With the above theorem we deduce

Theorem 1.0.3. [Thm. 2.3.15] *Every F-point on $S^+_{E,\mathfrak{n}}$ gives F-virtual Abelian varieties of* $\mathrm{GL}_2(E)$*-type. Conversely for an F-virtual Abelian variety A of* $\mathrm{GL}_2(E)$*-type there exists an isogenous Abelian variety A' which corresponds to an F-point on some $S^+_{E,\mathfrak{n}}$ where $\mathfrak{n}$ is determined as in the above proposition.*

There is also an analogous result for F-virtual Abelian varieties with endomorphism algebra D which is a quaternion algebra containing E.

Theorem 1.0.4. [Thm. 2.3.20] *Every F-point on the Shimura variety $S^+_{D,\mathfrak{n}}$ gives an F-virtual Abelian variety of* $\mathrm{GL}_2(E)$*-type. Conversely for any F-virtual Abelian variety A of* $\mathrm{GL}_2(E)$*-type s.t.* $\mathrm{End}^0 A = D$ *there is an isogenous Abelian variety A' of* $\mathrm{GL}_2(E)$*-type which corresponds to an F-rational point on $S^+_{D,\mathfrak{n}}$, a quotient of Shimura variety of PEL-type,.*

In [1] Elkies produced concrete examples of moduli space of $\mathbb{Q}$-elliptic curves and from that Quer[21] computed some equations for $\mathbb{Q}$-elliptic curves. In an attempt to give examples of F-virtual Abelian varieties we study the cases where E is a real quadratic field of narrow class number 1. Then the quotient of Shimura variety is a surface. Following Van der Geer's method in [28] we study the desingularity of $S^+_{E,\mathfrak{n}}$, estimate the Chern numbers and show in Thm. 2.5.20 with explicit conditions which ones are surfaces of general type:

Theorem 1.0.5. [Thm. 2.5.20] *The quotient of the Shimura variety is of general type if the discriminant D of E or $\mathbf{N}(\mathfrak{n})$ is sufficiently large.*

Then careful examination of configuration of (-1)-curves and (-2)-curves enables us to show in Chapter 2.5.4:

Example 1.0.6.

1. For $E = \mathbb{Q}(\sqrt{5})$ and $\mathfrak{n} = (2)$, the moduli space is a rational surface;
2. For $E = \mathbb{Q}(\sqrt{13})$ and $\mathfrak{n} = (4 + \sqrt{13})$, the moduli space is a rational surface;
3. For $E = \mathbb{Q}(\sqrt{13})$ and $\mathfrak{n} = (2)$, the moduli space is neither rational nor of general type.

In Chapter 3 we study the Rallis inner product formula. As it relies on a new case of regularized Siegel-Weil formula, we deduce the latter first.

Let k be a number field. Let V be a vector space of dimension m over k equipped with the quadratic form Q. Let $H = \mathrm{O}(V)$ and $G = \mathrm{Sp}(2n)$. Consider the dual reductive pair $H(\mathbb{A})$ and $\widetilde{G(\mathbb{A})}$ which is a metaplectic double cover of $G(\mathbb{A})$. Form the Eisenstein series $E(g, s, f_\phi)$ and the theta series $\theta(g, h, \phi)$ for $g \in \widetilde{G(\mathbb{A})}$ and $h \in H(\mathbb{A})$ and $\phi \in \mathcal{S}_0(V^n(\mathbb{A}))$. Here f_ϕ is the Siegel-Weil section associated to ϕ. Let I_{REG} denote the regularized theta integral, c.f. Chapter 3.2. Please see Chapter 3.1 for further notations. We prove a new case of regularized Siegel-Weil formula under some choice of Haar measures:

Theorem 1.0.7. [Thm. 3.1.1] *Let $m = n + 1$ and exclude the split binary case. We have for all $\phi \in \mathcal{S}_0(V^n(\mathbb{A}))$*

$$E(s, g, f_\phi)|_{s=0} = \kappa I_{\mathrm{REG}}(g, \phi)$$

where $\kappa = 2$.

Kudla and Rallis[17] introduced the regularized theta integral by using some differential operator at a real place. Then they showed for m even the leading term of the Eisenstein series is a scalar multiple of the regularized theta integral which involves the complementary space of V. Ichino[8] generalized Kudla and Rallis's regularization process by using Hecke operator and showed for $m > n+1$ with no parity restriction on m that the above statement holds.

Our method closely follows that of Kudla and Rallis[17] and Ichino's[8]. However some representation results were lacking in the metaplectic case. In Chapter 3.5 we prove the representations $R_n(V)$ are irreducible and nonsingular in the sense of Howe[7] and thus can produce an operator to kill the singular Fourier coefficients of the automorphic forms involved. Then we use Fourier-Jacobi coefficients (c.f. Chapter 3.4.1) to compare nonsingular Fourier coefficients of the Eisenstein series and the regularized theta integral inductively.

With this case of Siegel-Weil formula proved we are able to extend the Rallis inner product formula to the following case. Now let V be a quadratic space of dimension $m = 2n + 1$. Let π be a genuine irreducible cuspidal representation of $\widetilde{G(\mathbb{A})}$ and let $\Theta(h, f, \phi) = \int_{G(k)\setminus\widetilde{G(\mathbb{A})}} f(g)\theta(g, h, \phi)dg$. Then we have the regularized Rallis inner product formula:

Theorem 1.0.8. [Thm. 3.7.3] *Suppose* $m = 2n+1$. *Then*

$$\langle \Theta(f_1, \Phi_1), \Theta(f_2, \Phi_2) \rangle_{\mathrm{REG}} = \frac{L^S(\frac{1}{2}, \pi \otimes \chi)}{\widetilde{d}^S_{G_2}(0)} \cdot \langle \pi(\Xi_S) f_1, f_2 \rangle$$

where

$$\Xi_S(g) = \langle \omega_S(g) \Phi_{1,S}, \Phi_{2,S} \rangle.$$

Here $\widetilde{d}^S_{G_2}(s)$ is a product of some local zeta functions away from a finite set of places S. The local zeta integrals were computed by Li[19] in the unramified case. For m even this is the analogue of Kudla and Rallis[17, Thm. 8.7]. Combining our results with those of Ichino's[8], we get information on poles of L-function and the nonvanishing of theta lifts:

Theorem 1.0.9. [Thm. 3.7.2]

1. *The poles of* $L^S(s, \pi \otimes \chi)$ *in the half plane* $\operatorname{Re} s > 1/2$ *are simple and are contained in the set*
$$\left\{ 1, \frac{3}{2}, \frac{5}{2}, \ldots, \left[\frac{n+1}{2} \right] + \frac{1}{2} \right\}.$$

2. *Set* $m_0 = 4n + 2 - m$. *If* $4n + 2 > m > 2n + 1$ *then suppose* $L^S(s, \pi \otimes \chi)$ *has a pole at* $s = n + 1 - (m_0/2)$. *If* $m = 2n + 1$ *then suppose* $L^S(s, \pi \otimes \chi)$ *does not vanish at* $s = n + 1 - (m_0/2) = 1/2$. *Then there exists a quadratic space* U_0 *over* k *with dimension* m_0 *and* $\chi_{U_0} = \chi$ *such that* $\Theta_{U_0}(\pi) \neq 0$ *where* $\Theta_{U_0}(\pi)$ *denotes the space of automorphic forms* $\Theta(f, \Phi)$ *on* $\mathrm{O}(U_0)(\mathbb{A})$ *for* $f \in \pi$ *and* $\Phi \in \mathcal{S}(U_0(\mathbb{A})^n)$.

Chapter 2

F-virtual Abelian Varieties of GL_2-type

Let F be a number field. This chapter studies F-virtual Abelian varieties of GL_2-type. These Abelian varieties themselves are not necessarily defined over F but their isogeny classes are defined over F. They are generalization of Abelian varieties of GL_2-type defined over F which are in turn generalization of elliptic curves. The Galois representations of $\mathrm{Gal}(\overline{F}/F)$ associated to F-virtual Abelian varieties of GL_2-type are projective representations of dimension 2. Thus it is expected many techniques for GL_2-type can also be applied, such as modularity results and Gross-Zagier formula. Furthermore the study of virtual Abelian varieties of GL_2-type can possibly furnish evidence for the BSD conjecture.

The simplest case of F-virtual Abelian varieties of GL_2-type consists of $\mathbb{Q}$-elliptic curves. They were first studied by Gross[4] in the CM case and by Ribet[25][26] in the non-CM case. Also Ribet generalized the notion of $\mathbb{Q}$-elliptic curves to $\mathbb{Q}$-virtual Abelian varieties of GL_2-type. Elkies studied the quotients of modular curves $X^*(N)$ that parametrize $\mathbb{Q}$-elliptic curves and computed some explicit equations of these quotients[1]. Then González and Lario[3] described those $X^*(N)$ with genus zero or one. Based on the parametrization, Quer[21] computed explicit equations of some $\mathbb{Q}$-curves.

In this chapter first we study the ℓ-adic representations associated to F-virtual Abelian varieties of GL_2-type. Then we determine the quotients of Shimura varieties that parametrize

F-virtual Abelian varieties and classify them birationally in the case where the quotients are surfaces. It turns out that almost all of them are of general type. We also give examples of surfaces that are rational.

2.1 Abelian Varieties of GL_2-type

We start with the definition and some properties of the main object of this chapter.

Definition 2.1.1. Let A be an Abelian variety over some number field F and E a number field. Let $\theta : E \hookrightarrow \mathrm{End}^0 A = \mathrm{End}\, A \otimes_{\mathbb{Z}} \mathbb{Q}$ be an algebra embedding. Then the pair (A, θ) is said to be of $\mathrm{GL}_2(E)$-type if $[E : \mathbb{Q}] = \dim A$.

Remark 2.1.2. We will drop θ if there is no confusion. It is well-known that the Tate module $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a free $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 2. The action of $\mathrm{Gal}(\overline{F}/F)$ on $V_\ell(A)$ defines a representation with values in $\mathrm{GL}_2(E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$ and thus the nomenclature GL_2-type.

We can also define $\mathrm{GL}_n(E)$-type if we have $\theta : E \hookrightarrow \mathrm{End}^0(A)$ with $[E : \mathbb{Q}] = 2 \dim A/n$. Of course, we require $n | 2 \dim A$.

In the following when we say a field acting on an Abelian variety we mean the action up to isogeny.

Definition 2.1.3. An Abelian variety A over some number field F of dimension g is said to have sufficiently many complex multiplication (CM) if $\mathrm{End}^0(A_{\overline{F}})$, the endomorphism algebra of $A_{\overline{F}}$ contains a commutative $\mathbb{Q}$-algebra of degree $2g$. Also A over F of dimension g is said to have sufficiently many complex multiplication (CM) over F if $\mathrm{End}^0(A)$ contains a commutative $\mathbb{Q}$-algebra of degree $2g$.

Proposition 2.1.4. *Let A be an Abelian variety defined over a totally real field F. Then A does not have sufficiently many complex multiplication (CM) over F.*

Proof. Suppose the contrary. Fix an embedding of F into $\overline{\mathbb{Q}}$. Suppose $E \hookrightarrow \mathrm{End}^0 A$ with E a CM-algebra and $[E : \mathbb{Q}] = 2 \dim A$. Consider the CM-type coming from the action of E on $\mathrm{Lie}\, A_{\overline{\mathbb{Q}}}$. Then the reflex field E' of E is $\mathbb{Q}(\mathrm{tr}\, \Phi)$ and is a CM-field. Since E actually acts on $\mathrm{Lie}\, A/F$, we find $\mathrm{tr}\, \Phi \subset F$. Thus $E' = \mathbb{Q}(\mathrm{tr}\, \Phi)$ is contained in the totally real field F and we get a contradiction. $\square$

Remark 2.1.5. The proof, in particular, shows that if a field is embedded in $\mathrm{End}^0(A)$ for an Abelian variety A over a totally real field F, then it is at most of degree $\dim A$.

2.1.1 Decompostion over $\overline{F}$

Suppose that A is an Abelian variety of dimension g over some number field F of $\mathrm{GL}_2(E)$-type. Fix an embedding $F \hookrightarrow \overline{\mathbb{Q}}$. We consider the decomposition of $A_{\overline{\mathbb{Q}}}$. Since the embedding θ of a number field E into endomorphism algebra is given as part of the data, when we consider isogenies between Abelian varieties of GL_2-type we require compatibility with the given embeddings. More precisely we define:

Definition 2.1.6. Let θ_i be an embedding of a $\mathbb{Q}$-algebra D into the endomorphism algebra of an Abelian variety A_i/F, for $i = 1$ or 2. Then an isogeny μ between A_1 and A_2 is said to be D-equivariant or D-linear if $\mu \circ \theta_2(a) = \theta_1(a) \circ \mu$, for all $a \in D$.

To describe the factors of $A_{\overline{\mathbb{Q}}}$ we define F-virtuality:

Definition 2.1.7. Let A be an Abelian variety defined over $\overline{\mathbb{Q}}$ and suppose $\theta : D \xrightarrow{\sim} \mathrm{End}^0(A)$. Let F be a number field that embeds into $\overline{\mathbb{Q}}$. Then A is said to be F-virtual if $({}^\sigma\! A, {}^\sigma\theta)$ is D-equivariantly isogenous to (A, θ) for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Remark 2.1.8. Note that here we assumed that θ is an isomorphism.

This definition makes it precise what it means for an Abelian variety to have isogenous class defined over F.

If $\dim A = 1$ and $F = \mathbb{Q}$, A is what is known as a $\mathbb{Q}$-curve, c.f. [4] and [26].

We analyze the endomorphism algebra of $A_{\overline{\mathbb{Q}}}$. Notice the number field E also embeds into $\mathrm{End}^0(A_{\overline{\mathbb{Q}}})$ via $\mathrm{End}^0 A \hookrightarrow \mathrm{End}^0(A_{\overline{\mathbb{Q}}})$. Let C be the commutant of E in $\mathrm{End}^0(A_{\overline{\mathbb{Q}}})$. There are two possibilities: either $E = C$ so that we are in the non-CM case or $E \subsetneq C$ so that we are in the CM case.

2.1.1.1 Case CM

We have $E \subsetneq C$, so $[C : \mathbb{Q}] = 2 \dim A = 2g$. Hence A is CM. A priori, $A_{\overline{\mathbb{Q}}} \sim \prod B_i^{n_i}$ with B_i's pairwise nonisogenous simple Abelian varieties. Thus $\mathrm{End}^0 A_{\overline{\mathbb{Q}}} \cong \prod \mathrm{M}_{n_i}(L_i)$ where

$L_i = \mathrm{End}^0 B_i$. Then E embeds into $\prod P_i$ with P_i a maximal subfield of $\mathrm{M}_{n_i}(L_i)$. Consider projection to the i-th factor. We have $E \to P_i$. Note that the identity element of E is mapped to the identity element in $\prod P_i$. Hence $E \to P_i$ is not the zero map and so it is injective. Since $[E : \mathbb{Q}] = g$, P_i has degree greater than or equal to g. On the other hand $\sum [P_i : \mathbb{Q}] = 2g$. We are forced to have

$$E \hookrightarrow P_1$$

with $[P_1 : Q] = 2g$ or

$$E \hookrightarrow P_1 \times P_2$$

with $P_1 \cong P_2 \cong E$. Correspondingly we have either $A_{\overline{\mathbb{Q}}} \sim B_1^{n_1}$ or $A_{\overline{\mathbb{Q}}} \sim B_1^{n_1} \times B_2^{n_2}$. In the second case E must be a CM field.

Now we discuss what conditions on A make sure that $A_{\overline{\mathbb{Q}}}$ is isogenous to a power of a simple Abelian variety. Fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Proposition 2.1.9. *Suppose either that the g embeddings of E into $\mathbb{C}$ coming from the action of E on $\mathrm{Lie}(A_{\mathbb{C}})$ exhaust all possible embeddings of E into $\mathbb{C}$ or that A is defined over a totally real field F. Then $A_{\overline{\mathbb{Q}}} \sim B^n$ for some simple Abelian variety B.*

Proof. Suppose the contrary that A is not isogenous to a power of some simple Abelian variety, i.e., $A_{\overline{\mathbb{Q}}} \sim B_1^{n_1} \times B_2^{n_2}$. Let S be the centre of $\mathrm{End}^0 A_{\overline{\mathbb{Q}}}$. Then $S = L_1 \times L_2$ for two number fields L_1 and L_2. Denote by e_i the identity element of L_i. Let A_i be the image of A under e_i or rather he_i for some integer h such that $he_i \in \mathrm{End}\, A_{\overline{\mathbb{Q}}}$. Then $A \sim A_1 \times A_2$ and E acts on A_i. This gives rise to CM-types (E, Φ_i).

Suppose that the first part of the assumption holds; then $\Phi_1 \sqcup \Phi_2$ gives all possible embeddings of E into $\mathbb{C}$. Thus $\Phi_1 = \iota\Phi_2$ where ι denote the complex conjugation of $\mathbb{C}$. Then $\Phi_1 = \Phi_2\iota_E$ for the complex conjugation ι_E of E. If we change the embedding of E into $\mathrm{End}^0 A_2$ by $\iota_E \in \mathrm{Gal}(E/\mathbb{Q})$ then the action of E on A_2 has also type Φ_1. This means that $A_2 \sim A_1$ and we get a contradiction. Thus A decomposes over $\overline{\mathbb{Q}}$ into a power of a simple Abelian variety.

Now suppose that the second part of the assumption holds; then F is a totally real field. Thus the complex conjugate ${}^{\iota}A$ is isomorphic to A. If ${}^{\iota}A_1 \cong A_1$ then A_1 can be defined over $\mathbb{R}$. This is impossible by Prop. 2.1.4 since it is CM. Thus ${}^{\iota}A_1 \cong A_2$. Then for some automorphism σ of E, we have $\iota\Phi_1\sigma = \Phi_2$. Thus Φ_1 and Φ_2 are different by an automorphism $\iota_E\sigma$ of E. Again $A_1 \sim A_2$, a contradiction.

Thus A decomposes over $\overline{\mathbb{Q}}$ into a power of a simple Abelian variety. □

2.1.1.2 Case Non-CM

In this case $E = C$. In particular the centre L of $\mathrm{End}^0(A_{\overline{\mathbb{Q}}})$ is contained in E and hence is a field. Thus $\mathrm{End}^0(A_{\overline{\mathbb{Q}}}) \xleftarrow[\sim]{\theta} \mathrm{M}_n(D)$ for n some positive integer and D some division algebra. Correspondingly $A_{\overline{\mathbb{Q}}} \sim B^n$ with B some simple Abelian variety over $\overline{\mathbb{Q}}$ and $\mathrm{End}^0 B \cong D$. Let $e = [L : \mathbb{Q}]$ and $d = \sqrt{[D : L]}$. Then E is a maximal subfield of $\mathrm{M}_n(D)$. We know that $ed^2 | \frac{2g}{n}$ and that $g = [E : \mathbb{Q}] = nde$. This forces $d|2$. If $d = 1$, $\mathrm{End}^0 B \cong L$. If $d = 2$, $\mathrm{End}^0 B \cong D$ with D a quaternion algebra over L.

Since A is defined over F, we get for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$,

$$ {}^{\sigma}B^n \sim {}^{\sigma}A_{\overline{\mathbb{Q}}} \cong A_{\overline{\mathbb{Q}}} \sim B^n. $$

By the uniqueness of decomposition we find ${}^{\sigma}B \sim B$, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Furthermore the canonical isomorphism ${}^{\sigma}A \cong A$ is L-equivariant, since the endomorphisms in $\theta(L)$ are rational over F. Fix isogeny $A_{\overline{\mathbb{Q}}} \to B^n$. The actions of L on B^n and ${}^{\sigma}B^n$ are the pullbacks of the actions of L on A and ${}^{\sigma}A$, so ${}^{\sigma}B^n \sim B^n$ is L-equivariant. As the L-actions are just diagonal actions, we have ${}^{\sigma}B^n \sim B$ L-equivariantly.

Let P be a maximal subfield of D. Then B is an Abelian variety of $\mathrm{GL}_2(L)$-type for $d = 1$ or $\mathrm{GL}_2(P)$-type for $d = 2$. Even though we are using different letters L, E and D, they may refer to the same object. In this subsection all Abelian varieties are assumed to be without CM and this assumption is implicit in the lemmas and the propositions.

First we record a result that follows from the discussion.

Proposition 2.1.10. *The endomorphism algebra of an Abelian variety of* GL_2*-type has one of the following types: a matrix algebra over some number field or a matrix algebra over some quaternion algebra.*

The following lemma shows that L-equivariance is as strong as D-equivariance.

Lemma 2.1.11. *Let $B/\overline{\mathbb{Q}}$ be an Abelian variety of* GL_2*-type with $\theta : D \xrightarrow{\sim} \mathrm{End}^0(B)$. Let L be the centre of D. Suppose ${}^{\sigma}B$ is L-equivariantly isogenous to B for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then after modifying θ, we can make B into an F-virtual Abelian variety, i.e., there exist D-equivariant isogenies ${}^{\sigma}B \to B$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.*

Proof. Choose L-equivariant isogenies $\mu_\sigma : {}^{\sigma}B \to B$. This means that $\mu_\sigma \circ {}^{\sigma}\phi = \phi \circ \mu_\sigma$ for $\phi \in L$. Thus we have L-algebra isomorphisms:

$$\begin{aligned} D &\to D \\ \phi &\mapsto \mu_\sigma \circ {}^{\sigma}\phi \circ \mu_\sigma^{-1} \end{aligned}$$

By Skolem-Noether theorem there exists an element $\psi \in D^\times$ s.t. $\mu_\sigma \circ {}^{\sigma}\phi \circ \mu_\sigma^{-1} = \psi \circ \phi \circ \psi^{-1}$. Let $\mu'_\sigma = \psi^{-1} \circ \mu_\sigma$. Then μ'_σ gives a D-equivariant isogeny between ${}^{\sigma}B$ and B. □

Proposition 2.1.12. *An Abelian variety A/F of* GL_2*-type is $\overline{\mathbb{Q}}$-isogenous to a power of a $\overline{\mathbb{Q}}$-simple Abelian variety B. Moreover B is an F-virtual Abelian variety of* GL_2*-type.*

Proof. This follows from the discussion above and the lemma. □

2.1.2 F-virtual Abelian varieties and simple Abelian varieties over F

We consider the converse problem. Given a simple F-virtual Abelian variety B of GL_2-type we want to give an explicit construction of a simple Abelian variety A over F of GL_2-type such that B is a factor of A. We separate into two cases:

2.1.2.1 Case Non-CM

Theorem 2.1.13. *Let $A/\overline{\mathbb{Q}}$ be a simple abelian variety of* $\mathrm{GL}_2(E)$*-type. Suppose that ${}^{\sigma}A$ and A are E-equivariantly isogenous for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then there exists an F-simple Abelian variety B/F of* GL_2*-type s.t. A is a factor of $B_{\overline{\mathbb{Q}}}$.*

Proof. By Lemma 2.1.11, the Abelian variety A is actually F-virtual. Find a model of A over a number field K_1 such that K_1/F is Galois and all endomorphisms of A are defined over K_1. We identify ${}^\sigma A$ and A via the canonical isomorphism for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K_1)$. Let D denote the full endomorphism algebra of A. Choose D-equivariant $\overline{\mathbb{Q}}$-isogenies $\mu_\sigma : {}^\sigma A \to A$ for representatives σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)/\mathrm{Gal}(\overline{\mathbb{Q}}/K_1) \cong \mathrm{Gal}(K_1/F)$. The rest of the μ_σ's for all σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ are determined from these representatives and the canonical isomorphisms. Now let K be a field extension of K_1 such that K is Galois over F and that all μ_σ's for σ in $\mathrm{Gal}(K_1/F)$ are defined over K. Base change the model over K_1 to K and still call this model A. Instead of considering all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we only need to consider σ-twists of A for $\sigma \in \mathrm{Gal}(K/F)$.

Define $c(\sigma, \tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$. In the quaternion algebra case, note that

$$c(\sigma,\tau).\phi = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}.\phi = \mu_\sigma {}^\sigma\mu_\tau {}^{\sigma\tau}\phi \mu_{\sigma\tau}^{-1} = \mu_\sigma {}^\sigma\phi \mu_\tau \mu_{\sigma\tau}^{-1} = \phi \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$$

for $\phi \in D$. Thus c has values in $L^\times$.

It is easy to check that c is a 2-cocycle on $\mathrm{Gal}(K/F)$ with values in $L^\times$. By inflation we consider the class of c in $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), L^\times)$. It can be shown that $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{L}^\times)$ with the Galois group acting trivially on $\overline{L}^\times$ is trivial by a theorem of Tate as quoted as Theorem 6.3 in [26]. Hence there exists a locally constant function: $\alpha : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{L}^\times$ s.t. $c(\sigma,\tau) = \frac{\alpha(\sigma)\alpha(\tau)}{\alpha(\sigma\tau)}$. We will use α in the construction of an Abelian variety of GL_2-type.

Let $B = \mathrm{Res}_{K/F} A$ be the restriction of scalars of A from K to F. Then B is defined over F. We let $D \circ \mu_\sigma$ denote the set of isogenies $\{f \circ \mu_\sigma \mid \forall f \in D\}$. Then we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{End}^0 B \arrow[r, dash, "\sim"] \arrow[d, hook] & \prod_\sigma \mathrm{Hom}^0({}^\sigma A, A) \arrow[r, dash, "\sim"] \arrow[d, hook] & \prod_\sigma D \circ \mu_\sigma \\
\mathrm{End}^0 B_K \arrow[r, dash, "\sim"] & \prod_{\rho,\tau} \mathrm{Hom}^0({}^\tau A, {}^\rho A) &
\end{tikzcd}$$

where the right verticle arrow maps $f : {}^\sigma A \to A$ to ${}^\tau f : {}^{\tau\sigma}A \to {}^\tau A$, for all τ and the products are running over $\mathrm{Gal}(K/F)$. The multiplication of the ring $\prod_\sigma D \circ \mu_\sigma$ can be described as follows. Let $\phi,\ \psi \in D$. Then $(\phi \circ \mu_\sigma).(\psi \circ \mu_\tau) = \phi \circ \mu_\sigma \circ {}^\sigma\psi \circ {}^\sigma\mu_\tau = \phi \circ \psi \circ \mu_\sigma \circ {}^\sigma\mu_\tau = \phi \circ \psi \circ c(\sigma,\tau)\mu_{\sigma\tau}$ since μ_σ is D-equivariant. Thus $\prod_\sigma D \circ \mu_\sigma$ can be viewed as $D[\mathrm{Gal}(K/F)]$ twisted by the cocycle c.

Let the number field E embed into $\operatorname{End}^0 B$ via θ into the factor $\operatorname{Hom}^0(A, A)$ of $\operatorname{End}^0 B$. Let $\operatorname{End}^0_E B$ denote the commutant of E in $\operatorname{End}^0 B$. Since the μ_σ's are E-equivariant and the commutant of E in D is E, $\operatorname{End}^0_E B \cong \prod \theta(E) \circ \mu_\sigma$.

Let L_α(E_α resp.) denote the field L(E resp.) adjoined with values of α. Let $D_\alpha = D \otimes_L L_\alpha$. Take

$$\omega : \prod_\sigma D \circ \mu_\sigma \to D_\alpha$$

$$\phi \circ \mu_\sigma \mapsto \phi \otimes \alpha(\sigma).$$

Then ω is a D-algebra homomorphism. If we restrict ω to $\operatorname{End}^0_E B$ we find $\omega|_{\operatorname{End}^0_E B}$ has image E_α. Since $\operatorname{End}^0 B$ is a semisimple $\mathbb{Q}$-algebra, we have $\operatorname{End}^0 B \cong D_\alpha \oplus \ker \omega$. Being the commutant of E in a semisimple algebra $\operatorname{End}^0 B$, $\operatorname{End}^0_E B$ is semisimple and therefore we have $\operatorname{End}^0_E B \cong E_\alpha \oplus \ker \omega|_{\operatorname{End}^0_E B}$. Let $\pi \in \operatorname{End}^0 B$ be a projector to D_α. Let B_α be the image of π. Then B_α has action up to isogeny exactly given by D_α.

Before finishing the proof, we need the following:

Lemma 2.1.14. *Let R denote $\operatorname{End}^0_E B$. Then the Tate module $V_\ell(B) = V_\ell(B_K)$ is a free $R \otimes \mathbb{Q}_\ell$-module of rank 2.*

Proof. Note $B_K \cong \prod_\sigma {}^\sigma A$. Passing to Tate modules we have $V_\ell(B_K) \cong \oplus_\sigma V_\ell({}^\sigma A)$. We know that $V_\ell(A)$ is a free $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 2. Choose an $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-basis $\{e_1, e_2\}$ of $V_\ell(A)$. Then ${}^{\sigma^{-1}}\mu_\sigma\{e_1, e_2\}$ gives a basis for the free $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module $V_\ell({}^\sigma A)$ of rank 2, since μ_σ's are E-equivariant. Hence $V_\ell(B_K)$ is freely generated over R by $\{e_1, e_2\}$. $\square$

Thus $V_\ell(B_\alpha)$ is a free $E_\alpha \otimes \mathbb{Q}_\ell$-module of rank 2. Therefore $[E_\alpha : \mathbb{Q}] = \dim B_\alpha$. Now we consider 3 cases.

Case $D_\alpha = E_\alpha$. Then B_α is F-simple. We take $B = B_\alpha$ and A is a quotient of B over $\overline{\mathbb{Q}}$.

Case $D_\alpha \neq E_\alpha$ and D_α not split. Then D_α is a quaternion algebra over L_α. Then B_α is F-simple. We take $B = B_\alpha$ and A is a quotient of B over $\overline{\mathbb{Q}}$.

Case $D_\alpha \neq E_\alpha$ and D_α split. Then $D_\alpha \cong M_2(L_\alpha)$. B_α is F-isogenous to B^2 for some simple Abelian variety B and $\mathrm{End}^0 B \cong L_\alpha$. We check the degree of the endomorphism algebra: $[L_\alpha : \mathbb{Q}] = \frac{1}{2}\dim B_\alpha = \dim B'$. Thus A is a $\overline{\mathbb{Q}}$-quotient of a simple Abelian variety B of $\mathrm{GL}_2(L_\alpha)$-type. $\square$

We record a corollary to the proof of Thm. 2.1.13. Suppose μ_σ are given isogenies from ${}^\sigma A$ to A. Let $c(\sigma,\tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$ be the associated 2-cocycle. Fix a choice of α that trivializes c. Let E_α denote the field constructed from E by adjoining values of α.

Corollary 2.1.15. *Let A be a simple F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Then there exists an Abelian variety B_α/F that has A as an $\overline{F}$-factor, that has action by E_α and that is either simple or isogenous to the square of an F-simple Abelian variety of GL_2-type.*

2.1.2.2 Case CM

Next we consider a CM simple Abelian variety $A/\overline{\mathbb{Q}}$. Suppose (A,θ) has type (E,Φ). Then it is known from [27] that the isogeny class of (A,θ) is defined over $E^\sharp$, the reflex field of E, i.e., there exist E-equivariant isogenies $\mu_\sigma : {}^\sigma A \to A$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E^\sharp)$.

Theorem 2.1.16. *Let A be as above. Then there exists a simple Abelian variety A' over a totally real field F of GL_2-type with A a $\overline{\mathbb{Q}}$-quotient.*

Proof. Let $F = E^{\sharp +}$ be the maximal totally real subfield of $E^\sharp$. We will construct a simple Abelian variety over F of GL_2-type with A a $\overline{\mathbb{Q}}$-quotient. First we construct a simple CM Abelian variety over F whose endomorphisms are all defined over F. We proceed as in the proof of Theorem 2.1.13. Find a model of A over K, a number field Galois over $E^\sharp$ such that all endomorphisms of A is defined over K and that all the μ_σ's are defined over K. Let $B = \mathrm{Res}_{K/E^\sharp} A$. Also define $c(\sigma,\tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$. Then c has values in $E^\times$ and we can trivialize c by $\alpha : \mathrm{Gal}(\overline{\mathbb{Q}}/E^\sharp) \to \overline{E}^\times$. Again

$$\mathrm{End}^0 B \cong \prod \mathrm{Hom}^0({}^\sigma A, A) \cong \prod E \circ \mu_\sigma.$$

Similarly we find that $\mathrm{End}^0 B$ can be split into $\ker\omega \oplus E_\alpha$ where E_α is the field by adjoining E with values of α and $\omega : \mathrm{End}^0 B \to E_\alpha, \mu_\sigma \mapsto \alpha(\sigma)$. Hence corresponding to the decomposition of the endomorphism algebra we get an Abelian subvariety B_α of B . Consider $V_\ell(B)$

the Tate module of B. Similarly we find that $V_\ell(B)$ is a free $\prod(E\circ\mu_\sigma)\otimes\mathbb{Q}_\ell$-module of rank 1. Hence $V_\ell(B_\alpha)$ is a free $E_\alpha\otimes\mathbb{Q}_\ell$-module of rank 1. This shows that B_α has CM by E_α. Then we set $A'=\operatorname{Res}_{E^\sharp/F}B_\alpha$. Note that A' is F-simple. Otherwise $A'\sim A_1\times A_2$ for some Abelian varieties A_1 and A_2 defined over F, so $A'_{E^\sharp}\sim A_{1,E^\sharp}\times A_{2,E^\sharp}$. Yet $A'_{E^\sharp}\cong B_\alpha\times {}^\iota B_\alpha$ where ι is the nontrivial element in $\operatorname{Gal}(E^\sharp/F)$. Since B_α and ${}^\iota B_\alpha$ are $E^\sharp$-simple, we find $A_{1,E^\sharp}\sim B_\alpha$ or $A_{2,E^\sharp}\sim B_\alpha$. However Proposition 2.1.4 says B_α cannot be descended to a totally real field. We get a contradiction. Thus A' is F-simple.

Now $\operatorname{End}^0 A'\cong \operatorname{Hom}^0({}^\iota B_\alpha,B_\alpha)\times\operatorname{End}^0 B_\alpha$ as $\mathbb{Q}$-vector spaces. As $E_\alpha\hookrightarrow\operatorname{End}^0 A'$ we find A' is a simple Abelian variety over F of $\mathrm{GL}_2(E_\alpha)$-type with $\overline{\mathbb{Q}}$-quotient A. □

2.2 ℓ-adic Representations

In this section F denotes a totally real field. Let A be an Abelian variety over F of $\mathrm{GL}_2(E)$-type. Then the Tate module $V_\ell A$ is free of rank 2 over $E\otimes\mathbb{Q}_\ell$. Let G_F denote the Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}/F)$. The action of G_F on $V_\ell A$ induces a homomorphism

$$\rho_\ell: G_F\to\mathrm{GL}_2(E\otimes\mathbb{Q}_\ell).$$

For each prime λ of E lying above ℓ if we set $V_\lambda A=V_\ell A\otimes_{E\otimes\mathbb{Q}_\ell}E_\lambda$ then the action of G_F defines a λ-adic reprensentation on $V_\lambda A$:

$$\rho_\lambda: G_F\to\mathrm{GL}_2(E_\lambda)$$

We can also associate Galois representations to an F-virtual Abelian variety. Let B be an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type defined over $\overline{\mathbb{Q}}$. Define $\rho'_\ell(\sigma).P:=\mu_\sigma({}^\sigma P)$ for P in $V_\ell B$ and σ in G_F. This is not an action, since

$$\rho'_\ell(\sigma)\rho'_\ell(\tau)P=\mu_\sigma{}^\sigma\mu_\tau{}^{\sigma\tau}P.$$

The obstruction is given by $c(\sigma,\tau):=\mu_\sigma{}^\sigma\mu_\tau\mu_{\sigma\tau}^{-1}$. This is the same c we constructed in the proof of Thm. 2.1.13. We know that c is a 2-cocycle on G_F with values in $E^\times$, in fact, in

$L^\times$, if we let L denote the centre of the endomorphism algebra of B. Consider the class of c in $H^2(G_F, \overline{E}^\times)$. As is shown by Tate, $H^2(G_F, \overline{E}^\times) = 0$. Thus c can be trivialized:

$$c(\sigma,\tau) = \frac{\alpha(\sigma)\alpha(\tau)}{\alpha(\sigma\tau)}$$

by some locally constant map $\alpha : G_F \to \overline{E}^\times$. Then set

$$\rho_\ell(\sigma)P := \alpha^{-1}(\sigma)\mu_\sigma{}^\sigma P.$$

This gives an action of G_F on $V_\ell B$. We get a homomorphism

$$\rho_\ell : G_F \to \overline{E}^\times \operatorname{GL}_2(E \otimes \mathbb{Q}_\ell).$$

More precisely, ρ_ℓ actually factors through $E_\alpha^\times \operatorname{GL}_2(E \otimes \mathbb{Q}_\ell)$ where E_α denote the subfield of $\overline{\mathbb{Q}}$ generated over E by the values of α. If we fix a choice of α then to simplify notation we write E' for E_α. Let λ' be a prime of E' lying above λ a prime of E which in turn lies above ℓ. Then the representation ρ_ℓ gives rise to:

$$\rho_{\lambda'} : G_F \to {E'}^\times_{\lambda'} \operatorname{GL}_2(E_\lambda).$$

We can view $\rho_{\lambda'}$ as a representation of G_F on $V_\lambda \otimes_{E_\lambda} E'_{\lambda'}$.

Proposition 2.2.1. *Let A be a simple F-virtual Abelian variety of* $\operatorname{GL}_2(E)$*-type. Fix a choice of α and let A'/F denote the Abelian variety B_α as in Cor. 2.1.15. Then the field E' generated over E by the values of α acts up to isogeny on A'. Let λ' be a prime of E'. Associate the representations $\rho_{A,\lambda'}$ and $\rho_{A',\lambda'}$ to A and A' respectively. Then $\rho_{A,\lambda'} \cong \rho_{A',\lambda'}$ as $E'_{\lambda'}[G_F]$-module.*

Proof. Note that μ_σ's correspond to $\alpha(\sigma)$'s in the endomorphism algebra of A'. The proposition follows from the definiton of the λ'-adic representations. □

Because of this proposition we will focus on the Galois representations associated to Abelian varieties defined over F. We record some properties of the λ-adic representations.

Definition 2.2.2. Let E be a totally real field or CM field. A polarized Abelian variety of $\mathrm{GL}_2(E)$-type is an Abelian variety of $\mathrm{GL}_2(E)$-type with a polarization that is compatible with the canonical involution on E.

Remark 2.2.3. The canonical involution is just identity if E is totally real.

Proposition 2.2.4. *If A is a polarized simple F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type then the associated Abelian variety A'/F can also be endowed with a polarization compatible with E'.*

Proof. Let L' be the centre of the endomorphism algebra of A'. Then by construction E' is actually the composite of L' and E and hence is totally real or CM as L' is totally real by Prop. 2.2.12 and Prop. 2.2.14 (whose proofs depends solely on the analysis on polarized Abelian varieties over F and do not depend on this proposition) and E either CM or totally real. □

Let A be a polarized Abelian variety over F of $\mathrm{GL}_2(E)$-type. Let V_ℓ denote $V_\ell A$. Then $V_\ell = \oplus_{\lambda|\ell} V_\lambda$ corresponding to the decomposition of $E \otimes \mathbb{Q}_\ell = \oplus E_\lambda$ where λ's are primes of E lying above ℓ. We get the λ-adic representations ρ_λ of G_F on V_λ. The set of ρ_λ's for all λ forms a family of strictly compatible system of E-rational representations[25].

Let δ_λ denote $\det \rho_\lambda$ and χ_ℓ the ℓ-adic cyclotomic character.

Lemma 2.2.5. *There exists a character of finite order $\varepsilon : G_F \to E^*$ such that $\delta_\lambda = \varepsilon\chi_\ell$. Furthermore ε is unramified at primes which are primes of good reduction for A.*

Remark 2.2.6. ε is trivial if E is totally real, as can be seen from Prop. 2.2.9.

Proof. Since the δ_λ's arise from an Abelian variety, they are of the Hodge-Tate type. They are associated with an E-valued Grossencharacter of type A_0 of F. Thus they have to be of the form $\delta_\lambda = \varepsilon\chi_\ell^n$ for some E-valued character of finite order.

By the criterion of Néron-Ogg-Shafarevich ρ_λ is unramified at primes of F which are primes of good reduction for A and which do not divide ℓ. Then δ_λ is also unramified at those primes. Since χ_ℓ is unramified at primes not dividing ℓ, ε is unramified at primes of F which are primes of good reduction for A and which do not divide ℓ. Let ℓ vary and we find that ε is unramified at primes of F which are primes of good reduction for A.

Now consider the representation of G_F on $\det_{\mathbb{Q}_\ell} V_\ell$. It is known to be given by $\chi_\ell^{\dim A}$ which is equal to $\chi_\ell^{[E:Q]}$ since A is of $\mathrm{GL}_2(E)$-type. On the other hand it is also equal to $\prod_{\lambda|\ell} \mathbf{N}_{E_\lambda/\mathbb{Q}_\ell} \delta_\lambda = \mathbf{N}_{E/\mathbb{Q}} \varepsilon . \chi_\ell^{n[E:\mathbb{Q}]}$. Since χ_ℓ has infinite order and ε has finite order we are forced to have $n = 1$ and $\mathbf{N}_{E/\mathbb{Q}} \varepsilon = 1$. □

Lemma 2.2.7. *The character δ_λ is odd, i.e., δ_λ sends all complex conjugations to -1.*

Proof. For each embedding of fields $\overline{\mathbb{Q}} \to \mathbb{C}$ we have a comparison isomorphism $V_\lambda \cong H_1(A(\mathbb{C}), \mathbb{Q}) \otimes_E E_\lambda$. Via this isomorphism complex conjugation acts as $F_\infty \otimes 1$ where F_∞ comes from the action of complex conjugation on $A(\mathbb{C})$ by transport of structure. We need to show that $\det F_\infty$ is -1 where det is taken with respect to the E-linear action of F_∞ on $H_1(A(\mathbb{C}), \mathbb{Q})$.

Note that $H_1(A(\mathbb{C}), \mathbb{Q})$ is of dimension 2 over E. Since F_∞ is an involution on $H_1(A(\mathbb{C}), \mathbb{Q})$ its determinant is 1 if and only if F_∞ acts as a scalar. Since F_∞ permutes $H_{0,1}$ and $H_{1,0}$ in the Hodge decomposition of $H_1(A(\mathbb{C}), \mathbb{Q})$ it obviously does not act as a scalar. Thus $\det F_\infty$ is -1. □

Proposition 2.2.8. *For each λ, ρ_λ is an absolutely irreducible 2-dimensional representation of G_F over E_λ and $\mathrm{End}_{E_\lambda[G_F]} V_\lambda = E_\lambda$.*

Proof. By Faltings's results V_ℓ is a semisimple G_F-module and $\mathrm{End}_{E\otimes\mathbb{Q}_\ell[G_F]} V_\ell = E \otimes \mathbb{Q}$. Thus corresponding to the decomposition of $V_\ell = \oplus_{\lambda|\ell} V_\lambda$ we have $\mathrm{End}_{E_\lambda[G_F]} V_\lambda = E_\lambda$. This shows V_λ is simple over E_λ. Hence ρ_λ is absolutely irreducible. □

For prime v of F at which A has good reduction let $a_v = \mathrm{tr}_{E_\lambda}(\mathrm{Frob}_v |_{V_\lambda})$ if $v \nmid \ell$ for ℓ lying below λ. Let ι be the canonical involution on E if E is CM and identity if E is totally real.

Proposition 2.2.9. *For each place v of good reduction $a_v = \iota(a_v)\varepsilon(\mathrm{Frob}_v)$.*

Proof. For each embedding σ of E into $\overline{\mathbb{Q}}_\ell$ denote by $V_\sigma := V_\ell \otimes_{E\otimes\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ where the algebra homomorphism $E \otimes \mathbb{Q}_\ell \to \overline{\mathbb{Q}}_\ell$ is induced by σ.

Fix a polarization of A that is compatible with ι. We have a Weil pairing:

$$\langle\ ,\ \rangle : V_\ell \times V_\ell \to \mathbb{Q}_\ell(1)$$

such that $\langle ex, y\rangle = \langle x, \iota(e)y\rangle$ for e in E. This pairing is G_F-equivariant, i.e.,$\langle {}^\tau x, {}^\tau y\rangle = {}^\tau\langle x, y\rangle$ for all $\tau \in G_F$.

After extending scalar to $\overline{\mathbb{Q}}_\ell$ we get pairings between the spaces V_σ and $V_{\sigma\iota}$. Thus we have isomorphism of $\overline{\mathbb{Q}}_\ell[G_F]$-modules

$$V_{\sigma\iota} \cong \mathrm{Hom}(V_\sigma, \overline{\mathbb{Q}}_\ell(1)).$$

On $\det V_\sigma$, G_F acts by ${}^\sigma\varepsilon\chi_\ell$. As V_σ is 2-dimensional over $\overline{\mathbb{Q}}_\ell$

$$\mathrm{Hom}(V_\sigma, \overline{\mathbb{Q}}_\ell({}^\sigma\varepsilon\chi_\ell)) \cong V_\sigma$$

as G_F-modules. Thus we find that $V_{\sigma\iota}({}^\sigma\varepsilon) \cong V_\sigma$. As $\mathrm{tr}\,\mathrm{Frob}_v$ is $\sigma(a_v)$ on V_σ and is $\sigma\iota(a_v){}^\sigma\varepsilon(\mathrm{Frob}_v)$ on $V_{\sigma\iota}({}^\sigma\varepsilon)$ for v place of good reduction and v prime to ℓ we get

$$\sigma(a_v) = \sigma\iota(a_v){}^\sigma\varepsilon(\mathrm{Frob}_v).$$

It follows $a_v = \iota(a_v)\varepsilon(\mathrm{Frob}_v)$. □

Corollary 2.2.10. *ε is trivial when E is totally real.*

Proposition 2.2.11. *Let S be a finite set containing all the places of bad reduction. Then E is generated over $\mathbb{Q}$ by the a_v's with $v \notin S$.*

Proof. Again we consider the V_σ's corresponding to embeddings of E into $\overline{\mathbb{Q}}_\ell$ as in the proof of Prop. 2.2. As V_ℓ is semisimple, so is $V_\ell \otimes \overline{\mathbb{Q}}_\ell$. Since $V_\ell \otimes \overline{\mathbb{Q}}_\ell = \oplus_\sigma V_\sigma$ we get $\mathrm{End}_{E\otimes\overline{\mathbb{Q}}_\ell[G_F]}(\oplus_\sigma V_\sigma) = E \otimes \overline{\mathbb{Q}}_\ell = \prod_\sigma \overline{\mathbb{Q}}_\ell$. This shows that the V_σ's are simple and pairwise nonisomorphic. Thus their traces are pairwise distinct. Since the trace of Frob_v acting on V_σ is $\sigma(a_v)$ for v place of good reduction, by Cebotarev Density theorem the embeddings σ are pairwise distinct when restricted to the set of a_v's for $v \notin S$. Thus E is generated over $\mathbb{Q}$ by the set of a_v's for $v \notin S$. □

Proposition 2.2.12. *Let L be the subfield of E generated by $a_v^2/\varepsilon(\mathrm{Frob}_v)$ for $v \notin S$. Then L is a totally real field and E/L is Abelian.*

Proof. We check

$$\iota(\frac{a_v^2}{\varepsilon(\mathrm{Frob}_v)}) = \frac{\iota(a_v)^2}{\iota(\varepsilon(\mathrm{Frob}_v))} = a_v^2\varepsilon^{-2}(\mathrm{Frob}_v)\varepsilon(\mathrm{Frob}_v) = \frac{a_v^2}{\varepsilon(\mathrm{Frob}_v)}.$$

Thus L is totally real.

Since E is contained in the extension of L obtained by adjoining the square root of all of the $a_v^2/\varepsilon(\mathrm{Frob}_v)$'s and all roots of unity, E is an Abelian extension of L.

□

Now we consider the reductions of ρ_λ. Replace A by an isogenous Abelian variety so that $\mathcal{O}_E$ actually acts on A. Consider the action of G_F on the λ-torsion points $A[\lambda]$ of A and we get a 2-dimensional representation $\bar{\rho}_\lambda$ of G_F over $\mathbb{F}_\lambda$, the residue field at λ.

Lemma 2.2.13. *For almost all λ, the representation $\bar{\rho}_\lambda$ is absolutely irreducible.*

Proof. A result of Faltings implies that for almost all λ's $A[\lambda]$ is a semisimple $\mathbb{F}_\lambda[G_F]$-module whose commutant is $\mathbb{F}_\lambda$. The lemma follows immediately. □

Proposition 2.2.14. *Let A/F be a simple Abelian variety of $\mathrm{GL}_2(E)$-type. Let D denote the endomorphism algebra of $A_{\overline{\mathbb{Q}}}$ and L its centre. Then L is generated over $\mathbb{Q}$ by $a_v^2/\epsilon(\mathrm{Frob}_v)$.*

Proof. Let ℓ be a prime that splits completely in E. Then all embeddings of E into $\overline{\mathbb{Q}_\ell}$ actually factors through $\mathbb{Q}_\ell$. Suppose the isogenies in D are defined over a number field K. Fix an embedding of K into $\overline{\mathbb{Q}}$. Let H denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ which is an open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Shrink H if necessary so that H is contained in the kernel of ϵ. By a result of Faltings's, $D \otimes \mathbb{Q}_\ell \cong \mathrm{End}_{\mathbb{Q}_\ell[H]} V_\ell$. The centre of $D \otimes \mathbb{Q}_\ell$ is $L \otimes \mathbb{Q}_\ell$. By our choice of ℓ, the Tate module V_ℓ decomposes as $\oplus_\sigma V_\sigma$ where σ runs over all embedding of E into $\mathbb{Q}_\ell$ and where $V_\sigma := V_\ell \otimes_{E\otimes\mathbb{Q}_\ell} \mathbb{Q}_\ell$. Note that $\mathbb{Q}_\ell$ is viewed as an $E \otimes \mathbb{Q}_\ell$-module via σ. Each V_σ is a simple $\mathbb{Q}_\ell[H]$-module. Thus $\mathrm{End}_{\mathbb{Q}_\ell[H]} V_\sigma = \mathbb{Q}_\ell$.

For each prime w of K prime to ℓ and not a prime of bad reduction of A, we have a trace of Frobenius at w associated to the λ-adic representations. Denote it by b_w and it is in E. Then $\mathrm{tr}\,\mathrm{Frob}_w\,|V_\sigma = \sigma(b_w)$. The $\mathbb{Q}_\ell[H]$-modules V_σ and V_τ are isomorphic if and only if $\sigma(b_w) = \tau(b_w)$ for all w. If we let L' denote the field generated over $\mathbb{Q}$ by the b_w's, then the $\mathbb{Q}_\ell[H]$-modules V_σ and V_τ are isomorphic if and only if $\sigma|_{L'} = \tau|_{L'}$. Thus the centre of $D \otimes \mathbb{Q}_\ell$ is isomorphic to $L' \otimes \mathbb{Q}_\ell$. We have $L \otimes \mathbb{Q}_\ell = L' \otimes \mathbb{Q}_\ell$ with equality taken inside $E \otimes \mathbb{Q}_\ell$. Thus $L = L'$.

Now suppose that σ and τ agree on L so that $V_\sigma \cong V_\tau$ as $\mathbb{Q}_\ell[H]$-modules. There is a character $\varphi : G_F \to \mathbb{Q}_\ell^\times$ such that $V_\sigma \cong V_\tau \otimes \varphi$ as $\mathbb{Q}_\ell[G_F]$-modules. Taking traces we get $\sigma(a_v) = \varphi(\mathrm{Frob}_v)\tau(a_v)$ for all primes v of F which are prime to ℓ and not primes of bad reduction of A. Taking determinants we get ${}^\sigma\epsilon = \varphi^2\,{}^\tau\epsilon$. Thus σ and τ agree on a_v^2/ϵ for all good v prime to ℓ. This shows that a_v^2/ϵ is in L.

On the other hand, suppose $\sigma(a_v^2/\epsilon) = \tau(a_v^2/\epsilon)$ for all good v prime to ℓ. This implies by Chebotarev density theorem that $\mathrm{tr}^2/\det$ are the same for the representations of G_F on V_σ and V_τ. Since ϵ is trivial on H we get $\mathrm{tr}(h|V_\sigma) = \pm\,\mathrm{tr}(h|V_\tau)$. If we choose H small enough we will have $\mathrm{tr}(h|V_\sigma) = \mathrm{tr}(h|V_\tau)$. Then $V_\sigma \cong V_\tau$ as $\mathbb{Q}_\ell[H]$-modules. Thus L is contained in the field generated over $\mathbb{Q}$ by the a_v^2/ϵ's. □

Corollary 2.2.15. *Let A be a polarized simple F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Let L be the centre of $\mathrm{End}^0(A)$. Then L is generated over $\mathbb{Q}$ by $a_v^2/\epsilon(\mathrm{Frob}_v)$. In particular L is totally real.*

Proof. By 2.1.13 there exists a simple Abelian variety B/F of GL_2-type such that A is a $\overline{\mathbb{Q}}$-factor. The centre of the endomorphism algebra of A is just the centre of the endormorphism algebra of B by Prop 2.1.12. The corollary follows from the proposition. That L is totally real is established in Proposition 2.2.12. □

Proposition 2.2.16. *Suppose that 3 is totally split in E and suppose for $\lambda|3$ the representation $\bar\rho_\lambda$ is irreducible. Then $\bar\rho_\lambda$ is modular.*

Proof. In this case we have $\bar\rho_\lambda : G_F \to \mathrm{GL}_2(\mathbb{F}_3)$. We have already shown that $\bar\rho_\lambda$ is odd. Then the proposition follows from the theorem of Langlands and Tunnel. □

We record a modularity result first.

Theorem 2.2.17 (Shepherd-Barron, Taylor)**.** *Let ℓ be 3 or 5 and let F be a field of* *characteristic different from ℓ. Suppose $\rho : G_F \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is a representation such that* $\det(\rho) = \chi_\ell$. *Then there is an elliptic curve C defined over F such that $\rho \cong \bar{\rho}_{C,\ell}$.*

Then from our analysis we get:

Corollary 2.2.18. *Let ℓ be 3 or 5. Suppose that E is totally real and that ℓ is totally split in E. Then for $\lambda|\ell$ there exists an elliptic curve C/F such that $\bar{\rho}_{A,\lambda} \cong \bar{\rho}_{C,\ell}$.*

Proof. Since E is totally real, $\det(\rho) = \chi_\ell$. Then all conditions in the quoted theorem is satisfied. □

2.3 Moduli of F-virtual Abelian Varieties of GL_2-type

Now we consider the moduli space of F-virtual Abelian Varieties of GL_2-type. Roughly speaking, in the construction we will produce trees whose vertices are such Abelian varieties and whose edges represent isogenies. Then via graph theoretic properties of the trees, we can locate a nice Abelian variety that is isogenous to a given F-virtual Abelian Varieties of GL_2-type and that is represented by F-points on certain quotients of Shimura varieties. We exclude the CM case and consider the non-quaternion and quaternion cases separately.

2.3.1 Case $\mathrm{End}^0(A) \cong E$

Let $\mathcal{A}$ be the category of Abelian varieties over $\overline{\mathbb{Q}}$. Let $\mathcal{A}_0$ be the category of Abelian varieties over $\overline{\mathbb{Q}}$ up to isogeny. We consider the subcategory $\mathcal{B}$ of $\mathcal{A}$ defined as follows. The objects are pairs (A, ι) where A is an Abelian variety over $\overline{\mathbb{Q}}$ of dimension $[E : \mathbb{Q}]$ and $\iota : \mathcal{O}_E \to \mathrm{End}(A)$ is a ring isomorphism. The morphisms $\mathrm{Mor}_{\mathcal{B}}((A_1, \iota_1), (A_2, \iota_2))$ are those homomorphisms in $\mathrm{Hom}(A_1, A_2)$ that respect $\mathcal{O}_E$-action and is denoted by $\mathrm{Hom}_{\mathcal{O}_E}(A_1, A_2)$. Also define the categories $\mathcal{B}_\lambda$, where λ is a prime of $\mathcal{O}_E$, as follows. The objects are the same as in $\mathcal{B}$. The morphisms $\mathrm{Mor}_{\mathcal{B}_\lambda}((A_1, \iota_1), (A_2, \iota_2))$ are $\mathrm{Hom}_{\mathcal{O}_E}(A_1, A_2) \otimes_{\mathcal{O}_E} \mathcal{O}_{E,(\lambda)}$, where $\mathcal{O}_{E,(\lambda)}$ denotes the ring $\mathcal{O}_E$ localized at λ (but not completed). If no confusion arises the ι's will be omitted to simplify notation. Most often we will work in the category $\mathcal{B}$.

Let f be a morphism in $\mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$. Then f can be written as $g \otimes (1/s)$ with g in $\mathrm{Hom}(A, B)$ and s in $\mathcal{O}_E \setminus \lambda$. Define the kernel of f, $\ker f$ to be the λ part of the kernel of g

in the usual sense, $(\ker g)_\lambda$. Note here kernel is not in the sense of category theory. Suppose that we write f in another way by $g' \otimes (1/s')$. Then consider $f = gs' \otimes (1/ss')$ and we find $(\ker gs')_\lambda = (\ker g)_\lambda$ as s' is prime to λ and also $(\ker g's)_\lambda = (\ker g')_\lambda$. Since $g's = gs'$ we find our kernel well-defined.

We construct a graph associated to each $\mathcal{B}_\lambda$. Let the vertices be the isomorphism classes of $\mathcal{B}_\lambda$ modulo the relation $\approx$, where $A \approx B$ if there exists a fractional ideal $\mathfrak{A}$ of E such that

$$A \otimes_{\mathcal{O}_E} \mathfrak{A} \cong B$$

Denote the vertex associated to A by $[A]$ and connect $[A]$ and $[B]$ if there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathcal{O}_E/\lambda^{r+1} \oplus \mathcal{O}_E/\lambda^r$$

for some r. If we change A to $A/A[\lambda] = A \otimes_{\mathcal{O}_E} \lambda^{-1}$ then the kernel becomes $\mathcal{O}_E/\lambda^r \oplus \mathcal{O}_E/\lambda^{r-1}$ and as we quotient out more we will get to $\mathcal{O}_E/\lambda$. Obviously there is an $f' \in \mathrm{Mor}_{\mathcal{B}_\lambda}(B, A)$ such that $\ker f' \cong \mathcal{O}_E/\lambda^{s+1} \oplus \mathcal{O}_E/\lambda^s$ for some s.

Lemma 2.3.1. *Suppose that two vertices $[A]$ and $[B]$ can be connected by a path of length n. Then there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that $\ker f \cong \mathcal{O}_E/\lambda^n$ for some representatives A and B.*

Proof. For $n = 1$ the lemma is true. For $n = 2$ suppose we have $[A]$ connected to $[A_1]$ and then $[A_1]$ to $[B]$ and suppose we have $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, A_1)$ and $g \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, B)$ with $\ker f \cong \mathcal{O}_E/\lambda$ and $\ker g \cong \mathcal{O}_E/\lambda$. Then if $\ker g \circ f \cong \mathcal{O}_E/\lambda \oplus \mathcal{O}_E/\lambda$ we find $B \approx A$ and thus $[A] = [B]$. Then $[A]$ and $[A_1]$ are connected by two edges. However by the construction there is at most one edge between two vertices. Thus we must have $\ker g \circ f \cong \mathcal{O}_E/\lambda^2$.

Now suppose the lemma holds for all paths of length $n-1$. Suppose that $[A]$ and $[B]$ are connected via $[A_1], \cdots, [A_{n-1}]$ and that we have morphisms $A \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} B$ where all kernels are isomorphic to $\mathcal{O}_E/\lambda$. Then $f = f_{n-2} \circ \cdots \circ f_0$ and $g = f_{n-1} \circ \cdots \circ f_1$ have kernel isomorphic to $\mathcal{O}_E/\lambda^{n-1}$. If $g \circ f_0$ has kernel isomorphic to $\mathcal{O}_E/\lambda^{n-1} \oplus \mathcal{O}_E/\lambda$ then f must have kernel isomorphic to $\mathcal{O}_E/\lambda^{n-2} \oplus \mathcal{O}_E/\lambda$ and we get a contradiction. Thus $g \circ f_0$ has kernel isomorphic to $\mathcal{O}_E/\lambda^n$. This concludes the proof. $\square$

Proposition 2.3.2. *Each connected component of the the graph is a tree.*

Proof. We need to show that there is no loop. Suppose there is one. By the previous lemma we have a morphism $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, A')$ for some A' in the same $\approx$-equivalence class as A such that $\ker f \cong \mathcal{O}_E/\lambda^n$. This means we have a morphism $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, A)$ with kernel isomorphic to $\mathcal{O}_E/\lambda^{n+r} \oplus \mathcal{O}_E/\lambda^r$ for some r. This is impossible as any isogeny in $\mathrm{End}_{\mathcal{O}_E}(A)$ has kernel isomorphic to $(\mathcal{O}_E/\lambda^s)^2$ for some s. Thus there is no loop. $\square$

Definition 2.3.3. For an Abelian variety A and for each λ the tree that contains the vertex associated to A is called the λ-local tree associated to A.

Obviously the Galois group G_F acts on the graph. If a connect component has a vertex coming from an F-virtual Abelian variety, then automatically all vertices in this component come from F-virtual Abelian varieties. Also G_F preserves this component and we get another characterization of F-virtual Abelian varieties.

Lemma 2.3.4. *The G_F-orbit of an F-virtual Abelian variety is contained in the same λ-local tree.*

Now let A_0 be an F-virtual Abelian variety in $\mathcal{B}$. If considered as an object in $\mathcal{B}_\lambda$, it is mapped to a vertex in its λ-local tree. Also for all $\sigma \in G_F$ the vertex $[{}^\sigma A_0]$ is in the same λ-local tree. A priori, the Galois orbit of the vertex associated to A_0 is hard to describe. However for some special vertex in the tree the Galois orbit is essentially contained in the Atkin-Lehner orbit which we will describe below.

Definition 2.3.5. For a finite subset S of vertices of a tree the centre is the central edge or central vertex on any one of the longest paths connecting two vertices in S.

Remark 2.3.6. There are possibly more than one longest path, but they give the same centre. Thus the centre is well-defined.

Definition 2.3.7. The centre associated to A_0 in each local tree is defined to be the centre of the image of the set $\{{}^\sigma A_0 : \sigma \in G_F\}$ in the tree.

Obviously we have:

Proposition 2.3.8. *The associated centre of A_0 is fixed under the action of G_F. (Note if the centre is an edge it can be flipped.) Furthermore the vertices in the image of $\{{}^{\sigma}A_0 : \sigma \in G_F\}$ are at the same distance to the nearer vertex of the centre.*

Proposition 2.3.9. *The set of central edges associated to an F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type such that $\mathrm{End}^0 A \cong E$ is an E-linear isogeny invariant.*

Proof. Suppose in the λ-local tree the centre associated to A_0 is an edge. Then there is an element in G_F that exchanges the two vertices connected by the edge. Otherwise all Galois conjugates of A_0 will be on one side of the edge, contrary to the fact that this edge is central.

Once we have a fixed edge which is flipped under Galois action there can be no fixed vertices or other fixed edges in the tree. We take an Abelian variety B_0 which is E-linearly isogenous to A_0. Then the centre associated to B_0 is also fixed under Galois action and hence must be an edge. Furthermore it must coincide with the central edge associated to A_0. Thus central edges are E-linear isogeny invariants. □

Remark 2.3.10. Central vertices are not necessarily isogeny invariants. For example we can just take an Abelian variety A over F such that $\mathrm{End}^0 A = \mathcal{O}_E$ and take $B = A/C$ where C is a subgroup of A isomorphic to $\mathcal{O}_E/\lambda$. Then obviously the central vertex for A is $[A]$ and for B it is $[B]$ and they are not the same by construction.

Let Σ be the set of primes where the centre is an edge.

Lemma 2.3.11. *The set Σ is a finite set.*

Proof. The Abelian varieties ${}^{\sigma}A_0$ for σ in G_F end up in the same equivalence class as long as λ does not devide the degree of the isogenies μ_σ's between the Galois conjugates. Thus there are only finitely many λ's where the associated centre can be an edge. □

Remark 2.3.12. Also we note that for almost all λ's, $[A_0]$ is just its own associated central vertex.

For each central edge we choose one of the vertices and for central vertices we just use the central vertices. Then these vertices determine an Abelian variety up to $\approx$. Recall that $A \approx B$ if $A \otimes_{\mathcal{O}_E} \mathfrak{A}$ is isomorphic to B for some fractional ideal $\mathfrak{A}$.

Consider the Hilbert modular variety $Y_0(\mathfrak{n})$ classifying Abelian varieties with real multiplication E with level structure $K_0(\mathfrak{n})$ where $\mathfrak{n}$ is the product of the prime in Σ.

$$Y_0(\mathfrak{n}) := \mathrm{GL}_2(E) \setminus \mathcal{H}^n \times \mathrm{GL}_2(\widehat{E})/K_0(\mathfrak{n})$$

First we have $\operatorname{Pic} \mathcal{O}_E$ acting on $Y_0(\mathfrak{n})$ which is defined by:

$$\rho(\mathfrak{m}) : (A \to B) \mapsto (A \otimes_{\mathcal{O}_E} \mathfrak{m} \to B \otimes_{\mathcal{O}_E} \mathfrak{m}).$$

for $\mathfrak{m} \in \operatorname{Pic}(\mathcal{O}_E)$. This corresponds to $\approx$.

On the quotient $\operatorname{Pic}(\mathcal{O}_E) \setminus Y_0(\mathfrak{n})$ we have the Atkin-Lehner operators w_λ for $\lambda | \mathfrak{n}$ defined as follows. Suppose we are given a point on $\operatorname{Pic}(\mathcal{O}_E) \setminus Y_0(\mathfrak{n})$ represented by $(A \xrightarrow{f} B)$ and let C denote the kernel of f. Then w_λ sends this point to

$$(A/C[\lambda] \to B/f(A[\lambda])).$$

This operation of w_{λ_0} when viewed on the λ-tree sends one vertex on the central edge to the other one if $\lambda = \lambda_0$ or does nothing if $\lambda \neq \lambda_0$.

Definition 2.3.13. Let W be the group generated by w_λ's and let $\widetilde{W}$ be the group $W \ltimes \operatorname{Pic}(\mathcal{O}_E)$ acting on $Y_0(\mathfrak{n})$.

Thus we have:

Proposition 2.3.14. *The G_F-orbit of any Abelian variety coming the central vertices is contained in the $\widetilde{W}$-orbit.*

Denote $\widetilde{W} \setminus Y_0(\mathfrak{n})$ by $Y_0^+(\mathfrak{n})$. Then we have associated to A_0 a point in $Y_0^+(\mathfrak{n})(\overline{\mathbb{Q}})$. As this point is fixed by G_F this is actually an F-point.

On the other hand take an F-rational point of $Y_0^+(\mathfrak{n})$ and we get a set of Abelian varieties in $Y_0(\mathfrak{n})$ that lie above it. They are all isogenous. Take any one of them, say A. Then its Galois conjugates are still in the set and they are E-linearly isogenous to A by construction. We get an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. However we cannot rule out the possibility that it may have larger endomorphism algebra than E.

We have shown

Theorem 2.3.15. *Every F-point on the Hilbert modular variety $Y_0^+(\mathfrak{n})$ gives an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Conversely for any F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type there is an isogenous F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type A' which corresponds to an F-rational point on a Hilbert modular variety of the form $Y_0^+(\mathfrak{n})$ where $\mathfrak{n}$ is given as in the tree construction above.*

2.3.2 Case $\mathrm{End}^0 A = D$

We cannot simply follow Case $\mathrm{End}^0 A = E$, since we will get a graph with loop in that way. We will construct local trees in a slightly different way. Otherwise everything is the same as in Case $\mathrm{End}^0 A = E$. Let L denote the centre of D and $\mathcal{O}_D$ a maximal order of D containing $\mathcal{O}_E$.

Let $\mathcal{B}$ be the subcategory of $\mathcal{A}$ defined as above except for the requirements that $\iota : \mathcal{O}_D \to \mathrm{End}(A)$ is a ring isomorphism and that morphisms should respect $\mathcal{O}_D$-action. For the definition of $\mathcal{B}_\lambda$ we divide into 2 cases.

If D does not split at λ, let $\mathfrak{Q}$ be the prime of $\mathcal{O}_D$ lying above λ. We have $\mathfrak{Q}^2 = \lambda$. Then define $\mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be $\mathrm{Hom}_{\mathcal{O}_D}(A_1, A_2) \otimes_{\mathcal{O}_D} \mathcal{O}_{D,(\mathfrak{Q})}$. Define the kernel of $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be the λ-part of the usual kernel of g in some decomposition of $f = g \otimes (s^{-1})$. The equivalence relation $\approx$ on objects of $\mathcal{B}_\lambda$ is given as follows: $A \approx B$ if and only if there exists some $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathcal{O}_D/\lambda^r\mathcal{O}_D$$

for some r. Connect $[A]$ and $[B]$ if there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathcal{O}_D/\mathfrak{Q} \oplus \mathcal{O}_D/\lambda^r\mathcal{O}_D$$

for some r.

If D splits at λ, then define $\mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be $\mathrm{Hom}_{\mathcal{O}_D}(A_1, A_2) \otimes_{\mathcal{O}_L} \mathcal{O}_{L,(\lambda)}$. Still define the kernel of $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be the λ-part of the usual kernel of g in some decomposition of $f = g \otimes (s^{-1})$. The equivalence relation $\approx$ on objects of $\mathcal{B}_\lambda$ is given as follows: $A \approx B$ if and only if there exists some $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathrm{M}_2(\mathcal{O}_L/\lambda^r\mathcal{O}_L)$$

for some r. Connect $[A]$ and $[B]$ if there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ such that

$$\ker f \cong (\mathcal{O}_L/\lambda)^2 \oplus \mathrm{M}_2(\mathcal{O}_L/\lambda^r\mathcal{O}_L)$$

for some r.

Now we follow what we have done in the previous subsection. Similarly we get

Proposition 2.3.16. *Each connected component of the graph is a tree.*

Proposition 2.3.17. *The set of central edges associated to an F-virtual Abelian variety A of* $\mathrm{GL}_2(E)$*-type such that* $\mathrm{End}^0(A) \cong D$ *is a D-linear isogeny invariant.*

Let Σ denote the set of primes at which the associated centre is an edge. Note that Σ is again a finite set.

Proposition 2.3.18.

Now we consider the Shimura variety S_Σ parametrizing the quadruples $(A, *, \iota, C)$ where the level structure C is isomorphic to

$$(\oplus_{\substack{\lambda\in\Sigma \\ \lambda\nmid\mathrm{disc}(D)}} (\mathcal{O}_L/\lambda)^2) \oplus (\oplus_{\substack{\lambda\in\Sigma \\ \lambda\mid\mathrm{disc}(D)}} \mathcal{O}_{D_\lambda}/\mathfrak{Q}_\lambda).$$

Let $\widetilde{W}$ be the group acting on S_Σ generated by

$$w_\lambda : (A, *, \iota, C) \mapsto (A/C[\lambda], *', \iota', C + A[\lambda]/C[\lambda]) \quad \text{if } \lambda \in \Sigma \text{ and } \lambda \nmid \mathrm{disc}(D);$$

$$w_\lambda : (A, *, \iota, C) \mapsto (A/C[\mathfrak{Q}_\lambda], *', \iota', C/C[\mathfrak{Q}_\lambda] + (A/C[\mathfrak{Q}_\lambda])[\mathfrak{Q}_\lambda]) \quad \text{if } \lambda \in \Sigma \text{ and } \lambda\mid\mathrm{disc}(D);$$

$$w'_{\mathfrak{I}} : (A, *, \iota, C) \mapsto (A/A[\mathfrak{I}], *', \iota', C) \quad \text{if } \mathfrak{I} \text{ prime to } \Sigma.$$

Also we have

Lemma 2.3.19. *The group $\widetilde{W}$ is a finitely generated Abelian group.*

Denote $\widetilde{W} \setminus S_\Sigma$ by S_Σ^+. Similarly we have that the F-points of S_Σ^+ give isogeny classes of F-virtual Abelian varieties of $\mathrm{GL}_2(E)$-type.

Theorem 2.3.20. *Every F-point on the Shimura variety S_Σ^+ gives an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Conversely for any F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type s.t. $\mathrm{End}^0 A = D$ there is an isogenous Abelian variety A' of $\mathrm{GL}_2(E)$-type which corresponds to an F-rational point on S_Σ^+, a quotient of Shimura variety of PEL-type, where Σ is given above.*

2.4 Field of Definition

Now we consider the field of definition of F-virtual Abelian varieties of GL_2-type. Even though they are not necessarily defined over F we will show that in their isogenous class there exist ones that can be defined over some polyquadratic field extension of F. Here we follow essentially [6] except in the quaternion algebra case.

Proposition 2.4.1. *Let A be an F-virtual Abelian variety. Let c be the associated 2-cocycle on G_F. If c is trivial in $H^2(G_{F'}, L^\times)$, then A is isogenous to an Abelian variety defined over F'.*

Proof. Suppose $c(\sigma,\tau) = \frac{\alpha(\sigma)\alpha(\tau)}{\alpha(\sigma\tau)}$ where σ and τ are in $G_{F'}$. Let $\nu_\sigma = \mu_\sigma \alpha^{-1}(\sigma)$. Let K be Galois over F' such that A is defined over K, that all the isogenies involved are defined over K as we did in the proof of Thm 2.1.13 and α is constant on G_K . Let $B = \mathrm{Res}_{K/F'} A$. Then the assignment $\sigma \mapsto \nu_\sigma$ is a D-linear algebra isomorphism between $D[G_{K/F'}]$ and $\mathrm{End}^0 B$. Consider the D-algebra homomorphism $D[G_{K/F'}] \to D$ where σ is sent to 1. Since $D[G_{K/F'}]$ is semisimple, D is a direct summand of $D[G_{K/F'}]$. Since $e = \frac{1}{[K:F']} \sum_{\sigma \in G_{K/F'}} \sigma$ is an idempotent of $D[G_{K/F'}]$ which splits off D, then if we let $A' = Ne(B)$ where N is chosen so that Ne is an endomorphism of B, then A' is defined over F' and is isogenous to A. □

Proposition 2.4.2. *The two cocycle c is of 2-torsion in $H^2(G_F, L^\times)$ and c is trivial in $H^2(G_{F'}, L^\times)$ where F' can be taken as a polyquadratic extension.*

Proof. First consider the case when D is a quaternion algebra. Let Λ be the lattice in $\mathbb{C}^g$ that corresponds to A. Then $\Lambda \otimes \mathbb{Q}$ is isomorphic to D as left D-modules. Fix such an isomorphism. For ${}^{\sigma}A$ we get an induced isomorphism between ${}^{\sigma}\Lambda \otimes \mathbb{Q}$ and D as left D-modules. Then a D-linear isogeny f between ${}^{\sigma}A$ and ${}^{\tau}A$ induces a D-module isomorphism $D \to D$ which is just right multiplication by an element in $D^{\times}$. Let $d(f)$ denote the reduced norm of that element from D to L. Obviously $d(f) = d({}^{\sigma}f)$, for $\sigma \in G_F$. Furthermore $d(f \circ g) = d(f)d(g)$. Thus $d(c(\sigma,\tau)) = c^2(\sigma,\tau) = \frac{d(\mu_\sigma)d(\mu_\tau)}{d(\mu_{\sigma\tau})}$, which shows that c^2 is a coboundary.

Second consider the case when $D = E = L$. Then the lattice Λ corresponding to A is isomorphic to E^2 as E-vector space after tensoring with $\mathbb{Q}$. Fix such an isomorphism and correspondingly isomorphisms between ${}^{\sigma}\Lambda \otimes \mathbb{Q}$ and E^2. An E-linear isogeny f between ${}^{\sigma}A$ and ${}^{\tau}A$ then induces linear transformation. Let $d(f)$ denote the determinant of the linear transformation. Still d is multiplicative and $d(f) = d({}^{\sigma}f)$. Thus $d(c(\sigma,\tau)) = (c(\sigma,\tau))^2 = \frac{d(\mu_\sigma)d(\mu_\tau)}{d(\mu_{\sigma\tau})}$, which shows that c is 2-torsion in $H^2(G_F, L^{\times})$.

Now all that is left is to show that c is trivial after a polyquadratic extension of F.

Consider the split short exact sequence of G_F-modules

$$0 \to \mu_2 \to L^{\times} \to P \to 0$$

where $P \cong L^{\times}/\mu_2$. We get that $H^2(G_F, L^{\times}) \cong H^2(G_F, \mu_2) \times H^2(G_F, P)$. Since $H^2(G_F, \mu_2)$ corresponds to those quaternion algebra elements in $Br(F)$ it can be killed by a quadratic extension of F. To kill the 2-torsion elements in $H^2(G_F, P)$, consider the short exact sequence:

$$0 \to P \xrightarrow{\times 2} P \to P/2P \to 0$$

where group multiplication is written additively. We get a long exact sequence:

$$\mathrm{Hom}(G_F, P) \to \mathrm{Hom}(G_F, P/2P) \to H^2(G_F, P) \xrightarrow{\times 2} H^2(G_F, P) \to \cdots$$

Since P is torsion-free $\mathrm{Hom}(G_F, P)$ is trivial, we find $\mathrm{Hom}(G_F, P/2P) \cong H^2(G_F, P)[2]$. After a polyquadratic extension of F to F' we can make $\mathrm{Hom}(G_{F'}, P/2P)$ as well as

$H^2(G_F, \mu_2)$ trivial. Thus c is trivial in $H^2(G_{F'}, L^\times)$. $\square$

2.5 Classification of Hilbert Modular Surfaces

We will focus on the case where E is a real quadratic field with narrow class number 1 and study the Hilbert modular surfaces $Y_0^+(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of E. We have shown in Section 2.3 that the F-points of $Y_0^+(\mathfrak{p})$ represent F-virtual Abelian varieties. Suppose $E = \mathbb{Q}(\sqrt{D})$ where D is the discriminant. Because of the narrow class number 1 assumption, necessarily D is either a prime congruent to 1 modulo 4 or $D = 8$. We fix an embedding of E into $\mathbb{R}$. The conjugate of an element a in E is denoted by a^c. Since the class group of E is trivial, the group $\widetilde{W}$ in Definition 2.3.13 is just W, a group of order 2. The group $W\Gamma_0(\mathfrak{p})$ is in fact the normalizer of $\Gamma_0(\mathfrak{p})$ in $\mathrm{PGL}_2^+(E)$ and $Y_0^+(\mathfrak{p})$ is isomorphic to $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2$ where

$$\Gamma_0(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : c \equiv 0 \pmod{\mathfrak{p}} \right\}$$

In our case $\mathrm{PSL}_2(\mathcal{O}_E) = \mathrm{PGL}_2^+(\mathcal{O}_E)$. Let $X_0^+(\mathfrak{p})$ denote the minimal desingularity of $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$. The classification done in [28] does not cover our case. We will follow the line of [28] and show that most surfaces in question are of general type and will give examples of surfaces not of general type. Our method relies on the estimation of Chern numbers. To do so we must study the singularities on the surfaces.

2.5.1 Cusp Singularities

Obviously for $\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$ there are two inequivalent cusps 0 and ∞. They are identified via the Atkin-Lehner operator $w_\mathfrak{p} = \left(\begin{smallmatrix} 0 & 1 \\ -\varpi_\mathfrak{p} & 0 \end{smallmatrix}\right)$, where the prime ideal $\mathfrak{p}$ is equal to $(\varpi_\mathfrak{p})$ and $\varpi_\mathfrak{p}$ is chosen to be totally positive. This is possible as we assumed that the narrow class number of E is 1. The isotropy group of the unique inequivalent cusp ∞ in $W\Gamma_0(\mathfrak{p})$ is equal to that in $\mathrm{PSL}_2(\mathcal{O}_E)$, as $W\Gamma_0(\mathfrak{p})$ contains all those elements in $\mathrm{PSL}_2(\mathcal{O}_E)$ that are of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Thus the type of the cusp singularity is the same as that for

$\mathrm{PSL}_2(\mathcal{O})$ and the isotropy group is equal to

$$\begin{aligned}&\left\{\begin{pmatrix}\epsilon & \mu \\ 0 & \epsilon^{-1}\end{pmatrix} \in \mathrm{PSL}_2(E) : \epsilon \in \mathcal{O}_E^{\times}, \mu \in \mathcal{O}_E\right\} \\ \cong &\left\{\begin{pmatrix}\epsilon & \mu \\ 0 & 1\end{pmatrix} \in \mathrm{PGL}_2^{+}(E) : \epsilon \in \mathcal{O}_E^{\times +}, \mu \in \mathcal{O}_E\right\} \\ \cong &\ \mathcal{O}_E \rtimes \mathcal{O}_E^{\times +}.\end{aligned} \tag{2.5.1}$$

By [28, Chapter II] we have the minimal resolution of singularity resulting from toroidal embedding and the exceptional divisor consists of a cycle of rational curves.

2.5.2 Elliptic Fixed Points

Now consider the inequivalent elliptic fixed points of $W\Gamma_0(\mathfrak{p})$ on $\mathcal{H}^2$. More generally we consider the elliptic fixed points of $\mathrm{PGL}_2^{+}(E)$. Suppose $z = (z_1, z_2)$ is fixed by $\alpha = (\alpha_1, \alpha_2)$ in the image of $\mathrm{PGL}_2^{+}(E)$ in $\mathrm{PGL}_2^{+}(\mathbb{R})^2$. Then

$$\begin{pmatrix}a_j & b_j \\ c_j & d_j\end{pmatrix}.z_j = z_j$$

for $j = 1$ or 2 where $\alpha_j = \left(\begin{smallmatrix}a_j & b_j \\ c_j & d_j\end{smallmatrix}\right)$. Solving the equation we get

$$z_j = \frac{a_j - d_j}{2c_j} + \frac{1}{2|c_j|}\sqrt{\mathrm{tr}(\alpha_j)^2 - 4\det(\alpha_j)}.$$

Tranform z_j to 0 via möbius transformation $\zeta_j \mapsto \frac{\zeta_j - z_j}{\zeta_j - \bar{z}_j}$ then the isotropy group of (z_1, z_2) acts as rotation around 0 on each factor. For α_j the rotation factor is $r_j = e^{2i\theta_j}$ where

$$\cos\theta_j = \frac{\mathrm{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}}, \quad c_j \sin\theta_j > 0. \tag{2.5.2}$$

The isotropy group of an elliptic point is cyclic.

Definition 2.5.1. We say that the quotient singularity is of type $(n; a, b)$ if the rotation factor associated to a generator of the isotropy group acts as $(w_1, w_2) \mapsto (\zeta_n^a w_1, \zeta_n^b w_2)$ where

ζ_n is a primitive n-th root of 1.

Remark 2.5.2. We require that a and b are coprime to n respectively and we can make a equal to 1 by changing the primitive n-th root of 1. The method of resolution of singularity in [28, Section 6, Chapter II] depends only on the type.

Definition 2.5.3. Let $a_2(\Gamma)$ denote the number of Γ-inequivalent elliptic points of type $(2;1,1)$. Let $a_n^+(\Gamma)$ denote the number of Γ-inequivalent elliptic points of type $(n;1,1)$. Let $a_n^-(\Gamma)$ denote the number of Γ-inequivalent elliptic points of type $(n;1,-1)$.

From the expression for θ_j we get:

Lemma 2.5.4. *Assume $D > 12$. Then the elliptic elements of $\Gamma_0(\mathfrak{p})$ can only be of order 2 or 3.*

2.5.3 Estimation of Chern Numbers

Now we estimate the Chern numbers of $X_0^+(\mathfrak{p})$. We will use the following criterion (Prop. 2.5.5) found on page 171 of [28] to show that most of our surfaces are of general type. Let χ denote the Euler characteristic and c_i be the i-th Chern class. The Chern class of a surface S, $c_i(S)$, is the Chern class of the tangent bundle.

Proposition 2.5.5. *Let S be a nonsingular algebraic surface with vanishing irregularity. If $\chi > 1$ and $c_1^2 > 0$, then S is of general type.*

Definition 2.5.6. Let S be a normal surface with isolated singular points and let S' be its desingularization. Suppose p is a singular point on S and the irreducible curves $C_1, \ldots, C_m$ on S' form the resolution of p. Then the local Chern cycle of p is defined to be the unique divisor $Z = \sum_{i=1}^m a_i C_i$ with rational numbers a_i such that the adjunction formula holds:

$$ZC_i - C_iC_i = 2 - 2p_a(C_i).$$

Remark 2.5.7. For quotient singularity of type $(n;1,1)$, the exceptional divisor consists of one rational curve S_0 with $S_0^2 = -n$ and the local Chern cycle is $(1-2/n)S_0$. For quotient singularity of type $(n;1,-1)$, the exceptional divisor consists of $n-1$ rational curves $S_1, \dots, S_{n-1}$ with $S_i^2 = -2$, $S_{i-1}.S_i = 1$ and the rest of the intersection numbers involving these rational curves are 0. The local Chern cycle is 0. For cusp singularity, the exceptional divisor consists of several rational curves $S_0, \dots, S_m$ such that $S_{i-1}.S_i = 1$, $S_0.S_m = 1$, $S_i^2 \le -2$ and the rest of the intersection numbers involving these rational curves are 0. The local Chern cycle is $\sum_i S_i$.

We will make frequent comparison to the surface associated to the full Hilbert modular group $\mathrm{PSL}_2(\mathcal{O})$. As is computed on page 64 of [28] we have the following with a slight change of notation:

Theorem 2.5.8. *Let $\Gamma \subset \mathrm{PGL}_2^+(\mathbb{R})^2$ be commensurable with $\mathrm{PSL}_2(\mathcal{O})$ and let X_Γ be the minimal desingularization of $\overline{\Gamma \setminus \mathcal{H}^2}$. Then*

$$c_1^2(X_\Gamma) = 2\,\mathrm{vol}(\Gamma \setminus \mathcal{H}^2) + c + \sum a(\Gamma; n; a, b)c(n; a, b), \tag{2.5.3}$$

$$c_2(X_\Gamma) = \mathrm{vol}(\Gamma \setminus \mathcal{H}^2) + l + \sum a(\Gamma; n; a, b)(l(n; a, b) + \frac{n-1}{n}) \tag{2.5.4}$$

where

$$\begin{aligned}
a(\Gamma; n; a, b) =& \#\textit{quotient singularity of } \Gamma \setminus \mathcal{H}^2 \textit{ of type } (n; a, b),\\
c =& \textit{sum of the self-intersection number of the local Chern cycles of}\\
& \textit{cusp singularities},\\
c(n; a, b) =& \textit{self-intersection number of the local Chern cycle of a quotient}\\
& \textit{singularity of type } (n; a, b),\\
l =& \#\textit{curves in the resolution of cusps},\\
l(n; a, b) =& \#\textit{curves in the resolution of a quotient singularity of type}\\
& (n; a, b).
\end{aligned}$$

Lemma 2.5.9.

$$\mathrm{vol}(\mathrm{PSL}_2(\mathcal{O}_E) \setminus \mathcal{H}^2) = 2\zeta_E(-1). \tag{2.5.5}$$

Now we will estimate the chern numbers under the assumption that $D > 12$. This ensures that we only have $(2;1,1)$ or $(3;1,\pm 1)$ points for $\Gamma_0(\mathfrak{p})$ and hence only $(2;1,1)$, $(3;1,\pm 1)$, $(4;1,\pm 1)$ and $(6;1,\pm 1)$ points for $W\Gamma_0(\mathfrak{p})$. From [28, II. 6] as summarized in Remark 2.5.7, we know how the elliptic singularities are resolved and can compute the self-intersection number of local chern cycles. Thus after we plug in the values equation (2.5.3) reads

$$\begin{aligned} c_1^2(X(W\Gamma_0(\mathfrak{p}))) =& \frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}):\Gamma_0(\mathfrak{p})]4\zeta_E(-1) + c - \frac{1}{3}a_3^+ - a_4^+ - \frac{8}{3}a_6^+; \\ c_2(X(W\Gamma_0(\mathfrak{p}))) =& \frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}):\Gamma_0(\mathfrak{p})]2\zeta_E(-1) + l + (1+\frac{1}{2})a_2 + (1+\frac{2}{3})a_3^+ + (2+\frac{2}{3})a_3^- \\ & + (1+\frac{3}{4})a_4^+ + (3+\frac{3}{4})a_4^- + (1+\frac{5}{6})a_6^+ + (5+\frac{5}{6})a_6^-. \end{aligned} \tag{2.5.6}$$

We quote some results in [28, Section VII.5].

Lemma 2.5.10. *For all D a fundamental discriminant $\zeta_E(-1) > \frac{D^{3/2}}{360}$.*

This is equation (1) in [28, Section VII.5].

As a_2, $a_3^\pm$, $a_4^\pm$, $a_6^\pm$ and l are non-negative, $c_2(W\Gamma_0(\mathfrak{p})) > (\mathbf{N}\,\mathfrak{p}+1)\frac{D^{3/2}}{360}$. Thus if

$$(\mathbf{N}\,\mathfrak{p}+1)\frac{D^{3/2}}{360} > 12 \tag{2.5.7}$$

then $c_2(W\Gamma_0(\mathfrak{p})) > 12$.

Now we estimate $c_1^2(X(W\Gamma_0(\mathfrak{p})))$. Let n denote the index of $\Gamma_0(\mathfrak{p})$ in $\mathrm{PSL}_2(\mathcal{O}_E)$, which is equal to $\mathbf{N}\,\mathfrak{p}+1$. First the self-intersection number of the local Chern cycle at the cusp, c, is equal to that for $\mathrm{PSL}_2(\mathcal{O})$ as the isotropy group for the unique cusp in $W\Gamma_0(\mathfrak{p})$ is the same as that in $\mathrm{PSL}_2(\mathcal{O})$.

Lemma 2.5.11. *The local Chern cycle*

$$c = \frac{1}{2}\sum_{x^2<D,\, x^2\equiv D \pmod 4}\ \sum_{a>0,\, a\mid\frac{D-x^2}{4}} 1. \tag{2.5.8}$$

and if $D > 500$,

$$c \geq -\frac{1}{2}D^{1/2}\left(\frac{3}{2\pi^2}\log^2 D + 1.05\log D\right). \tag{2.5.9}$$

This is [28, Lemma VII.5.3].

Definition 2.5.12. Let $h(D)$ denote the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$ where D is a fundamental discriminant.

Lemma 2.5.13. *If $D > 0$ is a fundamental discriminant then $h(-D) \leq \frac{\sqrt{D}}{\pi} \log D$.*

This is [28, Lemma VII.5.2].

Lemma 2.5.14. *If $D > 12$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$, then*

$$\begin{aligned} a_2(\mathrm{PSL}_2(\mathcal{O})) =& h(-4D) \\ a_3^+(\mathrm{PSL}_2(\mathcal{O})) =& \frac{1}{2} h(-3D) \end{aligned} \tag{2.5.10}$$

Combining the above two lemmas we get:

Lemma 2.5.15.

$$\begin{aligned} a_2(\mathrm{PSL}_2(\mathcal{O})) &\leq \frac{\sqrt{4D}}{\pi} \log 4D \\ a_3^+(\mathrm{PSL}_2(\mathcal{O})) &\leq \frac{1}{2\pi} \sqrt{3D} \log(3D). \end{aligned} \tag{2.5.11}$$

Lemma 2.5.16. *If $D > 12$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$ then*

$$\begin{aligned} a_2(\Gamma_0(\mathfrak{p})) &\leq \frac{3\sqrt{4D}}{\pi} \log 4D \\ a_3^+(\Gamma_0(\mathfrak{p})) &\leq \frac{3}{2\pi} \sqrt{3D} \log(3D). \end{aligned} \tag{2.5.12}$$

Proof. Let z be an elliptic point of $\mathrm{PSL}_2(\mathcal{O}_E)$ with isotropy group generated by $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have coset decomposition of $\mathrm{SL}_2(\mathcal{O}_E) = \cup_\alpha \Gamma_0(\mathfrak{p})\delta_\alpha \cup \Gamma_0(\mathfrak{p})\delta_\infty$, where $\delta_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ with $\alpha \in \mathcal{O}_E$ running through a set of representatives of $\mathcal{O}_E/\mathfrak{p}$ and $\delta_\infty = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For each δ_α we need to check if $\delta_\alpha g \delta_\alpha^{-1}$ is in $\Gamma_0(\mathfrak{p})$, i.e., if $c + (a-d)\alpha - b\alpha^2$ is in $\mathfrak{p}$. In $\mathbb{F}_\mathfrak{p}$, the equation $c + (a-d)\alpha - b\alpha^2 = 0$ has at most two solutions unless $c, a-d, b \in \mathfrak{p}$. This cannot happen if g is elliptic. Indeed from $ad - bc = 1$ we get $a^2 \equiv 1 \pmod{\mathfrak{p}}$ and thus $a \equiv \pm 1 \pmod{\mathfrak{p}}$. Replace g by $-g$ if necessary we suppose that $a \equiv 1 \pmod{\mathfrak{p}}$. Suppose $a + d = t$ with $t = 0$

or ± 1 and suppose $a = 1 + v$ with $v \in \mathfrak{p}$. Then

$$\begin{aligned} 1 &= ad - bc \\ &= a(t-a) - bc \\ &\equiv a(t-a) \pmod{\mathfrak{p}^2} \\ &\equiv t - 1 + (t-2)v \pmod{\mathfrak{p}^2}. \end{aligned} \tag{2.5.13}$$

We find that $(t-2)(1+v) \pmod{\mathfrak{p}^2}$. By assumption $D > 12$ is a prime so (2) or (3) cannot ramify. We always have that $1 + v \equiv 0 \pmod{\mathfrak{p}}$, which is impossible.

Thus the number of elliptic points of any type increases to at most threefold that for $\mathrm{PSL}_2(\mathcal{O}_E)$. $\square$

Lemma 2.5.17. *Suppose $D > 12$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$. Then $a_6^+(W\Gamma_0(\mathfrak{p})) = 0$ unless (3) is inert in $\mathcal{O}_E$ and $\mathfrak{p} = (3)$ and*

$$2a_3^+(W\Gamma_0(\mathfrak{p})) + a_6^+(W\Gamma_0(\mathfrak{p})) \le \frac{2}{3\pi}\sqrt{3D}\log(3D); \tag{2.5.14}$$

and $a_4^+(W\Gamma_0(\mathfrak{p})) = 0$ unless (2) is inert and $\mathfrak{p} = (2)$ and

$$a_4^+(W\Gamma_0(\mathfrak{p})) \le \frac{3\sqrt{4D}}{\pi}\log 4D. \tag{2.5.15}$$

Proof. We check the rotation factor

$$\cos\theta_j = \frac{\mathrm{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}} \tag{2.5.16}$$

associated to an elliptic element α. In order to have a point with isotropy group of order 4 in $W\Gamma_0(\mathfrak{p})$ we must have $\cos\theta_j = \pm\frac{\sqrt{2}}{2}$. As $D > 12$, we need $\det(\alpha_j) = \varpi_{\mathfrak{p}}$ in $2\mathcal{O}_E^2$ and also $\mathfrak{p} = (2)$. In order to have a point with isotropy group of order 6 in $W\Gamma_0(\mathfrak{p})$ we must have $\cos\theta_j = \pm\frac{\sqrt{3}}{2}$. As $D > 12$, we need $\det(\alpha_j) = \varpi_{\mathfrak{p}}$ in $3\mathcal{O}_E^2$ and also $\mathfrak{p} = (3)$.

The Atkin-Lehner operator exchanges some of the $\Gamma_0(\mathfrak{p})$-inequivalent $(3;1,1)$-points which result in $(3;1,1)$-points and fixes the rest of the points which result in $(6;1,1)$-points.

All $(3;1,1)$- and $(6;1,1)$-points for $W\Gamma_0(\mathfrak{p})$ arise in this way. We have

$$2a_3^+(W\Gamma_0(\mathfrak{p})) + a_6^+(\Gamma_0(\mathfrak{p})) = a_3^+(\Gamma_0(\mathfrak{p})). \tag{2.5.17}$$

The Atkin-Lehner operator exchanges some of the $\Gamma_0(\mathfrak{p})$-inequivalent $(2;1,1)$-points which result in $(2;1,1)$-points and fixes the rest of the points which result in $(4;1,1)$-points. All $(4;1,1)$-points for $W\Gamma_0(\mathfrak{p})$ arise in this way, but we may get $(2;1,1)$-points not arising in this way. We have

$$a_4^+(\Gamma_0(\mathfrak{p})) \leq a_2^+(\Gamma_0(\mathfrak{p})). \tag{2.5.18}$$

Combining with Lemma 2.5.16 we prove this lemma. □

Lemma 2.5.18. *Suppose* $D > 12$ *and* $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$. *Then*

$$\frac{1}{3}a_3^+(W\Gamma_0(\mathfrak{p}) + 8a_6^+(W\Gamma_0(\mathfrak{p})) \leq \frac{8}{3}a_3^+(\Gamma_0(\mathfrak{p})) \tag{2.5.19}$$

if (3) *is inert and* $\mathfrak{p} = (3)$. *If* $\mathfrak{p} \neq (3)$ *then*

$$\frac{1}{3}a_3^+(W\Gamma_0(\mathfrak{p})) = \frac{1}{6}a_3^+(\Gamma_0(\mathfrak{p})). \tag{2.5.20}$$

Combining all these inequalities we find

Lemma 2.5.19. *Suppose* $D > 500$ *and* $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$. *Then*

$$\begin{aligned} c_1^2(X_0^+(\mathfrak{p})) > {} & \frac{nD^{3/2}}{180} - \frac{1}{2}D^{1/2}\left(\frac{3}{2\pi^2}\log^2 D + 1.05\log D\right) \\ & - \begin{cases} \frac{1}{4\pi}\sqrt{3D}\log(3D) & \text{if } \mathfrak{p} \neq (3) \\ \frac{4}{\pi}\sqrt{3D}\log(3D) & \text{if } \mathfrak{p} = (3) \end{cases} \\ & - \begin{cases} 0 & \text{if } \mathfrak{p} \neq (2) \\ \frac{3}{\pi}\sqrt{4D}\log(4D) & \text{if } \mathfrak{p} = (2). \end{cases} \end{aligned} \tag{2.5.21}$$

Now it is easy to estimate for what values of D and n we have $c_1^2(W\Gamma_0(\mathfrak{p})) > 0$. For small D we just compute c precisely by using Equation 2.5.8 instead of using the estimates.

Theorem 2.5.20. *Suppose* $D > 12$, $D \equiv 1 \pmod 4$ *and* $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$. *Then the Hilbert modular surface* $X_0^+(\mathfrak{p})$ *is of general type if* D *or* $n = \mathbf{N}\,\mathfrak{p} + 1$ *is suffiently large or more precisely if the following conditions on* D *and* n *are satisfied:*

$D \geq 853$ *or* $D = 193, 241, 313, 337, 409, 433, 457, 521, 569, 593, 601, 617, 641, 673, 769, 809$	n *can be arbitrary*
$D = 157, 181, 277, 349, 373, 397, 421, 509, 541, 557, 613, 653, 661, 677, 701, 709, 757, 773, 797, 821, 829$	$n \neq 5$, *i.e.*, $\mathfrak{p} \neq (2)$
$D = 137, 233, 281, 353, 449,$	$n \neq 10$, *i.e.*, $\mathfrak{p} \neq (3)$
$D = 149, 173, 197, 269, 293, 317, 389, 461$	$n \neq 5, 10$, *i.e.*, $\mathfrak{p} \neq (2), (3)$
$D = 113$	$n \geq 8$ *and* $n \neq 10$
$D = 109$	$n \geq 6$
$D = 101$	$n \geq 6$ *and* $n \neq 10$
$D = 97$	$n \geq 5$
$D = 89$	$n \geq 6$
$D = 73$	$n \geq 7$
$D = 61$	$n \geq 10$
$D = 53$	$n \geq 12$
$D = 41$	$n \geq 17$
$D = 37$	$n \geq 20$
$D = 29$	$n \geq 28$
$D = 17$	$n \geq 62$
$D = 13$	$n \geq 93$

Proof. First we note that the condition for (2) to split is that $D \equiv 1 \pmod 8$ and for (3) to split is that $D \equiv 1 \pmod 3$. We check for what values of D and n the inequality 2.5.7 is satisfied and the right hand side of the inequaltiy in Inequality 2.5.21 is greater than 0 by using a computer program. Then we have $c_1^2(X_0^+(\mathfrak{p})) > 0$ and $c_2(X_0^+(\mathfrak{p})) > 12$ and thus $\chi(X_0^+(\mathfrak{p})) > 1$. The surface is of general type by Prop. 2.5.5. $\square$

Remark 2.5.21. There are some values that, a priori, n cannot achieve.

2.5.4 Examples

The first two examples give rational surfaces and the third example is neither a rational surface nor a surface of general type.

2.5.4.1 $D = 5$

Consider the Hilbert modular surface $\mathrm{PSL}_2(\mathcal{O}_E) \setminus \mathcal{H}^2$ where $E = \mathbb{Q}(\sqrt{5})$. The cusp resolution is a nodal curve. We will focus on quotient resolutions to study the configurations of rational curves on the surface. Following the method in [5], we can locate all the $\mathrm{PSL}_2(\mathcal{O}_E)$-inequivalent elliptic points.

$\mathfrak{p} = (2)$ The $\Gamma_0(\mathfrak{p})$-inequivalent elliptic points and their types are summarized in the following table. As the coordinates of the points themselves are not important we only list a generator of the isotropy groups.

Type	Generator of Isotropy Group
$(2;1,1)$	$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$
$(2;1,1)$	$\begin{pmatrix} -1 & \frac{-1+\sqrt{5}}{2} \\ -1-\sqrt{5} & 1 \end{pmatrix}$
$(3;1,1)$	$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ 2 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$
$(3;1,1)$	$\begin{pmatrix} \frac{3+\sqrt{5}}{2} & -1 \\ 3+\sqrt{5} & -\frac{1+\sqrt{5}}{2} \end{pmatrix}$
$(3;1,-1)$	$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ -1-\sqrt{5} & \frac{1+\sqrt{5}}{2} \end{pmatrix}$
$(3;1,-1)$	$\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ -4-2\sqrt{5} & \frac{3+\sqrt{5}}{2} \end{pmatrix}$

The Atkin-Lehner operator fixes the two $(2;1,1)$-points respectively and exchanges the two $(3;1,1)$- (resp. $(3;1,-1)$-) points. We get one $(4;1,1)$-, one $(4;1,-1)$-, one $(3;1,1)$-

, one $(3;1,-1)$- and possibly some new $(2;1,1)$-points. We consider the $(4;1,-1)$-point represented by $(\frac{1+i}{1+\sqrt{5}}, \frac{1-i}{1-\sqrt{5}})$ and the $(3;1,-1)$-point represented by $(\frac{\sqrt{5}+i\sqrt{3}}{2(1+\sqrt{5})}, \frac{\sqrt{5}+i\sqrt{3}}{-2(1-\sqrt{5})})$. Consider the curve F_B on $W\Gamma_0(\mathfrak{p})\setminus\mathcal{H}^2$ defined as the image of the curve

$$\widetilde{F_B} = \left\{ (z_1, z_2) : \begin{pmatrix} z_2 & 1 \end{pmatrix} B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\} \tag{2.5.22}$$

and let F'_B denote the strict transform of F_B in $X_0^+(\mathfrak{p})$, the desingularity of $W\Gamma_0(\mathfrak{p})\setminus\mathcal{H}^2 \cup \mathbb{P}^1(E)$. Let $B = \begin{pmatrix} 0 & \frac{(1-\sqrt{5})\sqrt{5}}{2} \\ \frac{(1+\sqrt{5})\sqrt{5}}{2} & 0 \end{pmatrix}$. Then the two elliptic points noted above lie on F_B. The stabilizer Γ_B of B in $\Gamma_0(\mathfrak{p})$ is a degree 2 extension of the group

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{p}) : a, d \in \mathbb{Z}, c \in (1+\sqrt{5})\sqrt{5}\mathbb{Z}, b \in \frac{(1-\sqrt{5})\sqrt{5}}{2}\mathbb{Z} \right\} \tag{2.5.23}$$

generated by $\begin{pmatrix} \sqrt{5} & \sqrt{5}-1 \\ 1+\sqrt{5} & \sqrt{5} \end{pmatrix}$. The stabilizer of B in $W\Gamma_0(\mathfrak{p})$ is a degree 2 extension of Γ_B by $\begin{pmatrix} 2 & \frac{-(1-\sqrt{5})\sqrt{5}}{2} \\ -(1+\sqrt{5})\sqrt{5} & -4 \end{pmatrix}$. We find that the image of $\widetilde{F_B}$ in $W\Gamma_0(\mathfrak{p})\setminus\mathcal{H}^2$ is a quotient of $\Gamma_0(10\mathbb{Z})\setminus\mathcal{H}$ and thus F'_B is a rational curve in $X_0^+(\mathfrak{p})$. Consider the intersection of F'_B with the local Chern cycles. Following the method in [28, V.2] we find that the intersection number of F'_B with the cusp resolution is 2. Thus we have

$$c_1(X_0^+(\mathfrak{p})).F'_B = -2 \cdot \frac{1}{6} \cdot \frac{18}{4} + 2 + \frac{1}{3} \cdot n_3 + \frac{1}{2} \cdot n_4 \tag{2.5.24}$$

where n_3 is the number of $(3;1,1)$-points that F'_B passes through and n_4 is the number of $(4;1,1)$-points that F'_B passes through. As intersection number is an integer, we are force to have $n_3 = 0$ and $n_4 = 1$ and thus $c_1(X_0^+(\mathfrak{p})).F'_B = 1$. By Adjunction formula, $F'^2_B = -1$. We get a linear configuration of rational curves with self-intersection numbers -2, -1, -2, where the (-2)-curves come from desingularity of the $(3;1,-1)$- and the $(4;1,-1)$-points mentioned above. After blowing down F_B we acquire two intersecting (-1)-curves and this shows that the surface $X_0^+(\mathfrak{p})$ is a rational surface.

2.5.4.2 $D = 13$

Consider the Hilbert modular surface $\mathrm{PSL}_2(\mathcal{O}_E) \setminus \mathcal{H}^2$ where $E = \mathbb{Q}(\sqrt{13})$. The cusp resolution is of type $(5; 2, 2)$ and the rational curves are labelled as S_0, S_1 and S_2. Following the method in [5], we can locate all the $\mathrm{PSL}_2(\mathcal{O}_E)$-inequivalent elliptic points.

It is easy to find the $\Gamma_0(\mathfrak{p})$-inequivalent elliptic points from right coset decomposition $\mathrm{PSL}_2(\mathcal{O}_E) = \cup_\alpha \Gamma_0(\mathfrak{p}) g_\alpha \cup \Gamma_0(\mathfrak{p}) g_\infty$ where $g_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ and $g_\infty = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with α running over a set of representatives of $\mathcal{O}_E/\mathfrak{p}$.

$D = 13$, $\mathfrak{p} = (4 + \sqrt{13})$ Suppose $\mathfrak{p} = (4 + \sqrt{13})$. Then we list one generator of isotropy group for each $\Gamma_0(\mathfrak{p})$-inequivalent elliptic point:

Type	Generator of Isotropy Group
$(3; 1, 1)$	$\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$
$(3; 1, 1)$	$\begin{pmatrix} (-1+\sqrt{13})/2 & -2 \\ (5-\sqrt{13})/2 & (3-\sqrt{13})/2 \end{pmatrix}$
$(3; 1, -1)$	$\begin{pmatrix} 2 & (-1+\sqrt{13})/2 \\ -(1+\sqrt{13})/2 & -1 \end{pmatrix}$
$(3; 1, -1)$	$\begin{pmatrix} (5+\sqrt{13})/2 & (3+\sqrt{13})/2 \\ -(1+\sqrt{13}) & -(3+\sqrt{13})/2 \end{pmatrix}$

Since there cannot be any elliptic points with isotropy group of 6 for $W\Gamma_0(\mathfrak{p})$ acting on $\mathcal{H}^2$. The Atkin-Lehner operator exchanges the two $(3; 1, 1)$-points (resp. $(3; 1, -1)$-points).

Now consider the curve F_B on $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2$ defined as in (2.5.22) and set $B = \left(\begin{smallmatrix} 0 & 4-\sqrt{13} \\ -4-\sqrt{13} & 0 \end{smallmatrix}\right)$. Still let F_B denote the closure of F_B in $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$ and let F'_B denote the strict transform of F_B in $X_0^+(\mathfrak{p})$, the desingularity of $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$.

The elements of $W\Gamma_0(\mathfrak{p})$ that stabilize $\widetilde{F_B}$ are of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{Z}$, $c \in (4 + \sqrt{13})\mathbb{Z}$ and $b \in (4 - \sqrt{13})\mathbb{Z}$ and determinant 1. Thus we find that $F'_B \cong \Gamma_0(3\mathbb{Z}) \setminus \mathcal{H}$ which is of genus 0. Furthermore $c_1(X_0^+(\mathfrak{p})).F'_B = 2\,\mathrm{vol}(F'_B) + \sum Z_x.F'_B$ where Z_x is the local Chern cycle at x a singular point. We compute that

$$c_1(X_0^+(\mathfrak{p})).F'_B = -2 \cdot \frac{1}{6} \cdot 4 + 2 + \frac{1}{3} \cdot n \tag{2.5.25}$$

with n the number of $(3;1,1)$-points that F'_B passes through. As there is just one $(3;1,1)$-point, we are forced to have $n = 1$ and thus $c_1(X_0^+(\mathfrak{p})).F'_B = 1$. By Adjunction formula $F'^2_B = -1$. We also find that F'_B intersects with the cusp resolution: $F'_B.S_1 = F'_B.S_2 = 1$. Note that S_1 and S_2 have self-intersection number -2. After blowing down F'_B we get two intersecting (-1)-curves. Again by an algebraic geometry criterion, the surface $W\Gamma_0((4+\sqrt{13}))\setminus\mathcal{H}^2$ is a rational surface.

$D = 13$, $\mathfrak{p} = (2)$ Suppose $\mathfrak{p} = (2)$. We have two inequivalent $(2;1,1)$-points namely $((i+1)/2, (i+1)/2)$ and $((i+1)/(3+\sqrt{13}), (i-1)/(-3+\sqrt{13}))$, four inequivalent $(3;1,1)$-points and four inequivalent $(3;1,-1)$-points. The Atkin-Lehner operator fixes the $(2;1,1)$-points and exchanges $(3;1,1)$-points (resp. $(3;1,-1)$-points). It is easy to check that we get one $(4;1,1)$-, one $(4;1,-1)$-, two $(3;1,1)$- and two $(3;1,-1)$-points and some new $(2;1,1)$-points. We compute that $c_1(X_0^+(\mathfrak{p}))^2 = 2\cdot 2\cdot\frac{1}{6}\cdot\frac{5}{2} - \frac{1}{3}\cdot 2 - 1 = -3$ and $c_2(X_0^+(\mathfrak{p})) = 2\cdot\frac{1}{6}\cdot\frac{5}{2} + (1+\frac{1}{2})a_2 + (1+\frac{2}{3})2 + (2+\frac{2}{3})2 + (1+\frac{3}{4})1 + (3+\frac{3}{4})1 = 18 + \frac{3}{2}a_2$. The Euler characteristic $\chi(X_0^+(\mathfrak{p})) = (c_1(X_0^+(\mathfrak{p}))^2 + c_2(X_0^+(\mathfrak{p})))/12 = (15+\frac{3}{2}a_2)/12 \geq 2$. Thus $X_0^+(\mathfrak{p})$ cannot be a rational surface.

Consider the curve F'_B with

$$B = \begin{pmatrix} 0 & 4-\sqrt{13} \\ -4-\sqrt{13} & 0. \end{pmatrix} \tag{2.5.26}$$

The stabilizer Γ_B of $\widetilde{F_B}$ in $\Gamma_0(\mathfrak{p})$ consists of elements of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{Z}$, $c \in (2(4+\sqrt{13}))\mathbb{Z}$ and $b \in (4-\sqrt{13})\mathbb{Z}$ and determinant 1. The stabilizer of $\widetilde{F_B}$ in $W\Gamma_0(\mathfrak{p})$ is a degree 2 extension of Γ_B generated by $\left(\begin{smallmatrix} 2 & 4-\sqrt{13} \\ -2(4+\sqrt{13}) & -2 \end{smallmatrix}\right)$. Thus we find that $\Gamma_0(6\mathbb{Z})\setminus\mathcal{H}$ is a degree 2 cover of F'_B. Thus F'_B is of genus 0. We compute that

$$c_1(X_0^+(\mathfrak{p})).F'_B = -2\cdot\frac{1}{6}\cdot\frac{12}{2} + 2 + \frac{1}{3}\cdot n_3 + \frac{1}{2}\cdot n_4 \tag{2.5.27}$$

where n_3 is the number of $(3;1,1)$-points that F'_B passes through and n_4 is the number of $(4;1,1)$-points that F'_B passes through. As there are two $(3;1,1)$-points and one $(4;1,1)$-point on $X_0^+(\mathfrak{p})$ we are forced to have that $n_3 = 0$ and $n_4 = 0$. Thus $c_1(X_0^+(\mathfrak{p})) = 0$. By

Adjunction formula, $F_B'^2 = -2$. We have the configuration of (-2)-curves consisting of F'_B, S_1 and S_2 such that $F'_B.S_1 = F'_B.S_2 = S_1.S_2 = 1$. This cannot occur on a surface of general type by [28, Prop. VII.2.7]. Hence in this example we find a surface which is birationally equivalent to a K3, an Enrique surface or an honestly elliptic surface.

Chapter 3

Rallis Inner Product Formula

In this chapter we study the case of the Rallis inner product formula that relates the pairing of theta functions to the central value of Langlands L-function. We study the Siegel-Weil formula first, as it is a key ingredient in the proof.

We consider the dual reductive pair $H = \mathrm{O}(V)$ and $G = \mathrm{Sp}(2n)$ with n the rank of the symplectic group. We use $\widetilde{G(\mathbb{A})}$ to denote the metaplectic group which is a double cover of $G(\mathbb{A})$. Let V be a vector space over a number field k with the quadractic form Q and let m be the dimension of the vector space and r its Witt index. Set $s_0 = (m-n-1)/2$. Form the Siegel Eisenstein series $E(g, s, f_\Phi)$ and the theta integral $I(g, \Phi)$ for $g \in \widetilde{G(\mathbb{A})}$ and Φ in the Schwartz space $\mathcal{S}_0(V^n(\mathbb{A}))$. The Eisenstein series can be meromorphically continued to the whole s-plane. The theta integral is not necessarily convergent and we will use Ichino's[8] regularized theta integral $I_{\mathrm{REG}}(g, \Phi)$. Please see Sec. 3.1 for further notations. Roughly speaking, the Siegel-Weil formula gives the relation between the value or the residue at s_0 of the Siegel Eisenstein series and the regularized theta integral.

When the Eisenstein series and the theta integral are both absolutely convergent, Weil[30] proved the formula in great generality. In the case where the groups under consideration are orthogonal group and the metaplectic group, Weil's condition for absolute convergence for the theta integral is that $m - r > n + 1$ or $r = 0$. The Siegel Eisenstein series $E(g, s, f_\Phi)$ is absolutely convergent for $\mathrm{Re}\, s > (n+1)/2$, so if $m > 2n + 2$, $E(g, s, f_\Phi)$ is absolutely convergent at $s_0 = (m - n - 1)/2$. Then assuming only the absolute convergence of theta integral, i.e., $m - r > n + 1$ or $r = 0$, Kudla and Rallis in [15] and [16] proved that the analytic continued Siegel Eisenstein series is holomorphic at s_0 and showed that the Siegel-Weil formula holds between the value at s_0 of the Siegel Eisenstein series and the theta integral.

In [17] Kudla and Rallis introduced the regularized theta integral to remove the requirement of absolute convergence of the theta integral. The formula then relates the residue of Siegel Eisenstein series at s_0 with the leading term of the regularized theta integral. However they worked under the condition that m is even, in which case the metaplectic group splits. The regularized theta integral is actually associated to V_0, the complementary space of V if $n+1 < m \leq 2n$ and $m - r \leq n+1$ with V isotropic. In the case $m = n+1$ the Eisenstein series is holomorphic at $s_0 = 0$ and the formula relates the value of Eisenstein series at s_0 to the leading term of the regularized theta integral associated to V. Note that in the above summary we excluded the split binary case for clarity.

For m odd Ikeda in [10] proved an analogous formula. However his theta integral does not require regularization since he assumed that the complementary space V_0 of V is anisotropic in the case $n+1 < m < 2n+2$ or that V is anisotropic in the case $m = n+1$. The method for regularizing theta integral was generalized by Ichino[8]. Instead of using differential operator at a real place as in [17], he used a Hecke operator at a finite place and thus did away with the assumption that the ground field k has a real place. In Ichino's notation the Siegel-Weil formula is a relation between the residue at s_0 of the Siegel Eisenstein series and the regularized theta integral $I_{\mathrm{REG}}(g, \Phi)$ itself. He considered the case where $n+1 < m \leq 2n+2$ and $m - r \leq n+1$ with no parity restriction on m. The interesting case $m = n+1$ with m odd, however, is still left open.

The case of Rallis inner product formula we are concerned with involves the orthogonal group $\mathrm{O}(V)$ with V a quadratic space of dimension $2n'+1$ and the symplectic group $\mathrm{Sp}(2n')$ of rank n'. Via the doubling method, to compute the inner product we ultimately need to apply the Siegel-Weil formula with $m = 2n'+1$ and $n = 2n'$. Note that m is odd here. We show that the pairing of theta functions is related to the central value of an L-function.

The idea of proof originates from Kudla and Rallis's paper[17]. We try to show the identity by comparing the Fourier coefficients of the Siegel Eisenstein series and regularized theta integral. By showing that a certain representation is nonsingular (c.f. Section 3.5) we can find a Schwartz function on $\mathrm{Sym}_n(k_v)$ for some finite place v of k to kill the singular

Fourier coefficients of the automorphic form A which is the difference of $E(g,s,\Phi)|_{s=s_0}$ and $2I_{\mathrm{REG}}(g,\Phi)$. The constant 2 is with respect to some normalization of Haar measures. Then via the theory of Fourier-Jacobi coefficients we are able to show the nonsingular Fourier coefficients of A actually vanish. Then by a density argument we show that $A=0$.

Finally via the new case of Rallis inner product formula we show the relation between nonvanishing of L-value and the nonvanishing of theta lifts.

3.1 Notations and Preliminaries

Let k be a number field and $\mathbb{A}$ its adele ring. Let U be a vector space of dimension m over k with quadratic form Q. We view the vectors in U as column vectors. The associated bilinear form on U is denoted by $\langle,\rangle_Q$ and it is defined by $\langle x,y\rangle_Q = Q(x+y)-Q(x)-Q(y)$. Thus $Q(x)=\frac{1}{2}\langle x,x\rangle_Q$. Let r denote the Witt index of Q, i.e., the dimension of a maximal isotropic subspace of U. Let $H=O(U)$ denote the orthogonal group of (U,Q) and $G=\mathrm{Sp}(2n)$ the symplectic group of rank n. Let $\widetilde{G(\mathbb{A})}$ be the metaplectic group which is a double cover of $G(\mathbb{A})$ and fix a non-trivial additive character ψ of $\mathbb{A}/k$ and set $\psi_S(\cdot)=\psi(S\cdot)$ for $S\in k$. Locally the multiplication law of $\widetilde{G(k_v)}$ is given by

$$(g_1,\zeta_1)(g_2,\zeta_2)=(g_1g_2,c(g_1,g_2)\zeta_1\zeta_2).$$

where $\zeta_i \in \{\pm 1\}$ and $c(g_1,g_2)$ is Rao's 2-cocycle on $G(k_v)$ with values in $\{\pm 1\}$. The properties of c can be found in [24, Theorem 5.3]. There the factor $(-1,-1)^{\frac{j(j+1)}{2}}$ should be $(-1,-1)^{\frac{j(j-1)}{2}}$ as pointed out, for example, in [14, Remark 4.6].

Via the Weil representation ω, $\widetilde{G(\mathbb{A})}\times H(\mathbb{A})$ acts on the space of Schwartz functions

$\mathcal{S}(U^n(\mathbb{A}))$. Locally it is characterized by the following properties (see e.g., [14, Prop. 4.3]):

$$\omega_v((\begin{pmatrix} A & \\ & {}^{\mathrm{t}}A^{-1} \end{pmatrix}, \zeta))\Phi(X) = \chi_v(\det A, \zeta)|\det A|_v^{m/2}\Phi(XA),$$

$$\omega_v((\begin{pmatrix} 1_n & B \\ & 1_n \end{pmatrix}, \zeta))\Phi(X) = \zeta^m \psi_v(\frac{1}{2}\operatorname{tr}(\langle X, X\rangle_{Q_v} B))\Phi(X),$$

$$\omega_v((\begin{pmatrix} & -1_n \\ 1_n & \end{pmatrix}, \zeta))\Phi(X) = \zeta^m \gamma_v(\psi_v \circ Q_v)^{-n}\mathcal{F}\Phi(-X)$$

$$\omega(h)\Phi(X) = \Phi(h^{-1}X)$$

where $\Phi \in \mathcal{S}(U^n(k_v))$, $X \in U^n(k_v)$, $A \in \mathrm{GL}_n(k_v)$, $B \in \mathrm{Sym}_n(k_v)$, $\zeta \in \{\pm 1\}$ and $h \in H(k_v)$. Here γ_v is the Weil index of the character of second degree $x \mapsto \psi_v \circ Q_v(x)$ and has values in 8-th roots of unity. The matrix $\langle X, X\rangle_{Q_v}$ has $\langle X_i, X_j\rangle_{Q_v}$ as ij-th entry if we write $X = (X_1, \ldots, X_n)$ with X_i column vectors in $U(k_v)$. The Fourier transform of Φ with respect to ψ_v and Q_v is defined to be

$$\mathcal{F}\Phi(X) = \int_{U^n(k_v)} \psi_v(\operatorname{tr}\langle X, Y\rangle_{Q_v})\phi(Y)dY$$

and

$$\chi_v(a, \zeta) = \zeta^m (a, (-1)^{\frac{m(m-1)}{2}} \det \langle\ ,\ \rangle_{Q_v})_{k_v} \cdot \begin{cases} \gamma_v(a, \psi_{v,1/2})^{-1} & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases} \tag{3.1.1}$$

for $a \in k_v^\times$ and $\zeta \in \{\pm 1\}$. Here $(\ ,\ )_{k_v}$ denote the Hilbert symbol and $\det\langle\ ,\ \rangle_{Q_v}$ is the determinant of the symmetric bilinear form on $U(k_v)$.

Define the theta function

$$\Theta(g, h; \Phi) = \sum_{u \in U^n(k)} \omega(g, h)\Phi(u)$$

where $g \in \widetilde{G(\mathbb{A})}$, $h \in H(\mathbb{A})$ and $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ and consider the integral

$$I(g, \Phi) = \int_{H(k) \setminus H(\mathbb{A})} \Theta(g, h; \Phi) dh.$$

It is well-known that this integral is absolutely convergent for all $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ if either $r = 0$ or $m - r > n + 1$. Thus in the case considered in this paper we will need to regularize the theta integral unless Q is anisotropic.

Let P be the Siegel parabolic subgroup of G, N the unipotent part and $\widetilde{K_G}$ the standard maximal compact subgroup of $\widetilde{G(\mathbb{A})}$. For $g \in \widetilde{G(\mathbb{A})}$ write $g = m(A)nk$ with $A \in \mathrm{GL}_n(\mathbb{A})$, $n \in N$ and $k \in \widetilde{K_G}$. Set $a(g) = \det A$ in any such decomposition of g and it is well-defined. The Siegel-Weil section associate to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ is defined to be

$$f_\Phi(g, s) = |a(g)|^{s - s_0} \omega(g) \Phi(0),$$

where $s_0 = (m - n - 1)/2$. Then the Eisenstein series

$$E(g, s, f_\Phi) = \sum_{\gamma \in P(k) \setminus G(k)} f_\Phi(\gamma g, s)$$

is absolutely convergent for $\mathrm{Re}(s) > (n + 1)/2$ and has meromorphic continuation to the whole s-plane if Φ is $\widetilde{K_G}$-finite. In the case where $m = n + 1$, $E(g, s, f_\Phi)$ is holomorphic at $s = (m - n - 1)/2 = 0$[8, Page 216].

Let $\mathcal{S}_0(U^n(\mathbb{A}))$ denote the $\widetilde{K_G}$-finite part of $\mathcal{S}(U^n(\mathbb{A}))$. We will show, under some normalization of Haar measures, the following

Theorem 3.1.1. *Assume that $m = n + 1$ and exclude the split binary case. Then*

$$E(g, s, f_\Phi)|_{s=0} = 2 I_{\mathrm{REG}}(g, \Phi)$$

for all $\Phi \in \mathcal{S}_0(U^n(\mathbb{A}))$.

Remark 3.1.2. The regularized theta integral I_{REG} will be defined in Section 3.2.

3.2 Regularization of Theta Integral

The results concerning the regularization of theta integrals summarized in this section are due to Ichino[8, Section 1]. We consider the case where the theta integral is not necessarily absolutely convergent for all $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ i.e., if Q is isotropic and $m - r \leq n + 1$.

Take v a finite place of k and temporarily suppress it from notation. If $2 \nmid q$ then there is a canonical splitting of $\widetilde{G}$ over K_G, the standard maximal compact subgroup of G. Identify K_G with the image of the splitting. Let $\mathcal{H}_G$ and $\mathcal{H}_H$ denote the spherical Hecke algebras of $\widetilde{G}$ and H:

$$\mathcal{H}_G = \{\alpha \in \mathcal{H}(\widetilde{G}//K_G) | \alpha(\epsilon g) = \epsilon^m \alpha(g) \text{ for all } g \in \widetilde{G}\},$$
$$\mathcal{H}_H = \mathcal{H}(H//K_H)$$

where $\epsilon = (\mathbf{1}_{2n}, -1) \in \widetilde{G}$.

Proposition 3.2.1. *Assume $m \leq n + 1$ and $r \neq 0$. Fix $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ and choose a good place v for Φ. Then there exists a Hecke operator $\alpha \in \mathcal{H}_{G_v}$ satisfying the following conditions:*

1. *$I(g, \omega(\alpha)\Phi)$ is absolutely convergent for all $g \in \widetilde{G(\mathbb{A})}$;*

2. *$\theta(\alpha).\mathbf{1} = c_\alpha.\mathbf{1}$ with $c_\alpha \neq 0$.*

Remark 3.2.2. For the definition of good place please refer to [8, Page 209]. Here θ is an algebra homomorphism between the Hecke algebras $\mathcal{H}_{G_v}$ and $\mathcal{H}_{H_v}$ such that $\omega_Q(\alpha) = \omega_Q(\theta(\alpha))$ as in [8, Prop 1.1]. The trivial representation of H is denoted by $\mathbf{1}$ here.

Definition 3.2.3. Define the regularized theta integral by

$$I_{\mathrm{REG}}(g, \Phi) = c_\alpha^{-1} I(g, \omega(\alpha)\Phi).$$

Remark 3.2.4. Also write $I_{\mathrm{REG}}(g, \Phi) = I(g, \Phi)$ for Q anisotropic. The above definition is independent of the choice of v and α.

Let $\mathcal{S}(U^n(\mathbb{A}))_{\mathrm{abc}}$ denote the subspace of $\mathcal{S}(U^n(\mathbb{A}))$ consisting of Φ such that $I(g,\Phi)$ is absolutely convergent for all g. Then I defines an $H(\mathbb{A})$-invariant map

$$I : \mathcal{S}(U^n(\mathbb{A}))_{\mathrm{abc}} \to \mathcal{A}^\infty(G)$$

where $\mathcal{A}^\infty$ is the space of smooth automorphic forms on $\widetilde{G(\mathbb{A})}$ (left-invariant by $G(k)$) without the $\widetilde{K_G}$-finiteness condition.

Proposition 3.2.5. [8, Lemma 1.9] *Assume $m \leq n+1$. Then I_{REG} is the unique $H(\mathbb{A})$-invariant extension of I to $\mathcal{S}(U^n(\mathbb{A}))$.*

3.3 Siegel Eisenstein Series

Now we define the Siegel Eisenstein series. Let χ be a character of $\widetilde{P(\mathbb{A})}$. Let $I(\chi,s)$ denote the induced representation $\mathrm{Ind}_{\widetilde{P(\mathbb{A})}}^{\widetilde{G(\mathbb{A})}} \chi|\det|^s$. A function $f(g,s)$ on $\widetilde{G(\mathbb{A})} \times \mathbb{C}$ is said to be a holomorphic section of $I(\chi,s)$ if

1. $f(g,s)$ is holomorphic with respect to s for each $g \in \widetilde{G(\mathbb{A})}$,
2. $f(pg,s) = \chi(p)|a(p)|^{s+(n+1)/2} f(g,s)$ for $p \in \widetilde{P(\mathbb{A})}$ and $g \in \widetilde{G(\mathbb{A})}$ and
3. $f(\cdot,s)$ is $\widetilde{K_G}$-finite.

For f a holomorphic section of $I(\chi,s)$ we form the Siegel Eisenstein series

$$E(g,s,f) = \sum_{\gamma \in P(k)\setminus G(k)} f(\gamma g, s).$$

Note that $\widetilde{G(\mathbb{A})}$ splits over $G(k)$.

We will specialize to the case where χ is the character associated to the χ in (3.1.1):

$$(p,z) \mapsto \chi(a(p),z). \tag{3.3.1}$$

Still denote this character by χ. For $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ set

$$f_\Phi(g,s) = |a(g)|^{s - \frac{m-n-1}{2}} \omega(g)\Phi(0).$$

Then f_Φ is a holomorphic section of $I(\chi, s)$. The Eisenstein series $E(g, s, f_\Phi)$ is absolutely convergent for $\mathrm{Re}(s) > (n+1)/2$ and has meromorphic continuation to the whole s-plane. From [8, Page 216] we know that if $m = n+1$, $E(g, s, f_\Phi)$ is holomorphic at $s = 0$.

The following definition will be useful later.

Definition 3.3.1. A holomorphic section $f \in I(\chi, s)$ is said to be a weak SW section associated to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ if $f(g, \frac{m-n-1}{2}) = \omega(g)\Phi(0)$.

Define similarly $I_v(\chi_v, s)$ in the local cases. Fix one place v. For $w \neq v$, fix $\Phi_w \in \mathcal{S}(U^n(k_w))$ and let $f_w^0(g_w, s)$ be the associated holomorphic sections where we suppress the subscript Φ_w. Then if $m = n+1$ we have the map

$$\begin{aligned} &I_v \to \mathcal{A} \\ &f_v \mapsto E(g, s, f_v \otimes (\otimes_{w \neq v} f_w^0))|_{s=0}. \end{aligned} \tag{3.3.2}$$

Then by [16, Prop. 2.2] this map is $\widetilde{G_v}$-intertwining if v is finite or $(\mathfrak{g}_v, \widetilde{K_v})$-intertwining if v is archimedean.

3.4 Fourier-Jacobi Coefficients

A key step in the proof of Siegel-Weil formula is the comparison of the B-th Fourier coefficients of the Eisenstein series and the regularized theta integral where B is a nonsingular symmetric matrix. It is easy to show that the B-th Fourier coefficient of the Eisenstein series is a product of Whittaker functions. In the case where m is even by [29] and [11] the Whittaker functions can be analytically continued to the whole complex plane. Also true is the case where $n = 1$ and m arbitrary. However in the case m odd this is not fully known. To work around the problem Ikeda[10] used Fourier-Jacobi coefficients to initiate an induction process. The B-th Fourier coefficients can be calculated from the Fourier-Jacobi coefficients from lower dimensional objects.

We generalize the calculation done in [8] and in [10]. First we introduce some subgroups of G, describe Weil representation realized on some other space and then define the Fourier-

Jacobi coefficients. The exposition closely follows that in [8]. Put

$$V = \left\{ v(x,y,z) = \left(\begin{array}{cc|cc} 1 & x & z & y \\ & 1_{n-1} & {}^{\mathrm{t}}y & \\ \hline & & 1 & \\ & & -{}^{\mathrm{t}}x & 1_{n-1} \end{array} \right) \middle| \, x, y \in k^{n-1}, z \in k \right\},$$

$$Z = \{ v(0,0,z) \in V \},$$

$$W = \{ v(x,y,0) \in V \},$$

$$L = \{ v(x,0,0) \in V \},$$

$$G_1 = \left\{ \left(\begin{array}{cc|cc} 1 & & & \\ & a & & b \\ \hline & & 1 & \\ & c & & d \end{array} \right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{n-1} \right\},$$

$$N_1 = \left\{ \left(\begin{array}{cc|cc} 1 & & & \\ & 1_{n-1} & & n_1 \\ \hline & & 1 & \\ & & & 1_{n-1} \end{array} \right) \middle| \, n_1 \in \mathrm{Sym}_{n-1} \right\}.$$

Then $V = W \oplus Z$ is a Heisenberg group with centre Z and the symplectic form on W is set to be $\langle v(x_1, y_1, 0), v(x_2, y_2, 0) \rangle_W = 2(x_1 {}^{\mathrm{t}}y_2 - y_1 {}^{\mathrm{t}}x_2)$. Here the coefficient 2 is added to facilitate later computation. We set $\langle x, y \rangle = 2x{}^{\mathrm{t}}y$, for x and y row vectors of length $n-1$. Sometimes we identify L with row vectors of length $n-1$. The Schrödinger representation ω of $V(\mathbb{A})$ with central character ψ can be realized on the Schwartz space $\mathcal{S}(L(\mathbb{A}))$:

$$\omega(v(x,y,z))\phi(t) = \phi(t+x)\psi(z + \langle t, y \rangle + \frac{1}{2}\langle x, y \rangle)$$

for $\phi \in \mathcal{S}(L(\mathbb{A}))$. By the Stone-von Neumann theorem, ω is irreducible and unique up to isomorphism. Moreover the Schrödinger representation ω of $V(\mathbb{A})$ naturally extends to the Weil representation ω of $V(\mathbb{A}) \rtimes \widetilde{G_1(\mathbb{A})}$ on $\mathcal{S}(L(\mathbb{A}))$. Let $\widetilde{K_{G_1}}$ denote the standard maximal compact subgroup of $\widetilde{G_1(\mathbb{A})}$ and $\mathcal{S}_0(L(\mathbb{A}))$ the $\widetilde{K_{G_1}}$-finite vectors in $\mathcal{S}(L(\mathbb{A}))$.

For each $\phi \in \mathcal{S}(L(\mathbb{A}))$ define the theta function

$$\vartheta(vg_1, \phi) = \sum_{t \in L(k)} \omega(vg_1)\phi(t)$$

for $v \in V(\mathbb{A})$ and $g_1 \in \widetilde{G_1(\mathbb{A})}$. Suppose that A is an automorphic form on $\widetilde{G(\mathbb{A})}$. Then define a function on $G_1(k) \setminus \widetilde{G_1(\mathbb{A})}$ by

$$\mathrm{FJ}^{\phi}(g_1; A) = \int_{V(k) \setminus V(\mathbb{A})} A(vg_1)\overline{\vartheta(vg_1, \phi)}dv.$$

For $\beta \in \mathrm{Sym}_{n-1}(k)$, let $\mathrm{FJ}^{\phi}_{\beta}(g_1; A)$ be the β-th Fourier coefficient of $\mathrm{FJ}^{\phi}(g_1; A)$.

Suppose that the bilinear form $\langle\ ,\ \rangle_Q$ is equal to $\langle\ ,\ \rangle_S + \langle\ ,\ \rangle_{Q_1}$ where S and Q_1 are quadratic forms of dimension 1 and $n-1$ respectively. Decompose accordingly $U = k \oplus U_1$. Note that $\langle x, y\rangle_S = 2Sxy$. Let $H_1 = \mathrm{O}(U_1)$. With this setup we will use the character ψ_S in the Schrödinger model instead of ψ.

Lemma 3.4.1. [8, Lemma 4.1] *Let $S \in k^{\times}$ and $\beta \in \mathrm{Sym}_{n-1}(k)$. Let A be an automorphic form on $\widetilde{G(\mathbb{A})}$, and assume that $\mathrm{FJ}^{\phi}_{\beta}(g_1; \rho(f)A) = 0$ for all $\phi \in \mathcal{S}_0(L(\mathbb{A}))$ and all $f \in \mathcal{H}(\widetilde{G(\mathbb{A})})$. Then $A_B = 0$ for*

$$B = \begin{pmatrix} S & \\ & \beta. \end{pmatrix}$$

Proof. For $n \in N$ we set $b(n)$ to be the upper-right block of n and set $b_1(n)$ to be the lower-right block of size $(n-1)\times(n-1)$ of $b(n)$. We compute

$$\begin{aligned}
&\mathrm{FJ}^{\phi}_{\beta}(g_1, A)\\
&= \int_{N_1(k)\backslash N_1(\mathbb{A})}\int_{V(k)\backslash V(\mathbb{A})} A(vn_1g_1)\overline{\vartheta(vn_1g_1,\phi)}\psi(-\operatorname{tr}(b_1(n_1)\beta))dvdn_1\\
&= \int_{L(k)\backslash L(\mathbb{A})}\int_{N(k)\backslash N(\mathbb{A})} A(nxg_1)\overline{\vartheta(nxg_1)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(k)\backslash L(\mathbb{A})}\int_{N(k)\backslash N(\mathbb{A})}\sum_{t\in L(k)} A(nxg_1)\overline{\omega(tnxg_1)\phi(0)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(k)\backslash L(\mathbb{A})}\sum_{t\in L(k)}\int_{N(k)\backslash N(\mathbb{A})} A(ntxg_1)\overline{\omega(ntxg_1)\phi(0)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(\mathbb{A})}\int_{N(k)\backslash N(\mathbb{A})} A(nxg_1)\overline{\omega(nxg_1)\phi(0)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(\mathbb{A})}\int_{N(k)\backslash N(\mathbb{A})} A(nxg_1)\overline{\omega(g_1)\phi(x)\psi_S(z)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(\mathbb{A})}\int_{N(k)\backslash N(\mathbb{A})} A(nxg_1)\overline{\omega(g_1)\phi(x)}\psi(-\operatorname{tr}(b(n)B))dndx\\
&= \int_{L(\mathbb{A})} A_B(xg_1)\overline{\omega(g_1)\phi(x)}dx.
\end{aligned}$$

Since $\mathrm{FJ}^{\phi}_{\beta}(g_1, A) = 0$ for all $g_1 \in \widetilde{G_1(\mathbb{A})}$ we conclude that $A_B(g_1) = 0$ for all $g_1 \in \widetilde{G_1(\mathbb{A})}$. Then we apply a sequence of $f_i \in \mathcal{H}(\widetilde{G(\mathbb{A})})$ that converges to the Dirac delta at $g \in \widetilde{G(\mathbb{A})}$ to conclude that $A_B(g) = 0$ for all $g \in \widetilde{G(\mathbb{A})}$. □

3.4.1 Fourier-Jacobi coefficients of the regularized theta integrals

Now we consider the Fourier-Jacobi coefficients of the regularized theta integrals

$$\mathrm{FJ}^{\phi}(g_1; I_{\mathrm{REG}}(\Phi)) = c_\alpha^{-1}\int_{V(k)\backslash V(\mathbb{A})}\int_{H(k)\backslash H(\mathbb{A})}\Theta(vg_1, h; \omega(\alpha)\Phi)\overline{\vartheta(vg_1,\phi)}dhdv.$$

Put

$$\Psi(\Phi,\phi;u) = \int_{L(\mathbb{A})}\Phi\begin{pmatrix}1 & x\\ 0 & u\end{pmatrix}\overline{\phi(x)}dx$$

for $u \in U_1^{n-1}(\mathbb{A})$. Then the map

$$\mathcal{S}(U^n(\mathbb{A})) \otimes \mathcal{S}(L(\mathbb{A})) \to \mathcal{S}(U_1^{n-1}(\mathbb{A}))$$
$$\Phi \otimes \phi \mapsto \Psi(\Phi, \phi)$$

is $\widetilde{G_1(\mathbb{A})}$-intertwining, i.e.,

$$\omega(g_1)\Psi(\Phi, \phi) = \Psi(\omega(g_1)\Phi, \omega(g_1)\phi)$$

for $g_1 \in \widetilde{G_1(\mathbb{A})}$. Notice that on $\mathcal{S}(U^n(\mathbb{A}))$ and $\mathcal{S}(U_1^{n-1}(\mathbb{A}))$ the Weil representations are associated with the character ψ and on $\mathcal{S}(L(\mathbb{A}))$ the Weil representation is associated with the character ψ_S. Then we have:

Proposition 3.4.2. *Suppose that $\beta \in \mathrm{Sym}_{n-1}(k)$ with $\det(\beta) \neq 0$. Then*

$$\mathrm{FJ}_\beta^\phi(g_1; I_{\mathrm{REG}}(\Phi))$$

is equal to the absolutely convergent integral

$$\int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} I_{\mathrm{REG},\beta}(g_1, \Psi(\omega(h)\Phi, \phi))dh.$$

Proof. We need to compute the following integral.

$$\begin{aligned}&\mathrm{FJ}_\beta^\phi(g_1; I_{\mathrm{REG}}(\Phi)) \\ =&c_\alpha^{-1} \int_{N_1(k) \setminus N_1(\mathbb{A})} \int_{V(k) \setminus V(\mathbb{A})} \int_{H(k) \setminus H(\mathbb{A})} \theta(vn_1g_1, h_0, \omega(\alpha)\Phi)\overline{\vartheta(vn_1g_1, \phi)} \\ &\times \psi(-\operatorname{tr} b_1(n_1)\beta)dh_0dvdn_1.\end{aligned} \tag{3.4.1}$$

First we consider

$$\int_{V(k) \setminus V(\mathbb{A})} \theta(vg_1, h_0, \Phi)\overline{\vartheta(vg_1, \phi)}dv. \tag{3.4.2}$$

Suppose $v = v(x,0,0)v(0,y,z)$. Then

$$
\begin{aligned}
&\theta(vg_1, h_0, \Phi) \\
&= \sum_{t\in U^n(k)} \omega(vg_1, h_0)\Phi(t) \\
&= \sum_{t\in U^n(k)} \omega(v(0,y,z)g_1, h_0)\Phi(t \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \\
&= \sum_{t\in U^n(k)} \omega(g_1, h_0)\Phi(t \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\psi(\frac{1}{2}\operatorname{tr}\left(\left\langle t\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, t\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right\rangle_Q \begin{pmatrix} z & y \\ {}^{\mathrm{t}}y & \end{pmatrix}\right)) \\
&= \sum_{t=\left(\begin{smallmatrix} t_1 & t_2 \\ t_3 & t_4 \end{smallmatrix}\right)} \omega(g_1, h_0)\Phi(t \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\psi(\frac{1}{2}\left\langle \begin{pmatrix} t_1 \\ t_3 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_3 \end{pmatrix}\right\rangle_Q (z + 2x{}^{\mathrm{t}}y)) \\
&\times \psi(\left\langle \begin{pmatrix} t_1 \\ t_3 \end{pmatrix}, \begin{pmatrix} t_2 \\ t_4 \end{pmatrix}\right\rangle_Q {}^{\mathrm{t}}y)
\end{aligned}
$$

where $t_1 \in k$, $t_2 \in k^{n-1}$, $t_3 \in U_1(k)$ and $t_4 \in U_1^{n-1}(k)$. Also we expand

$$
\begin{aligned}
&\vartheta(vg_1, \phi) \\
&= \sum_{t\in L(k)} \omega(g_1)\phi(t+x)\psi_S(z + \langle x, y\rangle + \langle t, y\rangle) \\
&= \sum_{t\in L(k)} \omega(g_1)\phi(t+x)\psi_S(z + 2x{}^{\mathrm{t}}y + \langle t, y\rangle).
\end{aligned}
$$

Thus if we integrate against z the integral (3.4.2) vanishes unless

$$\left\langle \begin{pmatrix} t_1 \\ t_3 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_3 \end{pmatrix}\right\rangle_Q = 2S.$$

By Witt's theorem there exists some $h \in H(k)$ such that

$$\begin{pmatrix} t_1 \\ t_3 \end{pmatrix} = h^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that the stabilizer of $\binom{1}{0}$ in $H(k)$ is $H_1(k)$. After changing $\binom{t_2}{t_4}$ to $h^{-1}\binom{t_2}{t_4}$ we find that (3.4.2) is equal to

$$\int_{W(k)\setminus W(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{t_2,t_4} \sum_{t\in L(k)} \omega(g_1)\omega(\alpha)\Phi(h_0^{-1}h^{-1}\begin{pmatrix}1 & t_2\\ 0 & t_4\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})$$
$$\times\,\overline{\omega(g_1)\phi(t+x)}\psi(\mathrm{tr}\left\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}t_2\\t_4\end{pmatrix}\right\rangle_Q {}^{\mathrm{t}}y)\psi_S(-\langle t,y\rangle))dxdy$$
$$=\int_{W(k)\setminus W(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{t_2,t_4} \sum_{t\in L(k)} \omega(g_1)\Phi(h_0^{-1}h^{-1}\begin{pmatrix}1 & t_2\\ 0 & t_4\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})$$
$$\times\,\overline{\omega(g_1)\phi(t+x)}\psi(2St_2{}^{\mathrm{t}}y)\psi_S(-\langle t,y\rangle))dxdy.$$

Now the integration against y vanishes unless $t=t_2$ and we get

$$=\int_{L(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{t_4}\omega(g_1)\Phi(h_0^{-1}h^{-1}\begin{pmatrix}1 & t\\ 0 & t_4\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})\overline{\omega(g_1)\phi(t+x)}dx$$
$$=\sum_{h\in H_1(k)\setminus H(k)} \sum_{t\in U_1^{n-1}(k)} \omega(g_1)\Psi(t,\Phi,\phi).$$

Then we consider the integration over $N_1(k)\setminus N_1(\mathbb{A})$ in (3.4.1). This will kill those terms such that $\langle t,t\rangle_{Q_1}\neq\beta$. Thus (3.4.1) is equal to

$$c_\alpha^{-1}\int_{H(k)\setminus H(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{\substack{t\in U_1^{n-1}(k),\\ \langle t,t\rangle_{Q_1}=\beta}} \omega(g_1)\Psi(t,\omega(hh_0)\omega(\alpha)\Phi,\phi)dh_0.$$

We assume that $2Q_1$ represents β, since otherwise the lemma obviously holds. As $\mathrm{rk}\,\beta=n-1$, $\{t\in U_1^{n-1}(k)|\langle t,t\rangle_{Q_1}=\beta\}$ is a single $H_1(k)$-orbit. Fix a representative t_0 of this orbit. Since the stabilizer of t_0 in $H_1(k)$ is of order $\kappa=2$, $\mathrm{FJ}_\beta^\phi(g_1;I_{\mathrm{REG}}(\Phi))$ is equal to

$$\kappa^{-1}c_\alpha^{-1}\int_{H(k)\setminus H(\mathbb{A})} \sum_{h\in H(k)} \omega(g_1)\Psi(\omega(hh_0)\omega(\alpha)\Phi,\phi;t_0)dh_0.$$

The convergence Lemma in [8] holds also for $m=n+1$ which is recorded here as Lemma 3.4.3.

Thus we can exchange the orders of integration in $\mathrm{FJ}^{\phi}_{\beta}(g_1; I_{\mathrm{REG}}(\Phi))$ and continue the computation to get

$$\begin{aligned}
&\kappa^{-1}c_\alpha^{-1}\int_{H(\mathbb{A}}\omega(g_1)\Psi(\omega(h)\omega(\alpha)\Phi,\phi;t_0)dh\\
=&\kappa^{-1}c_\alpha^{-1}\sum_{\gamma\in H_1(k)}\int_{H_1(k)\setminus H(\mathbb{A}}\omega(g_1)\Psi(\gamma t_0,\omega(h)\omega(\alpha)\Phi,\phi)dh\\
=&c_\alpha^{-1}\int_{H_1(k)\setminus H(\mathbb{A})}\sum_{\langle t,t\rangle_{Q_1}=\beta}\omega(g_1)\Psi(\omega(\gamma h)\omega(\alpha)\Phi,\phi;t)dh\\
=&c_\alpha^{-1}\int_{H_1(\mathbb{A})\setminus H(\mathbb{A})}\int_{H_1(k)\setminus H_1(\mathbb{A})}\sum_{\langle t,t\rangle_{Q_1}=\beta}\omega(g_1,h_1)\Psi(\omega(h)\omega(\alpha)\Phi,\phi;t)dh_1dh\\
=&\int_{H_1(\mathbb{A})\setminus H(\mathbb{A})}I_{\mathrm{REG},\beta}(g_1,\Psi(\omega(h)\Phi,\phi))dh.
\end{aligned}$$

□

Lemma 3.4.3. *1. Let $t \in U^n(k)$. If $\operatorname{rk} t = n$ then $\int_{H(\mathbb{A})}\omega(h)\Phi(t)dh$ is absolutely convergent for any $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$.*

2. Let $t_1 \in U_1^{n-1}(k)$. If $\operatorname{rk} t_1 = n-1$ then

$$\int_{H(\mathbb{A})}\Psi(\omega(h)\Phi,\phi;t_1)dh = \int_{H(\mathbb{A})}\int_{L(\mathbb{A})}\omega(h)\Phi\begin{pmatrix}1 & x\\ 0 & t_1\end{pmatrix}\overline{\phi(x)}dxdh$$

is absolutely convergent for any $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ and $\phi \in \mathcal{S}(L(\mathbb{A}))$.

Proof. The argument in [17, pp. 59-60] also includes the case $m = n+1$ and it proves (1). For (2) consider the function on $U^n(\mathbb{A})$

$$\varphi(u) = \int_{L(\mathbb{A})}\Phi(u\begin{pmatrix}1 & x\\ & 1_{n-1}\end{pmatrix})\overline{\phi(x)}dx.$$

This integral is absolutely convergent and defines a smooth function on $U^n(\mathbb{A})$. Furthermore $\varphi \in \mathcal{S}(U^n(\mathbb{A}))$. Then we apply (1) to get (2). □

3.4.2 Fourier-Jacobi coefficients of the Siegel Eisenstein series

Now we compute the Fourier-Jacobi coefficients of the Siegel Eisenstein series

$$\mathrm{FJ}^{\phi}(g_1, E(f,s)) = \int_{V(k)\setminus V(\mathbb{A})} E(vg_1, f, s)\overline{\vartheta(vg_1, \phi)}dv.$$

Let χ_1 be the character associated to ψ and Q_1 defined similarly as in (3.1.1).

Proposition 3.4.4. *For $\phi \in \mathcal{S}_0(\widetilde{G_1(\mathbb{A})})$ we have*

$$\mathrm{FJ}^{\phi}(g_1, E(f,s)) = \sum_{\gamma \in P_1(k)\setminus G_1(k)} R(\gamma g_1, f, s, \phi)$$

where

$$R(g_1, f, s, \phi) = \int_{V(\mathbb{A})} f(w_n v w_{n-1} h)\overline{\omega(v w_{n-1} h)\phi(0)}dv.$$

is a holomorphic section of $Ind_{\widetilde{P_1(\mathbb{A})}}^{\widetilde{G_1(\mathbb{A})}}(\chi_1, s)$ for $\mathrm{Re}\, s >> 0$. Furthermore $R(g_1, f, s, \phi)$ is absolutely convergent for $\mathrm{Re}\, s > -(n-3)/2$ and can be analytically continued to the domain $\mathrm{Re}\, s > -(n-2)/2$.

Proof. This was proved in [9, Theorem 3.2 and Theorem 3.3]. □

Now we will relate $R(g_1, f_\Phi, s, \phi)$ to $\Psi(g_1, \Phi, \phi)$. First we need a lemma.

Lemma 3.4.5. *Let $n = 1$ and $S \in k^\times$. Assume $m \geq 5$ or $(m,r) = (4,0)$, $(4,1)$, $(3,0)$ or $(2,0)$. Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $s_0 = \frac{m}{2} - 1$. Then*

$$\int_{\mathbb{A}} f_\Phi(w \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, s)\overline{\psi_S(z)}dz \tag{3.4.3}$$

can be meromorphically continued to the whole s-plane and is holomorphic at $s = s_0$. Its value at $s = s_0$ is 0 if Q does not represent S. If $Q = \left(\begin{smallmatrix} S & \\ & Q_1 \end{smallmatrix}\right)$ then its value at $s = s_0$ is equal to the absolutely convergent integral

$$\kappa \int_{H_1(\mathbb{A})\setminus H(\mathbb{A})} \Phi(h^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix})dh$$

where $\kappa = 2$ for $(m, r) = (2, 0)$ and $\kappa = 1$ otherwise.

Proof. The cases excluded are those where the Eisenstein series has a pole at $s = s_0$ or when the theta integral is not absolutely convergent. Then (3.4.3) is the S-th Fourier coefficient of $E(g, s, f)$ and

$$\int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \Phi(h^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix})dh$$

is the S-th Fourier coefficient of $I(g, s)$. Thus the lemma follows from the known Siegel-Weil formula for $n = 1$. Please see [10] for details. □

Proposition 3.4.6. *Assume $m = n + 1$. Also assume that $m \geq 5$ or $(m, r) = (4, 0)$, $(4, 1)$, $(3, 0)$ or $(2, 0)$. Let $\phi \in \mathcal{S}_0(L(\mathbb{A}))$ and $f_\Phi(s)$ be a holomorphic section of $I(\chi, s)$ associated to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$. If Q does not represent S then $R(g_1, f_\Phi, s, \phi) = 0$. If*

$$Q = \begin{pmatrix} S & \\ & Q_1 \end{pmatrix}$$

then

$$\mathrm{FJ}^\phi(g_1; E(s, f_\Phi))|_{s=0} = \int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} E(g_1, f_{\Psi(\omega_Q(h)\Phi, \phi)}(s))dh.$$

Proof. First we simplify $R(g_1, f_\Phi, s, \phi)$. We will suppress the subscript Φ. Suppose $v = v(x, 0, 0)v(0, y, z)$. Then $R(g_1, f, s, \phi)$ is equal to

$$\begin{aligned} &\int_{V(\mathbb{A})} f(w_n v(0, y, z)w_{n-1}g_1, s)\overline{\omega(w_{n-1}g_1)\phi(x)\psi_S(z + \langle x, y\rangle)}dv \\ = &\int_{V(\mathbb{A})} \int_{L(\mathbb{A})} f(w_n v(0, y, z)w_{n-1}g_1, s)\overline{\omega(g_1)\phi(t)\psi_S(\langle -x, t\rangle)\psi_S(z + \langle x, y\rangle)}dtdv. \end{aligned}$$

Integration against x vanishes unless $y = t$. Thus we continue

$$= \int_{\mathbb{A}} \int_{L(\mathbb{A})} f(w_n v(0, y, z)w_{n-1}g_1, s)\overline{\omega(g_1)\phi(y)\psi_S(z)}dydz.$$

Embed $\mathrm{Sp}(2)$ into $G = \mathrm{Sp}(2n)$ by

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & 1_{n-1} & & 0_{n-1} \\ c & & d & \\ & 0_{n-1} & & 1_{n-1} \end{pmatrix}$$

and denote this embedding by ι. Also denote the lift $\widetilde{\mathrm{Sp}(2)} \to \widetilde{\mathrm{Sp}(2n)}$ by ι. Then as a function of $g_0 \in \widetilde{\mathrm{Sp}(2)}$,

$$f(\iota(g_0) w_{n-1} \left(\begin{array}{c|cc} 1_n & \begin{matrix} 0 & y \\ {}^t y & 0_{n-1} \end{matrix} \\ \hline 0_n & 1_n \end{array}\right) w_{n-1} g_1)$$

is a weak SW section associated to

$$u \mapsto \omega(w_{n-1} \left(\begin{array}{c|c} 1_n & \begin{matrix} 0 & y \\ {}^t y & 0_{n-1} \end{matrix} \\ \hline 0_n & 1_n \end{array}\right) w_{n-1} g_1)) \Phi(u, 0),$$

a Schwartz function in $\mathcal{S}(U(\mathbb{A}))$. Then by lemma 3.4.5, if Q does not represent S then $R(g_1, f, 0, \phi) = 0$. If $Q = \left(\begin{smallmatrix} S & \\ & Q_1 \end{smallmatrix} \right)$ then by Lemma 3.4.5 $R(g_1, f, 0, \phi)$ is equal to

$$\int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \int_{L(\mathbb{A})} \omega(w_{n-1} \left(\begin{array}{c|c} 1_n & \begin{matrix} 0 & y \\ {}^t y & 0_{n-1} \end{matrix} \\ \hline 0_n & 1_n \end{array}\right) w_{n-1} g_1) \Phi(h^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \overline{\omega(g_1)\phi(-y)} dy dh.$$

Note the part

$$\omega(w_{n-1} \left(\begin{array}{c|c} 1_n & \begin{matrix} 0 & y \\ {}^t y & 0_{n-1} \end{matrix} \\ \hline 0_n & 1_n \end{array}\right) w_{n-1} g_1) \Phi(h^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \omega(g_1) \Phi(h^{-1} \begin{pmatrix} 1 & -y \\ 0 & 0 \end{pmatrix}).$$

Thus we find

$$R(g_1, f, 0, \phi) = \int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \omega(g_1)\Psi(0, \omega(h)\Phi, \phi)dh.$$

The calculation relies on the Siegel-Weil formula in the case $n = 1$ and $m - r > 2$ or $r = 0$ arbitrary. Thus we have to exclude certain cases where the Eisenstein series may have a pole at $s_0 = (m - 2)/2$. □

Remark 3.4.7. The cases not covered above are $(m, r) = (4, 2)$, $(3, 1)$ and $(2, 1)$. The anisotropic cases and the m even cases of the Siegel-Weil formula were dealt with in [17]. Thus we cannot go down only when we reach the $(m, r) = (3, 1)$ case.

3.5 Some Representation Theory

Now we want to study irreducible submodules of the induced representations and show that it is nonsingular in the sense of Howe[7]. In Section 3.6 we will interpret the difference $A(g, \Phi) = E(g, s, f_\Phi)|_{s=0} - 2I(g, \Phi)$ as an element in an irreducible nonsingular submodules. This forces the B-th Fourier coefficients of A to vanish if B is not of full rank.

Fix v a finite place of k and suppress it from notation. Thus k is a nonarchimedean local field for the present. We consider the various groups over k. Let χ be a quasicharacter of $\widetilde{P}$ trivial on N and form the normalised induced representation $I(\chi) = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\chi)$. Define maximal parabolic subgroups of GL_n:

$$Q_r = \left\{ \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \middle| a \in \mathrm{GL}_{n-r}, b \in \mathrm{GL}_r \right\}.$$

Here r is not related to the Witt index of Q.

Lemma 3.5.1. *The Jacquet module of $I(\chi)_N$ has an $\widetilde{M}$-stable filtration*

$$I(\chi)_N = I^0 \supset I^1 \supset \cdots \supset I^n \supset I^{n+1} = 0$$

with successive quotients

$$Z^r(\chi) = I^r / I^{r+1} \cong \mathrm{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}}(\xi_r)$$

where ξ_r is the quasicharacter of Q_r given by

$$\xi_r\left(\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}, \zeta\right)\right) = \chi\left(\left(m\left(\begin{pmatrix} a & \\ & {}^{t}b^{-1} \end{pmatrix}\right), \zeta\right)\right) |\det a|^{\frac{n+1-r}{2}} |\det b|^{\frac{r+1}{2}}.$$

Proof. We follow the proof in [18]. Choose double coset decomposition representatives w_r for $\widetilde{P} \backslash \widetilde{G} / \widetilde{P}$: for $0 \le r \le n$, let

$$w_r = \left(\begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & -1_r & 0 & 0 \end{pmatrix}, 1\right).$$

Then the relative Bruhat decomposition holds $\widetilde{G} = \amalg_{j=0}^{n} \widetilde{P} w_j \widetilde{P}$. Let $J^0 = I(\chi)$ and for $1 \le r \le n+1$, set $J^r = \left\{ f \in J^0 \middle| f = 0 \text{ on } \widetilde{P} w_{r-1} \widetilde{P} \right\}$. Alternatively set $J^{n+1} = 0$ and $J^r = \left\{ f \in I(\chi) \middle| \operatorname{supp}(f) \subset \amalg_{j=r}^{n} \widetilde{P} w_j \widetilde{P} \right\}$. Also define

$$N_r = \left\{ n\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathrm{Sym}_r(k) \right\}.$$

We check that we have a $\widetilde{P}$-intertwining map

$$\begin{aligned} J^r &\to \operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}}(\xi_r), \\ \Phi &\mapsto \left\{ \Psi : (m(a), \zeta) \mapsto \int_{N_r} \Phi(w_r n(m(a), \zeta)) dn \right\}. \end{aligned}$$

Obviously the map factors through J^{r+1} if the above is well-defined. Notice that the properties of Rao's 2-cocycle[24, Theorem 5.3] implies that $w_r n(m, \zeta) = (m', \zeta) w_r n'$ for some other elements $m' \in M$ and $n' \in N$. Standard computation then shows that Ψ is in the space of $\operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}}(\xi_r)$.

Gustafson checked in Sp_n case that the integral converges and that the map is surjective and that the kernel of the map $J^r/J^{r+1} \to \operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}} \xi_r$ is $J^r/J^{r+1}(N)$. By exactness of

Jacquet functor we get an $\widetilde{M}$-module isomorphism of J_N^r/J_N^{r+1} with the space $\mathrm{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}} \xi_r$. Setting $I^r = J_N^r$ for each r finishes the proof. □

We are interested in the case where $\chi(m(a), \zeta)$ is the one in (3.3.1).

Lemma 3.5.2. *Suppose that $\pi \subset I(\chi)$ is a $\widetilde{G}$-submodule. Then*

$$\dim \mathrm{Hom}_{\widetilde{G}}(\pi, I(\chi)) \leq 2.$$

In particular, $I(\chi)$ has at most two irreducible submodules.

Proof. The centre $\widetilde{Z}$ of $\widetilde{\mathrm{GL}_n}$ consists of elements of the form

$$(aI_n, \zeta).$$

Also note that we can view χ as a character on $\widetilde{\mathrm{GL}_n}$. Given π

$$\mathrm{Hom}_{\widetilde{G}}(\pi, I(\chi)) = \mathrm{Hom}_{\widetilde{\mathrm{GL}_n}}(\pi_N, \chi|\ |^{(n+1)/2}).$$

Now we consider the generalized eigenspaces of π_N and of $I(\chi)_N$ with respect to the action of $\widetilde{Z}$, where the eigencharacter of interest is

$$\mu(aI_n, \zeta) = \chi(aI_n, \zeta)|a|^{n(n+1)/2}.$$

On the other hand the central characters of the successive quotients $Z_r(\chi)$ of $I(\chi)_N$ are

$$(aI_n, \zeta) \mapsto \chi\left(\begin{pmatrix} aI_{n-r} & \\ & aI_r \end{pmatrix}, \zeta\right)|a|^{\frac{n^2+n-2nr-2r}{2}}.$$

If $r = 0$ then one of these coincides with μ. If $r = n$ and since $\chi(a^n, \zeta) = \chi(a^{-n}, \zeta)$ we get one more solution. Thus we get the bound

$$\dim \mathrm{Hom}_{\widetilde{G}}(\pi, I(\chi)) \leq 2.$$

□

Let $R_n(U)$ denote the image of the map

$$\mathcal{S}(U^n) \to I(\chi)$$
$$\Phi \mapsto \omega(g)\Phi(0).$$

This map induces an isomorphism $\mathcal{S}(U^n)_H \cong R_n(U)$ by [22]. Let U' be the quadratic space with the same dimension and determinant with U but with opposition Hasse invariant.

Lemma 3.5.3. *The $\widetilde{G}$-modules $R_n(U)$ and $R_n(U')$ are irreducible. Furthermore $I(\chi) \cong R_n(U) \oplus R_n(U')$,*

Proof. We have an intertwining operator

$$M : I(\chi) \to I(\chi)$$
$$f \mapsto (g \mapsto \int_N f(w_n n g) dn).$$

Thus $I(\chi)$ is unitarizable and hence completely reducible. Also by [16, Prop. 3.4] we know that $R_n(U)$ and $R_n(U')$ are inequivalent and by [16, Lemmas 3.5 and 3.6] it cannot happen that one is contained in the other. These combined with Lemma 3.5.2 force $I(\chi) \cong R_n(U) \oplus R_n(U')$ with $R_n(U)$ and $R_n(U')$ irreducible. □

Lemma 3.5.4. *Assume $m = n + 1$. Then $R_n(U)$ is a nonsingular representation of $\widetilde{G}$ in the sense of Howe[7].*

Proof. This follows from [16, Prop 3.2(ii)]. □

3.6 Proof of Siegel-Weil Formula

Combining the results above we are ready to show the Siegel-Weil formula. Note the assumption that $m = n + 1$. We will focus on the cases where metaplectic double cover of $\mathrm{Sp}(2n)$ must be considered so in the proofs we only deal with the cases where m is odd. For the cases where m is even please refer to [17]. Set $A(g, \Phi) = E(g, s, f_\Phi)|_{s=0} - 2I(g, \Phi)$.

Proposition 3.6.1. *Assume $m \geq 3$ or $m = 2$ and V anisotropic. Then for $B \in \mathrm{Sym}_n(k)$ with rank n, the Fourier coefficients $A_B = 0$.*

Proof. Without loss of generality suppose $B = \left(\begin{smallmatrix} S & \\ & \beta \end{smallmatrix}\right)$ for some $S \in k^\times$ and some nonsingular $\beta \in \mathrm{Sym}_{n-1}(k)$. First we prove the anisotropic case. The base case $m = 2$ and $n = 1$ was proved in [23, Chapter 4]. Now for $m = n+1$, if Q does not represent S then by Prop. 3.4.2 and Prop. 3.4.6 we obviously have $A_B = 0$. If Q represents S then we can just assume that $Q = \left(\begin{smallmatrix} S & \\ & Q_1 \end{smallmatrix}\right)$. Note that Q_1 is still anistropic. Again by Prop. 3.4.2 and Prop. 3.4.6 and the induction hypothesis we conclude that A_B=0.

Secondly we assume Q to be isotropic and $m \geq 4$, so Q represents S. We can just assume that $Q = \left(\begin{smallmatrix} S & \\ & Q_1 \end{smallmatrix}\right)$.

From Section 3.4 we get by Prop. 3.4.2 and Prop. 3.4.6 and the m even case $\mathrm{FJ}^{\phi}_{\beta}(A) = 0$ for all $\phi \in \mathcal{S}(L(\mathbb{A}))$ if the rank of β is $n-1$. Then by Lemma 3.4.1, A_B vanishes for $B \in \mathrm{Sym}^n(k)$ such that $\det B \neq 0$.

Finally assume that Q is isotropic and $m = 3$. By the expression for $E_B(g, s, f_\Phi)$ in Remark 4.1 of [23] we know that $E_B(g, s, f_\Phi)$ is analytic at $s = 0$. Thus Prop 4.2 of [23] holds: $E_B(g, 0, f_\Phi) = cI_B(g, \Phi)$ where c does not depend on Φ or B. Now we consider objects in dimension $m = 4$ and $n = 3$. Here $E_B(g, 0, f_\Phi) = 2I_B(g, \Phi)$. By Prop. 3.4.2 and Prop. 3.4.6 and the independence of c on Φ we conclude that $c = 2$ and this finishes the proof of the lemma. $\square$

Remark 3.6.2. For the split binary case please refer to [17] and note that the Eisenstein series vanishes at 0, so the Siegel-Weil formula takes a different form.

In the above proof we the argument dealing with the case $(m, r) = (3, 1)$ can also be used to prove other cases. We use two methods for the record.

Proof of Theorem 3.1.1. Fix a finite place v of k and fix for each place w not equal to v a $\Phi^0_w \in \mathcal{S}_0(U^n(\mathbb{A}))$. Consider the map A_v which sends $\Phi_v \in \mathcal{S}_0(U^n_v)$ to $A(g, \Phi_v \otimes (\otimes_{w \neq v} \Phi^0_w))$. By invariant distribution theorem $R_n(U_v) \cong \mathcal{S}(U^n_v)_{H_v}$. Thus A_v is a $\widetilde{G_v}$-intertwining operator

$$\mathcal{S}(U^n_v) \to \mathcal{A}$$

which actually factors through $R_n(U_v)$. As $R_n(U_v)$ is nonsingular in the sense of [7] by Lemma 3.5.4 take $f \in \mathcal{S}(\mathrm{Sym}^n(k_v))$ such that its Fourier transform is supported on non-singular symmetric matrices. Then for all $g \in G(\mathbb{A})$ with $g_v = 1$ and all $B \in \mathrm{Sym}_n(k)$ we have

$$\begin{aligned}
(\rho(f).A(\Phi))_B(g) &= \int_{\mathrm{Sym}_n(k)\setminus\mathrm{Sym}_n(\mathbb{A})}\int_{\mathrm{Sym}_n(k_v)} f(c)A(\Phi)(ngn(c))\psi(-\operatorname{tr}(Bb))dcdb \\
&= \int_{\mathrm{Sym}_n(k)\setminus\mathrm{Sym}_n(\mathbb{A})}\int_{\mathrm{Sym}_n(k_v)} f(c)A(\Phi)(nn(c)g)\psi(-\operatorname{tr}(Bb))dcdb \\
&= \int_{\mathrm{Sym}_n(k)\setminus\mathrm{Sym}_n(\mathbb{A})}\int_{\mathrm{Sym}_n(k_v)} f(c)A(\Phi)(ng)\psi(-\operatorname{tr}(B(b-c)))dcdb \\
&= \hat{f}(-B)A(\Phi)_B(g).
\end{aligned}$$

The above is always 0, since $\hat{f}(B) = 0$ if $\operatorname{rk} B < n$ and $A(\Phi)_B \equiv 0$ if $\operatorname{rk} B = n$. Thus $\rho(f)A(\Phi) = 0$ as $G(k)\prod_{w\neq v} G_w$ is dense in $G(\mathbb{A})$. Since f does not act by zero and $R_n(U_v)$ is irreducible we find that in fact $A(\Phi) = 0$ and this concludes the proof. $\square$

3.7 Inner Product Formula

We will apply Theorem 3.1.1 to show a case of Rallis Inner Product formula via the doubling method. We will also deduce the location of poles of Langlands L-function from information on the theta lifting.

Let G_2 denote the symplectic group of rank $2n$, P_2 its Siegel parabolic and $\widetilde{G_2(\mathbb{A})}$ the metaplectic group. Let $H = \mathrm{O}(U, Q)$ with (U, Q) a quadratic space of dimension $2n + 1$. Let π be a genuine irreducible cuspidal automorphic representation of $\widetilde{G(\mathbb{A})}$. For $f \in \pi$ and $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ define

$$\Theta(h; f, \Phi) = \int_{G(k)\setminus\widetilde{G(\mathbb{A})}} f(g)\Theta(g, h; \Phi)dg.$$

Consider the mapping

$$\iota_0 : \widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})} \to \widetilde{G_2(\mathbb{A})}$$
$$((g_1,\zeta_1),(g_2,\zeta_2)) \mapsto (\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}, \zeta_1\zeta_2)$$

if $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$. For $g \in G$ set

$$\check{g} = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}.$$

Then let $\iota((g_1,\zeta_1),(g_2,\zeta_2)) = \iota_0((g_1,\zeta_1),(\check{g}_2,\zeta_2))$. In fact we are just mapping $((g_1,\zeta_1),(g_2,\zeta_2))$ to

$$(\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & -b_2 \\ c_1 & & d_1 & \\ & -c_2 & & d_2 \end{pmatrix}, \zeta_1\zeta_2).$$

With this we find $\Theta(\iota(g_1,g_2),h;\Phi) = \Theta(g_1,h;\Phi_1)\overline{\Theta(g_2,h;\Phi_2)}$ if we set $\Phi = \Phi_1 \otimes \overline{\Phi_2}$ for $\Phi_i \in \mathcal{S}(U^n(\mathbb{A}))$. Suppose the inner product

$$\langle \Theta(f_1,\Phi_1), \Theta(f_2,\Phi_2) \rangle$$
$$= \int_{H(k)\setminus H(\mathbb{A})} \int_{(G(k)\times G(k)\setminus(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\Theta(g_1,h;\Phi_1)\overline{f_2(g_2)\Theta(g_2,h;\Phi_2)}dg_1dg_2dh$$

is absolutely convergent. Then it is equal to

$$\begin{aligned}
&\int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}\left(\int_{H(k)\,\backslash\,H(\mathbb{A})}\Theta(g_1,h;\Phi_1)\overline{\Theta(g_2,h;\Phi_2)}dh\right)dg_1dg_2\\
&=\int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}\left(\int_{H(k)\,\backslash\,H(\mathbb{A})}\Theta(\iota(g_1,g_2),h;\Phi)\right)dg_1dg_2\\
&=\int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}I(\iota(g_1,g_2);\Phi)dg_1dg_2.
\end{aligned}$$

Thus we define the regularized inner product by

$$\begin{aligned}
&\langle\Theta(f_1,\Phi_1),\Theta(f_2,\Phi_2)\rangle_{\mathrm{REG}}\\
&\qquad=\int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}I_{\mathrm{REG}}(\iota(g_1,g_2);\Phi)dg_1dg_2. \quad (3.7.1)
\end{aligned}$$

We could apply the Siegle-Weil formula now, but then we would not be able to use the basic identity in [20] directly. Thus we will follow Li[19] to continue the computation. Now consider G to be the group of isometry of the $2n$-dimensional space V with symplectic form $(\ ,\ )$ and suppose $V = X \oplus Y$ with X and Y maximal isotropic subspaces. Then the Weil representation ω considered up till now is in fact realised on $\mathcal{S}(U \otimes X(\mathbb{A}))$. Let $V_2 = V \oplus V$ be endowed with the split form $(\ ,\ ) - (\ ,\ )$. The space $U \otimes V_2$ has two complete polarizations $U \otimes V_2 = (U \otimes (X \oplus X)) \oplus (U \otimes (Y \oplus Y))$ and $U \otimes V_2 = (U \otimes V^d) \oplus (U \otimes V_d)$, where $V^d = \{(v,v)|v \in V\}$ and $V_d = \{(v,-v)|v \in V\}$. There is an isometry

$$\delta : \mathcal{S}((U \otimes (X \oplus X))(\mathbb{A})) \to \mathcal{S}((U \otimes V_d)(\mathbb{A}))$$

intertwining the action of $\widetilde{G_2(\mathbb{A})}$. We have as in [19, Eq. (13)]

$$\delta(\Phi_1 \otimes \overline{\Phi_2})(0) = \langle\Phi_1,\Phi_2\rangle.$$

Then (3.7.1) is equal to

$$\int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}I_{\mathrm{REG}}(\iota(g_1,g_2);\delta\Phi)dg_1dg_2.$$

Note here the theta function is associated to the Weil representation realised on $\mathcal{S}(U \otimes V_d)(\mathbb{A}))$. Now we apply the regularized Siegel-Weil formula and get

$$2^{-1} \int_{(G(k)\times G(k)\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}E(\iota(g_1, g_2), s, F_{\delta\Phi})|_{s=0} dg_1 dg_2$$

where to avoid conflict of notation we use $F_{\delta\Phi}$ to denote the Siegel-Weil section associated to $\delta\Phi$.

Set the zeta function to be

$$Z(f_1, f_2, s, F) = 2^{-1} \int_{(G(k)\times G(k)\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)}E(\iota(g_1, g_2), s, F) dg_1 dg_2 \qquad (3.7.2)$$

and we will deduce some of its properties.

By the basic identity in [20] generalized to the metaplectic case and by [19, Eq. (25)] $Z(f_1, f_2, s, F)$ is equal to

$$\begin{aligned}
&2^{-1} \int_{\widetilde{G(\mathbb{A})}} F(\iota(g, 1), s) \int_{G(k)\backslash\widetilde{G(\mathbb{A})}} f_1(g_2 g)\overline{f_2(g)} dg_2 dg \\
&= 2^{-1} \int_{\widetilde{G(\mathbb{A})}} F(\iota(g, 1), s).\langle \pi(g) f_1, f_2\rangle dg \\
&= \int_{G(\mathbb{A})} F(\iota(g, 1), s)\langle \pi(g) f_1, f_2\rangle dg.
\end{aligned}$$

The last equation holds since we are dealing with genuine representations.

Suppose F and f_i are factorizable. Then the above factorizes into a product of local zeta integrals

$$Z(f_{1,v}, f_{2,v}, s, F_v) = \int_{G_v} F_v(\iota(g_v, 1), s)\langle \pi_v(g_v) f_{1,v}, f_{2,v}\rangle dg_v.$$

Let S be a finite set of places of k containing all the archimedean places, even places, outside which π_v is an unramified principal series representation, f_i spherical and normalized, F normalized spherical Siegel-Weil section and ψ_v unramified. Notice $\pi_v \otimes \chi_v$ can be viewed as a representation of G_v rather than $\widetilde{G_v}$. Then by [19, Prop. 4.6] the local integral

$Z(f_{1,v}, f_{2,v}, s, F_v)$ is equal to

$$\frac{L(s+\frac{1}{2}, \pi_v \otimes \chi_v)}{\widetilde{d}_{G_{2,v}}(s)}$$

where $L(s+\frac{1}{2}, \pi_v \otimes \chi_v)$ is the Langlands L-function associated to $\pi \otimes \chi$ and

$$\widetilde{d}_{G_{2,v}}(s) = \zeta_v(s+\frac{1}{2}) \cdot \prod_{i=1}^{n} \zeta_v(2s+2i).$$

Note here we normalize the Haar measure on $\widetilde{G_v}$ so that K_{G_v} has volume 1.

Proposition 3.7.1. *The poles of $L^S(s, \pi \otimes \chi)$ in $\mathrm{Re}(s) > 1/2$ are simple and are contained in the set*

$$\{1, \frac{3}{2}, \frac{5}{2}, \cdots, n+\frac{1}{2}\}.$$

Proof. By [17, Prop. 7.2.1] we deduce that the poles of $L^S(s+\frac{1}{2}, \pi \otimes \chi)$ are contained in the set of poles of $\widetilde{d}^S_{G_2}(s)E(s, \iota(g_1, g_2), F)$. The poles of the Eisenstein series in $\mathrm{Re}(s) > 0$ are simple and are contained in $\{1, 2, \ldots, n\}$, c.f. [8, Page 216]. From this we get the proposition. ☐

Our result combined with that of Ichino's[8] gives an analogue of Kudla and Rallis's [17, Thm. 7.2.5]. Let $m_0 = 4n+2-m$ be the dimension of the complementary space U_0 of U.

Theorem 3.7.2.

1. *The poles of $L^S(s, \pi \otimes \chi)$ in the half plane $\mathrm{Re}\, s > 1/2$ are simple and are contained in the set*

$$\left\{1, \frac{3}{2}, \frac{5}{2}, \ldots, \left[\frac{n+1}{2}\right] + \frac{1}{2}\right\}.$$

2. *If $4n+2 > m > 2n+1$ then suppose $L^S(s, \pi \otimes \chi)$ has a pole at $s = n+1-(m_0/2)$. If $m = 2n+1$ then suppose $L^S(s, \pi \otimes \chi)$ does not vanish at $s = n+1-(m_0/2) = 1/2$. Then there exists a quadratic space U_0 over k with dimension m_0 and $\chi_{U_0} = \chi$ such that $\Theta_{U_0}(\pi) \neq 0$ where $\Theta_{U_0}(\pi)$ denotes the space of automorphic forms $\Theta(f, \Phi)$ on $\mathrm{O}_{U_0}(\mathbb{A})$ for $f \in \pi$ and $\Phi \in \mathcal{S}(U_0(\mathbb{A})^n)$.*

Proof. Consider the residue of $L^S(s, \pi \otimes \chi)$ at $s_0 + \frac{1}{2}$ with $s_0 \in \{1, 2, \ldots, n\}$. Then it vanishes if the residue of $Z(f_1, f_2, s, F_{\delta\Phi})$ vanishes at s_0. Note that for some choice of

Schwartz function Φ, F is the normalized spherical standard Siegel-Weil section. We apply the Siegel-Weil formula of Ichino's[8] and ours and get

$$\mathrm{Res}_{s=s_0} Z(f_1, f_2, s, F_{\delta\Phi}) = \int_{(G(k)\times G(k)\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I_{\mathrm{REG}}(\iota(g_1, g_2), \Phi) dg_1 dg_2$$

or

$$Z(f_1, f_2, 0, F_{\delta\Phi}) = \int_{(G(k)\times G(k)\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I_{\mathrm{REG}}(\iota(g_1, g_2), \Phi) dg_1 dg_2$$

which is exactly the regularized pairing of theta liftings $\Theta(f_1, \Phi_1)$ and $\Theta(f_2, \Phi_2)$. Then if the residue of $L^S(s, \pi \otimes \chi)$ at $s_0 + \frac{1}{2}$ does not vanish or $L^S(s, \pi \otimes \chi)$ does not vanish at $\frac{1}{2}$ then the space of theta lifting does not vanish and we prove 2).

On the other hand the space of theta lifting vanishes if $m_0 < n$ by [17, Lemma 7.2.6]. This means $s_0 > (n+1)/2$, so $L^S(s, \pi \otimes \chi)$ can only have poles for $s \leq \frac{n+2}{2}$ and we prove 1). $\square$

Finally we set s to 0 in the zeta function and get the Rallis inner product formula:

Theorem 3.7.3. *Suppose $m = 2n+1$. Then*

$$\langle \Theta(f_1, \Phi_1), \Theta(f_2, \Phi_2) \rangle_{\mathrm{REG}} = \frac{L^S(\frac{1}{2}, \pi \otimes \chi)}{\widetilde{d}^S_{G_2}(0)} \cdot \langle \pi(\Xi_S) f_1, f_2 \rangle$$

where

$$\Xi_S(g) = \langle \omega_S(g) \Phi_{1,S}, \Phi_{2,S} \rangle.$$

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