An Algebraic Circle Method

Thibaut PUGIN

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2011

ABSTRACT

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In this thesis we present an adaptation of the Hardy-Littlewood Circle Method to give estimates for the number of curves in a variety over a finite field. The key step in the classical Circle Method is to prove that some cancellation occurs in some exponential sums. Using a theorem of Katz, we reduce this to bounding the dimension of some singular loci. The method is fully carried out to estimate the number of rational curves in a Fermat hypersurface of low degree and some suggestions are given as to how to handle other cases. We draw geometrical consequences from the main estimates, for instance the irreducibility of the space of rational curves on a Fermat hypersurface in a given degree range, and a bound on the dimension of the singular locus of the moduli space.

Contents

Acknowledgements

It is a pleasure to thank my advisor Johan de Jong for his incredible generosity with his time and ideas. For his dedication, knowledge and ability in Mathematics, he is an inspiration to me. I would also like to thank him for being so understanding for the many times when my dedication, knowledge and ability have failed me.

I would like to thank Akshay Venkatesh and Jordan Ellenberg for letting me carry out the ideas presented in this work. The idea and the setup of the Algebraic Circle Method is due to them entirely. I hope that I have adequately attributed their ideas in this manuscript.

I would also like to thank Robert Friedman, Davesh Maulik, Andrew Obus and Michael Thaddeus for agreeing to be on my dissertation committee.

Finally, I would like to thank Terrance Cope, without whom the Mathematics department at Columbia University would simply cease to exist.

I apologize in advance for the undoubtedly many errors in this text, for which I alone am responsible.

To Marja, Hugo and his little sister

Notations

We collect various notations used throughout the text.

If x is a real number, $\lceil x \rceil$ denotes the smallest integer that is at least x, and $\lfloor x \rfloor$ denotes the biggest integer that is at most x.

The function e denotes the complex exponential $e(z) = \exp(i\tau z)$, where $\tau = 2\pi$ (see [9]).

We use Landau's "big O" and "little o" notation, that is, if f and g are two functions of the (say real) variable r, then we write $f = O(g)$ (resp. $f = o(g)$) as $r \to \infty$ if the quotient $f(r)/g(r)$ is bounded (resp. has limit 0) as $r \to \infty$.

The symbol $^\vee$ denotes the dual of a vector space.

The symbols $\mathbb{F}_q$ and k denote a finite field of size q.

If a projective or affine space is denoted without index, then it is understood that it is over the finite field k, e.g. $\mathbb{P}^n = \mathbb{P}^n_k$.

We write $P_r = H^0(\mathbb{P}^1, \mathcal{O}(r))$, where $\mathcal{O}(r)$ denotes the Serre twisted sheaf.

Introduction

There are two aspects to this thesis. The first is to present a method to estimate the number of points of some mapping spaces which is inspired by the Hardy-Littlewood Circle Method. The second is to try and prove the irreducibility of the moduli space of rational curves on a variety. These two aspects are intertwined throughout these notes. We now describe the contents of each chapter.

In Chapter 1, we recall briefly the standard Circle Method in the context of Waring's problem. The key step in the method is to prove that some cancellation occurs in some exponential sums (the minor arcs). There are typically two ways that such cancellation is proven. The first is to use tools from analytic number theory. These may be mere tricks, equidistribution results or just heavy integral mongering. The second is to use ℓ-adic cohomology to express these sums as the trace of Frobenii and then use the works of Weil and Deligne to estimate the size of the eigenvalues of those Frobenii. We collect one theorem of Katz which bounds "singular" exponential sums in terms of the dimension of some singular locus. Later on, we use this theorem essentially as a black box so that nonspecialists can still favorably use the method for their own purposes. Finally, we introduce some mapping spaces of interest. One such type of mapping space describes the rational curves on a variety, which encompass the rational connectedness of that variety. One big motivation to study the results presented in this thesis was to be able to prove that the space of rational curves in, say, a specific hypersurface is irreducible. There are many theorems stating the irreducibility of such mapping spaces, but they typically only prove a generic statement. We take this opportunity to give a brief survey of those results. It was interesting to us to be able to write down an equation and have a method to try to prove the result directly.

In Chapter 2, we present an algebraic version of the Circle Method that allows us to estimate the number of points of a mapping space of the type previously introduced. The method is carried out in the case of rational curves on a smooth hypersurface and eventually refined for a Fermat hypersurface of low degree. The main result is theorem 2.4.1 which gives the estimate for the number of rational curves on a smooth hypersurface conditionally to some dimension bounds for singular loci. The latter are expected to hold when the hypersurface has low enough degree (compared to the number of variables) and proved to hold in the Fermat case.

In Chapter 3, we give some ways in which the method can be adapted. We prove that the estimates hold for cubic hypersurfaces in enough variables (more precisely, enough so that it admits a linear section that is a Fermat in twelve variables). We prove a similar result for arbitrary smooth hypersurfaces. We make a few remarks regarding more general target or source spaces. Then we present a version of the method for estimating the size of the singular locus of the mapping space. In particular, we prove that in the cases where the method of chapter 2 applies, the singular locus has high codimension.

Finally, in Chapter 4 we list some consequences of the estimates obtained in Chapter 2. We prove the irreducibility of the space of rational curves on some hypersurfaces. We also explain how the predictions of the method fit in with the geometric picture in characteristic 0 using rational homotopy models.

Chapter 1

Preliminaries

1.1 The Circle Method

1.1.1 Analytic Circle Method

We give a brief introduction to the classical Circle Method. The idea originated in works of Hardy and Ramanujan (in [6]) and the Circle Method appeared in the paper [5] by Hardy and Littlewood. The modern version of the method is based on a further modification of Vinogradov and is one of the standard tools in analytic number theory. For more information on the Circle Method, see [21].

A set B of integers is called an *additive basis* if every integer is a sum of elements of B. It is *of finite order* s if each integer is the sum of at most s elements of B.

Question 1.1.1 (Waring's problem)**.** *Let k be a positive integer and $B = B_k$ the set of k^{th} powers. Is B_k a basis of finite order? What is the smallest possible order $g(k)$ for which this holds?*

The first question was answered positively by Hilbert in [10] although his solution did not address the second question, which the Circle Method was designed to study. More precisely, it gives estimates for the number $r(n, s, k)$ of representations of n as a sum of s k^{th} powers. More generally, the method considers the following "additive" number theory problem.

Question 1.1.2. *Given a set B of integers and a positive integer s, what integers can be written as the sum of s elements of B and in how many ways?*

We denote $r(n, s, B)$ the number of representations of n as a sum of s elements of B. We start by expressing it as an integral. Consider the generating function

$$g_B(z) = \sum_{b \in B} e(bz) \quad \text{and its truncation} \quad g_B(z; N) = \sum_{b \in B_N} e(bz)$$

where $e(z) = \exp(i\tau z)$, N is an integer and $B_N = B \cap [0, N]$. Then we have the identity, for any integer $s \geqslant 1$

$$g_B(z; N)^s = \sum_{n=0}^{N} r(n, s, B_N) z^n.$$

Recall orthogonality of characters in the following sense

$$\int_0^1 e(-n\alpha)\mathrm{d}\alpha = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

so that for $n \leqslant N$ we have

$$r(n, s, B) = r(n, s, B_N) = \int_0^1 g_B(\alpha; N)^s e(-n\alpha)\mathrm{d}\alpha.$$

The role of the truncation is simply to guarantee convergence throughout. We now explain the heuristics of the Circle Method. The quantity $g_B(z;N)$ is a sum of $b_N = \#B_N$ complex numbers of modulus 1, a so-called exponential sum. If the fractional part of the bz for $b \in B_N$ are somewhat equidistributed (or symmetrically distributed) then some cancellation is expected in this sum. Typically, $g_B(z;N)$ should be roughly of size $\sqrt{b_N}$ and this should happen for most values of $z \in [0,1]$. The set of such values is denoted $\mathfrak{m}$ and called the *minor arcs*. For the remaining values of $z \in \mathcal{M} = [0,1]\backslash\mathfrak{m}$, the cancellation will not occur and $g_B(z;N)$ should be roughly of size b_N. These are called the *major arcs*. We then have

$$r(n,s,B) = \int_{\mathfrak{m}} g_B(\alpha;N)^s e(-n\alpha)\mathrm{d}\alpha + \int_{\mathcal{M}} g_B(\alpha;N)^s e(-n\alpha)\mathrm{d}\alpha. \tag{1.1}$$

The main contribution to $r(n,s,B)$ comes from the major arcs and should be computable. The remaining contribution, of the lower arcs, ought to be of smaller order. This is typically technical and hard to prove and requires knowledge of equidistribution of the bz. It also involves a lot of computational analytic number theory tricks. The definition of the major and minor arcs depends on the problem at hand and the value of N.

Remark 1.1.3. Instead of a generating function using exponentials, one can form a generating power series. Via the correspondence between Fourier series and complex functions on the unit circle, $r(n,s,B)$ can be expressed as a Cauchy integral on the unit circle. Major and minor arcs are then unions of actual arcs on the unit circle, which explains the terminology. In practice, the major arcs are a finite union of small arcs around special values on the unit circle and the sum of their length is much smaller than that of the minor arcs.

For the case of Waring's problem, we have

$$g_k(z;N) = g_{B_k}(z;N) = \sum_{m=0}^{\lfloor N^{1/k} \rfloor} e(zm^k).$$

If z is "close" to a reduced fraction of the form $\frac{a}{b}$ where the denominator b is "small" then it is not hard to show that

$$g_k\left(z;N\right) \sim \frac{N^{1/k}}{b} \sum_{m=1}^{b} e\left(\frac{am^k}{b}\right)$$

in an interval centered at $\frac{a}{b}$. If $\mathcal{M}$ is a suitable union of such intervals then

$$\int_{\mathcal{M}} g_k(\alpha;N)^s e(-n\alpha)\mathrm{d}\alpha \sim C(n)n^{s/k-1}$$

for some function $C(n)$ that is bounded above and below by a positive value. On the other hand, it can be shown that

$$\int_{m} |g_k(\alpha;N)|^s \,\mathrm{d}\alpha \leqslant \left(\sup_{m} |g_k(\alpha;N)|\right)^{s-2t} \int_0^1 |g_k(\alpha;N)|^{2t} \,\mathrm{d}\alpha = O(n^{s/k-1-\varepsilon})$$

for some $\varepsilon > 0$ and for s big enough. Again, it should be emphasized that this last step is hard and uses equidistribution results. In our adaptation of the Circle Method, we use some input to bound the minor arcs. We explain in the next section what those inputs are.

Let us conclude this section with a few remarks on Waring's problem.

- As mentioned before, Hilbert proved that $g(k)$ exists for all k. Thanks to the Circle Method, the value of $g(k)$ is known for all k and equal to $2^k + [(3/2)^k] - 2$ for all but finitely many k which are at least 4×10^8. For instance, one has $g(2) = 4$, $g(3) = 9$, $g(4) = 19$, $g(5) = 37$, $g(17) = 132\,055$, etc.

- It turns out that the value of $g(k)$ decreases significantly if we omit finitely many integers. More precisely, one defines the number $G(k)$ to be the smallest integer s such that every sufficiently large integer n is a sum of s k^{th} powers. The only known values of G are $G(2) = 4$ and $G(4) = 16$. There are upper bounds for $G(k)$, mostly obtained via the Circle Method, but they remain rather far from the conjectured values. For more detail, see the survey article [22]

1.1.2 Exponential Sums

Estimating the minor arcs usually involves proving that some cancellation occurs in the corresponding exponential sums. An *exponential sum* is any finite sum of complex numbers of modulus 1. As mentioned before, the general yoga is that some cancellation will occur if the arguments of the summands are equally or symmetrically distributed, in an appropriate sense. Then the sum should behave roughly as the square root of the number of summands. For instance, consider an exponential sum of the form

$$S_N = \sum_{i=1}^{N} e(x_i)$$

where the x_i are independent uniform random variables on $[0, 1]$. Then the central limit theorem implies that

$$\lim_{N\to\infty} \mathbf{P}\left(a < \frac{S_N}{\sqrt{N}} < b\right) = \frac{1}{\sqrt{\tau}} \int_a^b e^{-x^2/2} \mathrm{d}x$$

suggesting not only that the square-root philosophy is justified, but that it is in a sense optimal. In the case of exponential sums over a finite field, the arguments take a discrete set of values on the unit circle and counting averages of such sums gives more evidence for the square-root principle.

There are very few known cases where exponential sums can be computed exactly. When it can be, this is done via elementary techniques and yields strong consequences. Determining the value of some simple Gauss sums, for instance, is equivalent to proving the Quadratic Reciprocity Theorem. A general method to bound the size of exponential sums stems from the works of Deligne on the Weil Conjectures. The main idea is to express an exponential sum as the trace of a Frobenius operator acting on the cohomology of some algebraic variety. The Weil conjectures then predict the size of the eigenvalues of Frobenius and provide an upper bound for the sum. This reduces the estimation of the sums to some cohomological computations on algebraic varieties. These computations involve perverse sheaves and are usually quite technical, but they provide geometric justification for the estimates. For more information, see [14]. In his investigations of the Weil Conjectures [4], Deligne proved the following result.

Theorem 1.1.4. *Let k be a finite field of order q, $d \geqslant 1$ an integer prime to q, $\psi : k \to \mathbb{C}^\times$ a nontrivial additive character and f a polynomial in $n+1$ variables with coefficients in k. Write $f = f_0 + \cdots + f_d$ where f_i is homogeneous of degree i. Assume* *that f_d defines a nonsingular hypersurface in $\mathbb{P}^n$. Then we have the estimate*

$$\left|\sum_{x\in k^{n+1}} \psi(f(x))\right| \leqslant (d-1)^{n+1} q^{\frac{n+1}{2}}.$$

An appropriate generalization was given by Katz [13] and Laumon [17].

Theorem 1.1.5. *Let k be a finite field of order q, $d \geqslant 1$ an integer prime to q and $\psi : k \to \mathbb{C}^{\times}$ a nontrivial additive character. Let X be a projective, nonsingular, geometrically connected k-scheme of dimension $n \geqslant 1$ together with a projective embedding. Let $z \in H^0(X, \mathcal{O}(1))$ and $h \in H^0(X, \mathcal{O}(d))$ and write H and Z their zero loci in X. Assume that*

- *$Z \cap X$ is nonsingular of codimension 1 in X, and*
- *$X \cap Z \cap H$ is nonsingular of codimension 2 in X.*

Consider the smooth affine variety $V = X - X \cap Z$ of dimension n and the function $f = \dfrac{h}{z^d} : V \to \mathbb{A}^1_k$. Then there exists a constant C depending only on X, its projective embedding and d such that

$$\left|\sum_{x\in V(k)} \psi(f(x))\right| \leqslant C q^{\frac{n}{2}}.$$

In fact, C is a topological constant and is given by

$$C = \left|\int_X \frac{c(X)}{(1+L)(1+dL)}\right|$$

where $c(X)$ (resp. L) is the total Chern class of X (resp. $\mathcal{O}(1)$). In [12], Katz further generalized the result to drop the hypotheses of nonsingularity. The estimates suffer surprisingly little.

Theorem 1.1.6. *Let k be a finite field of order q, $d \geqslant 1$ an integer prime to q and $\psi : k \to \mathbb{C}^\times$ a nontrivial additive character. Let X be a projective k-scheme together with a projective embedding. Assume that*

(H) X is geometrically irreducible and integral.

Let $n = \dim X$ and assume that $n \geqslant 1$. Let $z \in H^0(X, \mathcal{O}(1))$ and $h \in H^0(X, \mathcal{O}(d))$ and write H and Z their zero loci in X. Assume that

- *$X \cap Z \cap H$ has codimension 2 in X.*

Let δ be the dimension of the singular locus of $X \cap Z \cap H$. Consider the function $f = \dfrac{h}{z^d} : V \to \mathbb{A}^1_k$. Then there exists a constant C depending only on X, its projective embedding and d such that

$$\left| \sum_{x \in V(k)} \psi(f(x)) \right| \leqslant C q^{\frac{n+2+\delta}{2}}. \tag{1.2}$$

A few remarks are in order.

Remark 1.1.7. Since k is a perfect field, the singular locus of $X \cap Z \cap H$ is the locus of points where the local ring is not regular.

Remark 1.1.8. One can replace hypothesis *(H)* by *(H') : X is Cohen-Macaulay and equidimensional*.

Remark 1.1.9. The constant C is explicit. If we assume that X is definable (scheme-theoretically) in $\mathbb{P}^N$ by m homogeneous equations of degrees $d_1, \dots, d_m$, then we can take C to be the Bombieri constant $C = (4 \sup(d_1 + 1, \dots, d_m + 1, d) + 5)^{N+m}$.

Remark 1.1.10. In fact, the "correct" exponent should be $\dfrac{n+1+\delta}{2}$. The exponent in estimate (1.2) can be lowered by $\dfrac{1}{2}$ in the two following cases:

- Let ε be the dimension of the singular locus of $X \cap Z$. Then it is easy to show $\varepsilon \leqslant \delta + 1$. If it is the case that $\varepsilon \leqslant \delta$, then the estimate holds with the correct exponent.

- If we replace C with a slightly bigger (explicit) constant and assume that the characteristic of k is bigger than some ineffective constant, then estimate (1.2) holds with the correct exponent.

We won't be needing those cases. We now turn to an unrelated topic, discussing mapping spaces and rational connectedness.

1.2 Mapping Spaces

There are many interesting versions of the moduli spaces that parametrize morphisms from a scheme into another, even in the case of rational curves. We consider the simplest one, in some sense.

1.2.1 Parametrizing Morphisms

Let k be a field and $\mathbb{P}^1_k = \mathrm{Proj}(k[u, v])$. We write $P_r(k) = H^0(\mathbb{P}^1_k, \mathcal{O}(r))$ for the set of homogeneous polynomials in u, v of degree r. Let $\mathscr{X}$ be a closed subscheme of $\mathbb{P}^n_k$ defined by homogeneous equations $f_1, \ldots, f_m$. A *rational curve on* $\mathscr{X}$ is simply a nonconstant morphism φ from $\mathbb{P}^1_k$ to $\mathscr{X}$. Such a morphism is given by a collection

$$\varphi = (\varphi_0(u,v), \dots, \varphi_n(u,v))$$

of $n+1$ homogeneous polynomials of the same degree r with no nonconstant common factor in $k[u,v]$, such that $f_i(\varphi_0, \dots, \varphi_n) = 0$ in $P_{rd}(k)$ for all $i = 1, \dots, m$. The morphism φ is said to have degree r. We now explain how these are parametrized by a quasi-projective scheme.

We first consider the case where $\mathscr{X} = \mathbb{P}^n_k$. Let $(\varphi_0, \dots, \varphi_n) \in P_r(k)^{n+1}$. Then these polynomials have no nonconstant common factor in $k[u,v]$ if and only if they have no nonconstant common factor in $\bar{k}[u,v]$, where $\bar{k}$ is an algebraic closure of k. By Hilbert's Nullstellensatz, this happens if and only if the ideal $(\varphi_0, \dots, \varphi_n)$ in $\bar{k}[u,v]$ contains some power of the maximal ideal (u,v). That is, if and only if, for some integer t, the map

$$\begin{array}{ccc} P_r(\bar{k})^{n+1} & \longrightarrow & P_{r+t}(\bar{k}) \\ (x_0, \dots, x_n) & \longmapsto & \displaystyle\sum_{i=0}^{n} \varphi_i x_i \end{array}$$

is surjective. But since this map is linear and defined over k, it is not surjective if and only if all the $(n+1)$-minors of some universal matrix whose entries are linear integral combinations of the coefficients of the φ_i vanish. This defines a closed subscheme of $\mathrm{Proj}((\mathrm{Sym}^r k^2)^{n+1})$ which complement is an open subscheme denoted $\mathscr{M}_r(\mathbb{P}^1_k, \mathbb{P}^n_k)$ which parametrizes rational curves in $\mathbb{P}^n_k$.

Now, if $\mathscr{X}$ is the closed subscheme of $\mathbb{P}^n_k$ defined by $f_1, \dots, f_m$, then $\mathscr{M}_r(\mathbb{P}^1_k, \mathscr{X})$ is the closed subcheme of $\mathscr{M}_r(\mathbb{P}^1_k, \mathbb{P}^n_k)$ defined by the equations $f_i(\varphi_0, \dots, \varphi_n) =$

$0, i = 1, \ldots, m$. The moduli space of all rational curves on $\mathscr{X}$ is $\mathscr{M}(\mathbb{P}^1_k, \mathscr{X}) = \coprod_r \mathscr{M}_r(\mathbb{P}^1_k, \mathscr{X})$. Note that this space comes equipped with a natural evaluation map

$$\begin{array}{ccc} \mathbb{P}^1_k \times \mathscr{M}(\mathbb{P}^1, \mathscr{X}) & \longrightarrow & \mathscr{X} \\ ((u,v), \varphi) & \longmapsto & \varphi(u,v). \end{array}$$

1.2.2 Rational Curves and Irreducibility

The study of rational curves in algebraic geometry is the pendant to the study of connectedness in topology. For instance, an algebraic variety is rationally connected if for any pair of points on it, there is a rational curve containing them. To take the analogy a little further, one can ask what it would mean for an algebraic variety to be simply connected. One way to formulate simple connectedness in topology is by requiring that the space of paths is itself connected. One can thus define an algebraic variety to be simply connected if the space of rational curves on it is itself rationally connected. To make this statement precise, one needs to decide which parameter space to work with and verify that it is an algebraic variety in its own right. Once that is done (for instance, consider the space introduced in the previous section, but there are many other versions) one can ask all sorts of fundamental questions: under appropriate assumption on the scheme $\mathscr{X}$, can we compute the dimension of the space of rational curves on $\mathscr{X}$? is this space irreducible? nonsingular? While there are many other questions that one can ask, we will focus on these three.

The simplest type of varieties in algebraic geometry is that of homogeneous spaces, for which the space of rational curves is generally understood and well behaved (see

[15]). Another interesting class of varieties are smooth projective hypersurfaces. Even in this case, there is much to be done to achieve a full understanding of the moduli space. However, we know enough not to expect too much regularity in general. A general philosophy is that $\mathscr{M}(\mathbb{P}^1, \mathscr{X})$ is well behaved when $\mathscr{X}$ is a smooth hypersurface in $\mathbb{P}^n$ of degree d small compared to n. It is conjectured (see [2] and [20]) that $d < n$ is enough to guarantee that each irreducible component of $\mathscr{M}(\mathbb{P}^1, \mathscr{X})$ is of the expected dimension, but it is not known in general.

Due to the techniques of algebraic geometry (and in particular, to the generic smoothness theorem) many of the results that answers questions about the space of rational curves on a hypersurface (smooth, projective and of low degree, say) have some "genericity" assumption (see for instance [7, 8]). That is, the result is known to hold only in a noneffective Zariski open set of the mapping space. So while we may know for instance that for almost all hypersurfaces of degree 4 in $\mathbb{P}^6$, the space of rational curves is irreducible of the expected dimension, these theorems do not allow us to write a single equation down. The method that we develop in chapter 2 allows the study of a particular hypersurface and thus provides new categories of examples. The geometric consequences of that method are discussed in chapter 4.

Chapter 2

Algebraic Circle Method

This chapter is the core of this report. We present a method that allows to find an estimate for the number of points of a mapping space. We present the method through an example. Our goal is to count the number of rational curves on a smooth projective hypersurface $\mathscr{X}$ of low degree, that is, to estimate $\#\mathscr{M}_r(\mathbb{P}^1_{\mathbb{F}_q}, \mathscr{X})(\mathbb{F}_q)$, using the notation of 1.2.1. As explained in 1.1.1, the Circle Method was devised to study "additive" problems in number theory. Naturally, its adaptation is suitable for equations of "additive" type. This is not quite a precise notion. For now, let us agree that the Fermat hypersurface of degree d, defined by $f(x_0, \ldots, x_n) = \sum_{i=0}^{n} x_i^d$, is certainly additive. Later on (see 3.1.2), we will give a modification of the method that suggests what we need from our equations.

The main result is theorem 2.4.1, which gives estimates for $\#\mathscr{M}_r(\mathbb{P}^1_{\mathbb{F}_q}, \mathscr{X})(\mathbb{F}_q)$ provided that some singular locus has high enough codimension. We then prove that this holds for the Fermat hypersurface, and in the next chapter we investigate other cases.

2.1 Setup

Notation 2.1.1. Let k be a field with q elements. If j is an integer, we write $\mathbb{P}^j$ for $\mathbb{P}^j_k$, $\mathbb{A}^j$ for $\mathbb{A}^j_k$ and P_j for $P_j(k)$. Let f be a homogeneous form of degree d in $n+1$ variables. Let $\mathscr{X} \subseteq \mathbb{P}^n$ (resp. $X \subseteq \mathbb{A}^{n+1}$) be the projective (resp. affine) hypersurface defined by f. Let $\mathscr{M}_r = \mathscr{M}_r(\mathbb{P}^1, \mathscr{X})$ be the space of degree r rational curves on $\mathscr{X}$. We are interested in the number of points of $\mathscr{M}_r$.

We denote u, v the coordinates on $\mathbb{P}^1$ and $P_r = P_r(k)$. A naive affine version of (the cone over) $\mathscr{M}_r$, suitable for our enumerating purposes, is the space

$$M_r = \{(x_0, \ldots, x_n) \in P_r^{n+1} \ : \ f(x_0, \ldots, x_n) = 0\}$$

so that $\mathscr{M}_r$ is a quotient of an open subvariety of M_r by $\mathbb{G}_m$.

Counting points

Notation 2.1.2. If V is a scheme of finite type over a finite field k and R is a finite k-algebra, we denote

$$[V]_R = \frac{\#V(R)}{\#R^{\dim_e V}}$$

where $\dim_e V$ denotes the *expected* dimension of V. Since this is not a well defined notion, whenever we use the notation we will state explicitly what this dimension is. We let $[V] = [V]_k$.

The quantity $[V]_R$ should be close to 1. We observe that if S is another finite k-algebra, then

$$[V]_{R\times S} = [V]_R[V]_S.$$

Divisors

Since $\mathbb{P}^1$ has finitely many points in each degree, we can construct a sequence $\{D_i\}_{i\geqslant 1}$ of effective divisors such that $D_i \leqslant D_{i+1}$ and for any divisor D, there exists i such that $D \leqslant D_i$. Such a sequence is called a *cofinal* sequences of divisors.

Notation 2.1.3. Let φ be a function from effective divisors of $\mathbb{P}^1$ to $\mathbb{R}$ and let $\{D_i\}$ be a cofinal sequence of effective divisors. We write $\lim\limits_{D\to\infty} \varphi(D)$ for the limit of the sequence $\{\varphi(D_i)\}$ if it exists and is independent of the choice of the sequence $\{D_i\}$.

Notation 2.1.4. If D is an effective divisor on $\mathbb{P}^1$, we write $\mathcal{O}_D = \Gamma(D, \mathcal{O}(N)(-D))$, which is isomorphic to $\Gamma(D, \mathcal{O}(N)|_D)$. We will fix such an isomorphism and identify the two vector spaces.

Exponential Sums

Notation 2.1.5. Let $e : k \to \mathbb{C}^*$ be a nontrivial additive character, $\alpha \in P_{rd}^\vee$ and $x \in P_r^{n+1}$. We consider the sums

$$E^x = \sum_{\beta \in P_{rd}^\vee} e \circ \beta \circ f(x) \quad \text{and} \quad E_\alpha = \sum_{y \in P_r^{n+1}} e \circ \alpha \circ f(y)$$

and the corresponding averages

$$e^x = q^{-(rd+1)} E^x \quad \text{and} \quad e_\alpha = q^{-(r+1)(n+1)} E_\alpha.$$

We can think of f as defining a map $P_r^{n+1} \to P_{rd}$, in which case M_r corresponds to the fiber above 0. Using the previous definitions, we can express $[M_r]$ as an exponential sum. The expected dimension of M_r is $(r+1)(n+1) - (rd+1)$.

Lemma 2.1.6. $[M_r] = \sum_{\alpha \in P_{rd}^{\vee}} e_\alpha.$

Proof. Observe that, since α is a nontrivial character, we have

$$e^x = \begin{cases} 0 & \text{if } f(x) \neq 0, \\ 1 & \text{if } f(x) = 0 \end{cases}$$

so that

$$\sum_{x \in P_r^{n+1}} e^x = \sum_{x : f(x)=0} 1 = \# M_r(k).$$

It follows that

$$\begin{aligned}
\sum_{\alpha \in P_{rd}^{\vee}} e_\alpha &= q^{-(r+1)(n+1)} \sum_{\alpha \in P_{rd}^{\vee}} E_\alpha \\
&= q^{-(r+1)(n+1)} \sum_{x \in P_r^{n+1}} E^x \\
&= q^{-(r+1)(n+1)} q^{rd+1} \sum_{x \in P_r^{n+1}} e^x \\
&= q^{rd+1-(r+1)(n+1)} \# M_r(k) \\
&= [M_r].
\end{aligned}$$

$\square$

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To estimate $[M_r]$, we take our cue from the classical Circle Method and we break the exponential sum into major arcs and minor arcs. The major arcs should correspond to those α which yield a "big" contribution e_α. That happens, in turn, when the kernel of

α is “big”. We make the following definition.

Definition 2.1.7. Let N be a positive integer, $\alpha \in P_N^\vee$ and D an effective divisor on $\mathbb{P}^1$. We say that D *dominates* α and write $D > \alpha$ (or $\alpha < D$) if

$$\Gamma(\mathbb{P}^1, \mathcal{O}(N)(-D)) \subseteq \operatorname{Ker} \alpha.$$

The *index* of α is the integer $\operatorname{ind}(\alpha) = \min\{\deg D \ : \ D > \alpha\}$.

In other words, $\alpha < D$ is it factorizes uniquely through

$$P_N(-D) = P_N / \Gamma(\mathbb{P}^1, \mathcal{O}(N)(-D)).$$

We will denote $\bar{\alpha} : P_N(-D) \to k$ the factorization of α. This notation is abusive, as the source of $\bar{\alpha}$ depends on D, but it should not cause any confusion. If $x \in P_N$, we write $\bar{x}$ (or x mod D if we want to emphasize D) for its image in $P_N(-D)$. Observe that when $N \geqslant r$, we have $P_N(-D) = \mathcal{O}_D$.

One also has the following cohomological interpretation. Consider the exact sequence

$$\Gamma(\mathbb{P}^1, \mathcal{O}(N)(-D)) \xrightarrow{i_D} P_N \xrightarrow{\text{res}} \Gamma(D, \mathcal{O}(N)\big|_D) = \mathcal{O}_D.$$

where res is the natural restriction map. Let $\omega = \mathcal{O}(-2)$ be the dualizing sheaf on $\mathbb{P}^1$. By Serre duality, the dual of the previous sequence is the exact sequence

$$H^1(\mathbb{P}^1, \omega(D)(-N)) \xleftarrow{i_D^\vee} H^1(\mathbb{P}^1, \omega(-N)) \xleftarrow{\text{res}^\vee} \mathcal{O}_D^\vee.$$

Saying that α is dominated by D amounts to saying that its restriction to $\mathcal{O}_D^\vee$ is trivial, *i.e.* that the corresponding cohomology class $\mathrm{res}^\vee(\alpha)$ belongs to the kernel of $i_D^\vee$.

Lemma 2.1.8. *For all* $\alpha \in P_N^\vee$, $\operatorname{ind}\alpha \leqslant \frac{N}{2} + 1$.

Proof. Let $\alpha \in P_N^\vee$ and a, b be integers satisfying $a + b = N$. The natural pairing of divisors combined with α yields a map $\Gamma(\mathbb{P}^1, \mathcal{O}(a)) \to \Gamma(\mathbb{P}^1, \mathcal{O}(b))^\vee$ which is injective if and only if $\operatorname{ind}\alpha > a$. In particular, if $a > b$, this cannot be the case for dimensional reasons. The lemma follows. $\square$

Summing over $\alpha \in P_{rd}^\vee$, we can now write

$$[M_r] = \sum_{\alpha \in \mathcal{M}} e_\alpha + \sum_{\alpha \in \mathcal{m}} e_\alpha \tag{2.1}$$

where $\mathcal{M} = \{\alpha \in P_{rd}^\vee : \operatorname{ind}\alpha \leqslant c\}$ and $\mathcal{m} = \{\alpha \in P_{rd}^\vee : \operatorname{ind}\alpha \leqslant c\}$ where c is some integer that depends on the problem at hand and r. We say that $\alpha \in P_{rd}^\vee$ has *high* (respectively *low*) *index* if $\operatorname{ind}\alpha > c$ (resp. $\operatorname{ind}\alpha \leqslant c$). Although this equality has virtually no content, it is the counterpart to (1.1). For our current purposes where $\mathscr{X}$ is a hypersurface, we take $c = r$.

2.2 Major Arcs: Low Indices

It should be noted that very few of the α have low index. Yet the corresponding sum accounts for the main value of $[M_r]$.

Proposition 2.2.1. *Let* D *be an effective divisor on* $\mathbb{P}^1$ *such that* $\deg D \leqslant r$ *and write*

$N = rd$. *Summing over* $\alpha \in P_N^\vee$, *we have*

$$\sum_{\alpha < D} e_\alpha = [X]_{\mathcal{O}_D}.$$

Proof. If $\alpha < D$, we have

$$\begin{aligned} E_\alpha &= \sum_{x:\operatorname{div} f(x) \geqslant D} e(0) + \sum_{x:\operatorname{div} f(x) \not\geqslant D} e(\alpha(f(x))) \\ &= \#\{x : f(x) = 0 \bmod D\} + \sum_{x:\overline{f(x)} \neq 0} e(\bar{\alpha}(\overline{f(x)})). \end{aligned}$$

Summing over $\alpha < D$, we get

$$\begin{aligned} \sum_{\alpha < D} E_\alpha &= \#P_N(-D)^\vee \#\{x : \overline{f(x)} = 0\} + \sum_{x:\overline{f(x)} \neq 0} \sum_{\bar{\alpha} \in P_{rd}(D)^\vee} e(\bar{\alpha}(\overline{f(x)})) \\ &= \#P_N(-D)^\vee \#\{x : \overline{f(x)} = 0\}. \end{aligned}$$

Observing that for $r \geqslant \deg D$, P_r surjects onto $\mathcal{O}_D$ with kernel of size $q^{r+1-\deg D}$, we see that

$$\begin{aligned} \sum_{\alpha < D} E_\alpha &= \#P_N(-D)^\vee \#\{x \in \mathcal{O}_D^{n+1} : f(x) = 0\} q^{(n+1)(r+1-\deg D)} \\ &= q^{(n+1)(r+1)} \#X(\mathcal{O}_D) \#\mathcal{O}_D^{-\dim_e X} \\ &= q^{(n+1)(r+1)} [X]_{\mathcal{O}_D} \end{aligned}$$

and the claim follows. $\square$

In turn, this implies that

$$\sum_{\alpha\in\mathscr{M}} e_\alpha = \prod_{x:\deg x\leqslant r} [X]_{\mathcal{O}_x/\pi_x^{\lfloor \frac{r}{\deg x}\rfloor}\mathcal{O}_x}$$

where $\mathcal{O}_x$ is the local ring at x and π_x is a uniformizer at x. This is slightly cumbersome, so the final statement will use the limit of the right-hand side as $r \to \infty$. We now turn to the majoration of the minor arcs.

2.3 Minor Arcs: High Indices

Proposition 2.3.1. *Let $\alpha \in P_{rd}^\vee$ and δ_α be the dimension of the singular locus*

$$S_\alpha = \left\{ s \in P_r^{n+1} \ : \ \forall i = 0, \ldots, n, \quad \frac{\partial f}{\partial x_i}(s) \in \operatorname{Ker}\alpha^\vee \right\}.$$

Then there exists a constant C depending only on n and d such that we have the estimate

$$\left|\sum_{\alpha\in m} e_\alpha\right| \leqslant C^r \sqrt{q}^{\delta_m - r(n+1-2d)} \tag{2.2}$$

for any integer r, where $\delta_m = \sup_{\alpha\in m} \delta_\alpha$.

Remark 2.3.2. In this chapter, we will use C to denote any constant that depends only on n and d. In particular the constant may change from one line to the next.

Proof. Applying theorem 1.1.6 with $X = P_r^{n+1}$ which is naturally a closed subset of projective space, $\psi = e$ and $f = \alpha \circ f$, we get

$$|e_\alpha| \leqslant C^r \sqrt{q}^{2+\delta_\alpha - (n+1)(r+1)}$$

Summing over α, we get

$$\left|\sum_{\alpha\in m} e_\alpha\right| \leqslant \frac{C^r}{\sqrt{q}^{3-n}}\sqrt{q}^{\delta_m - r(n+1-2d)}$$

where C depends only on d and n. The result follows. $\square$

Again, it should be noted that the proof is misleadingly simple, as theorem 1.1.6 is hard. As explained earlier, bounding the minor arcs is the key step in the Circle Method and usually involves some computational trick, which is hard to generalize. This step has now been reduced to bounding the dimension of some singular locus, which is a much more manageable task.

Estimate (2.2) remains true if we sum over all nontrivial α. The dimension δ_α, however, can only be favorably controlled when α has high index. Namely, if we can prove that $\dfrac{\delta_\alpha}{r} < n+1-2d$ when $r \gg 0$, then equation (2.2) implies that $\sum\limits_{\alpha\in m} e_\alpha = o(1)$ as $r \to \infty$. If that were true for all nontrivial α, we would merely get $[M_r] = 1$. That would be the case if, say, all the fibers of f had the same size, but there is a correcting factor of $\lim\limits_{D\to\infty}[X]_{\mathcal{O}_D}$.

2.4 Generating Functions

We put together all the results obtained so far. The main result of the Circle Method for a hypersurface follows. We recall all the data for clarity.

Theorem 2.4.1. *Let k be a finite field of size q. Let f be a polynomial of degree d in $n+1$ variables such that the hypersurface $\mathscr{X}$ defined by f in $\mathbb{P}^n_k$ is smooth and $\mathscr{M}_r$*

the space of rational curves of degree r on $\mathscr{X}$. Define the quantity

$$\varepsilon(r) = \varepsilon(\mathbb{P}^1, \mathscr{X}, q\,; r) = \left| [\mathscr{M}_r] - \frac{1}{1-q^{-1}} \prod_x (1 - q_x^{-1}) [\mathscr{X}]_{\kappa(x)} \right|$$

where the product runs over the closed points x of $\mathbb{P}^1_k$, $\kappa(x)$ denotes the residue field of x and $q_x = \#\kappa(x)$. If $\mathscr{X}$ is smooth, we have the estimate

$$\varepsilon(r) = O(C^r q^{\gamma r}) \quad \textit{as} \quad r \to \infty \tag{2.3}$$

where C is independent of q and r, $\gamma = \max(d-n, \delta)$ and

$$\delta = \sup \left\{ \frac{\delta_\alpha}{r} + 2d - (n+1) : \alpha \in P^\vee_{rd},\ \mathrm{ind}(\alpha) > r \right\}.$$

Consequently if $d < n$ and $\delta < 0$ then $\gamma < 0$ and $\varepsilon(r) = o(1)$. In particular, in this case we have

$$[\mathscr{M}_r] \sim \lim_{D \to \infty} [\mathscr{X}]_{\mathcal{O}_D} \quad \textit{as} \quad q \to \infty \tag{2.4}$$

for all $r \gg 0$.

The theorem is uninteresting without a way to estimate the quantity δ and prove that it is negative. This will be our next task after we prove the theorem. We start by computing $\lim_{D\to\infty} [X]_{\mathcal{O}_D}$. By multiplicativity, we can work one point at a time.

Lemma 2.4.2. *Let $R_e = k[t]/(t^{e+1})$. Then*

$$[X]_{R_e} = \frac{1}{1 - q^{d-(n+1)}} [X] + O(q^{(d-(n+1))(\lfloor \frac{e}{d} \rfloor + 1)}) \quad \textit{as} \quad e \to \infty.$$

Proof. We define $X_{e,i}^j = \{(t^i x_0, \ldots, t^i x_n) \in X(R_e) : x_j \notin (t)\}$ so that

$$X(R_e) = \coprod_{i=0}^{e+1} X_{e,i} \quad \text{where} \quad X_{e,i} = \coprod_{j=0}^{n} X_{e,i}^j.$$

Furthermore, as long as $di < e+1$, we have

$$(t^i x_0, \ldots, t^i x_n) \in X(R_e) \iff (x_0, \ldots, x_n) \bmod t^{e+1-di} \in X(R_{e-di}).$$

Therefore, if we write $X^* = X - \{0\}$, we have

$$\begin{aligned} \#X(R_e) &= \sum_{i=0}^{\lfloor e/d \rfloor} q^{(n+1)(d-1)i} \#X^*(R_{e-di}) + \sum_{i=\lfloor e/d \rfloor+1}^{e} q^{(n+1)(e-i)}(q^{n+1}-1) + 1 \\ &= \sum_{i=0}^{\lfloor e/d \rfloor} q^{(n+1)(d-1)i} \#X^*(R_{e-di}) + q^{(n+1)(e-\lfloor e/d \rfloor)}. \end{aligned}$$

It follows that

$$[X]_{R_e} = \sum_{i=0}^{\lfloor e/d \rfloor} q^{-i(n+1-d)} [X^*]_{R_{e-di}} + q^{-(\lfloor e/d \rfloor+1)(n+1)+e+1}.$$

Now, since $\mathscr{X}$ and so X^* is smooth, we have

$$[X]_{R_e} = \left(\sum_{i=0}^{\lfloor e/d \rfloor} q^{-i(n+1-d)} \right) [X^*] + q^{-(\lfloor e/d \rfloor+1)(n+1)+e+1}$$

and the lemma follows. $\square$

The construction of the classical Circle Method uses generating functions. They are convenient in the proof of theorem 2.4.1. We consider the function a_D defined on effective divisors by

$$\sum_{T \leqslant D} a_T = [X]_{\mathcal{O}_D}$$

and the associated generating function $A(z) = \sum_D a_D z^{\deg D}$. Then the Euler product expansion of A reads

$$A(z) = \prod_x a_x(z) \quad \text{where} \quad a_x(z) = \left(1 + \sum_{i=1}^{\infty} a_{ix} z^{i \deg x}\right)$$

where the product is taken over the closed points of $\mathbb{P}^1$. Remark that lemma 2.4.2 implies that

$$a_x(1) = \frac{1}{1 - q_x^{d-(n+1)}} [X]_{\kappa(x)} \quad \text{where} \quad q_x = \#\kappa(x).$$

Also, A is holomorphic in a complex disk centered at the origin and of radius $Cq^{\frac{n+1}{2}}$ where C depends only on n and d.

Proposition 2.4.3.

$$[M_r] = A(1) + O(C^r \sqrt{q}^{\delta_\alpha - r(n+1-2d)}, q^{r(d-n)}).$$

Proof. Using the definition of the coefficients a_D and proposition 2.2.1, we see that

$$\sum_{\deg D \leqslant r} a_D = \sum_{\alpha \in \mathcal{M}} e_\alpha$$

so that

$$[M_r] - A(1) = \sum_{\alpha \in m} e_\alpha - \sum_{\deg D > r} a_D.$$

The first term is bounded by $C^r\sqrt{q}^{\delta_\alpha - r(n+1-2d)}$ by proposition 2.3.1. To bound the second one, we use lemma 2.4.2 to deduce that, say,

$$\sum_{\deg D > r} a_D = O(q^{r(d-n)}).$$

The proposition follows. $\square$

To complete the proof, we need to switch from affine to projective schemes. We write

$$\zeta(t) = \zeta_{\mathbb{P}^1_k}(t) = \frac{1}{(1-t)(1-qt)}$$

for the zeta function of the projective line.

Lemma 2.4.4.

- *If x is a closed point of $\mathbb{P}^1$, then $[X^*]_{\kappa(x)} = (1-q_x^{-1})[\mathscr{X}]_{\kappa(x)}$, and*
- $[\mathscr{M}_r] = (1-q^{-1})^{-1}\zeta(q^{d-n})^{-1}[M_r] + O(q^{r(d-(n+1))})$ *as $r \to \infty$.*

Proof. The first statement is clear. To prove the second, observe first that an element $x = (x_0, \ldots, x_n) \in M_r$ defines an actual morphism from $\mathbb{P}^1$ to $\mathscr{X}$ if and only if the x_i have no common zero. In fact, writing h for the common factor of the x_i, we can write $x = (hy_0, \ldots, hy_n)$ with $h \in P_s$ and $y = (y_0, \ldots y_n) \in \mathscr{M}_{r-\deg h}$. Note that h and y are determined uniquely from x up to the action of k^*. Considering $D = \operatorname{div} h$ instead and accounting for dimension, we can thus write

$$[M_r] = (1-q^{-1}) \sum_{\deg D \leqslant r} [\mathscr{M}_{r-\deg D}].$$

Expanding into an Euler product, we get

$$[M_r] = (1-q^{-1})[\mathscr{M}_r]\zeta(q^{d-n}) + O(q^{r(d-(n+1))})$$

whence the lemma. $\square$

Finally, we can prove theorem 2.4.1. Using lemma 2.4.4 we have

$$\begin{aligned}
&[\mathscr{M}_r] - \frac{1}{1-q^{-1}}\prod_x (1-q_x^{-1})[\mathscr{X}]_{\kappa(x)} \\
= \;& \left(1-q^{-1}\right)^{-1}\left(\zeta(q^{d-(n+1)})^{-1}[M_r] - \prod_x [X]_{\kappa(x)}\right) + O(q^{r(d-(n+1))}) \\
= \;& \left(1-q^{-1}\right)^{-1}\zeta(q^{d-(n+1)})^{-1}\left([M_r] - A(1)\right) + O(q^{r(d-(n+1))}) \\
= \;& O(C^r q^{\gamma r})
\end{aligned}$$

by proposition 2.4.3, where $\gamma = \max\{d-n, \sup\{\frac{\delta_\alpha}{r} - (n+1) + 2d : \operatorname{ind}\alpha > r\}\}$.

2.5 The Fermat Hypersurface

Assume that $f(x_0,\dots,x_n) = x_0^d + \cdots + x_n^d$. Then

$$S_\alpha = \left\{x \in P_r^{n+1} : \alpha\circ f(x) = 0 \text{ and } \forall i \in [0,n]\ \alpha(dx_i^{d-1}-) = 0 \text{ in } P^\vee_{r(d-1)}\right\}.$$

Lemma 2.5.1. *If f is the Fermat hypersurface of degree d and $\alpha \in P^\vee_{rd}$ satisfies $\operatorname{ind}\alpha > r$, then*

$$\delta_\alpha \leqslant \frac{d-2}{d-1} r(n+1).$$

In particular, (2.4) holds whenever $n+1>2d(d-1)$.

Proof. We consider the map $\alpha^\vee : P_{r(d-1)} \to P_r^\vee$ and we define

$$\mathfrak{S}_\alpha = \{x \in P_r : \alpha^\vee(x^{d-1}) = 0\},$$

so that $S_\alpha \subseteq \mathfrak{S}_\alpha \cap V(\alpha \circ f)$. Let $\sigma_\alpha = \dim \mathfrak{S}_\alpha$. Then $\delta_\alpha \leqslant (n+1)\sigma_\alpha$. Consider the composition

$$P_r \xrightarrow{\Delta} P_r^{d-1} \xrightarrow{m} P_{r(d-1)} \xrightarrow{\alpha^\vee} P_r^\vee$$

where Δ is the diagonal map and m is the multiplication map $m(x_1,\dots,x_d) = x_1 \cdots x_d$. Then $\mathfrak{S}_\alpha$ is the inverse image of 0 under this composition. Then $\sigma_\alpha \leqslant \frac{1}{d-1}\dim\ker\alpha^\vee$. Besides, using our ongoing notion of high index we can give the following characterization:

Lemma 2.5.2. *Let* $\alpha \in P_{rd}^\vee$. *Then* $\operatorname{ind}\alpha > r$ *if and only if the map* $P_{r(d-1)} \to P_r^\vee$ *induced by* α *is surjective.*

This is just a restatement of the definition. It follows that $\dim\ker\alpha^\vee \leqslant r(d-2)$. Therefore

$$\delta_\alpha \leqslant (n+1)\sigma_\alpha \leqslant \frac{d-2}{d-1} r(n+1)$$

as claimed. Now, (2.4) holds as soon as $\delta_\alpha < r(n+1-2d)$. This is true whenever $\dfrac{d-2}{d-1}r(n+1) < r(n+1-2d)$, which gives $n+1 > 2d(d-1)$, whence the lemma. $\square$

Finally, we can write down the conclusion

Theorem 2.5.3. *Let* $\mathscr{X} = \mathscr{X}_{n,d}$ *be the degree* d *Fermat hypersurface in* $\mathbb{P}^n$ *and assume* *that* $n \geqslant 2d(d-1)$. *Then there exists a constant* $C = C_{n,d}$ *such that*

$$[\mathscr{M}_r] = (1-q^{-1}) \prod_x (1-q_x^{-1})[\mathscr{X}]_{\kappa(x)} + O((Cq^{\gamma})^r)$$

where $\gamma = \max\left\{d-n, 2d - \dfrac{n+1}{d-1}\right\}$. *In particular* $[\mathscr{M}_r] = \lim_{D\to\infty} [\mathscr{X}_f]_{\mathcal{O}_D}$.

Remark 2.5.4. We note that the statement and its proof remained unchanged if we replace $\mathscr{X}$ by a diagonal hypersurface, of the form

$$f(x_0, \dots, x_n) = a_0 x_0^d + \cdots + a_n x_n^d$$

where $a_0, \dots, a_n$ are nonzero elements of k.

Chapter 3

Variations

3.1 Specializations and Generalizations

3.1.1 Cubic Hypersurfaces

In this section we prove that an *arbitrary* smooth cubic hypersurface of sufficiently low degree satisfies the estimate of theorem 2.4.1.

Proposition 3.1.1. *Let k be a finite field and $\mathscr{X} \subseteq \mathbb{P}^n_k$ the hypersurface defined by a nonsingular cubic form f. There exists an integer Ψ such that if $n > \Psi$, then $\delta \leqslant r(n-5)$. In particular, estimate (2.3) holds.*

Before proving the proposition, we collect a theorem of Birch (see [1]).

Theorem 3.1.2. *Let $h \geqslant 1$ and $m \geqslant 1$ be integers, and let $r_1, \cdots r_h$ be odd positive integers. Let K be a number field. Then there exists a number $\Psi = \Psi(r_1, \ldots, r_h; m, K)$ such that if $n \geqslant \Psi$ and $f_{r_1}(x), \ldots, f_{r_h}(x)$ are any forms over K of degrees $r_1, \ldots, r_h$ respectively in the n variables $x_1, \ldots, x_n$, there is an m-dimensional linear space over*

K on which $f_{r_1}(x) = 0, \dots, f_{r_h}(x) = 0$.

Remark 3.1.3. The proof of this theorem consists merely in looking at the polar forms of the forms $f_{r_1}, \dots, f_{r_h}$ and requiring that the mixed polar forms vanish so that the forms become diagonal. The theorem then boils down to whether a diagonal form in enough variables properly represents zero or not. Of course, over the rationals that is not true, whence the assumption that the degree be odd. But for a finite field, this condition is superfluous. In particular, the theorem is true for K a finite field and without assumptions on the degrees of the forms. Interestingly enough, we use this theorem to prove that an arbitrary nonsingular form can be reduced to a Fermat form on a large linear subspace.

Lemma 3.1.4. *Let k be a finite field and $\mathscr{X} \subseteq \mathbb{P}^n_k$ the hypersurface defined by a nonsingular cubic form f. Then there exists an integer Ψ such that if $n > \Psi$, then there exists a linear subspace $\mathbb{P} \subseteq \mathbb{P}^n_k$ of dimension 11 such that if $\mathscr{X}_k \cap L$ is a Fermat cubic hypersurface in $\mathbb{P}$.*

Proof. Since f is nonsingular, we can find a change of variables so that f has the form

$$f(x_0, \dots, x_n) = a_0 x_0^3 + x_0 Q_1(x_1, \dots, x_n) + C_1(x_1, \dots, x_n)$$

where $a_0 \in k$ is nonzero and Q_1, C_1 are forms of degree $2, 3$ respectively. By theorem 3.1.2 and remark 3.1.3, there exists an integer $\Psi_0 = \Psi(2; N_0, k)$ such that if $n+1 \geqslant \Psi_0$ then

$$Q_1(x_1, \dots, x_n) = 0$$

on a linear subspace H_0 of $\mathbb{P}^n$ of dimension N_0, where N_0 is an integer to be fixed later, and C_1 is nonsingular. Applying if necessary another change of variables to $x_1, \ldots, x_n$ we may assume that

$$C_1(x_1, \ldots, x_n) = a_1 x_1^3 + x_1 Q_2(x_2, \ldots, x_n) + C_2(x_2, \ldots, x_n)$$

where $a_1 \neq 0$, Q_2, C_2 are forms of degree $2, 3$ respectively and C_2 is nonsingular. By theorem 3.1.2 and remark 3.1.3, there exists an integer $\Psi_1 = \Psi(2; N_1, k)$ such that if $N_0 \geqslant \Psi_1$ then

$$Q_2(x_1, \ldots, x_n) = 0$$

on a linear subspace H_1 of H_0 of dimension N_1, where N_1 is an integer to be fixed later. Repeating the process, we can find linear subspaces $H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{12}$ of respective dimensions $N_0 \geqslant N_1 \geqslant \cdots \geqslant N_{12}$ such that

$$f(x_0, \ldots, x_n) = a_0 x_0^3 + \cdots + a_i x_i^3 + C_{i+1}(x_{i+1}, \ldots, x_n)$$

with $a_0, \ldots, a_i \neq 0$, whenever $(x_0, \ldots, x_n) \in H_i$. By 3.1.2 and remark 3.1.3 again, there exists an integer $\Psi_{13} = \Psi(3; 12, k)$ such that if $N_{12} \geqslant \Psi_{13}$ then

$$C_{13}(x_{13}, \ldots, x_n) = 0$$

on a linear subspace $\mathbb{P}$ of H_{12} of dimension 12. We thus see that we can fix the N_i to be big enough by backward induction, and therefore there exists an integer Ψ such that if $n > \Psi$ then there exists a linear subspace $\mathbb{P}$ of $\mathbb{P}^n$ of (projective) dimension 12 such that

$$f(x_0, \cdots, x_n) = a_0 x_0^3 + \cdots + a_{12} x_{12}^3$$

for all $(x_0, \dots, x_n) \in \mathbb{P}$. $\square$

Proof of proposition 3.1.1. Let $\mathscr{X}$ and f be as in the statement of the proposition and $\alpha \in P_{3r}^{\vee}$. We have to evaluate $\delta_\alpha = \dim S_\alpha$. By lemma 3.1.4, there is an integer Ψ such that if $n > \Psi$ then there exists a linear subspace of dimension 11 of $\mathbb{P}_k^n$ for which $\mathscr{X} \cap \mathbb{P} = \mathscr{X}_{11,3}$ is the Fermat cubic in 12 variables. Observe that

$$\delta_\alpha \leqslant \dim S_\alpha \cap \mathbb{P} + \operatorname{codim} \mathbb{P}$$

since all schemes pass through the origin. In addition, $S_\alpha \cap \mathbb{P}$ is the corresponding singular locus for a diagonal hypersurface in 12 variables. By lemma 2.5.1 and remark 2.5.4, we thus get

$$\delta_\alpha \leqslant 6r + r(n-11) = r(n-5).$$

In particular, this implies that $\delta < 0$ in theorem 2.4.1. $\square$

3.1.2 Smooth Hypersurfaces

In this section, we present two results. The first is the mere observation that the proof given for cubic hypersurfaces in the last section has nothing specific to cubics, so the result extends to hypersurfaces of arbitrary degree. Then we discuss the quantification of the number of variables. The second result stems from the proof of the main estimate in the Fermat case. It takes a closer look at the method and suggests ways it can be adapted for a particular equation.

Specialization to Diagonal Hypersurfaces

As we just remarked, the proof of proposition 3.1.1 is not specific to cubic hypersurfaces. Therefore the result holds in arbitrary degree.

Proposition 3.1.5. *Let k be a finite field and $\mathscr{X} \subseteq \mathbb{P}^n_k$ a nonsingular hypersurface. There exists an integer Ψ such that if $n > \Psi$, then $\delta < r(n+1-2d)$ when r is big enough. In particular, estimate (2.3) holds.*

We sketch the argument.

Sketch of proof. The first step is to find a linear subspace $\mathbb{P}$ of $\mathbb{P}^n_k$ of some dimension j such that the defining equation f of $\mathscr{X}$ reduces to a diagonal equation in $j+1$ variables, where j will be specified later. This is possible, provided that n is greater than some integer Ψ depending on d and j. This follows from theorem 3.1.2 of Birch, or by looking directly at the necessary vanishing of the mixed polar forms (which is the proof of the theorem).

The second step is to use the estimate for δ already proven in the diagonal case. Namely, using 2.5.1, we can write for $\alpha \in P^\vee_{rd}$

$$\delta_\alpha \leqslant \dim S_\alpha \cap \mathbb{P} + \operatorname{codim} \mathbb{P} \leqslant \frac{d-2}{d-1} r(j+1) + (n-j)(r+1).$$

For theorem 2.4.1 to be effective, it is therefore sufficient that

$$\frac{d-2}{d-1} r(j+1) + (n-j)(r+1) < r(n+1-2d)$$

which holds for $j \geqslant 2d(d-1)$ when $r \to \infty$. Unsurprisingly, this is what we found for the Fermat itself. $\square$

The proof of Birch's theorem does not yield an efficient, or even reasonable estimate for the number Ψ. Using a more effective diagonalization process, Wooley obtains some nontrivial bounds (see [23] and [24]). He proves that for the field of rational numbers and for equations of odd degree, the bounds are, in his terms, "not even astronomical". For instance, he proves that one can take for h rational quintic forms,

$$\Psi(5,\ldots,5;m,\mathbb{Q}) < (90h)^8(\log(27h))^5(m+1)^5.$$

Essentially, the bounds are exponential in h and m. It is quite possible that the bounds for finite fields are better. Again, this is the condition that guarantees that the forms can be appropriately diagonalized.

In comparison, Starr considers in [19], for a smooth complex hypersurface $\mathscr{X}$ of $\mathbb{P}^n$, the rational transformation

$$\begin{array}{rccc} \Phi : & \mathbb{G}(m,n) & \dashrightarrow & \mathbb{P}^{N_d} /\!\!/ \mathbf{PGL}_{m+1} \\ & \mathbb{P} & \longmapsto & [\mathbb{P} \cap \mathscr{X}] \end{array}$$

from the Grassmanian parametrizing linear m-dimensional subspaces of $\mathbb{P}^n$ to the moduli space $\mathbb{P}^{N_d} /\!\!/ \mathbf{PGL}_{m+1}$ of degree d semistable hypersurfaces in $\mathbb{P}^m$, that maps an m-plane $\mathbb{P}$ to the (equivalence class of the) intersection $\mathbb{P} \cap \mathscr{X}$. He shows the following

Theorem 3.1.6. *If $\mathscr{X}$ is a smooth hypersurface of degree d in $\mathbb{P}^n_{\mathbb{C}}$, then Φ is dominant* *as soon as* $n \geqslant \binom{d+m-1}{m} + m - 1.$

By a mere dimension count, this map can only be dominant if

$$n \geqslant \frac{1}{m+1}\binom{d+m}{m} - 1,$$

which is roughly a factor of m smaller than the above bound. This result is not specific enough for our purposes. We need to find one special point in the moduli space, so the dominance of Φ may be irrelevant, but it suggests that, unless something really special happens for the diagonal hypersurfaces, the method will require n to be exponentially larger than j, which is roughly $2d^2$.

Other examples

So far, all the cases that we have seen reduce the estimate of δ to the estimate that was obtained in lemma 2.5.1 for a Fermat (or diagonal) hypersurface. Instead of using this result directly, we try to adapt its proof to fit other cases without having to take linear sections. There are two keys aspects of the Fermat hypersurface which make it suitable for that method. We highlight those now.

Let k be a finite field and f a nonsingular form of degree d over k. Let also $\alpha \in P_{rd}^{\vee}$ be a linear functional with $\operatorname{ind} \alpha > r$. Recall that this means that $\dim \ker \alpha^{\vee} \leqslant r(d-2)$. The first favorable case is when the singular locus S_α can be written as a product of other singular loci involving fewer variables. In the diagonal case, $S_\alpha = \mathfrak{S}_\alpha^{n+1}$ is completely split, as each factor involves only one variable. If, say, the first j partial derivatives of f involve only the first j variables and no other derivative involves any of these variable, then $S_\alpha = \mathfrak{S}_\alpha(0,\dots,j-1)\times\mathfrak{S}_\alpha(j,\dots,n)$ where

$$\mathfrak{S}_\alpha(a,b) = \left\{ s \in P_{rd}^{b-a+1} :\ \forall i = a,\dots,b,\ \frac{\partial f}{\partial x_i}(s) \in \ker \alpha^\vee \right\}.$$

This makes the problem easier to deal with, although it does not necessarily improve the bounds.

The second step is to deal with each individual term $\mathfrak{S}_\alpha$. Here one has to be creative. In the case of a diagonal hypersurface, it sufficed to consider a $(d-1)$-fold product map $P_r^{d-1} \to P_{r(d-1)}$. In general, matters are more complex, as $\mathfrak{S}_\alpha$ involves more than one variable. One way to try and bound its dimension is to take further derivatives, in the following sense: we look at

$$\mathfrak{S}_\alpha^{(t)}(a,b) = \left\{ (s, s^1, \dots, s^t) \in (P_r^{b-a+1})^t :\ \forall i = a,\dots,b,\ \sum_{|J|=t} \frac{\partial^t \partial f}{\partial x_J \partial x_i}(s) s^J \in \ker \alpha^\vee \right\}$$

where J denotes a subset of $\{a, a+1, \dots, b\}$, s^j are upper indices (not exponents) and $s^J = s^1_{j_1}\cdots s^t_{j_t}$ if $J = \{j_1, \dots j_t\}$. Typically, one expects that $\sigma_\alpha^{(t)}(a,b) = \sigma_\alpha^{(t+1)}(a,b)$, where $\sigma_\alpha^{(t)}(a,b) = \dim \mathfrak{S}_\alpha^{(t)}(a,b)$. It is usually convenient to take linear sections of the $\mathfrak{S}_\alpha^{(t)}(a,b)$ to use power maps again.

An example is in order.

Proposition 3.1.7. *Let k be a finite field of characteristic prime to 7 and define the equation*

$$f(x_0, y_0, z_0, \dots, x_n, y_n, z_n) = \sum_{i=0}^n (x_i^7 y_i + y_i^7 z_i + z_i^7 x_i).$$

Then f defines a nonsingular hypersurface of degree 8 in $\mathbb{P}^{3(n+1)-1}$ and we have the *estimate*

$$\delta_\alpha < r(3n-13) \quad \textit{for} \quad r \gg 0$$

for $\alpha \in P_{8r}^\vee$ *with* $\operatorname{ind}\alpha > r$*, provided that* $n \geqslant 56$*. In particular, estimate (2.3) holds.*

Proof. It is easy to check that f is nonsingular. We fix $\alpha \in P_{8r}^\vee$ with $\operatorname{ind}\alpha > r$. Then $S_\alpha = \mathfrak{S}_\alpha^{n+1}$ where

$$\mathfrak{S}_\alpha = \left\{(x,y,z) \in P_r^3 :\ 7x^6y+z^7, 7y^6z+x^7 \text{ and } 7z^6x+y^7 \in \ker\alpha^\vee\right\}.$$

To bound σ_α, we can simply take the linear section $x=0$ to write

$$\sigma_\alpha \leqslant 1 + \dim\left\{s \in P_r^2 :\ s_1^7, s_2^7 \in \ker\alpha^\vee\right\}$$

which we can bound similarly as in the Fermat case by $(1+\frac{12}{7})r+1$. This is proportional to $\frac{19}{7}r$, so it is certainly less than $r(3(n+1)-16)$ when r is big enough. A computation shows that this holds when $n > 55$ and $r \to \infty$. $\square$

The terminology of additive number theory is usually reserved for the study of diagonal equations. We see that the above method is optimal in that case, but that it still works for equations that are "close" to being additive.

3.1.3 Remarks

Varying the Source

Although the case of interest for us is that of rational curves, it is conceivable to use the method for curves of higher genus. The essential difference is that P_r is replaced with the global sections on a sheaf on the curve C, $P_r = H^0(C, \mathcal{F}(r)(-D))$. One can still estimate the dimension of P_r using Riemann-Roch to prove a result analogous to 2.4.1, but the computations using S_α are much less explicit.

Varying the Target

Alternatively, one can replace the hypersurface $\mathscr{X}$ with another space of interest. The case of homogoneous spaces should be easy, but the mapping space is already well understood. In the case of a complete intersection, we see no obstruction to a theorem of the form of 2.4.1, and computations should be manageable in some examples.

3.2 A Singular Version

In this section we present a variant of the method that allows to bound the dimension of the singular locus of $\mathscr{M}_r$ from above. We work out the case of chapter 2, of rational curves on a Fermat hypersurface. We use the notation from that chapter, in particular $f(x) = x_0^d + \cdots + x_n^d$. More precisely, we want to show that for some $\varepsilon > 0$, the number of singular points of the mapping space is bounded above by $q^{(1-\varepsilon)\dim \mathscr{M}_r}$ as $r \gg 0$. This guarantees in particular that the singularities are in low codimension when r is big.

Unstable locus

Definition 3.2.1. Let $x \in P_r^{n+1}$. We write $\nabla f(x) = \left(\dfrac{\partial f}{\partial x_0}(x), \ldots, \dfrac{\partial f}{\partial x_n}(x)\right)$ the gradient of f. Let D_x be the effective divisor on $\mathbb{P}^1$ of common zeroes of $\dfrac{\partial f}{\partial x_0}(x), \ldots, \dfrac{\partial f}{\partial x_n}(x)$.

Then we say that x is *unstable* if the map

$$\begin{aligned} P_r^{n+1} &\longrightarrow P_{rd}(-D_x) \\ h &\longmapsto \nabla f(x) \cdot h \end{aligned}$$

given by dot product, is not surjective.

If the point x corresponds to an actual map $\mathbb{P}^1 \to \mathscr{X}$, then it is unstable if and only if it is not a smooth point of $\mathscr{M}_r$.

Notation 3.2.2. Since we are in fact analyzing the unstable locus, we introduce the scheme $Y \subseteq \mathbb{A}^{n+1} \times \mathbb{A}^{n+1}$ defined by the equations $f(x) = 0$, $\nabla f(x) \cdot h = 0$. It has dimension $2n$. We also introduce $Z \subseteq \mathbb{A}^{n+1} \times \mathbb{A}^{n+1}$ defined by the single equation $\nabla f(x) \cdot h = 0$. It has dimension $2n + 1$.

We first collect a lemma about the size of Y.

Lemma 3.2.3. *Let $x \in P_r^{n+1}$ be unstable and write $d_x = \deg D_x$. Let*

$$m \geqslant \frac{(rd - d_x) - 2r - 2}{n - 1}$$

be an integer. Then there exists a nonzero vector $h \in P_m^{n+1}$ such that $\nabla f(x) \cdot h = 0$.

Proof. Consider the map of locally free sheaves induced by $\nabla f(x)$ and denote $\mathcal{K}$ its kernel

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^1}(r)^{n+1} \to \mathcal{O}_{\mathbb{P}^1}(rd - d_x) \to 0. \tag{3.1}$$

Since all locally free sheaves on $\mathbb{P}^1$ decompose as a direct sum of line bundles, we can write $\mathcal{K} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ where the a_i are integers satisfying $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Since the above sequence is exact, we have $\sum_{i=1}^{n} a_i = r(n+1) - (rd - d_x)$. By assumption, the vector x is unstable, which means that we loose surjectivity in the sequence (3.1) when we take global sections. This implies in turn that $H^1(\mathbb{P}^1, \mathcal{K})$ is nonzero and so necessarily $a_1 \leqslant -2$. Therefore,

$$\sum_{i=2}^{n} a_i \geqslant r(n+1) - (rd - d_x) + 2.$$

It follows that $a_n \geqslant \dfrac{r(n+1) - (rd - d_x) + 2}{n-1}$ and so using the assumption on m, we have

$$a_n + m - r \geqslant \frac{r(n+1) - (rd - d_x) + 2}{n-1} + \frac{(rd - d_x) - 2r - 2}{n-1} - r = 0.$$

In particular, if we twist the sequence (3.1) by $m - r$, the sheaf $\mathcal{K}(m-r)$ has global sections and we can find h as in the statement of the lemma. □

Exponential Sums

Notation 3.2.4. Let m be the smallest integer satisfying $m \geqslant \dfrac{rd - 2r - 2}{n-1}$, $\alpha \in P_{rd}^{\vee}$ and $\beta \in P_{r(d-1)+m}^{\vee}$. We introduce the exponential sums

$$e_{\alpha,\beta} = q^{-(r+1)(n+1)} q^{-(m+1)(n+1)} \sum_{x \in P_r^{n+1}} \sum_{h \in P_m^{n+1}} e(\alpha(f(x)) + \beta(\nabla f(x) \cdot h)).$$

We would like to bound the size of the unstable locus in terms of these exponential sums. Summing over $\alpha \in P_{rd}^{\vee}$ and $\beta \in P_{r(d-1)+m}^{\vee}$ we have

$$\sum_{\alpha}\sum_{\beta} e_{\alpha,\beta} = q^{-(r+1)(n+1)} q^{-(m+1)(n+1)} \sum_{\alpha}\sum_{\beta}\sum_{x}\sum_{h} e(\alpha(f(x)) + \beta(\nabla f(x)\cdot h))$$

Similarly to before, the sum for fixed x and h is nonzero if and only if x is unstable and h satisfies $\nabla f(x)\cdot h = 0$. That is, if (x,h) is a point of Y. Counting dimensions, we get

$$\sum_{\alpha}\sum_{\beta} e_{\alpha,\beta} = q^{rd+1} q^{r(d-1)+m+1} q^{-(r+1)(n+1)} q^{-(m+1)(n+1)} \# Y(P_r \times P_m).$$

Now observe that lemma 3.2.3 implies that

$$\# Y(P_r \times P_m) \geqslant q \# \{x \in P_r^{n+1} : f(x) = 0 \text{ and } x \text{ is unstable}\}$$

so finally we get

$$\#\{x \in P_r^{n+1} : f(x) = 0 \text{ and } x \text{ is unstable}\} \leqslant q^{r(2d-1-(n+1))-(n+1)(m+2)+m+3} \sum_{\alpha}\sum_{\beta} e_{\alpha,\beta}.$$

Our goal is to show that there exists $\varepsilon > 0$ such that for big enough r

$$\#\{x \in P_r^{n+1} : f(x) = 0 \text{ and } x \text{ is unstable}\} \leqslant q^{(1-\varepsilon)\dim \mathscr{M}_r}.$$

Using the fact that $m \geqslant \frac{rd-2r-2}{n-1}$, it suffices to show that there exists $\varepsilon > 0$ such that

$$\sum_{\alpha}\sum_{\beta} e_{\alpha,\beta} \leqslant q^{r\left(\frac{n+1-d}{n-1}-\varepsilon\right)+C} \tag{3.2}$$

where C is a constant that depends on n and d only.

Major Arcs : Low Indices

Notation 3.2.5. If D_1, D_2 are two effective divisors on $\mathbb{P}^1$, we use the notation

$$[X \times Z]_{D_1,D_2} = \left(\#X(\mathcal{O}_{D_1}) \times_{\mathbb{A}^{n+1}(\mathcal{O}_{D_1\cap D_2})} Z(\mathcal{O}_{D_2})\right) \frac{(\#\mathcal{O}_{D_1\cap D_2})^{n+1}}{(\#\mathcal{O}_{D_1})^{\dim X}(\#\mathcal{O}_{D_2})^{\dim Z}}.$$

In particular, we have that

$$[X \times Z]_{0,0} = 1, \quad [X \times Z]_{D_1,0} = [X]_{\mathcal{O}_{D_1}} \quad \text{and} \quad [X \times Z]_{0,D_2} = [Z]_{\mathcal{O}_{D_2}}.$$

Proposition 3.2.6. *Let D_1, D_2 be two effective divisors on $\mathbb{P}^1$. Assume that $\deg(D_2) \leqslant m$ and $\deg(D_1 \cup D_2) \leqslant r$. Then*

$$\sum_{\alpha < D_1}\sum_{\beta < D_2} e_{\alpha,\beta} = [X \times Z]_{D_1,D_2}.$$

Proof. This is completely analogous to proposition 2.2.1. Write

$$Y(D_1, D_2) = \{(x,h) \in P_r^{n+1} \times P_m^{n+1} : f(x) = 0 \bmod D_1 \text{ and } \nabla f(x) \cdot h = 0 \bmod D_2\}.$$

Then

$$\sum_{\alpha<D_1}\sum_{\beta<D_2} e_{\alpha,\beta} = q^{\deg D_1-(n+1)(r+1)+\deg D_2-(n+1)(m+1)}\#Y(D_1,D_2).$$

By the assumptions on the degrees, the maps $P_r \to \mathcal{O}_{D_1\cup D_2}(r)$ and $P_m \to \mathcal{O}_{D_2}(m)$ are surjective and so there is a surjective map

$$Y(D_1,D_2) \longrightarrow X(\mathcal{O}_{D_1}) \times_{\mathbb{A}^{n+1}(\mathcal{O}_{D_1\cap D_2})} Z(\mathcal{O}_{D_2})$$

which kernel has size $q^{(n+1)((r+1-\deg D_1)+(m+1-\deg D_2)-\deg D_1\cap D_2)}$. The result follows.

$\square$

This gives the main contribution to the size of the unstable locus.

Minor Arcs: High Indices

This is very similar to chapter 2. We need only apply theorem 1.1.6.

Proposition 3.2.7. *Let $\delta_{\alpha,\beta}$ be the dimension of the singular locus $S_{\alpha,\beta}$ consisting of pairs $(s,t)\in P_r^{n+1}\times P_m^{n+1}$ such that for all $(\sigma,\tau)\in P_r\times P_m$*

$$\alpha\left(\frac{\partial f}{\partial x_i}(s)\sigma\right)+\beta\left(\frac{\partial\nabla f}{\partial x_i}(s)\sigma\cdot t\right)=0 \quad \textit{and} \quad \beta\left(\frac{\partial f}{\partial x_i}(s)\tau\right)=0.$$

Then there exists a constant C depending only on n and d such that we have the estimate

$$\left|\sum_{\operatorname{ind}\alpha>r-m}\sum_{\beta} e_{\alpha,\beta}+\sum_{\alpha}\sum_{\operatorname{ind}\beta>m} e_{\alpha,\beta}\right| \leqslant C^r\sqrt{q}^{\tilde{\delta}-r(n-5(d-1))} \tag{3.3}$$

where $\tilde{\delta} = \max\left(\sup_{\alpha}\sup_{\operatorname{ind}\beta>m}\delta_{\alpha,\beta}, \sup_{\operatorname{ind}\alpha>r-m}\sup_{\beta}\delta_{\alpha,\beta}\right)$.

Note that the description of $S_{\alpha,\beta}$ is very concrete. In particular it is well adapted to computation as soon as we pick a particular hypersurface. For instance, in the case of the Fermat hypersurface, we have $(s,t) \in S_{\alpha,\beta}$ if and only if for all $(\sigma,\tau) \in P_r \times P_m$

$$\alpha(s_i^{d-1}\sigma) + \beta((d-1)s_i^{d-2}t_i\sigma) = 0 \quad \text{and} \quad \beta(s_i^{d-1}\tau) = 0.$$

Generating Functions

We now prove the analogue of lemma 2.4.2.

Notation 3.2.8. As before, we denote $R_e = k[t]/(t^{e+1})$. For $0 \leqslant s_1, s_2 \leqslant e+1$, we define

$$Y_{e,s_1,s_2} = \left\{(x,h) \in R_e^{n+1} \times R_e^{n+1} : f(x) \in (t^{s_1}) \text{ and } \nabla f(x)\cdot h \in (t^{s_2})\right\}$$

and its expected dimension is $\dim_e Y_{e,s_1,s_2} = 2(n+1)(e+1) - s_1 - s_2$.

Lemma 3.2.9. *Assume that* d *is prime to* q. *Then*

$$\begin{aligned}[Y_{e,s_1,s_2}] &= \sum_{i=0}^{\lceil \frac{s_1}{d}\rceil - 1} q^{-(n+1-d)i+s_2-\max\{0,s_2-(d-1)i\}}[X^*] \\ &\quad + \sum_{i=\lceil \frac{s_1}{d}\rceil}^{e} q^{-i(n+1)+s_1+s_2-\max\{0,s_2-(d-1)i\}}(1-q^{-(n+1)}) + q^{-(n+1)(e+1)+s_1+s_2}.\end{aligned}$$

Proof. We write

$$Y^{i,j}_{e,s_1,s_2} = \left\{(x,h) \in Y_{e,s_1,s_2} : x = t^i y \text{ for some } y \in R^{n+1}_{e-i} \text{ such that } y_j \notin (t)\right\}.$$

Then we have

$$Y_{e,s_1,s_2} = \coprod_{i=0}^{e+1} \coprod_{j=0}^{n} Y^{i,j}_{e,s_1,s_2}.$$

Furthermore, we remark that

$$(x,h) = (t^i y, h) \in Y^{i,j}_{e,s_1,s_2} \iff (\bar{y}, \bar{h}) \in Y^{0,j}_{e-i,\max\{0,s_1-di\},\max\{0,s_2-(d-1)i\}}$$

where the bar indicates the reduction map $R_e \to R_{e-i}$. Note that the kernel of this map is of size $q^{(n+1)i}$, so that

$$\#Y_{e,s_1,s_2} = \sum_{i=0}^{e} \sum_{j=0}^{n} q^{(n+1)i} \#Y^{0,j}_{e-i,\max\{0,s_1-di\},\max\{0,s_2-(d-1)i\}} + q^{(n+1)(e+1)}.$$

Now observe that since q is prime to d, the partials of f have no nontrivial common zero in $\mathbb{A}^{n+1}$. Let $y \in R^{n+1}_s$ such that $y_j \notin (t)$. Then provided that $s_1 - di > 0$ we have

$$\#\{h \in R^{n+1}_s : \nabla f(x) \cdot h \in (t^{s_2})\} = q^{(n+1)(s+1)-s_2}$$

from which we deduce

$$\begin{aligned}
&\sum_{j=0}^{n} \#Y^{0,j}_{e-i,\max\{0,s_1-di\},\max\{0,s_2-(d-1)i\}} \\
&= q^{(n+1)((e-i)-(s_1-di-1)+(e-i+1))-\max\{0,s_2-(d-1)i)\}}\#X^*(R_{s_1-di-1}) \\
&= q^{(n+1)((e-i)-(s_1-di-1)+(e-i+1))-\max\{0,s_2-(d-1)i)\}+n(s_1-di-1)}\#X^*(k) \\
&= q^{(n+1)((e-i)+(e-i+1))-s_1+di+1)-\max\{0,s_2-(d-1)i)\}}\#X^*(k).
\end{aligned}$$

Similarly, if $s_1 - di \leqslant 0$ and $i < e + 1$ then

$$\sum_{j=0}^{n} \#Y^{0,j}_{e-i,\max\{0,s_1-di\},\max\{0,s_2-(d-1)i\}} = q^{(n+1)((e-i)+(e-i+1))-\max\{0,s_2-(d-1)i\}}(q^{n+1}-1)$$

and so finally,

$$\begin{aligned}
\#Y_{e,s_1,s_2} &= \sum_{i=0}^{\lceil \frac{s_1}{d} \rceil - 1} q^{(n+1)(2e-i+1)-s_1+di+1-\max\{0,s_2-(d-1)i\}}\#X^*(k) \\
&+ \sum_{i=\lceil \frac{s_1}{d} \rceil}^{e} q^{(n+1)(2e-i+1)-\max\{0,s_2-(d-1)i\}}(q^{n+1}-1) + q^{(n+1)(e+1)}
\end{aligned}$$

and the lemma follows by dividing by $q^{2(n+1)(e+1)-s_1-s_2}$. $\square$

We now introduce some generating functions useful for the calculation.

Notation 3.2.10. Fixing effective divisors D_1 and D_2, we define the coefficients a_{T_1,T_2} by

$$\sum_{T_1 \leqslant D_1} \sum_{T_2 \leqslant D_2} a_{T_1,T_2} = [X \times Z]_{D_1,D_2}$$

and the corresponding generating function $A(z_1, z_2) = \sum_{D_1} \sum_{D_2} a_{D_1,D_2} z_1^{\deg D_1} z_2^{\deg D_2}$.

We will also use the coefficients b_{D_1,D_2} defined by $b_{D_1,D_2} = \sum_{T_1 \leqslant D_1} a_{T_1,D_2}$.

Lemma 3.2.11. *Let e_1 and e_2 be two nonnegative integers such that $e_2 \leqslant m$ and $e_1 + e_2 \leqslant r$. Let also D_1 be an effective divisor of degree at most r. Then*

$$\sum_{\deg T_2 \leqslant e_2} b_{D_1,T_2} = \sum_{\alpha < D_1} \sum_{\operatorname{ind} \beta \leqslant e_2} e_{\alpha,\beta}$$

and

$$\sum_{\deg T_1 \leqslant e_1} \sum_{\deg T_2 \leqslant e_2} a_{T_1,T_2} = \sum_{\operatorname{ind} \alpha \leqslant e_1} \sum_{\operatorname{ind} \beta \leqslant e_2} e_{\alpha,\beta}.$$

Proof. We only prove the first equality, the second one is similar. Observe first that if $\operatorname{ind} \beta \leqslant \frac{m}{2}$ then there exists a unique divisor D_2 of degree $\operatorname{ind} \beta$ that dominates β. Using lemma 3.2.6 we have

$$\begin{aligned} \sum_{\deg T_2 \leqslant e_2} b_{D_1,T_2} &= \sum_{\deg T_2 \leqslant e_2} [X \times Z]_{D_1,T_2} - \sum_{\deg T_2 \leqslant e_2} \sum_{S_2 < T_2} b_{D_1,S_2} \\ &= \sum_{\alpha < D_1} \sum_{\deg D_2 \leqslant e_2} \sum_{\beta < D_2} e_{\alpha,\beta} - \sum_{\deg T_2 \leqslant e_2} \sum_{S_2 < T_2} b_{D_1,S_2}. \end{aligned}$$

Now it is easy to see that $\sum_{\deg T_2 \leqslant e_2} \sum_{S_2 < T_2} b_{D_1,S_2}$ is a sum of exponential sums of the form $e_{\alpha,\beta}$ with $\alpha < D_1$ and $\operatorname{ind} \beta < e_2$ where each term occurs

$$\#\{(S_2, T_2) : \deg D_2 \leqslant e_2, S_2 < T_2 \text{ and } \beta < S_2\}$$

times. This is exactly one less than the multiplicity with which it occurs in

$$\sum_{\alpha < D_1} \sum_{\deg D_2 \leqslant e_2} \sum_{\beta < D_2} e_{\alpha,\beta}$$

thereby proving the formula. □

From here on, the conclusion is, yet again, very similar to chapter 2. By expanding the function as an Euler product and computing the local terms, we find that

$$\lim_{e_1,e_2\to\infty}[X\times Z]_{R_{e_1}\times R_{e_2}} = \frac{1}{1-q^{2d-n-2}}[X].$$

Since we don't really need an estimate for the number of unstable points of the mapping space, but rather we are merely looking for an upper bound on its dimension, we omit the details of the computation with the local terms. In light of (3.2), we have the following conclusion

Theorem 3.2.12. *Let k be a finite field and $\mathscr{X}\subset\mathbb{P}^n_k$ a smooth projective hypersurface of degree d. Let $\mathscr{M}_r$ be the space of rational curves of degree r on $\mathscr{X}$. Assume that $\delta<0$ so that estimate (2.3) holds. Assume analogously that for $\beta\in P^\vee_{r(d-1)+m}$ the dimension δ_β of*

$$S_\beta = \left\{ s\in P_r^{n+1} : \forall i=0,\dots,n, \forall\tau\in P_m,\ \beta\left(\frac{\partial f}{\partial x_i}(s)\tau\right)=0\right\}$$

satisfies $\delta_\beta < r(n-5d+7)$ when $r\gg 0$. Then there exists $\varepsilon>0$ such that the dimension of the unstable vectors satisfies

$$\dim\mathscr{M}_r^{sing}\leqslant q^{(1-\varepsilon)\dim\mathscr{M}_r}.$$

Proof. The major arcs are bounded by lemma 3.2.6. Working out the cutoff for $\tilde{\delta}$ using

(3.2) and proposition 3.2.7, we see that the conclusion holds whenever

$$\tilde{\delta} < r\,\frac{n(n-5d+6)+3(d-1)}{n-1} + C$$

where C depends only on n and d. In other words it suffices to show that

$$\frac{\tilde{\delta}}{r} < n - 5d + 7 - \varepsilon_{n,d} \quad \text{where} \quad \varepsilon_{n,d} = \frac{2d-4}{n-1}$$

when $r \gg 0$. When $\operatorname{ind}\alpha > r$ then we can use the fact that $\delta < 0$ and the fact that $S_{\alpha,\beta} \cap \{t = 0\} = S_\alpha$ to obtain the bound on $\delta_{\alpha,\beta}$ (we need to require that the difference $n-d$ be greater here than in theorem 2.4.1) and when $\operatorname{ind}\beta > m$ we can use the bound on δ_β in the theorem to get the desired bound for $\delta_{\alpha,\beta}$. $\square$

Observe that the condition on δ_β is very similar to the original condition on δ_α in theorem 2.4.1. In particular, for the Fermat equation, the trick used in lemma 2.5.1 works to bound δ_β as well. The same should go for every computable case.

Chapter 4

Some Consequences

4.1 Irreducibility

This short section explains why when (2.3) holds, we can deduce the irreducibility of the mapping space.

Proposition 4.1.1. *Let k be a finite field of size q and $\mathscr{X}$ a smooth hypersurface of degree d in $\mathbb{P}^n_k$. Assume that $\delta < 0$ so that (2.3) holds. Then the mapping space $\mathscr{M}_r(\mathbb{P}^1, \mathscr{X})$ is irreducible of dimension $r(n+1-d)+n-1$.*

Proof. By looking at the proof of theorem 2.4.1, we see that $\lim\limits_{D\to\infty}[\mathscr{X}]_{\mathcal{O}_D}$ is nonzero. This implies that $\mathscr{M}_r(\mathbb{P}^1, \mathscr{X})$ has the expected dimension. Furthermore, by a result of Kollar [15, Theorem II.1.2/3] all irreducible components are of dimension $r(n+1-d)+n-1$ or bigger. So all irreducible components are of the expected dimension. In addition, for all $r \gg 0$, we have

$$[\mathscr{M}_r(\mathbb{P}^1, \mathscr{X})] \sim \lim_{D\to\infty}[\mathscr{X}]_{\mathcal{O}_D} \quad \text{as} \quad q \to \infty$$

in particular, $\lim_{q\to\infty}[\mathscr{M}_r(\mathbb{P}^1,\mathscr{X})] = 1$ so $\mathscr{M}_r(\mathbb{P}^1,\mathscr{X})$ is irreducible according to the Lang-Weil estimate [16]. □

In particular this holds for the examples seen in chapter 3. For cubic hypersurfaces this was proven by Deland over the complex numbers [3] in the better range $n \geqslant 9$. In general, the result was only known for general smooth hypersurfaces.

4.2 Rational Homotopy

It is natural to ask if the point counts we obtain are the result of a general mechanism or if they are merely circumstancial. We fix a smooth hypersurface $\mathscr{X}$ of low degree d in $\mathbb{P}^n$. One can think of all the data as being defined over a finitely generated ring R over which $\mathscr{X}$ is spread out in the usual way. In their simplest version, the estimates we have are of the form

$$\#\mathscr{M}_r(\mathbb{P}^1,\mathscr{X})(\mathbb{F}_q) \sim cq^{\dim_e \mathscr{M}_r} \text{ as } r \to \infty.$$

There is a conjecture of Batyrev and Manin (see for instance [11, appendix F]) that predicts the growth of the number of points of specified height on some special varieties. More precisely, the conjecture predicts that

$$\{x \in \mathscr{X}(\mathbb{k}) : \mathrm{ht}(x) \leqslant N\} \sim cN^a(\log N)^b \quad \text{as} \quad N \to \infty$$

where $\mathbb{k}$ is a global field, b, c are constants that depend only on $\mathscr{X}$ and $\mathbb{k}$ and a is a constant that depends only on $\mathscr{X}$. For a smooth projective hypersurface of low degree, these estimates where shown to hold by Birch. While the Batyrev-Manin estimates deal with global fields, we can think of the above estimates as an analogue for function fields. In particular, there should be a geometric reason for those estimates to hold true.

Another case where the Batyrev-Manin estimates are known to hold both for global fields and function fields is the case where $\mathscr{X} = \mathbb{P}^n$. The global field case was proven by Schenoel and the function field case was proven by Segal (see [18]). In fact, Segal's approach yields interesting corollaries. He considers the natural forgetful map

$$\mathscr{M}_r(\mathbb{P}^1, \mathscr{X}) \longrightarrow \text{Maps}(S^2, \mathscr{X}(\mathbb{C})) = \Omega^2\mathbb{C}$$

that sends an algebraic morphism to its topological (smooth) self. The space on the right hand side is the topological space parametrizing smooth morphisms from the 2-sphere into (the complex points of) $\mathscr{X}$. He then showed that this map is a homology equivalence in degrees up to $\gamma(r)$, where $\gamma(r)$ grows (at least linearly) with r. Loosely, this means that the space of holomorphic functions approximates that of smooth functions. This has the consequence that the cohomology $H^i(\mathscr{M}_r(\mathbb{P}^1, \mathscr{X}))$ stabilizes in the range $0 < i < \gamma(r)$.

Taking our cue from Segal's theorem, we explain how the Circle Method estimates would follow from a stabilization property. In fact, the space $\mathscr{M}_r(\mathbb{P}^1, \mathscr{X})$ is quite big and difficult to work with. A popular alternative is to consider the space of pointed morphisms with specified curve class. More precisely, assume for simplicity that $H_2(\mathscr{X}, \mathbb{Z}) = \mathbb{Z}$, let $\beta \in H_2(\mathscr{X}, \mathbb{Z})$ and x be a closed point of $\mathscr{X}$. Consider the space

$$\mathscr{M}_\beta^*(\mathscr{X}) = \{f \in \mathscr{M}(\mathbb{P}^1, \mathscr{X}) : f(0) = x \text{ and } f_*[\mathbb{P}^1] = \beta\}.$$

It is a quasi-projective scheme. Note that if $x \in \mathscr{X}(\mathbb{C})$ is very general then $\mathscr{M}_\beta^*$ is smooth of the expected dimension (but not necessarily irreducible). For an arbitrary point x the scheme $\mathscr{M}_\beta^*$ can be quite singular and even nonreduced. We also introduce the topological version

$$\text{Maps}_\beta^*(S^2, \mathscr{X}) = \{f \in \text{Maps}(S^2, \mathscr{X}(\mathbb{C})) : f(0) = x \text{ and } f_*[S^2] = \beta\}.$$

Assume the following:

(HL) There exists an integer $\gamma(\beta)$ such that $\mathscr{M}_\beta^*$ is irreducible of expected dimension m_β, the codimension of the singular locus of $\mathscr{M}_\beta^*$ is at least $\gamma(\beta)$, the map $(\mathscr{M}_\beta^*)^{\text{nonsing}} \to \mathscr{M}_\beta^*$ induces an isomorphism on cohomology in degree up to $\gamma(\beta)$, and the canonical map $\mathscr{M}_\beta^* \to \text{Maps}_\beta^*(S^2, \mathscr{X})$ induces an isomorphism on rational homotopy types up to dimension $\gamma(\beta)$, where "$\gamma(\beta)$ grows with β".

This condition guarantees that Poincaré duality holds in the sense that

$$H_c^{2m_\beta - i}(\mathscr{M}_\beta^*) = \text{Hom}(H^i(\mathscr{M}_\beta^*), \mathbb{Q}(-m_\beta)) \tag{4.1}$$

in the range $0 \leqslant i \leqslant \gamma(\beta)$ for the (possibly singular) space $\mathscr{M}_\beta^*$. Then by the Lefschetz trace formula, we have

$$\begin{aligned}
\#\mathscr{M}_\beta^*(\mathbb{F}_q) &= \sum_{i=0}^{\infty}(-1)^i \mathrm{Tr}\left(F|_{H_c^{2m_\beta-i}(\mathscr{M}_\beta^*)}\right) \\
&= \sum_{i=0}^{\infty} q^{m_\beta}(-1)^i \mathrm{Tr}\left(F|_{\mathrm{Hom}(H^i(\mathscr{M}_\beta^*),\mathbb{Q})}\right) \\
&= q^{m_\beta}\sum_{i=0}^{\infty}(-1)^i \mathrm{Tr}\left(F^{-1}|_{H^i(\mathscr{M}_\beta^*)}\right).
\end{aligned}$$

Let $(\Lambda(V), d)$ with $V = \{V^p\}_{p\geqslant 1}$ be a minimal Sullivan model for $\mathscr{X}$. In particular, $\pi_i(\mathscr{X}) \otimes \mathbb{Q} = \mathrm{Hom}(V^i, \mathbb{Q})$. Since $\mathscr{X}$ is smooth and projective, it is formal and we can compute V in terms of $H^*(\mathscr{X})$. We have

$$H^*(\mathrm{Maps}_\beta^*(S^2, \mathscr{X})) = \Lambda(W)$$

where $W^i = V^{i+2}(1)$ for $i \geqslant 1$. In particular,

$$\begin{aligned}
q^{m_\beta}\sum_{i=0}^{\infty} \mathrm{Tr}\left(F^{-1}|_{\Lambda(W)^i}\right) &= q^{m_\beta}\prod_{i \text{ even}} \mathrm{Tr}(F^{-1}|_{\mathrm{Sym}^*(W^i)})\prod_{i \text{ odd}} \mathrm{Tr}(F^{-1}|_{\Lambda^*(W^i)}) \\
&= q^{m_\beta}\prod_{i \text{ even}} \det(1-F^{-1}|_{W^i})^{-1}\prod_{i \text{ odd}} \det(1-F^{-1}|_{W^i})
\end{aligned}$$

and since $W^i = \mathrm{Hom}(\pi_{i+2}(\mathscr{X}), \mathbb{Q})(1)$, this is equal to

$$q^{m_\beta}\prod_i \det(1-qF|_{\pi_{i+2}(\mathscr{X})})^{(-1)^{i+1}}.$$

In conclusion, since the minimal Sullivan model is quasi-isomorphic to the cohomology, we have the following:

Assume that $\mathscr{X}$ satisfies condition (HL) above. Consider the quantity

$$\varepsilon(r) = \varepsilon(\mathbb{P}^1, \mathscr{X}, q\,; r) = \left| [\mathscr{M}_r] - \frac{1}{1-q^{-1}} \prod_x (1 - q_x^{-1})[\mathscr{X}]_{\kappa(x)} \right|.$$

Then there exists a constant C depending on n and d and a positive constant γ depending on the function $\gamma(\beta)$ such that

$$\varepsilon(r) = O(C^r q^{-\gamma r}) \quad \textit{as} \quad r \to \infty.$$

This gives a geometric statement that would yield the point count estimates that we proved earlier. In fact, the condition (HL) is clearly too strong, and we merely need some isomorphism that establishes (4.1) to guarantee that both predictions agree.

Bibliography

[1] Bryan Birch, *Homogeneous Forms of Odd Degree in a Large Number of Variables*, Mathematatika (1957), no. 4, 102–105.

[2] Olivier Debarre, *Higher-dimensional Algebraic Geometry*, Universitext, Springer-Verlag, 2001.

[3] Matthew Deland, *Geometry of Rational Curves on Algebraic Varieties*, Ph.D. thesis, Columbia University, 2009.

[4] Pierre Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.

[5] Godfrey Hardy and John Littlewood, *Some problems of "Partitio Numerorum" I: A new solution of Waring's problem*, Gottingen Nachr. (1920), 33–54.

[6] Godfrey Hardy and Srinivasa Ramanujan, *Asymptotic formulae in combinatorial analysis*, Proc. London Math. Soc. (1918), no. 17, 75–115.

[7] Joe Harris, Mike Roth, and Jason Starr, *Rational Curves on Hypersurfaces of Low Degree*, arXiv:math/0203088.

[8] Joe Harris and Jason Starr, *Rational Curves on Hypersurfaces of Low Degree II*, arXiv:math/0203088.

[9] Michael Hartl, *The Tau Manifesto*, `http://tauday.com/`.

[10] David Hilbert, *Beweis fur die Darstellbarkeit der ganzen zahlen durch eine feste Anzahlnter Potenzen (Waringsches problem)*, Mat. Annalen (1909), no. 67, 281–300.

[11] Marc Hindry and Joseph Silverman, *Diophantine Geometry: an Introduction*, GTM, no. 201, Springer-Verlag, 2000.

[12] Nicholas Katz, *Estimates for "Singular" Exponential Sums*, International Mathematics Research Notices (1999), no. 16, 875–899.

[13] ______, *Sommes exponentielles*, no. 79, Astérisque, 2001.

[14] Reinhardt Kiehl and Reiner Weissauer, *Weil Conjectures, Perverse Sheaves and ℓ-adic Fourier Transform*, no. 42, Springer-Verlag, 1980.

[15] János Kollár, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer-Verlag, 1996.

[16] Serge Lang and André Weil, *Number of Points of Varieties in Finite Fields*, Amer. J. Math. (1954), no. 76, 819–827.

[17] Gérard Laumon, *Majoration de sommes trigonométriques (d'après P. Deligne et N. Katz)*, Caractéristique d'Euler-Poincaré (Séminaire de l'ENS 1978/9) (Soc. Math. France, ed.), no. 82-83, Astérisque, 1981, pp. 221–258.

[18] Graeme Segal, *The topology of spaces of rational functions*, Acta Math. **1-2** (1979), no. 143, 972, 3.

[19] Jason Starr, *Fano Varieties and Linear Sections of Hypersurfaces*, arXiv:math/0607133v1.

[20] Jason Starr and Izzet Coskun, *Rational Curves on Smooth Cubic Hypersurfaces*, Int. Math. Res. Not. (2009), no. 24, 46264641, IMRN.

[21] Robert Vaughan, *The Hardy-Littlewood Method*, Cambridge University Press, 1981.

[22] Robert Vaughan and Trevor Wooley, *Waring's Problem: A Survey*, Number Theory for the Millenium (Bennett *et al.*, ed.), vol. III, A. K. Peters, 2002, pp. 301–340.

[23] Trevor Wooley, *Forms in Many Variable*, Analytic Number Theory: Proceedings of the 39th Taniguchi International Symposium, Kyoto, May 1996 (Y. Motohashi, ed.), London Math. Soc. Lecture Note Ser. 247, Cambridge University Press, 1997, pp. 361–376.

[24] ______, *An Explicit Version of Birch's Theorem*, Acta Arithmetica (1998), no. LXXXV.1, 79–95.