Approximate Converse Theorem

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ABSTRACT

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The theme of this thesis is an "approximate converse theorem" for globally unramified cuspidal representations of $PGL(n, \mathbb{A})$, $n \geqslant 2$, which is inspired by [19] and [3]. For a given set of Langlands parameters for some places of $\mathbb{Q}$, we can compute $\epsilon > 0$ such that there exists a genuine globally unramified cuspidal representation, whose Langlands parameters are within ϵ of the given ones for finitely many places.

CONTENTS

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Chapter 1

INTRODUCTION

1.1 Cuspidal representations and Maass forms

Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$. Let $n \geqslant 2$ be an integer and π be a cuspidal automorphic representation for $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$. By the tensor product theorem ([11], [17], [8]), there exists an irreducible admissible generic unitary local representation π_v of $\mathbb{Q}_v^\times \backslash GL(n, \mathbb{Q}_v)$ for each place $v \leqslant \infty$ of $\mathbb{Q}$, such that $\pi \cong \otimes_v' \pi_v$. Here the local representation π_v is spherical except at finitely many places. Define

$$\mathfrak{a}_{\mathbb{C}}^*(n) := \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \alpha_j = 0 \right\}.$$

Fix a place $v \leqslant \infty$. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathfrak{a}_{\mathbb{C}}^*(n)$, there exists an unramified character χ_α (for the minimal parabolic subgroup of $GL(n, \mathbb{Q}_v)$) defined by

$$\chi_\alpha\left(\begin{pmatrix} x_1 & & * \\ & \ddots & \\ & & x_n \end{pmatrix}\right) := \prod_{j=1}^{n} |x_j|_v^{\frac{1}{2}(n-2j+1)+\alpha_j}, \qquad \left(\text{for } \begin{pmatrix} x_1 & & * \\ & \ddots & \\ & & x_n \end{pmatrix} \in GL(n, \mathbb{Q}_v)\right).$$

For each place $v \leqslant \infty$, if π_v is spherical, then there exists $\sigma_v \in \mathfrak{a}_{\mathbb{C}}^*(n)$ such that $\pi_v \cong \pi_v(\sigma_v)$, where $\pi_v(\sigma_v)$ is the irreducible spherical principal series representation (or the irreducible spherical subquotient of the reducible principal series representation), associated to the character χ_{σ_v}. We call σ_v the Langlands parameter associated to π_v. For a cuspidal automorphic representation $\pi \cong \otimes_v \pi_v$ define

$$\sigma := \left\{ \sigma_v \in \mathfrak{a}_{\mathbb{C}}^*(n) \;\middle|\; \begin{array}{c} \pi_v \text{ is spherical and} \\ \pi_v \cong \pi_v(\sigma_v) \end{array} \right\}.$$

Then σ is called the automorphic parameter for π. By the multiplicity one theorem (first proved by Casselman [5] for $GL(2)$ in 1975, the strong version for $GL(2)$ proved in [17] by Jacquet and Langlands in 1970, and generalized separately by Shalika [25], Piatetski-Shapiro [23] and by Gelfand and Kazhdan [10]), it follows that the automorphic parameter σ is uniquely determined by π.

At the Conference on *Analytic number theory in higher rank groups*, P. Sarnak suggested the following problem:

Given a positive number X, a set S of places and a representation π_v of $GL(n, \mathbb{Q}_v)$ (for $v \in S$), give an algorithm to determine whether or not there is a global automorphic representation π with $c(\pi) < X$ and σ_v within ϵ of π_v for $v \in S$ (in whatever reasonable sense). Here $c(\pi)$ is the analytic conductor of π.

In this thesis, this problem is solved for the globally unramified case.

Let $\pi \cong \otimes_v \pi_v$ be a globally unramified cuspidal representation of $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$. Then π_v is spherical for every $v \leqslant \infty$, and $\sigma = \{\sigma_v \in \mathfrak{a}^*_{\mathbb{C}}(n) \mid \pi_v \cong \pi_v(\sigma_v), \ \forall v \leqslant \infty\}$ is the automorphic parameter of π such that $\pi \cong \otimes'_v \pi_v(\sigma_v) =: \pi(\sigma)$. There exists a unique (up to constant) Hecke-Maass form f_σ (associated to $\pi(\sigma)$) on $SL(n, \mathbb{Z})$ for the generalized upper half plane $\mathbb{H}^n \cong \mathbb{R}^\times \backslash GL(n, \mathbb{R})/O(n, \mathbb{R})$. The Whittaker-Fourier coefficients of f_σ are determined by the automorphic parameter σ. Moreover, there is a one-to-one correspondence between unramified cuspidal representation of $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$ and Hecke-Maass forms on $SL(n, \mathbb{Z})$.

The existence of Maass forms on $SL(2, \mathbb{Z})$ was first proved by Selberg [24] in 1956. He used the trace formula as a tool to obtain Weyl's law, which gives an asymptotic count for the number of Maass forms with Laplacian eigenvalue $|\lambda| \leqslant X$ as $X \to \infty$. In 2001, Selberg's method was extended by Miller [21] to obtain Weyl's law for Maass forms on $SL(3,\mathbb{Z})$. In 2004, Müller [22] further extended Selberg's method to obtain Weyl's law for Maass forms on $SL(n,\mathbb{Z})$, $n \geqslant 2$.

More recently, in 2007, Lindenstrauss and Venkatesh obtained Weyl's law for Maass forms on $G(\mathbb{Z})\backslash G(\mathbb{R})/K_\infty$ [19] where G is a split semisimple group over $\mathbb{Q}$ and $K_\infty \subset G$ is the maximal compact subgroup. In the Appendix [19], they explain a constructive proof of the *existence of Maass forms*. Our work is inspired by this proof; for a given set of Langlands parameters, we give an explicit bound, which ensures that there exists a genuine unramified cuspidal representation within the boundary of the finite subset of the given parameters.

1.2 Approximate converse theorem

The converse theorem of Cogdell and Piatetski-Shapiro ([6], [7]) proves that an L-function is the Mellin transform of an automorphic form on $GL(n)$ if it satisfies a certain infinite class of twisted functional equations. In this thesis we introduce the approximate converse theorem whose main aim is to prove that an L-function is the Mellin transform of a function which is very close to an actual automorphic form provided a finite set of conditions are satisfied. We now explicitly describe these conditions and quantify the notion of closeness in this context.

Let M be a set of places of $\mathbb{Q}$ including ∞. Let $n \geqslant 2$ be an integer. Define

$$\ell_M := \left\{ \ell_v \in \mathfrak{a}^*_{\mathbb{C}}(n) \;\middle|\; \begin{array}{c} \pi_v(\ell_v) \text{ is an irreducible unitary spherical} \\ \text{generic representation for } \mathbb{Q}_v^\times \backslash GL(n, \mathbb{Q}_v), \ (v \in M) \end{array} \right\}.$$

Then ℓ_M is called the quasi-automorphic parameter for M. For example, the automorphic parameter for a cuspidal automorphic representation is a quasi-automorphic parameter.

We use the usual quasi-mode construction for a given quasi-automorphic parameter ℓ_M. The Whittaker-Fourier coefficient can be constructed from the parameters $\ell_v \in \ell_M$, for each $v \in M$. By summing these constructed coefficients, we define a function $F_{\ell_M}(z)$ on the upper half plane $\mathbb{H}^n \cong \mathbb{R}^\times \backslash GL(n, \mathbb{R})/O(n, \mathbb{R})$ which is essentially a finite Whittaker-Fourier expansion. The function F_{ℓ_M} is called a quasi-Maass form associated to ℓ_M. In general the quasi-Maass form is not automorphic; but it is an eigenform of the Casimir operators $\Delta_n^{(j)}$ (for $j = 1, 2, \ldots, n-1$), such that

$$\Delta_n^{(j)} F_{\ell_M}(z) = \lambda_\infty^{(j)}(\ell_\infty) \cdot F_{\ell_M}(z)$$

with eigenvalues $\lambda_\infty^{(j)}(\ell_\infty) \in \mathbb{C}$. Also, for each positive integer $N \geqslant 1$, the quasi-Maass form is an eigenfunction of the Hecke operator T_N, such that

$$T_N F_{\ell_M}(z) = A_{\ell_M}(N) \cdot F_{\ell_M}(z)$$

with eigenvalues $A_{\ell_M}(N) \in \mathbb{C}$. For each $j = 1, \ldots, n-1$, and a prime $q \in M$, define the Hecke operators

$$T_q^{(j)} = \sum_{k=0}^{j-1} (-1)^k T_{q^{k+1}} T_q^{(j-k-1)}, \qquad \left(T_{q^r}^{(1)} = T_{q^r}, \text{ for any integer } r \geqslant 0 \right).$$

Then

$$T_q^{(j)} F_{\ell_M}(z) = \lambda_q^{(j)}(\ell_q) \cdot F_{\ell_M}(z)$$

for $\lambda_q^{(j)}(\ell_q) \in \mathbb{C}$.

Let M and M' be sets of places of $\mathbb{Q}$ including ∞ and let ℓ_M and $\sigma_{M'}$ be quasi-automorphic parameters for M and M', respectively. Let $S \subset M \cap M'$ be a finite subset including ∞. Let $\epsilon > 0$. The quasi-automorphic parameters ℓ_M and $\sigma_{M'}$ are ϵ-close for S if

$$\sum_{j=1}^{n-1} \left|\lambda_\infty^{(j)}(\ell_\infty) - \lambda_\infty^{(j)}(\sigma_\infty)\right|^2 + \sum_{\substack{q \in S, \\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left|\lambda_q^{(j)}(\ell_q) - \lambda_q^{(j)}(\sigma_q)\right|^2 \;<\; \epsilon.$$

Fix the fundamental domain $\mathfrak{F}^n \cong SL(n,\mathbb{Z})\backslash\mathbb{H}^n$ (described in Proposition 2.8; and based on [14]). Define an automorphic lifting

$$\widetilde{F}_{\ell_M}(z) = F_{\ell_M}(\gamma z),$$

for $z \in \mathbb{H}^n$ and $\gamma \in SL(n,\mathbb{Z})$, which is uniquely determined by $\gamma z \in \mathfrak{F}^n$. Then $\widetilde{F}_{\ell_M}$ is automorphic for $SL(n,\mathbb{Z})$, and square-integrable. But it is neither smooth nor cuspidal in general. We can get the distance between the given quasi-automorphic parameter ℓ_M and a genuine automorphic parameter, by determining the distance between the quasi-Maass form F_{ℓ_M} and its automorphic lifting.

For $\delta > 0$ and a finite set S of places of $\mathbb{Q}$ including ∞, let $B^n(\delta; S)$ be a region bounded by δ and finite primes in S, around the neighborhood of the boundary of the fundamental domain $\mathfrak{F}^n$, as described in (5.1). Let H_δ be a smooth compactly supported, bi-$(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$-invariant function on $\mathbb{R}^\times\backslash GL(n,\mathbb{R})$, which is given in §4.3.

Theorem 1.1. (Approximate converse theorem) *Let $n \geqslant 2$ be an integer, M be a set of places of $\mathbb{Q}$ including ∞, and let $\ell_M = \{\ell_v \in \mathfrak{a}_{\mathbb{C}}^*(n),\ v \in M\}$ be a quasi-automorphic parameter. Let F_{ℓ_M} be a quasi-Maass form associated to ℓ_M. Define*

$$\widehat{\mathfrak{h}}_p^n(\ell_\infty, \ell_p) := \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \prod_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \prod_{1 \leqslant i_1 < \cdots i_k \leqslant n} \left(1 - p^{-(\ell_{\infty,i_1} + \cdots + \ell_{\infty,i_k}) - (\ell_{p,j_1} + \cdots + \ell_{p,j_k})}\right)$$

and assume that $\widehat{\natural}_p^n(\ell_\infty, \ell_p) \neq 0$ *for some prime* $p \in M$. *Assume that* $\widehat{H}_\delta(\ell_\infty) \neq 0$ *where* $\widehat{H}_\delta$ *is the spherical transform of* H_δ. *Let* S *be a finite subset of* M *including* ∞.

Then there exists a genuine unramified cuspidal automorphic representation $\pi(\sigma)$ *for* $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$ *with an automorphic parameter* $\sigma = \{\sigma_v \in \mathfrak{a}_{\mathbb{C}}^*(n),\ v \leqslant \infty\}$ *such that* ℓ_M *and* σ *are* ϵ*-close for* S *where*

$$\epsilon \ := \ \frac{\sup\limits_{B^n(\delta;S)} \left| \widetilde{F}_{\ell_M} - F_{\ell_M} \right|^2 \cdot C_p(n, \delta; S)}{\left| \widehat{H}_\delta(\ell_\infty) \right|^2 \cdot \left| \widehat{\natural}_p^n(\ell_\infty, \ell_p) \right|^2 \cdot \int_T^\infty \cdots \int_T^\infty |W_J(y; \ell_\infty)|^2 \ d^*y}$$

for some

$$0 < \delta \leqslant \frac{1}{2} \ln \left(\left[\max_{j=1,2,\ldots,n-1} \left\{ \left| \lambda_\infty^{(j)}(\ell_\infty) \right| \right\} \cdot \max_{0 \leqslant t \leqslant 1} \left\{ \int\limits_{\substack{\mathbb{H}^n, \\ u(z)=t}} 1 \ d^*z \right\} \right]^{-1} + 1 \right)$$

where $C_p(n, \delta; S) > 0$ *is a constant and* $W_J(*; \ell_\infty)$ *is the Whittaker function on* $\mathbb{H}^n$. *Here* $T > 1$ *is a positive constant determined by* δ, *the prime* p *and* n.

In this theorem, we see that the closeness for the given quasi-automorphic parameter and a geuniue automorphic parameter mainly depends on the difference between the quasi-Maass form of the given quasi-automorphic parameter and its automorphic lifting on the neighborhood of the boundary of the fundamental domain $\mathfrak{F}^n$. This theorem does not give uniqueness. However, by Remark 8 in [4], if the difference between $\widetilde{F}_{\ell_M}$ and F_{ℓ_M} is small enough when S is sufficiently large, then the cuspidal representation should be uniquely determined. The neighborhood for the boundary of the fundamental domain becomes much larger as the primes in S becomes bigger. The formula for $C_p(n, \delta; S)$ is given in (5.5). A more general result is described in Theorem 5.1, where we take arbitrary $\delta > 0$ and an arbitrary compactly supported bi-$(\mathbb{R}^\times \cdot O(n, \mathbb{R}))$-invariant smooth function H_δ, such that

$\widehat{H_\delta}(\ell_\infty) \neq 0$ for the given Langlnads parameter ℓ_∞ at ∞. It is an interesting problem to choose H_δ so that the ϵ in Theorem 1.1 (or in Theorem 5.1) is as small as possible.

The constant $\widehat{\natural}_p^n(\ell_\infty, \ell_p)$ turns out to be an eigenvalue of the annihilating operator $\natural_p^n$, which maps $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ to cuspidal functions, i.e., $\natural_p^n\left(\widetilde{F}_{\ell_M} * H_\delta\right)$ is a smooth cuspidal automorphic function. The annihilating operator $\natural_p^n$ plays an important role in the proof of the approximate converse theorem. It is constructed by following Lindenstrauss and Venkatesh [19]. They observe that there are strong relations between the spectrum of the Eisenstein series at different places. From this observation, they construct the convolution operator $\aleph$, whose image is purely cuspidal. They use this operator $\aleph$ to get Weyl's law for cusp forms in [19]. For example, for automorphic functions on $SL(2,\mathbb{Z})$, for any prime p,

$$\aleph = T_p - p^{\sqrt{\frac{1}{4}-\Delta}} - p^{-\sqrt{\frac{1}{4}-\Delta}}$$

and it also has a rigorous interpretation in terms of convolution operators. More detailed explanation and explicit description of $\natural_p^n$ are given in chapter 4.

In the 1970's a number of authors considered the problem of computing Maass forms on $PSL(2,\mathbb{Z})$ numerically. The first notable algorithms for computing Maass forms on $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$ are due to Stark in [26] and Hejhal in [15]. In [27], Hejhal's algorithm was used by Then to compute large Laplace eigenvalues on $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$.

In [3], Booker, Strömbergsson and Venkatesh compute the Laplace and Hecke eigenvalues for Maass forms, to over 1000 decimal places, for the first few Maass forms on $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$. Their paper is another inspiration and source for the approximate converse theorem. In particular, we followed the method for verification of their computation in

Proposition 2, [3]. The ϵ in the approximate converse theorem may recover (38) in [3] weakly, with good choices for δ and H_δ, for the case $n = 2$ and $S = \infty$. In [3] they choose $\delta \leqslant \frac{1}{4\sqrt{\lambda}}$ for a given Laplacian eigenvalue λ, and H_δ as

$$H_\delta(z) = \begin{cases} 3\left(1 - 2\delta^{-2}\left(\frac{1}{2}\left(y + \frac{x^2}{y} + \frac{1}{y}\right) - 1\right)\right)^2, & \text{if } \frac{1}{2}\left(y + \frac{x^2}{y} + \frac{1}{y}\right) - 1 \leqslant \frac{\delta^2}{2}\ ; \\ 0, & \text{otherwise,} \end{cases}$$

where $z = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{H}^2$. See [3] for more details.

Recently, Booker and his student Bian computed the Laplace and Hecke eigenvalues for Maass forms on $PSL(3, \mathbb{Z})\backslash\mathbb{H}^3$ [2], [1]. Moreover, Mezhericher presented an algorithm for evaluating a (quasi-)Maass form for $SL(3, \mathbb{Z})$ in his thesis [20]. We expect that we might use the approximate converse theorem to certify Bian's computations.

1.3 Format of Thesis

The main theorem is stated and proved in chapter 5. In chapter 2, we review the theory of automorphic forms for $SL(n, \mathbb{Z})\backslash\mathbb{H}^n$ and introduce notations. The main reference for this chapter is [12]. In chapter 3, we review the theory of automorphic cuspidal representation for $\mathbb{A}^\times\backslash GL(n, \mathbb{A})$. The main reference for this chapter is [13]. In §3.5, we define the quasi-automorphic parameter and the quasi-Maass form of the given quasi-automorphic parameter. The annihilating operator $\natural_p^n$ is defined in chapter 4. Several properties of the annihilating operator are proved in §4.2.

Chapter 2

AUTOMORPHIC FUNCTIONS FOR $SL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/\left(O(n,\mathbb{R})\cdot\mathbb{R}^\times\right)$

2.1 Parabolic subgroups

Let $n \geqslant 1$ be an integer. For an integer $1 \leqslant r \leqslant n$, define an ordered partition of n to be a set of integers $(n_1,\ldots,n_r)$ where $1 \leqslant n_1,\ldots,n_r \leqslant n$ and $n_1+\cdots+n_r = n$.

Definition 2.1. (Parabolic subgroups) *Fix an integer $n \geqslant 1$ and let R be a commutative ring with identity 1. A subgroup P of $GL(n,R)$ is said to be parabolic if there exists an ordered partition $(n_1,\ldots,n_r)$ of n and an element $g \in GL(n,R)$ such that $P = gP_{n_1,\ldots,n_r}(R)g^{-1}$ where $P_{n_1,\ldots,n_r}(R)$ is the standard parabolic of $GL(n,R)$ associated to the partition $(n_1,\ldots,n_r)$ defined by*

$$P_{n_1,\ldots,n_r}(R) := \left\{ \begin{pmatrix} A_1 & & * \\ & \ddots & \\ & & A_r \end{pmatrix} \in GL(n,R) \;\middle|\; A_i \in GL(n_i,R),\ 1 \leqslant i \leqslant r \right\}. \tag{2.1}$$

The integer r is termed the rank of the parabolic subgroup $P_{n_1,\ldots,n_r}(R)$. Define

$$M_{n_1,\ldots,n_r}(R) := \left\{ \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix} \;\middle|\; A_i \in GL(n_i,R),\ 1 \leqslant i \leqslant r \right\} \tag{2.2}$$

to be the standard Levi subgroup of $P_{n_1,\ldots,n_r}(R)$ and call $gM_{n_1,\ldots,n_r}(R)g^{-1}$ a Levi factor of P. Define

$$U_{n_1,\ldots,n_r}(R) := \left\{ \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_r} \end{pmatrix} \in GL(n,R) \right\}, \tag{2.3}$$

where I_k is the $k \times k$ identity matrix for an integer $k \geqslant 1$, to be the unipotent radical of $P_{n_1,\ldots,n_r}(R)$. Call $gU_{n_1,\ldots,n_r}(R)g^{-1}$ the unipotent radical of P.

Two standard parabolic subgroups $P_{n_1,\ldots,n_r}(R)$ and $P_{n'_1,\ldots,n'_r}(R)$ of $GL(n,R)$ corresponding to the partitions $n = n_1 + \cdots + n_r = n'_1 + \cdots + n'_r$ are said to be associated if $\{n_1,\ldots,n_r\} = \{n'_1,\ldots,n'_r\}$. We write $P_{n_1,\ldots,n_r}(R) \sim P_{n'_1,\ldots,n'_r}(R)$, if $P_{n_1,\ldots,n_r}$ and $P_{n'_1,\ldots,n'_r}$ are associated.

Let $n \geqslant 1$ be an integer and fix an ordered partition $(n_1,\ldots,n_r)$ of n. For each $j = 1,\ldots,r$ define a map

$$\mathfrak{m}_{n_j} : P_{n_1,\ldots,n_r}(R) \to GL(n_j,R), \quad \text{such that} \tag{2.4}$$
$$g = \begin{pmatrix} \mathfrak{m}_{n_1}(g) & * & \cdots & * \\ & \mathfrak{m}_{n_2}(g) & \cdots & * \\ & & \ddots & \vdots \\ & & & \mathfrak{m}_{n_r}(g) \end{pmatrix} \in P_{n_1,\ldots,n_r}(R) \qquad (\mathfrak{m}_{n_j}(g) \in GL(n_j,R)).$$

The standard parabolic subgroup associated to the partition $n = n_1 + \cdots + n_r$ (denoted by $P_{n_1,\ldots,n_r}(R)$) is defined to be the group of all matrices of the form

$$g = \begin{pmatrix} \mathfrak{m}_{n_1}(g) & * & \cdots & * \\ & \mathfrak{m}_{n_2}(g) & \cdots & * \\ & & \ddots & \vdots \\ & & & \mathfrak{m}_{n_r}(g) \end{pmatrix} \in GL(n,R) \tag{2.5}$$

where $\mathfrak{m}_{n_i}(g) \in GL(n_i,R)$ for $i = 1,\ldots,r$.

Let $n \geqslant 1$ be an integer. For $r = n$, let

$$N(n,R) := U_{1,1,\dots,1}(R) \tag{2.6}$$
$$= \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{1-n,n} \\ & & & & 1 \end{pmatrix} \middle| \; x_{i,j} \in R \text{ for } 1 \leqslant i < j \leqslant n \right\} \subset GL(n,R),$$
$$A(n,R) := M_{1,1,\dots,1}(R) = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \middle| \; 0 \neq a_j \in R \text{ for } j = 1, \dots, n \right\} \subset GL(n,R),$$

and

$$P(n,R) = P_{1,1,\dots,1}(R) = N(n,R) \cdot A(n,R).$$

Here $P(n,R)$ is called the minimal parabolic subgroup of $GL(n,R)$.

2.2 Coordinates for $GL(n,\mathbb{R})$

Definition 2.2. (Generalized Upper half plane) *Let $n \geqslant 2$ be an integer. Define the generalized upper half plane $\mathbb{H}^n$ to be a set of matrices $z \in GL(n,\mathbb{R})$ and $z = xy$ such that*

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \in N(n,\mathbb{R}) \tag{2.7}$$

and

$$y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \in A(n,\mathbb{R}), \qquad (y_1, \dots, y_{n-1} > 0). \tag{2.8}$$

By the Iwasawa Decomposition,

$$GL(n,\mathbb{R}) = N(n,\mathbb{R})A(n,\mathbb{R})O(n,\mathbb{R}), \tag{2.9}$$

so $\mathbb{H}^n \cong GL(n,\mathbb{R})/(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$, i.e., for any $g \in GL(n,\mathbb{R})$ there exist unique $x \in N(n,\mathbb{R})$ as in (2.7), and $y \in A(n,\mathbb{R})$ as in (2.8), some $k \in O(n,\mathbb{R})$ and a positive real number d, such that

$$g = d \cdot xy \cdot k = d \cdot z \cdot k, \qquad (z = xy \in \mathbb{H}^n). \tag{2.10}$$

Remark 2.3. *Let W_n denote the Weyl group of $GL(n,\mathbb{R})$, consisting of all $n \times n$ matrices in $SL(n,\mathbb{Z}) \cap O(n,\mathbb{R})$ which have exactly one ± 1 in each row and column. The Weyl group W_n acts on the diagonal matrices as a permutation group. For any $w \in W_n$ there exists a unique permutation σ_w on n symbols such that*

$$w. \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} := w \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} w^{-1} = \begin{pmatrix} a_{\sigma_w(1)} & & \\ & \ddots & \\ & & a_{\sigma_w(n)} \end{pmatrix} \tag{2.11}$$

for any diagonal matrix $\left(\begin{smallmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{smallmatrix}\right)$ with $a_i \in \mathbb{R}$ (or $a_i \in \mathbb{C}$).

By the Cartan decomposition,

$$GL(n,\mathbb{R}) = O(n,\mathbb{R})A(n,\mathbb{R})O(n,\mathbb{R}). \tag{2.12}$$

So for any $g \in GL(n,\mathbb{R})$ there exist $k_1, k_2 \in O(n,\mathbb{R})$ and a unique $A(g) \in A^1(n,\mathbb{R}^+)$ (up to the conjugation by the Weyl group W_n) such that

$$g = |\det g|^{\frac{1}{n}} \cdot k_1 \cdot A(g) \cdot k_2, \tag{2.13}$$

where

$$A^1(n,\mathbb{R}^+) = \left\{ a \in A(n,\mathbb{R}^+) \ \middle|\ \det a = 1 \right\}. \tag{2.14}$$

For an integer $n \geqslant 1$, define the set

$$\mathfrak{a}(n) := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \mid \alpha_1 + \cdots + \alpha_n = 0\}. \tag{2.15}$$

Definition 2.4. *Let $n \geqslant 2$ be an integer. For $g \in GL(n, \mathbb{R})$ define $\ln : GL(n, \mathbb{R}) \to \mathfrak{a}(n)$ such that*

$$\ln(A(g)) := (\ln a_1, \ldots, \ln a_n) \in \mathfrak{a}(n) \tag{2.16}$$

where $A(g) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ with $a_1, \ldots, a_n > 0$ as in (2.13). This $\ln(A(g))$ is uniquely determined up to the Weyl group action, i.e., permutation for $A(g)$. Moreover, $\ln a_1 + \cdots + \ln a_n = 0$ since $\det A(g) = 1$.

Conversely, define $\exp : \mathfrak{a}(n) \to A^1(n, \mathbb{R}^+)$ such that

$$\exp(h) := \begin{pmatrix} e^{h_1} & & \\ & \ddots & \\ & & e^{h_n} \end{pmatrix} \in A^1(n, \mathbb{R}^+) \tag{2.17}$$

for any $h = (h_1, \ldots, h_n) \in \mathfrak{a}(n)$.

For any $g \in GL(n, \mathbb{R})$ define

$$|| \ln A(g) || := \sqrt{(\ln a_1)^2 + \cdots + (\ln a_n)^2} \tag{2.18}$$

for $A(g) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ with $a_1, \ldots, a_n > 0$ and $a_1 \cdots a_n = 1$ as in (2.13).

Lemma 2.5. (Relations between the coordinates for generalized upper half plane and Cartan decomposition) *Let $n \geqslant 2$ be an integer. For any $g \in GL(n, \mathbb{R})$, by the Iwasawa decomposition and Cartan decomposition, we have*

$$g = d \cdot xy \cdot k_{\mathrm{Iwa}} = (|\det g|)^{\frac{1}{n}} k_1 \begin{pmatrix} e^{\alpha_1} & & \\ & \ddots & \\ & & e^{\alpha_n} \end{pmatrix} k_2, \qquad (k_{\mathrm{Iwa}}, k_1, k_2 \in O(n, \mathbb{R})),$$

where $d > 0$ *with* $d^n \cdot \det y = |\det g|$,

$$x = \begin{pmatrix} 1 & & x_{i,j} \\ & \ddots & \\ & & 1 \end{pmatrix} \in N(n, \mathbb{R}), \qquad y = \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in A(n, \mathbb{R}^+),$$

and $\ln(A(g)) = (\alpha_1, \ldots, \alpha_n) \in \mathfrak{a}(n)$. *Then*

$$\begin{aligned} & e^{-2||\ln A(g)||} \leqslant y_1 \leqslant e^{2||\ln A(g)||}, \\ & e^{-4||\ln A(g)||} \leqslant y_j \leqslant e^{4||\ln A(g)||}, \qquad (\textit{for } j = 2, \ldots, n-1). \end{aligned} \tag{2.19}$$

Proof. Let $z := xy \in \mathbb{H}^n$. Then for $g \in GL(n, \mathbb{R})$,

$$z = xy = (\det y)^{\frac{1}{n}} k_1 \begin{pmatrix} e^{\alpha_1} & & \\ & \ddots & \\ & & e^{\alpha_n} \end{pmatrix} k_2 k_{\mathrm{Iwa}}^{-1},$$

and

$$z \cdot {}^t z = xy^2\, {}^t x = (\det y)^{\frac{2}{n}} k \begin{pmatrix} e^{2\alpha_1} & & \\ & \ddots & \\ & & e^{2\alpha_n} \end{pmatrix} {}^t k.$$

So

$$xy^2 (\det y)^{-\frac{2}{n}}\, {}^t x = k \begin{pmatrix} e^{2\alpha_1} & & \\ & \ddots & \\ & & e^{2\alpha_n} \end{pmatrix} {}^t k,$$

for $k = k_1 \in O(n, \mathbb{R})$. Compare the diagonal parts. For the left hand side, for $j = 1, \ldots, n$, we have

$$y'_j + y'_{j+1} x_{j,j+1}^2 + \cdots + y'_n x_{j,n}^2, \qquad (x_{n,n} = 1)$$

on the diagonal, where $y'_j = (\det y)^{-\frac{2}{n}} (y_1 \cdots y_{n-j})^2$ and $y'_n = (\det y)^{-\frac{2}{n}}$. For the right hand side, for $j = 1, \ldots, n$, we have

$$k_{j,1}^2 e^{2\alpha_1} + \cdots + k_{j,n}^2 e^{2\alpha_n}$$

on the diagonal, where

$$k = \begin{pmatrix} k_{1,1} & \dots & k_{1,n} \\ k_{2,1} & \dots & k_{2,n} \\ \vdots & \dots & \vdots \\ k_{n,1} & \dots & k_{n,n} \end{pmatrix} \in O(n, \mathbb{R}).$$

So for any $j = 1, \dots, n$,

$$y'_j \leqslant y'_j + y'_{j+1}x^2_{j,j+1} + \cdots + y'_n x^2_{j,n} = k^2_{j,1}e^{2\alpha_1} + \cdots + k^2_{j,n}e^{2\alpha_n}$$
$$\leqslant (k^2_{j,1} + \cdots + k^2_{j,n})e^{2||\ln A(g)||} = e^{2||\ln A(g)||}.$$

Since $||\ln A(g)|| = ||\ln A(g^{-1})||$, we also have

$$y'^{-1}_j \leqslant e^{2||\ln A(g)||}, \qquad (\text{ for } j = 1, \dots, n)$$

so

$$e^{-2||\ln A(g)||} \leqslant y'_j \leqslant e^{2||\ln A(g)||}, \qquad (\text{ for } j = 1, \dots, n).$$

For $j = 1, \dots, n-1$, we have

$$-||\ln A(g)|| \leqslant -\frac{1}{n}\ln(\det y) + \ln y_1 + \cdots + \ln y_{n-j} \leqslant ||\ln A(g)||$$

and

$$-||\ln A(g)|| \leqslant -\frac{1}{n}\ln(\det y) \leqslant ||\ln A(g)||.$$

Therefore, we have

$$-2||\ln A(g)|| \leqslant \ln y_1 \leqslant 2||\ln A(g)|||$$

and

$$-4||\ln A(g)|| \leqslant \ln y_j \leqslant 4||\ln A(g)||, \qquad (\text{ for } j = 1, \dots, n-1).$$

$\square$

Definition 2.6. **(Siegel Sets)** *Fix $a, b \geqslant 0$. We define the Siegel set $\Sigma_{a,b} \subset \mathbb{H}^n$ to be the set of all matrices of the form*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \in \mathbb{H}^n,$$

with $|x_{i,j}| \leqslant b$ for $1 \leqslant i < j \leqslant n$ and $y_i > a$ for $1 \leqslant i \leqslant n-1$.

Definition 2.7. **(Fundamental Domain)** *For $n \geqslant 2$, we define $\mathfrak{F}^n$ to be the subset of the Siegel set $\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}$, satisfying:*

- *for any $z \in \mathbb{H}^n$, there exists $\gamma \in SL(n, \mathbb{Z})$ such that $\gamma z \in \mathfrak{F}^n$,*
- *for any $z \in \mathfrak{F}^n$, $\gamma z \notin \mathfrak{F}^n$ for any $\gamma \in SL(n, \mathbb{Z})$ (with $\gamma \neq I_n$).*

Then $\mathfrak{F}^n$ becomes a fundamental domain for $SL(n, \mathbb{Z})$ and

$$\mathfrak{F}^n \cong SL(n, \mathbb{Z}) \backslash \mathbb{H}^n.$$

We introduce the partial Iwasawa decomposition for $\mathbb{H}^n$ to describe the fundamental domain explicitly. Let $n \geqslant 2$ be an integer. For any $z \in \mathbb{H}^n$, we may write

$$\begin{aligned} z &= \begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & & & \\ & y_1 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \\ &= \begin{pmatrix} & & & x_{1,n} \\ & I_{n-1} & & \vdots \\ & & & x_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.20}$$

where

$$z' = \begin{pmatrix} 1 & \cdots & x_{1,n-1} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \begin{pmatrix} y_2 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathbb{H}^{n-1}.$$

The following proposition is an interpretation of §2 in [14].

Proposition 2.8. (Explicit Description of the Fundamental domain) *Let $n \geqslant 2$ be an integer and $\overline{\mathfrak{F}^n}$ be the closure of the fundamental domain $\mathfrak{F}^n$.*

(1) *for $n = 2$, the closure of the fundamental domain $\overline{\mathfrak{F}^2}$ is the set of $z = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{H}^n$ for $x, y \in \mathbb{R}$ and $y > 0$ satisfying*

$$x^2 + y^2 \geqslant 1 \text{ and } |x| \leqslant \frac{1}{2}.$$

(2) *for $n > 2$, the closure of the fundamental domain $\overline{\mathfrak{F}^n}$ is the set of*

$$z = \begin{pmatrix} & & x_1 \\ & I_{n-1} & \vdots \\ & & x_{n-1} \\ 0 & \dots\ 0 & 1 \end{pmatrix}\begin{pmatrix} & & 0 \\ & y_1 z' & \vdots \\ & & 0 \\ 0 & \dots\ 0 & 1 \end{pmatrix}$$

for $x_1, \dots, x_{n-1} \in \mathbb{R}$ and $y_1 > 0$ satisfying the following conditions:

(i) $z' \in \overline{\mathfrak{F}^{n-1}}$;

(ii) *for any $\left(\begin{smallmatrix} & & b_1 \\ & * & \vdots \\ & & b_{n-1} \\ c_1 & \dots\ c_{n-1} & a \end{smallmatrix}\right) \in GL(n, \mathbb{Z})/\{\pm I_n\}$, we have*

$$(a + c_1 x_1 + \cdots + c_{n-1} x_{n-1})^2 + y_1^2 (c_1\ \dots\ c_{n-1}) z'\, {}^t z' \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \geqslant 1;$$

i.e., if $z = \left(\begin{smallmatrix} 1 & x_{1,2} & \dots & x_{1,n-1} & x_1 \\ & 1 & \dots & x_{2,n-1} & x_2 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & x_n \\ & & & & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y_1 \cdots y_{n-1} & & & \\ & y_1 \cdots y_{n-2} & & \\ & & \ddots & \\ & & & 1 \end{smallmatrix}\right) \in \mathbb{H}^n$, *then*

$$\begin{aligned} &(a + c_1 x_{1,n} + \cdots + c_{n-1} x_{n-1,n})^2 \\ &\quad + y_1^2 \left[c_1^2 (y_2 \cdots y_{n-1})^2 + (c_1 x_{1,2} + c_2)^2 (y_2 \cdots y_{n-2})^2 + \cdots \right. \\ &\qquad + (c_1 x_{1,j} + \cdots + c_j x_{j-1,j} + c_j)^2 (y_1 \cdots y_{n-j})^2 + \cdots \\ &\qquad \left. + (c_1 x_{1,n-1} + \cdots + c_{n-2} x_{n-2,n-1} + c_{n-1})^2 \right] \geqslant 1; \end{aligned}$$

(iii) $|x_j| \leqslant \frac{1}{2}$ *for* $j = 1, \ldots, n-1$.

Proof. The proof is again an interpretation of §2 in [14]. Let $\mathcal{SP}_n$ denote the space of quadratic forms of determinant 1, which is identified by

$$\begin{array}{ccc} O(n,\mathbb{R})\backslash GL(n,\mathbb{R}) & \to & \mathcal{SP}_n \\ O(n,\mathbb{R})\cdot g & \mapsto & {}^t g\cdot g =: Z. \end{array}$$

Then $\gamma \in GL(n,\mathbb{Z})/\{\pm I_n\}$ acts on $\mathcal{SP}_n$ discontinuously by $Z \mapsto Z[\gamma] := {}^t\gamma Z\gamma$. Every $Z \in \mathcal{SP}_n$ can be represented as

$$Z = \begin{pmatrix} y^{-1} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & y^{\frac{1}{n-1}} Z' & \\ 0 & & & \end{pmatrix} \left[\begin{pmatrix} 1 & x_{n-1} & \ldots & x_1 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{pmatrix} \right]$$

with $y > 0,\ Z' \in \mathcal{SP}_{n-1}$ and $x_1, \ldots, x_{n-1} \in \mathbb{R}$ by the partial Iwasawa decomposition (2.20). By repeating this, we can get the Iwasawa decompositon for $Z \in \mathcal{SP}_n$, namely

$$Z = y^{-1} \begin{pmatrix} 1 & & & & \\ & y_1^2 & & & \\ & & (y_1y_2)^2 & & \\ & & & \ddots & \\ & & & & (y_1\cdots y_{n-1})^2 \end{pmatrix} \left[\begin{pmatrix} 1 & & x_{i,j} & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right]$$

with $y_1, \ldots, y_{n-1}, y > 0$, $x_{i,j} \in \mathbb{R}$, (for $1 \leqslant i < j \leqslant n$). In [14], by using the partial Iwasawa decomposition, the fundamental domain $\mathcal{F}_n \cong \mathcal{SP}_n/\left(GL(n,\mathbb{Z})/\{\pm I_n\}\right)$ is described as the set of all $Z = \begin{pmatrix} y^{-1} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & y^{\frac{1}{n-1}} Z' & \\ 0 & & & \end{pmatrix} \left[\begin{pmatrix} 1 & x_{n-1} & \ldots & x_1 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{pmatrix} \right] \in \mathcal{SP}_n$ satisfying:

(i) $Z' \in \mathcal{F}_n$;

(ii) for any $\begin{pmatrix} a & {}^tb \\ c & D \end{pmatrix} \in GL(n,\mathbb{Z})/\{\pm I_n\},\ a \in \mathbb{Z},\ b, c = \begin{pmatrix} c_{n-1} \\ \vdots \\ c_1 \end{pmatrix} \in \mathbb{Z}^{n-1}$ and $D \in Mat(n-1,\mathbb{Z})$, we must have

$$(a + x_{n-1}c_{n-1} + \cdots x_1c_1)^2 + y^{\frac{n}{n-1}} \begin{pmatrix} c_{n-1} & \ldots & c_1 \end{pmatrix} Z' \begin{pmatrix} c_{n-1} \\ \vdots \\ c_1 \end{pmatrix} \geqslant 1;$$

(iii) for $j = 1, 2, \ldots, n-2$, we have

$$0 \leqslant x_{n-1} < \frac{1}{2}, \qquad |x_j| \leqslant \frac{1}{2}.$$

Let

$$w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & 1 \\ w_{n-1} & \end{pmatrix} = \begin{pmatrix} & w_{n-1} \\ 1 & \end{pmatrix}. \tag{2.21}$$

We may identify $\mathcal{SP}_n$ by $\mathbb{H}^n$ via

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\sim} & \mathcal{SP}_n \\ z & \mapsto & w_n\left((\det z)^{-\frac{1}{n}} z\right) \cdot {}^t\left((\det z)^{-\frac{1}{n}} z\right) w_n =: Z. \end{array}$$

Then for any $z = \begin{pmatrix} & x_{1,n} \\ I_{n-1} & \vdots \\ & x_{n-1,n} \\ 0 \ \ldots \ 0 & 1 \end{pmatrix} \begin{pmatrix} & 0 \\ y_1 z' & \vdots \\ & 0 \\ 0 \ \ldots \ 0 & 1 \end{pmatrix} \in \mathbb{H}^n$ and $z' \in \mathbb{H}^{n-1}$,

$$\begin{aligned} Z &= w_n\left((\det z)^{-\frac{1}{n}} z\right) \cdot {}^t\left((\det z)^{-\frac{1}{n}} z\right) w_n \\ &= \begin{pmatrix} (\det z)^{-\frac{2}{n}} & \\ & (\det z)^{\frac{2}{n(n-1)}} Z' \end{pmatrix} \left[\begin{pmatrix} 1 & x_{n-1} & \ldots & x_1 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{pmatrix} \right] \in \mathcal{SP}_n, \end{aligned}$$

where

$$Z' = w_{n-1}\left((\det z')^{-\frac{1}{n-1}} z'\right) \cdot {}^t\left((\det z')^{-\frac{1}{n-1}} z'\right) w_{n-1} \in \mathcal{SP}_{n-1}.$$

So for $\gamma \in SL(n, \mathbb{Z})$,

$$\gamma z \mapsto w_n\left((\det z)^{-\frac{1}{n}} \gamma z\right) \cdot {}^t\left((\det z)^{-\frac{1}{n}} \gamma z\right) w_n = Z[w_n\, {}^t\gamma w_n],$$

and $w_n\, {}^t\gamma w_n \in SL(n, \mathbb{Z})$. Therefore by using the fundamental domain $\mathcal{F}_n$ for $\mathcal{SP}_n$, we can get the explicit description for the fundamental domain $\mathfrak{F}^n \cong SL(n, \mathbb{Z})\backslash\mathbb{H}^n$. $\square$

Consider $GL(n, \mathbb{R})$ or $GL(n, \mathbb{R})/\mathbb{R}^\times$, with Haar measure dg, which is normalized as

$$\int_{O(n,\mathbb{R})} dk = \int_{O(n,\mathbb{R})/\mathbb{R}^\times} dk = 1.$$

By [12], the left invariant $GL(n,\mathbb{R})$-measure d^*z on $\mathbb{H}^n$ can be given explicitly by the formula

$$d^*z = d^*x \; d*y, \tag{2.22}$$

$$\text{where} \quad d^*x = \prod_{1\leqslant i<j\leqslant n} dx_{i,j}, \quad d^*y = \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k.$$

2.3 Invariant differential operators

Let $n \geqslant 2$ be an integer and $\mathfrak{gl}(n,\mathbb{R})$ be the Lie algebra of $GL(n,\mathbb{R})$ with the Lie bracket $[,]$ given by $[\alpha,\beta] = \alpha\beta - \beta\alpha$ for $\alpha,\beta \in \mathfrak{gl}(n,\mathbb{R})$. The universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$ can be realized as an algebra of differential operators D_α acting on smooth functions $f : GL(n,\mathbb{R}) \to \mathbb{C}$. The action is given by

$$D_\alpha f(g) := \frac{\partial}{\partial t} f\left(g\cdot \exp(t\alpha)\right)\bigg|_{t=0} = \frac{\partial}{\partial t} f\left(g + tg\alpha\right)\bigg|_{t=0} \tag{2.23}$$

for $\alpha \in \mathfrak{gl}(n,\mathbb{R})$. For any $\alpha,\beta \in \mathfrak{gl}(n,\mathbb{R})$, $D_{\alpha+\beta} = D_\alpha + D_\beta$ and $D_{[\alpha,\beta]} = [D_\alpha, D_\beta] = D_\alpha \circ D_\beta - D_\beta \circ D_\alpha$. Here $\circ$ is the composition of differential operators. The differential operators D_α with $\alpha \in \mathfrak{gl}(n,\mathbb{R})$ generate an associative algebra $\mathcal{D}^n$ defined over $\mathbb{R}$.

For $1 \leqslant i,j \leqslant n$, let $E_{i,j} \in \mathfrak{gl}(n,\mathbb{R})$ be the matrix with 1 at the i,jth entry and 0 elsewhere. Let $D_{i,j} = D_{E_{i,j}}$ for $1 \leqslant i,j \leqslant n$.

Definition 2.9. (Casimir operators) *Let $n \geqslant 2$ be an integer. For $j = 1,\ldots,n-1$, we define Casimir operators $\Delta_n^{(j)}$ given by*

$$\Delta_n^{(j)} = -\frac{1}{j+1}\sum_{i_1=1}^{n}\cdots\sum_{i_{j+1}}^{n} D_{i_1,i_2}\circ D_{i_2,i_3}\circ\cdots\circ D_{i_{j+1},i_1}. \tag{2.24}$$

Let $\Delta_n := \Delta_n^{(1)}$ be the Laplace operator.

For $n \geqslant 2$, define $\mathcal{Z}(\mathcal{D}^n)$ to be the center of the algebra of differential operators $\mathcal{D}^n$. It is well known that the Casimir operators $\Delta_n^{(1)}, \ldots, \Delta_n^{(n-1)} \in \mathcal{Z}(\mathcal{D}^n)$. Moreover every differential operator which lies in $\mathcal{Z}(\mathcal{D}^n)$ can be expressed as a polynomial (with coefficients in $\mathbb{R}$) in the Casimir operators $\Delta_n^{(1)}, \ldots, \Delta_n^{(n-1)}$, i.e.,

$$\mathcal{Z}(\mathcal{D}^n) \cong \mathbb{R}\left[\Delta_n^{(1)}, \ldots, \Delta_n^{(n-1)}\right]$$

(see [12]).

There is a standard procedure to construct simultaneous eigenfunctions of all differential operators of $D \in \mathcal{Z}(\mathcal{D}^n)$. Let $n \geqslant 2$ be an integer and $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. Define

$$I_\nu(g) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j} \tag{2.25}$$

where $g = |\det g|^{\frac{1}{n}} \begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,n} \\ & 1 & \cdots & x_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} k \in GL(n, \mathbb{R})$ for $x_{i,j}, y_i \in \mathbb{R}$, $y_i > 0$, with $1 \leqslant i < j \leqslant n$ and $k \in O(n, \mathbb{R})$. Here

$$b_{i,j} = \begin{cases} ij & \text{if } i + j \leqslant n \\ (n-i)(n-j) & \text{if } i + j \geqslant n \end{cases}.$$

For $j = 1, \ldots, n-1$, let

$$B_j(\nu) := j \cdot \sum_{k=1}^{n-j} k \cdot \nu_k + (n-j) \cdot \sum_{k=1}^{j-1} k \cdot \nu_{n-k}. \tag{2.26}$$

Then

$$I_\nu\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}\right) = \prod_{j=1}^{n-1} y_j^{B_j(\nu)}.$$

Clearly, $B_j(\nu + \mu) = B_j(\nu) + B_j(\mu)$ for any $\nu, \mu \in \mathbb{C}^{n-1}$.

For $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, the function I_ν is an eigenfunction of $\mathcal{Z}(\mathcal{D}^n)$. Define $\lambda_\nu : \mathcal{Z}(\mathcal{D}^n) \to \mathbb{C}$ to be the character such that

$$DI_\nu = \lambda_\nu(D) I_\nu \tag{2.27}$$

for any $D \in \mathcal{Z}(\mathcal{D}^n)$. For any $D_1, D_2 \in \mathcal{Z}(\mathcal{D}^n)$, we have

$$\lambda_\nu(D_1 \circ D_2) = \lambda_\nu(D_1) \cdot \lambda_\nu(D_2),$$

$$\lambda_\nu(D_1 + D_2) = \lambda_\nu(D_1) + \lambda_\nu(D_2).$$

The Weyl group W_n (defined in Remark 2.3) acts on $\nu = (\nu_1, \ldots, \nu_{n-1})$ in the following way. For any $w \in W_n$,

$$w.\nu := \mu = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{C}^{n-1} \text{ if and only if } I_{\nu - \frac{1}{n}}(y) = I_{\mu - \frac{1}{n}}(wy), \tag{2.28}$$

for $y = \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}$, $y_1, \ldots, y_{n-1} > 0$. Here $\nu - \frac{1}{n} = (\nu_1 - \frac{1}{n}, \ldots, \nu_{n-1} - \frac{1}{n})$. Then for any $w \in W_n$, we have $\lambda_\nu = \lambda_{w.\nu}$ for any $\nu \in \mathbb{C}^{n-1}$, i.e., I_ν and $I_{w.\nu}$ have the same eigenvalues for $\mathcal{Z}(\mathcal{D}^n)$.

Let $\mathfrak{a}^*(n) = \mathfrak{a}(n)$ as in (2.15) and

$$\begin{aligned} \mathfrak{a}^*_{\mathbb{C}}(n) &:= \mathfrak{a}^*(n) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \{\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{C}^n \mid \ell_1 + \cdots + \ell_n = 0\}. \end{aligned} \tag{2.29}$$

Definition 2.10. *Let $n \geqslant 2$ and $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. The Langlands parameter for ν is defined to be*

$$\ell_\infty(\nu) = (\ell_{\infty,1}(\nu), \ldots, \ell_{\infty,n}(\nu)) \in \mathfrak{a}^*_{\mathbb{C}}(n),$$

where

$$\ell_{\infty,j}(\nu) := \begin{cases} -\frac{n-1}{2} + B_{n-1}(\nu), & \text{for } j = 1, \\ \frac{-n+2j-1}{2} + B_{n-j}(\nu) - B_{n-j+1}(\nu) & \text{for } 1 < j < n, \\ \frac{n-1}{2} - B_1(\nu), & \text{for } j = n. \end{cases} \tag{2.30}$$

Here B_j (for $j = 1, \dots, n-1$) are defined in (2.26). Then $\ell_{\infty,1}(\nu) + \cdots + \ell_{\infty,n}(\nu) = 0$.

Remark 2.11. **(i)** *The Langlands parameter is the same parameter defined in p.314-315, [12].*

(ii) *For any $\nu = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{C}^{n-1}$,*

$$\prod_{j=1}^{n-1} (y_1 \cdots y_{n-j})^{\ell_{\infty,j}(\nu) + \frac{n-2j+1}{2}} = \prod_{j=1}^{n} y_j^{\sum_{k=1}^{n-j}\left(\ell_{\infty,k}(\nu) + \frac{n-2k+1}{2}\right)} = \prod_{j=1}^{n-1} y_j^{B_j(\nu)}.$$

(iii) *Let $f : \mathbb{H}^n \to \mathbb{C}$ be an eigenfunction of $\mathcal{Z}(\mathcal{D}^n)$ of type ν. Define*

$$\ell_\infty(f) := \ell_\infty(\nu). \tag{2.31}$$

(iv) *The Weyl group W_n acts on the Langlands parameter as a permutation group. For any $w \in W_n$ there exists a permutation σ_w on n symbols such that*

$$\ell_\infty(w.\nu) = \sigma_w(\ell_\infty(\nu)) = (\ell_{\infty,\sigma_w(1)}(\nu), \dots, \ell_{\infty,\sigma_w(n)}(\nu)). \tag{2.32}$$

(v) *Since (2.26) and (2.30) are linear, from the given $\ell_\infty = (\ell_{\infty,1}, \dots, \ell_{\infty,n}) \in \mathfrak{a}^*_{\mathbb{C}}(n)$ with $\ell_{\infty,1} + \cdots + \ell_{\infty,n} = 0$, we can get $\nu \in \mathbb{C}^{n-1}$, satisfying (2.30). For $j = 1, \dots, n-1$, let*

$$\nu_j(\ell_\infty) := \frac{1}{n}\left(\ell_{\infty,j} - \ell_{\infty,j+1} + 1\right), \qquad \nu(\ell_\infty) := (\nu_1(\ell_\infty), \dots, \nu_{n-1}(\ell_\infty)). \tag{2.33}$$

Then $\ell_\infty(\nu) = \nu\left(\ell_\infty(\nu)\right)$. For $j = 1, \dots, n-1$, define

$$\lambda_\infty^{(j)}(\ell_\infty) := \lambda_{\nu(\ell_\infty)}(\Delta_n^{(j)}). \tag{2.34}$$

The following Lemma is given in [12].

Lemma 2.12. (Eigenvalues for $D_{i,j}$) *Let $n \geqslant 2$ and $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. For $1 \leqslant i, j \leqslant n$ and $k = 1, 2, \ldots$, we have*

$$D_{i,j}^k I_\nu = \begin{cases} \left(\frac{n-2j+1}{2} + \ell_{\infty,j}(\nu)\right)^k I_\nu, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.35}$$

Here $\ell_\infty(\nu) = (\ell_{\infty,1}(\nu), \ldots, \ell_{\infty,n}(\nu)) \in \mathfrak{a}_{\mathbb{C}}^*(n)$ is defined in (2.30).

Proposition 2.13. (The Laplace eigenvalue) *Let $n \geqslant 2$ and $\nu = (\nu_1, \ldots, \nu_{n-1})$. The Laplace eigenvalue is*

$$\lambda_\nu(\Delta_n) = \frac{1}{24}\left(n^3 - n\right) - \frac{1}{2}\left(\ell_{\infty,1}(\nu)^2 + \cdots + \ell_{\infty,n}(\nu)^2\right), \tag{2.36}$$

where Δ_n is the Laplace operator in Definition 2.9. Here $\ell_\infty(\nu) = (\ell_{\infty,1}(\nu), \ldots, \ell_{\infty,n}(\nu))$ is the Langlands parameter for ν, defined in (2.30).

Proof. For any $y = \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in A(n, \mathbb{R}^+)$ with $y_1, \ldots, y_{n-1} > 0$, consider $\Delta_n I_\nu(y)$. Then

$$\begin{aligned} \Delta_n I_\nu(y) &= -\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n D_{i,j} \circ D_{j,i} I_\nu(y) \\ &= -\frac{1}{2}\left\{\sum_{j=1}^n D_{j,j} \circ D_{j,j} I_\nu(y) + \sum_{1 \leqslant i < j \leqslant n} \left(D_{i,j} \circ D_{j,i} + D_{j,i} \circ D_{i,j}\right) I_\nu(y)\right\}. \end{aligned}$$

For $1 \leqslant i, j, i', j' \leqslant n$, we have $[E_{i,j}, E_{i',j'}] = \delta_{i',j} E_{i,j'} - \delta_{i,j'} E_{i',j}$ where $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases}$.
So

$$[D_{i,j}, D_{i',j'}] = D_{[E_{i,j}, E_{i',j'}]} = \delta_{i',j} D_{i,j'} - \delta_{i,j'} D_{i',j}.$$

For $1 \leqslant i < j \leqslant n$, we have

$$D_{i,j} \circ D_{j,i} + D_{j,i} \circ D_{i,j} = 2 D_{i,j} \circ D_{j,i} + D_{j,j} - D_{i,i},$$

and

$$\begin{aligned} & D_{i,j} \circ D_{j,i} I_\nu(y) \\ & \quad = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} I_\nu \left(\left. \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & t_1 \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ t_2 & \ddots & \\ & & 1 \end{pmatrix} \right) \right|_{t_1 = t_2 = 0} . \end{aligned}$$

For $1 \leqslant i < j \leqslant n$, we have

$$\begin{aligned} & (I_2 + t_1 E_{i,j})(I_2 + t_2 E_{j,i}) \\ & \quad = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & \dots & t_1 & \\ & & & \ddots & \vdots & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & (t_2^2+1)^{-\frac{1}{2}} & \dots & \frac{t_2}{(t_2^2+1)^{\frac{1}{2}}} & \\ & & & \ddots & \vdots & \\ & & & & (t_2^2+1)^{\frac{1}{2}} & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \\ & \quad = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & (t_2^2+1)^{-\frac{1}{2}} & \dots & \frac{t_2}{(t_2^2+1)^{\frac{1}{2}}} + t_1 (t_2^2+1)^{\frac{1}{2}} & \\ & & & \ddots & \vdots & \\ & & & & (t_2^2+1)^{\frac{1}{2}} & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \mod O(n, \mathbb{R}), \end{aligned}$$

so $D_{i,j} \circ D_{j,i} I_\nu(y) = 0$ and

$$\Delta_n I_\nu(y) = -\frac{1}{2} \left\{ \sum_{j=1}^{n} D_{j,j}^2 I_\nu(y) + \sum_{1 \leqslant i < j \leqslant n} (D_{j,j} - D_{i,i}) I_\nu(y) \right\} .$$

Here

$$\begin{aligned} \sum_{1 \leqslant i < j \leqslant n} (D_{j,j} - D_{i,i}) &= \sum_{j=2}^{n} \left(\sum_{i=1}^{j-1} D_{j,j} - D_{i,i} \right) = \sum_{j=2}^{n} (j-1) D_{j,j} - \sum_{i=1}^{n-1} (n-i) D_{i,i} \\ &= -\sum_{j=1}^{n} (n - 2j + 1) D_{j,j} . \end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_n I_\nu(y) &= -\frac{1}{2}\left\{\sum_{j=1}^{n}\left(D_{j,j}^2-(n-2j+1)D_{j,j}\right)I_\nu(y)\right\}\\ &= -\frac{1}{2}\sum_{j=1}^{n}\left\{\left(\frac{n-2j+1}{2}+\ell_{\infty,j}(\nu)\right)^2\right.\\ &\qquad\left.-(n-2j+1)\left(\frac{n-2j+1}{2}+\ell_{\infty,j}(\nu)\right)\right\}I_\nu(y)\\ &= -\frac{1}{2}\sum_{j=1}^{n}\left\{\ell_{\infty,j}(\nu)^2-\frac{(n-2j+1)^2}{4}\right\}I_\nu(y)\\ &= \left\{\frac{1}{24}(n^3-n)-\frac{1}{2}\sum_{j=1}^{n}\ell_{\infty,j}(\nu)^2\right\}I_\nu(y).\end{aligned}$$

□

2.4 Maass forms

Definition 2.14. (Automorphic Function) *For an integer $n \geqslant 2$, an automorphic function for $SL(n,\mathbb{Z})$ is a function $f:\mathbb{H}^n\to\mathbb{C}$ such that*

$$f(\gamma z) = f(z)$$

for any $\gamma \in SL(n,\mathbb{Z})$ and $z\in\mathbb{H}^n$.

Consider $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^2\right)$ to be the space of automorphic functions $f:\mathbb{H}^n\to\mathbb{C}$ satisfying

$$||f||_2^2 := \int_{SL(n,\mathbb{Z})\backslash\mathbb{H}^n}|f(z)|^2\ d^*z < \infty.$$

For $f_1,\ f_2\in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$, define the inner product

$$\langle f_1, f_2\rangle := \int_{SL(n,\mathbb{Z})\backslash\mathbb{H}^n} f_1(z)\overline{f_2(z)}\ d^*z. \tag{2.37}$$

Definition 2.15. (Cuspidal function) *Let $n \geqslant 2$ be an integer and let $f : \mathbb{H}^n \to \mathbb{C}$ be an automorphic function for $SL(n,\mathbb{Z})$. The function f is cuspidal if*

$$\int\limits_{\left(SL(n,\mathbb{Z})\cap U_{n_1,\ldots,n_r}(\mathbb{Z})\right)\backslash U_{n_1,\ldots,n_r}(\mathbb{R})} f(uz)\,du = 0 \tag{2.38}$$

for any partition $n_1 + \cdots + n_r = n$ and $r \geqslant 2$. Here $U_{n_1,\ldots,n_r}$ is the unipotent radical defined in (2.3).

Let $L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ denote the space of automorphic cuspidal functions. Let $C^\infty\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ (resp. $C^\infty_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$) denote the space of smooth automorphic functions (resp. smooth automorphic cuspidal functions) and $C^\infty \cap L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ (resp. $C^\infty \cap L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$) denote the space of smooth automorphic functions (resp. smooth automorphic cuspidal functions), which are square integrable.

Let $n \geqslant 2$ be an integer and $f \in L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$. By Theorem 5.3.2 [12], f has the following Fourier expansion:

$$\begin{aligned} &f(z) \\ &= \sum_{\gamma\in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty}\sum_{m_{n-1}\neq 0} W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z; (m_1,\ldots,m_{n-1})\right) \\ &= \sum_{\gamma\in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty}\sum_{m_{n-1}\neq 0} W_f(y^\gamma; (m_1,\ldots,m_{n-1})) \\ &\qquad\qquad \times e^{2\pi i(m_1 x^\gamma_{n-1,n}+\cdots+m_{n-2}x^\gamma_{2,3}+m_{n-1}x^\gamma_{1,2})} \end{aligned} \tag{2.39}$$

where $\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right) z = x^\gamma y^\gamma \in \mathbb{H}^n$ for $z = xy \in \mathbb{H}^n$ with x, y, x^γ, y^γ given as in (2.7) and (2.8).

Here the sum is independent of the choice of coset representative γ. Further

$$W_f\left(z;(m_1,\ldots,m_{n-1})\right) \tag{2.40}$$
$$:= \int_{(N(n,\mathbb{R})\cap SL(n,\mathbb{Z}))\backslash N(n,\mathbb{R})} f(uz)e^{-2\pi i(m_1u_{n-1,n}+\cdots+m_{n-2}u_{2,3}+m_{n-1}u_{1,2})}\ d^*u$$
$$= \int_{\mathbb{Z}\backslash\mathbb{R}} \cdots \int_{\mathbb{Z}\backslash\mathbb{R}} f(uz)e^{-2\pi i(m_1u_{n-1,n}+\cdots+m_{n-2}u_{2,3}+m_{n-1}u_{1,2})}\ d^*u$$

with $u = \begin{pmatrix} 1 & u_{1,2} & \ldots & u_{1,n} \\ & \ddots & \ddots & \vdots \\ & & 1 & u_{n-1,n} \\ & & & 1 \end{pmatrix} \in N(n,\mathbb{R})$ and $d^*u = \prod_{1\leqslant i<k\leqslant n} du_{i,j}$ as in (2.22). Here $N(n,\mathbb{R})$ is the minimal unipotent radical defined in (2.6).

Definition 2.16. **(Maass forms)** *Let $n \geqslant 2$ be an integer and $\nu = (\nu_1,\ldots,\nu_{n-1}) \in \mathbb{C}^{n-1}$. A smooth automorphic function $f : \mathbb{H}^n \to \mathbb{C}$ is a Maass form of type ν if:*

(i) *f is an eigenform of $\mathcal{Z}(\mathcal{D}^n)$ i.e., for $j = 1,\ldots,n-1$,*

$$\Delta_n^{(j)} f = \lambda_\nu(\Delta_n^{(j)})f,$$

where λ_ν is the Harish-Chandra character of type ν defined in (2.27);

(ii) *f is square-integrable, i.e.,*

$$||f||_2^2 = \int_{SL(n,\mathbb{Z})\backslash\mathbb{H}^n} |f(z)|^2\ d^*z < \infty;$$

(iii) *f is cuspidal.*

Definition 2.17. **(Jacquet's Whittaker function)** *Let $n \geqslant 2$ be an integer. For each $\nu \in \mathbb{C}^{n-1}$ and $\epsilon = \pm 1$, we define a function $W_J(\ ;\nu,\epsilon) : \mathbb{H}^n \to \mathbb{C}$ such that*

$$W_J(z;\nu,\epsilon) = \int_{N(n,\mathbb{R})} I_\nu\left(\left(\begin{pmatrix} & & & (-1)^{\lfloor \frac{n}{2}\rfloor} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} \cdot \begin{pmatrix} 1 & u_{1,2} & \ldots & u_{1,n} \\ & 1 & \ldots & u_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} z\right)\right) \tag{2.41}$$
$$\cdot \exp\left(-2\pi i\left(\epsilon u_{1,2} - u_{2,3} - \cdots - u_{n-1,n}\right)\right)\ d^*u,$$

for $\int_{N(n,\mathbb{R})} d^*u = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leqslant i < j \leqslant n} du_{i,j}$. *The function* $W_J(z;\nu,\epsilon)$ *is called Jacquet's Whittaker function or Whittaker function of type* ν.

Remark 2.18. *The Whittaker function of type* ν *is an eigenfunction of* $\mathcal{Z}(\mathcal{D}^n)$ *of type* ν*, i.e., for any* $D \in \mathcal{Z}(\mathcal{D}^n)$*,*

$$DW_J(z;\nu,\epsilon) = \lambda_\nu(D) \cdot W_J(z;\nu,\epsilon).$$

Let f be a Maass form of type $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. Then by (9.1.2) [12], f has a Fourier-Whittaker expansion of the form

$$f(z) = \sum_{\gamma \in N(n-1,\mathbb{Z}) \backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0}^{\infty} \frac{A_f(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} \\ \times W_J\left(\begin{pmatrix} m_1 \cdots |m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z; \nu, \frac{m_{n-1}}{|m_{n-1}|} \right) \tag{2.42}$$

where $A_f(m_1, \ldots, m_{n-1}) \in \mathbb{C}$. Here $A_f(m_1, \ldots, m_{n-1})$ is called the $(m_1, \ldots, m_{n-1})$th Fourier coefficient of f for each $1 \leqslant m_1, \ldots, m_{n-2} \in \mathbb{Z}$ and nonzero $m_{n-1} \in \mathbb{Z}$.

2.5 Hecke operators and Hecke-Maass forms

Recall the general definition of Hecke operators from [12]. Let X be a topological space. Consider a group G that acts continuously on X. Let Γ be a discrete subgroup of G, and set

$$C_G(\Gamma) := \left\{ g \in G \;\middle|\; [\Gamma : (g^{-1}\Gamma g) \cap \Gamma] < \infty \text{ and } [g^{-1}\Gamma g : (g^{-1}\Gamma g) \cap \Gamma] < \infty \right\}$$

to be the commensurator group of Γ in G. For any $g \in C_G(\Gamma)$, we have a decomposition of a double coset into disjoint right cosets of the form

$$\Gamma g \Gamma = \bigcup_i \Gamma \alpha_i.$$

For each such $g \in C_G(\Gamma)$, the Hecke operator $T_g : L^2(\Gamma\backslash X) \to L^2(\Gamma\backslash X)$ is defined by

$$T_g f(x) = \sum_i f(\alpha_i x),$$

where $f \in L^2(\Gamma\backslash X),\ x \in X.$ Fix a semiring Δ where $\Gamma \subset \Delta \subset C_G(\Gamma).$ The Hecke ring consists of all formal sums

$$\sum_k c_k T_{g_k}$$

for $c_k \in \mathbb{Z}$ and $g_k \in \Delta.$ Since two double cosets are either identical or totally disjoint, it follows that unions of double cosets are associated to elements in the Hecke ring. If there exists an antiautomorphism $g \mapsto g^*$ satisfying $(gh)^* = h^* g^*$, $\Gamma^* = \Gamma$ and $(\Gamma g \Gamma)^* = \Gamma g \Gamma$ for every $g \in \Delta$, the Hecke ring is commutative.

Definition 2.19. **(Hecke Operator T_N)** *Let $f : \mathbb{H}^n \to \mathbb{C}$ be a function. For each integer $N \geqslant 1$, we define a Hecke operator*

$$T_N f(z) := \frac{1}{N^{\frac{n-1}{2}}} \sum_{\substack{\prod_{j=1}^n c_j = N, \\ 0 \leqslant c_{i,j} < c_j \ (1 \leqslant i < j \leqslant n)}} f\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} z\right). \tag{2.43}$$

Clearly T_1 is the identity operator.

For $n = 2$, the Hecke operators are self-adjoint with respect to the inner product (2.37), i.e., for any $f_1, f_2 \in L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$ and any integer $N \geqslant 1$, we have $\langle T_N f_1, f_2 \rangle = \langle f_1, T_N f_2 \rangle$. For $n \geqslant 3$, the Hecke operator is no longer self-adjoint, but the adjoint operator is again a Hecke operator and the Hecke operator commutes with its adjoint, so it is a normal operator. The following Theorem is proved in Theorem 9.3.6, [12].

Theorem 2.20. *Let $n \geqslant 2$ be an integer. Consider the Hecke operators T_N for any integer $N \geqslant 1$, defined in (2.43). Let T_N^* be the adjoint operator which satisfies*

$$\langle T_N f, g \rangle = \langle f, T_N^* g \rangle \tag{2.44}$$

for all $f, g \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$. Then T_N^* is another Hecke operator which commutes with T_N so that T_N is a normal operator. Explicitly,

$$T_N^* f(z) = \frac{1}{N^{\frac{n-1}{2}}} \sum_{\substack{\prod_{j=1}^n c_j = N, \\ 0 \leqslant c_{i,j} < c_j (1 \leqslant i < j \leqslant n)}} f\left(\begin{pmatrix} N & & & \\ & \ddots & & \\ & & N & \\ & & & N \end{pmatrix} \begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix}^{-1} \cdot z \right). \tag{2.45}$$

Definition 2.21. (Hecke-Maass form) *Let $n \geqslant 2$. A Maass form f is called a Hecke-Maass form if it is an eigenfunction of all Hecke operators T_N for $N \geqslant 1$.*

Assume that f is a Hecke-Maass form then f has the Fourier-Whittaker expansion as in (2.42). Let $A_f(m_1, \ldots, m_{n-1}) \in \mathbb{C}$ be the $(m_1, \ldots, m_{n-1})$th Fourier coefficients for $0 \neq m_1, \ldots, m_{n-1} \in \mathbb{Z}$. Since f is a Hecke-Maass form, $A_f(1, 1, \ldots, 1) \neq 0$. Assume that $A_f(1, \ldots, 1) = 1$. Then we have the following (multiplicative) relations (see [12]):

- $T_N f = A_f(N, 1, \ldots, 1) f$ for any integer $N \geqslant 1$;
- for $(m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, we have

$$A_f(m_{n-1}, \ldots, m_1) = \overline{A_f(m_1, \ldots, m_{n-1})};$$

- for $(m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, and a nonzero integer m, we have

$$\begin{aligned} &A_f(m, 1, \ldots, 1) A_f(m_1, \ldots, m_{n-1}) \\ &\quad = \sum_{\substack{\prod_{j=1}^n c_j = m, \\ c_1 \mid m_1, \ldots, c_{n-1} \mid m_{n-1}}} A_f\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots, \frac{m_{n-1} c_{n-2}}{c_{n-1}} \right). \end{aligned} \tag{2.46}$$

Let p be a prime. Then for any $k = 1, 2, \ldots,$

$$\begin{aligned} &A_f(p^k, 1, \ldots, 1) A_f(\underbrace{1, \ldots, 1, p}_{r}, 1, \ldots, 1) \\ &\quad = A_f(\underbrace{p^k, 1, \ldots, 1, p}_{r}, 1, \ldots, 1) + A_f(\underbrace{p^{k-1}, 1, \ldots, 1, p}_{r+1}, 1, \ldots, 1), \end{aligned} \tag{2.47}$$

for $r = 1, \ldots, n-2$, and

$$A_f(p^k, 1, \ldots, 1)A_f(1, \ldots, 1, p) = A_f(p^k, 1, \ldots, 1, p) + A_f(p^{k-1}, 1, \ldots, 1).$$

Definition 2.22. *Let $n \geqslant 2$ and fix a prime p. For $j = 1, \ldots, n-1$, define*

$$T_p^{(j)} = \sum_{k=0}^{j-1} (-1)^k T_{p^{k+1}} T_p^{(j-k-1)} \tag{2.48}$$

and $T_{p^r}^{(1)} = T_{p^r}$ for any integer $r \geqslant 0$ and $T_p^{(0)}$ is an identity operator.

Lemma 2.23. (Eigenvalues of Hecke operators $T_p^{(r)}$) *Let $n \geqslant 2$ be an integer and f be a Maass form for $SL(n, \mathbb{Z})$. Then f has the Fourier-Whittaker expansion as in (2.42) and let $A_f(m_1, \ldots, m_{n-1}) \in \mathbb{C}$ be the $(m_1, \ldots, m_{n-1})$th Fourier coefficients for $0 \neq m_1, \ldots, m_{n-1} \in \mathbb{Z}$. Assume that f is an eigenfunction for T_{p^j} for $j = 0, \ldots, n$ and $A_f(1, \ldots, 1) = 1$. Then for $r = 1, \ldots, n-1$,*

$$T_p^{(r)} f = A_f(\underbrace{1, \ldots, 1, p}_{r}, 1, \ldots, 1) f$$

for any prime p.

Proof. By using the definition of $T_p^{(r)}$ (for $r = 1, \ldots, n-1$) and the multiplicative relations in (2.47), we get the eigenvalue of $T_p^{(r)}$ (for $r = 1, \ldots, n-1$). □

Definition 2.24. *Let $n \geqslant 2$ be an integer and fix a prime p. Let $f : \mathbb{H}^n \to \mathbb{C}$ be an eigenfunction for $T_p^{(j)}$ for $j = 1, \ldots, n-1$ as in Definition 2.22, i.e., for $j = 1, \ldots, n-1$ there exists $\lambda_p^{(j)}(f) \in \mathbb{C}$ such that*

$$T_p^{(j)} f(z) = \lambda_p^{(j)}(f) \cdot f(z), \qquad (z \in \mathbb{H}^n)$$

Define the parameters

$$\ell_p(f) := (\ell_{p,1}(f), \ldots, \ell_{p,n}(f)) \in \mathfrak{a}_{\mathbb{C}}^*(n) \tag{2.49}$$

such that

$$1+\sum_{j=1}^{n-1}(-1)^j\lambda_p^{(j)}(f)x^j+(-1)^nx^n=\prod_{j=1}^{n}\left(1-p^{-\ell_{p,j}(f)}x\right).$$

Here $\mathfrak{a}_{\mathbb{C}}^*(n)\subset\mathbb{C}^n$ *is the complex vector space defined in (2.29). So, for* $j=1,\ldots,n-1$,

$$\begin{aligned}\lambda_p^{(j)}(f) &= \sum_{1\leqslant k_1<\cdots<k_j\leqslant n} p^{-(\ell_{p,k_1}(f)+\cdots+\ell_{p,k_j}(f))} \\ &=: \lambda_p^{(j)}(\ell_p(f)).\end{aligned} \tag{2.50}$$

Remark 2.25. **(i)** *Let* f *be a Hecke-Maass form with* $(m_1,\ldots,m_{n-1})$*th Fourier coefficients* $A_f(m_1,\ldots,m_{n-1})\in\mathbb{C}$ *as in (2.42). Assume that* $A_f(1,\ldots,1)=1$. *Then we have*

$$A_f(\underbrace{1,\ldots,1,p}_{r},1,\ldots,1)=\sum_{1\leqslant j_1<\ldots<j_r\leqslant n}p^{-(\ell_{p,j_1}(f)+\cdots+\ell_{p,j_r}(f))}=\lambda_p^{(j)}(\ell_p(f))$$

for any $1\leqslant r\leqslant n-1$.

(ii) *The parameter is given by the equation*

$$\begin{aligned}&1-A_f(p,1,\ldots,1)x+A_f(1,p,1,\ldots,1)+\cdots \\ &\quad+(-1)^rA_f(\underbrace{1,\ldots,1,p}_{r},1,\ldots,1)x^r+\cdots+(-1)^nx^n=0\end{aligned} \tag{2.51}$$

and it has solutions $p^{-\ell_{p,1}(f)},\ldots,p^{-\ell_{p,n}(f)}$. *This equation comes from the* p*th factor of the* L*-function of the Hecke-Maass form* f. *For* $s\in\mathbb{C}$ *with* $\Re(s)>1$, *let*

$$\begin{aligned}L_p(f;s) &:= \sum_{k=0}^{\infty}A_f(p^k,1,\ldots,1)p^{-ks} \\ &= (1-A_f(p,1\ldots,1)p^{-s}+\cdots+(-1)^rA_f(\underbrace{1,\ldots,1,p}_{r},1,\ldots,1)p^{-rs}+\cdots \\ &\quad+(-1)^np^{-ns})^{-1} \\ &= \prod_{j=1}^{n}\left(1-p^{-\ell_{p,j}(f)-s}\right)^{-1}\end{aligned} \tag{2.52}$$

(see [12]). Conversely, if the parameters $\ell_p(f) \in \mathfrak{a}_{\mathbb{C}}^*(n)$ *is given then we can determine* $A_f(\underbrace{1, \ldots, 1, p}_{r}, 1, \ldots, 1)$ *for each* $r = 1, \ldots, n-1$.

Recall the definition of dual Maass forms and the properties from Proposition 9.2.1, [12].

Proposition 2.26. (Dual Maass forms) *Let* $\phi(z)$ *be a Maass form of type* $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. *Then*

$$\widetilde{\phi}(z) := \phi\left(w \cdot {}^t(z^{-1}) \cdot w\right), \qquad w = \begin{pmatrix} & & & (-1)^{\lfloor \frac{n}{2} \rfloor} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

is a Maass form of type $\widetilde{\nu} = (\nu_{n-1}, \ldots, \nu_1)$ *for* $SL(n, \mathbb{Z})$. *The Maass form* $\widetilde{\phi}$ *is called the dual Maass form. If* $A(m_1, \ldots, m_{n-1})$ *is the* $(m_1, \ldots, m_{n-1})$*th Fourier coefficient of* ϕ *then* $A(m_{n-1}, \ldots, m_1)$ *is the corresponding Fourier coefficient of* $\widetilde{\phi}$. *If* $\phi = \widetilde{\phi}$, *then the Maass form* ϕ *is called the self-dual Maass form.*

Remark 2.27. *Let* f *be a Hecke-Maass form of type* ν. *Then the dual Maass form* $\widetilde{f}$ *of type* $\widetilde{\nu}$ *has the following Langlands parameters.*

(i) $v = \infty$*: Since* $B_j(\widetilde{\nu}) = B_{n-j}(\nu)$ *for* $j = 1, \ldots, n-1$*, we have*

$$\ell_\infty(\widetilde{f}) = \ell_\infty(\widetilde{\nu}) = -\ell_{\infty,n-j+1}(\nu) = -\ell_{\infty,n-j+1}(f), \qquad (\text{for } j = 1, \ldots, n)$$

so

$$\ell_\infty(\widetilde{f}) = -\ell_\infty(f).$$

(ii) $v = p$*, prime: Since*

$$A_{\widetilde{f}}(\underbrace{1, \ldots, 1, p}_{j}, 1, \ldots, 1) = A_f(\underbrace{1, \ldots, 1, p}_{n-j}, 1, \ldots, 1), \qquad (\text{for } j = 1, \ldots, n-1),$$

we have

$$\ell_p(\widetilde{f}) = -\ell_p(f).$$

2.6 Eisenstein series

We defined parabolic subgroups, their Levi parts and unipotent radicals in Definition 2.1. Then for each partition $n = n_1 + \cdots + n_r$ with rank $1 < r \leqslant n$, we have the factorization

$$P_{n_1,\ldots,n_r}(\mathbb{R}) = U_{n_1,\ldots,n_r}(\mathbb{R}) \cdot M_{n_1,\ldots,n_r}(\mathbb{R}).$$

It follows that for any $g \in P_{n_1,\ldots,n_r}(\mathbb{R})$, we have

$$g \in U_{n_1,\ldots,n_r}(\mathbb{R}) \cdot \begin{pmatrix} \mathfrak{m}_{n_1}(g) & 0 & \ldots & 0 \\ & \mathfrak{m}_{n_2}(g) & \ldots & 0 \\ & & \ddots & \vdots \\ & & & \mathfrak{m}_{n_r}(g) \end{pmatrix}$$

where $\mathfrak{m}_{n_i}(g) \in GL(n_i, \mathbb{R})$ for $i = 1, \ldots, r$.

Let $n \geqslant 2$ be an integer and fix a partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1, \ldots, n_r < n$. For each $i = 1, \ldots, r$, let ϕ_i be either a Maass form for $SL(n_i, \mathbb{Z})\backslash\mathbb{H}^{n_i}$ of type $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,n_i-1}) \in \mathbb{C}^{n_i-1}$ or a constant with $\mu_i = (0, \ldots, 0)$. For $t = (t_1, \ldots, t_r) \in \mathbb{C}^r$ with $n_1t_1 + \cdots + n_rt_r = 0$, define a function

$$I_{P_{n_1,\ldots,n_r}}(*; t; \phi_1, \ldots, \phi_r) : P_{n_1,\ldots,n_r}(\mathbb{R}) \to \mathbb{C}$$

by the formula

$$I_{P_{n_1,\ldots,n_r}}(g; t; \phi_1, \ldots, \phi_r) := \prod_{i=1}^{r} \phi_i(\mathfrak{m}_{n_i}(g)) \cdot |\det(\mathfrak{m}_{n_i}(g))|^{t_i} \quad \text{for } g \in P_{n_1,\ldots,n_r}(\mathbb{R}). \tag{2.53}$$

For each $i = 1, \ldots, r$, let $\phi_i(\mathfrak{m}_{n_i}(gk)) = \phi_i(\mathfrak{m}_{n_i}(g))$ and $|\det(\mathfrak{m}_{n_i}(gk))| = |\det(\mathfrak{m}_{n_i}(g))|$ where $k \in O(n, \mathbb{R})$. So $I_{P_{n_1,\ldots,n_r}}(g; t; \phi_1, \ldots, \phi_r) = I_{P_{n_1,\ldots,n_r}}(z; t; \phi_1, \ldots, \phi_r)$ for $g = d{\cdot}z{\cdot}k$ with $z \in \mathbb{H}^n$, $d \in \mathbb{R}^\times$ and $k \in O(n, \mathbb{R})$. Let $\eta_1 = 0$ and $\eta_i = n_1 + \cdots + n_{i-1}$ for $i = 2, \ldots, r$. There exists a unique $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ (up to the action of the Weyl group W_n)

such that $I_{P_{n_1,\ldots,n_r}}(*;t;\phi_1,\ldots,\phi_r)$ is an eigenfunction for $\mathcal{Z}(\mathcal{D}^n)$ of type ν. Furthermore,

$$\begin{aligned} I_\nu(y) &= \prod_{k=1}^{n-1} y_k^{B_k(\nu)} \\ &= \prod_{j=1}^{r}\left(\big(y_1\cdots y_{n-\eta_j-n_j}\big)^{n_j t_j}\cdot \prod_{k=1}^{n_j-1} y_{n-\eta_j-n_j+k}^{B_k(\mu_j)+(n_j-k)t_j}\right)\end{aligned} \tag{2.54}$$

for any $y = \begin{pmatrix} y_1\cdots y_{n-1} & & \\ & \ddots & \\ & & 1\end{pmatrix} \in A(n,\mathbb{R}^+)$ (see Proposition 10.9.1, [12]). Then for $1 \leqslant i \leqslant n-1$,

$$B_i(\nu) = \begin{cases} B_{i-(n-n_1)}(\mu_1) + (n-i)t_1, \\ \qquad\qquad \text{if } n-n_1+1 \leqslant i \leqslant n-1 \\ n_1t_1+\cdots+n_{j-1}t_{j-1} + B_{i-(n-\eta_j-n_j)}(\mu_j) + (n-\eta_j-i)t_j \\ \qquad\qquad \text{if } 2 \leqslant j \leqslant r \text{ and } n-\eta_j-n_j+1 \leqslant i \leqslant n-\eta_j-1 \\ n_1t_1+\cdots+n_jt_j, \\ \qquad\qquad \text{if } i = n-\eta_j-n_j. \end{cases} \tag{2.55}$$

Therefore, by (2.30), for $1 \leqslant j \leqslant r$ and $\eta_j + 1 \leqslant i \leqslant \eta_j + n_j$, we have

$$\ell_{\infty,i}(\nu) = \Big(\frac{-n+n_j}{2} + t_j + \eta_j\Big) + \ell_{\infty,i-\eta_j}(\mu_j). \tag{2.56}$$

Definition 2.28. (Eisenstein series) *Let $n \geqslant 2$ be an integer and fix an ordered partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1,\ldots,n_r < n$. For each $i = 1,\ldots,r$, let ϕ_i be either a Maass form for $SL(n_i,\mathbb{Z})\backslash\mathbb{H}^{n_i}$ of type $\mu_i = (\mu_{i,1},\ldots,\mu_{i,n_i-1}) \in \mathbb{C}^{n_i-1}$ or a constant with $\mu_i = (0,\ldots,0)$. Let $t = (t_1,\ldots,t_r) \in \mathbb{C}^r$ with $n_1t_1+\cdots+n_rt_r = 0$. Define the Eisenstein series by the infinite series*

$$\begin{aligned} &E_{P_{n_1,\ldots,n_r}}(z;t;\phi_1,\ldots,\phi_r) \\ &:= \sum_{\gamma\in\left(P_{n_1,\ldots,n_r}(\mathbb{Z})\cap SL(n,\mathbb{Z})\right)\backslash SL(n,\mathbb{Z})} I_{P_{n_1,\ldots,n_r}}(\gamma z;t;\phi_1,\ldots,\phi_r)\end{aligned} \tag{2.57}$$

for $z \in \mathbb{H}^n$.

Remark 2.29. **(i)** *Since $I_{P_{n_1,\dots,n_r}}(*;t;\phi_1,\dots,\phi_r)$ is actually a function on $\mathbb{H}^n$, the Eisenstein series (2.57) are well-defined on $\mathbb{H}^n$, but they are not square-integrable.*

(ii) *Eisenstein series are automorphic, i.e., for any $\gamma \in SL(n,\mathbb{Z})$, we have*

$$E_{P_{n_1,\dots,n_r}}(\gamma z;t;\phi_1,\dots,\phi_r) = E_{P_{n_1,\dots,n_r}}(z;t;\phi_1,\dots,\phi_r), \qquad (z \in \mathbb{H}^n).$$

(iii) *Each Eisenstein series is an eigenfunction of type ν of $\mathcal{Z}(\mathcal{D}^n)$ where ν is given by the formula (2.54).*

The Fourier coefficients for Eisenstein series are given in Proposition 10.9.3 [12].

Proposition 2.30. *Let $n \geqslant 2$ and fix a partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1,\dots,n_r < n$. For each $i = 1,\dots,r$, let ϕ_i be either a Hecke-Maass form for $SL(n_i,\mathbb{Z})\backslash\mathbb{H}^{n_i}$ or a constant. Let $t = (t_1,\dots,t_r) \in \mathbb{C}^r$ with $n_1t_1 + \cdots + n_rt_r = 0$. Then the Eisenstein series $E_{P_{n_1,\dots,n_r}}(z;t;\phi_1,\dots,\phi_r)$ is an eigenfunction of the Hecke operators T_N (for any $N \geqslant 1$) with eigenvalues*

$$A_{t;\phi_1,\dots,\phi_r}(N) = N^{-\frac{n-1}{2}} \sum_{\substack{C_1\cdots C_r=N,\\ 1\leqslant C_j\in\mathbb{Z}}} \prod_{j=1}^{r} \left(A_{\phi_j}(C_j) C_j^{\frac{n_j-1}{2}+t_j+\eta_j} \right) \tag{2.58}$$

$$= N^{-\frac{n-1}{2}} \sum_{\substack{C_1\cdots C_r=N,\\ 1\leqslant C_j\in\mathbb{Z}}} A_{\phi_1}(C_1)\cdots A_{\phi_r}(C_r)\cdot C_1^{\frac{n_1-1}{2}+t_1} C_2^{\frac{n_2-1}{2}+t_2+\eta_2} \cdots C_r^{\frac{n_r-1}{2}+t_r+\eta_r},$$

where $\eta_1 = 0$ and $\eta_j = n_1 + \cdots + n_{j-1}$ (for $j = 2,\dots,r$). Here $A_{\phi_j}(C_j)$ is the Hecke eigenvalue of T_{C_j} for ϕ_j.

Remark 2.31. *If ϕ_j is a constant, then*

$$T_{C_j}\phi_j = \left(C_j^{-\frac{n_j-1}{2}} \sum_{d_1\cdots d_n=C_j} \left(\prod_{k=1}^{n_j-1} d_k^{k-1} \right) \right) \cdot \phi_j \tag{2.59}$$

$$\text{and } A_{\phi_j}(C_j) = C_j^{-\frac{n_j-1}{2}} \sum_{d_1\cdots d_n=C_j} \left(\prod_{k=1}^{n_j-1} d_k^{k-1} \right).$$

Now, we can extend the parameters defined in Definition 2.24. Let $n \geqslant 2$ and fix a partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1, \ldots, n_r < n$. For each $i = 1, \ldots, r$, let ϕ_i be either a Maass form for $SL(n_i, \mathbb{Z})\backslash \mathbb{H}^{n_i}$ or a constant. Let $t = (t_1, \ldots, t_r) \in \mathbb{C}^r$ with $n_1 t_1 + \cdots + n_r t_r = 0$. Let $E(z) := E_{P_{n_1,\ldots,n_r}}(z; t; \phi_1, \ldots, \phi_r)$ then

$$A_E(p^k) = p^{-\frac{k(n-1)}{2}} \sum_{\substack{C_1 \cdots C_r = p^k, \\ 1 \leqslant C_j \in \mathbb{Z}}} \prod_{j=1}^{r} A_{\phi_j}(C_j) C_j^{\frac{n_j-1}{2}+t_j+\eta_j}$$

for a prime p and $k \geqslant 0$. By (2.52), define

$$\ell_p(E) := (\ell_{p,1}(E), \ldots, \ell_{p,n}(E)) \in \mathfrak{a}^*_{\mathbb{C}}(n) \tag{2.60}$$

in the following way. For $\Re(s) > 0$, $s \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} A_E(p^k) p^{-ks} &= \prod_{j=1}^{r} \left(\sum_{k_j=0}^{\infty} \left(A_{\phi_j}(p^{k_j}) p^{k_j\left(\frac{n_j-n}{2}+t_j+\eta_j\right) - k_j s} \right) \right) \\
&= \prod_{j=1}^{r} \prod_{k=1}^{n_j} \left(p^{-\ell_{p,k}(\phi_j) + \frac{n_j-n}{2} + t_j + \eta_j} - p^{-s} \right)^{-1} \\
&= \prod_{k=1}^{n} \left(p^{-\ell_{p,k}(E)} - p^{-s} \right)^{-1}.
\end{aligned}$$

Then for $i = 1, \ldots, r$ and $\eta_i + 1 \leqslant j \leqslant \eta_i + n_i$, it follows that

$$\ell_{p,j}(E) = \ell_{p,j-\eta_i}(\phi_i) - \left(\frac{n_i - n}{2} + t_i + \eta_i \right). \tag{2.61}$$

Lemma 2.32. *Let $n \geqslant 2$ and fix a partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1, \ldots, n_r < n$. For each $i = 1, \ldots, r$, let ϕ_i be either a Hecke-Maass form for $SL(n_i, \mathbb{Z})\backslash \mathbb{H}^{n_i}$ or a constant. Let $t = (t_1, \ldots, t_r) \in \mathbb{C}^r$ with $n_1 t_1 + \cdots + n_r t_r = 0$. Let $E := E_{P_{n_1,\ldots,n_r}}(z; t; \phi_1, \ldots, \phi_r)$. By (2.56) and (2.61), for $i = 1, \ldots, r$ and $\eta_i + 1 \leqslant j \leqslant \eta_i + n_i$, we have*

$$\ell_{v,j}(E) = (-1)^{\epsilon} \left(\frac{n_i - n}{2} + t_i + \eta_i \right) + \ell_{v,j-\eta_i}(\phi_i)$$

where $\epsilon = \begin{cases} 0, & \text{if } v = \infty; \\ 1, & \text{if } v < \infty, \end{cases}$ *and* $\eta_i = n_1 + \cdots + n_{i-1}$ *and* $\eta_1 = 0$.

Chapter 3

AUTOMORPHIC CUSPIDAL REPRESENTATIONS FOR $\mathbb{A}^{\times}\backslash GL(n, \mathbb{A})$

3.1 Local representations for $GL(n, \mathbb{Q}_v)$

Let G be a group and let V be a complex vector space. A representation of G on V is a pair of (π, V) where

$$\pi : G \to End(V) = \{ \text{ set of all linear maps: } V \to V\}$$

is a homomorphism. We let $\pi(g).v$ denote the action of $\pi(g)$ on v and $\pi(g'g'') = \pi(g').\pi(g'')$ for all $g',\ g'' \in G$. The vector space V is called the space of the representation (π, V). If the group G and the vector space V are equipped with topologies, then we shall also require the map $G \times V \to V$ given by $(g, v) \to \pi(g).v$ to be continuous. A representation (π, V) is said to be irreducible if $V \neq 0$ and V has no closed π-invariant subspace other than 0 and V.

Let V be a space of functions $f : G \to \mathbb{C}$ and π^R be the action given by right translation,

$$(\pi^R(h)f)(g) = f(gh), \qquad (\forall g, h \in G).$$

Then (π^R, V) is a representation of G.

In this section we review the properties of local representations of $GL(n, \mathbb{Q}_v)$. The main reference is [13].

Let $n \geqslant 1$ be an integer. Consider the archimedean case with $v = \infty$. Let V be a complex vector space equipped with a positive definite Hermitian form $(\ ,\ ) : V \times V \to \mathbb{C}$.

A unitary representation of $GL(n,\mathbb{R})$ consists of V and a homomorphism $\pi : GL(n,\mathbb{R}) \to GL(V)$ such that the function $(g,v) \mapsto \pi(g).v$ is a continuous function $GL(n,\mathbb{R})\times V \to V$, and

$$(\pi(g).v, w) = (v, \pi(g^{-1}).w), \qquad (\text{ for all } v, w \in V,\ g \in GL(n,\mathbb{R})).$$

The representation (π, V) has the trivial central character if $\pi\left(\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}\right).v = v$ for any $a \in \mathbb{R}^\times$, $v \in V$.

As in §2.3, for an integer $n \geqslant 1$, the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ consists of the additive vector space of $n \times n$ matrices with coefficients in $\mathbb{R}$ with Lie bracket given by $[\alpha,\beta] = \alpha\beta - \beta\alpha$ for any $\alpha, \beta \in \mathfrak{gl}(n,\mathbb{R})$. The universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$ is an associative algebra which contains $\mathfrak{gl}(n,\mathbb{R})$. The Lie bracket and the associative product $\circ$ on $U(\mathfrak{gl}(n,\mathbb{R}))$ are compatible, in the sense that $[\alpha,\beta] = \alpha\circ\beta - \beta\circ\alpha$ for all $\alpha,\beta \in U(\mathfrak{gl}(n,\mathbb{R}))$. The universal enveloping algebra $U(\mathfrak{gl}(n,\mathbb{R}))$ can be realized as an algebra of differential operators acting on smooth functions $F : GL(n,\mathbb{R}) \to \mathbb{C}$ as in (2.23). Set $i = \sqrt{-1}$. For any $\alpha,\beta \in \mathfrak{gl}(n,\mathbb{R})$ we define a differential operator $D_{\alpha+i\beta}$ acting on F by the rule

$$D_{\alpha+i\beta} := D_\alpha + iD_\beta.$$

The differential operators $D_{\alpha+i\beta}$ generate an algebra of differential operators which is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ where $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$.

Fix an integer $n \geqslant 1$. Let $K_\infty = O(n,\mathbb{R})$. We define a $(\mathfrak{g}, K_\infty)$-module to be a complex vector space V with actions

$$\pi_{\mathfrak{g}} : U(\mathfrak{g}) \to End(V), \qquad \pi_{K_\infty} : K_\infty \to GL(V), \tag{3.1}$$

such that for each $v \in V$ the subspace of V spanned by $\{\pi_{K_\infty}(k).v \mid k \in K_\infty\}$ is finite dimensional, and the actions $\pi_{\mathfrak{g}}$ and π_{K_∞} satisfy the relations

$$\pi_{\mathfrak{g}}(D_\alpha)\pi_{K_\infty}(k) = \pi_{K_\infty}(k)\pi_{\mathfrak{g}}(D_{k^{-1}\alpha k})$$

for all $\alpha \in \mathfrak{g}$ and $k \in K_\infty$. Further, we require that

$$\pi_{\mathfrak{g}}(D_\alpha).v = \lim_{t \to 0} \frac{1}{t} \left(\pi_{K_\infty}(\exp(t\alpha)).v - v\right)$$

for all $\alpha \in \mathfrak{gl}(n, \mathbb{R})$ such that $\exp(\alpha) \in O(n, \mathbb{R})$. We denote this $(\mathfrak{g}, K_\infty)$-module as (π, V) where $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty})$.

Let (π, V) be the $(\mathfrak{g}, K_\infty)$-module. For each $v \in V$ define a vector space $W_v \subset V$ to be the span of $\{\pi_{K_\infty}(k).v \mid k \in K_\infty\}$ and define a homomorphism $\rho_v : K_\infty \to GL(W_v)$ given by $\rho_v(k).w = \pi_{K_\infty}(k).w$ for all $k \in K_\infty$ and $w \in W_v$. Then (π, V) is admissible, if for each finite dimensional representation (ρ, W) of K_∞, the span of $\{v \in V \mid (\rho_v, W_v) \cong (\rho, W)\}$ is finite dimensional. Let (π, V) be a $(\mathfrak{g}, K_\infty)$-module. Then it is said to be unramified or spherical if there exists a nonzero vector $v^\circ \in V$ such that

$$\pi_{K_\infty}(k).v^\circ = v^\circ \qquad (\text{ for all } k \in K_\infty).$$

Otherwise, it is said to be ramified.

The $(\mathfrak{g}, K_\infty)$-module (π, V) is said to be unitary if there exists a positive definite Hermitian form $(\ ,\ ) : V \times V \to \mathbb{C}$ which is invariant in the sense that

$$(\pi_{K_\infty}(k).v, w) = \left(v, \pi_{K_\infty}(k^{-1}).w\right), \;\; (\pi_{\mathfrak{g}}(D_\alpha).v, w) = -\left(v, \pi_{\mathfrak{g}}(D_\alpha).w\right),$$

for all $v, w \in V$, $k \in K_\infty$ and $\alpha \in \mathfrak{gl}(n, \mathbb{R})$.

Theorem 3.1. *Fix an integer* $n \geqslant 1$.

(i) *If* (π, V) *is a unitary representation of* $GL(n, \mathbb{R})$ *then there is a dense subspace* $V_{(\mathfrak{g},K_\infty)} \subset V$ *such that* $((\pi_{\mathfrak{g}}, \pi_{K_\infty}), V_{(\mathfrak{g},K_\infty)})$ *is a unitary* $(\mathfrak{g}, K_\infty)$*-module called the underlying* $(\mathfrak{g}, K_\infty)$*-module of* (π, V).

(ii) *If* $((\pi_{\mathfrak{g}}, \pi_{K_\infty}), V)$ *is a unitary* $(\mathfrak{g}, K_\infty)$*-module, then there exists a unitary representation* $(\pi, V_{GL(n,\mathbb{R})})$ *of* $GL(n, \mathbb{R})$ *such that* $((\pi_{\mathfrak{g}}, \pi_{K_\infty}), V)$ *is isomorphic to the underlying* $(\mathfrak{g}, K_\infty)$*-module of* $(\pi, V_{GL(n,\mathbb{R})})$.

(iii) *A unitary representation of* $GL(n, \mathbb{R})$ *is irreducible if and only if its underlying* $(\mathfrak{g}, K_\infty)$*-module is irreducible. Moreover, two irreducible unitary representations of* $GL(n, \mathbb{R})$ *are isomorphic if and only if their underlying* $(\mathfrak{g}, K_\infty)$*-modules are isomorphic.*

Proof. Theorem 14.8.11 in [13]. □

Let $n \geqslant 1$ be an integer. Consider a prime $v = p < \infty$, and $G = GL(n, \mathbb{Q}_p)$. A representation of $GL(n, \mathbb{Q}_p)$ is a pair of (π, V) where V is a complex vector space and $\pi : GL(n, \mathbb{Q}_p) \to GL(V)$ is a homomorphism. Such a representation is smooth if for any vector $\xi \in V$ there exists an open subgroup $U_\xi \subset GL(n, \mathbb{Q}_p)$ such that $\pi(g)\xi = \xi$ for any $g \in U_\xi$. It is admissible if for any $r \geqslant 1$, the space

$$\{\xi \in V \mid \pi(k)\xi = \xi, \text{ for all } k \in K_r\}$$

is finite dimensional where

$$K_r = \{k \in GL(n, \mathbb{Z}_p) \mid k - I_n \in p^r \cdot Mat(n, \mathbb{Z}_p)\}.$$

If (π, V) is an irreducible smooth representation of $GL(n, \mathbb{Q}_p)$ then there exists a unique multiplicative character $\omega_\pi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ such that $\pi\left(\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}\right)\xi = \omega_\pi(a)\xi$ for any $a \in \mathbb{Q}_p^\times$ and $\xi \in V$. This character ω_π is called the central character associated to the representation (π, V). A smooth representation (π, V) of $GL(n, \mathbb{Q}_p)$ is said to be unitary if V is equipped with a positive definite Hermitian form $(,) : V \times V \to \mathbb{C}$ and

$$(\pi(g)v, \pi(g)w) = (v, w)\,, \qquad (\forall g \in GL(n, \mathbb{Q}_p)).$$

A representation (π, V) of $GL(n, \mathbb{Q}_p)$ is termed unramified or spherical if there exists a nonzero $GL(n, \mathbb{Z}_p)$ fixed vector $\xi^\circ \in V$. Otherwise it is said to be ramified.

3.2 Adelic automorphic forms and automorphic representations

Fix an integer $n \geqslant 1$. Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$ and

$$K(n, \mathbb{A}) := O(n, \mathbb{R}) \prod_p GL(n, \mathbb{Z}_p)$$

be the standard maximal compact subgroup of $GL(n, \mathbb{A})$. In this section we review adelic automorphic forms and automorphic representations for $(\mathbb{A}^\times \cdot GL(n, \mathbb{Q}))\backslash GL(n, \mathbb{A})$. As in the previous section, the main reference is [13].

Definition 3.2. *Let $n \geqslant 1$ be an integer and $\phi : GL(n, \mathbb{A}) \to \mathbb{C}$ be a function.*

(i) *Smoothness: A function ϕ is said to be smooth if for every fixed $g_0 \in GL(n, \mathbb{A})$, there exists an open set $U \subset GL(n, \mathbb{A})$, containing g_0 and a smooth function $\phi_\infty^U : GL(n, \mathbb{R}) \to \mathbb{C}$ such that $\phi(x) = \phi_\infty^U(x_\infty)$ for all $x = \{x_\infty, x_2, \ldots, x_p, \ldots\} \in U$.*

(ii) *Moderate growth: For each place v of $\mathbb{Q}$ define a norm function $||\ ||_v$ on $GL(n, \mathbb{Q}_v)$ by $||g||_v := \max\left(\{|g_{i,j}|_v,\ 1 \leqslant i, j \leqslant n\} \cup \{|\det g|_v\}\right)$. Define a norm function $||\ ||$ on $GL(n, \mathbb{A})$ by $||g|| := \prod_v ||g_v||_v$. Then we say a function ϕ is of moderate growth if there exist constants $C, B > 0$ such that $|\phi(g)| < C||g||^B$ for all $g \in GL(n, \mathbb{A})$.*

(iii) *$K(n, \mathbb{A})$-finiteness: A function ϕ is said to be right $K(n, \mathbb{A})$-finite if the set $\{\phi(gk) \mid k \in K(n, \mathbb{A})\}$, of all right translates of $\phi(g)$ generates a finite dimensional vector space.*

(iv) *$Z(U(\mathfrak{g}))$-finiteness: Let $Z(U(\mathfrak{g}))$ denote the center of the universal enveloping algebra of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then we say a function ϕ is $Z(U(\mathfrak{g}))$-finite if the set $\{D\phi(g) \mid D \in Z(U(\mathfrak{g}))\}$ generates a finite dimensional vector space.*

Definition 3.3. (Adelic automorphic form on $GL(n, \mathbb{A})$ with trivial central character) *Let $n \geqslant 1$ be an integer. An automorphic form for $GL(n, \mathbb{A})$ with trivial central character is a smooth function $\phi : GL(n, \mathbb{A}) \to \mathbb{C}$ which satisfies the following five properties:*

(i) $\phi(\gamma g) = \phi(g), \ \forall g \in GL(n, \mathbb{A}), \ \gamma \in GL(n, \mathbb{Q})$*;*

(ii) $\phi(zg) = \phi(g), \ \forall g \in GL(n, \mathbb{A}), \ z \in \mathbb{A}^\times$*;*

(iii) *ϕ is right $K(n, \mathbb{A})$-finite;*

(iv) *ϕ is $Z(U(\mathfrak{g}))$-finite;*

(v) *ϕ is of moderate growth.*

An adelic automorphic form ϕ is a said to be a cusp form (or cuspidal) if

$$\varphi_P(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\, du = 0$$

for any proper parabolic subgroups $P(\mathbb{A})$ of $GL(n, \mathbb{A})$ and for all $g \in GL(n, \mathbb{A})$. Here U is the unipotent radical of the parabolic subgroup P defined in Definition 2.1.

Let $\mathcal{A}(\mathbb{A}^\times \backslash GL(n, \mathbb{A}))$ denote the $\mathbb{C}$-vector space of all adelic automorphic forms for $GL(n, \mathbb{A})$ with the trivial central character. Let $\mathcal{A}_{\text{cusp}}(\mathbb{A}^\times \backslash GL(n, \mathbb{A}))$ denote the $\mathbb{C}$-vector space of all adelic cuspidal forms for $GL(n, \mathbb{A})$ with the central character.

Let $\mathbb{A}_{\text{finite}}$ denote the finite adeles. For an integer $n \geqslant 1$, let $GL(n, \mathbb{A}_{\text{finite}})$ denote the multiplicative subgroup of all $a_{\text{finite}} \in GL(n, \mathbb{A})$ of the form $a_{\text{finite}} = \{I_n, a_2, a_3, \ldots, a_p, \ldots\}$

where $a_p \in GL(n, \mathbb{Q}_p)$ for all finite primes p and $a_p \in GL(n, \mathbb{Z}_p)$ for all but finitely many primes p. We define the action

$$\pi_{\text{finite}} : GL(n, \mathbb{A}_{\text{finite}}) \to GL(\mathcal{A}(\mathbb{A}^\times \backslash GL(n, \mathbb{A})))$$

as follows. For $\phi \in \mathcal{A}(\mathbb{A}^\times \backslash GL(n, \mathbb{A}))$ let

$$\pi_{\text{finite}}(a_{\text{finite}}).\phi(g) := \phi(g a_{\text{finite}}),$$

for all $g \in GL(n, \mathbb{A}), \ a_{\text{finite}} \in GL(n, \mathbb{A}_{\text{finite}})$.

Definition 3.4. *Let $n \geqslant 1$ be an integer. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $K_\infty = O(n, \mathbb{R})$. We define a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module to be a complex vector space V with actions*

$$\pi_{\mathfrak{g}} : U(\mathfrak{g}) \to End(V), \quad \pi_{K_\infty} : K_\infty \to GL(V), \quad \pi_{\text{finite}} : GL(n, \mathbb{A}_{\text{finite}}) \to GL(V),$$

such that $(\pi_{\mathfrak{g}}, \pi_{K_\infty})$ and V form a $(\mathfrak{g}, K_\infty)$-module, and the actions $(\pi_{\mathfrak{g}}, \pi_{K_\infty})$ and π_{finite} commute. The ordered pair $(((\pi_{\mathfrak{g}}, \pi_{K_\infty}), \pi_{\text{finite}}), V)$ is said to be a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module.

(i) *The representation $(((\pi_{\mathfrak{g}}, \pi_{K_\infty}), \pi_{\text{finite}}), V)$ is smooth if every vector $v \in V$ is fixed by some open compact subgroup of $GL(n, \mathbb{A}_{\text{finite}})$ under the action π_{finite}.*

(ii) *The representation $(((\pi_{\mathfrak{g}}, \pi_{K_\infty}), \pi_{\text{finite}}), V)$ is admissible if it is smooth and for any fixed open compact subgroup $K' \subset GL(n, \mathbb{A}_{\text{finite}})$, and any fixed finite-dimensional representation ρ of $SO(n, \mathbb{R})$, the set of vectors in V fixed by K' and generate a subrepresentation under the action of $SO(n, \mathbb{R})$ (which is isomorphic to ρ) spans a finite dimensional space.*

(iii) *The representation $(((\pi_{\mathfrak{g}}, \pi_{K_\infty}), \pi_{\text{finite}}), V)$ is irreducible if it is nonzero and has no proper nonzero subspace preserved by the actions π.*

Definition 3.5. (Automorphic representation) *Let $n \geqslant 1$ be an integer. An automorphic (resp. cuspidal) representation with the trivial central character is an irreducible smooth $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module which is isomorphic to a subquotient of $\mathcal{A}(\mathbb{A}^\times \backslash GL(n, \mathbb{A}))$ (resp. $\mathcal{A}_{\text{cusp}}(\mathbb{A}^\times \backslash GL(n, \mathbb{A}))$).*

3.3 Principal series for $GL(n, \mathbb{Q}_v)$

Again the main reference for this section is [13].

Definition 3.6. (Modular quasi-character) *Let $n \geqslant 2$ and fix a prime $v \leqslant \infty$. The modular quasi-character of the minimal (standard) parabolic subgroup $P(n, \mathbb{Q}_v)$ is defined as*

$$\delta_v \left(\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \right) := \prod_{j=1}^{n} |a_j|_v^{n-2j+1} \tag{3.2}$$

for any $\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \in P(n, \mathbb{Q}_v)$. Here $P(n, \mathbb{Q}_v)$ is the minimal parabolic subgroup defined in (2.6).

Let $\chi : A(n, \mathbb{Q}_v) \to \mathbb{C}^\times$ be a character. Then we can extend the character χ to the minimal parabolic $P(n, \mathbb{Q}_v)$ as

$$\chi \left(\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \right) = \chi \left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \right)$$

where $\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \in P(n, \mathbb{Q}_v)$.

Definition 3.7. (Principal series) *Let $n \geqslant 2$ be an integer and fix a prime $v \leqslant \infty$. Let χ be* *a character of $A(n,\mathbb{Q}_v)$. Denote*

$$\begin{aligned}&\mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi) && (3.3)\\ &:= \left\{ f : GL(n,\mathbb{Q}_v) \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ is locally constant, } f(umg) = \delta_v^{\frac{1}{2}}(m)\chi(m)f(g), \\ \text{for all } u \in N(n,\mathbb{Q}_v),\ m \in P(n,\mathbb{Q}_v), g \in GL(n,\mathbb{Q}_v) \end{array} \right\}.\end{aligned}$$

Define a homomorphism $\pi^R : GL(n,\mathbb{Q}_v) \to GL\left(\mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi)\right)$ where $\left(\pi^R(h)f\right)(g) = f(gh)$ for any $g,h \in GL(n,\mathbb{Q}_v)$ and $f \in \mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}$. Then $\left(\pi^R, \mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi)\right)$ is called the principal series representation of $GL(n,\mathbb{Q}_v)$ associated to χ.

For each $v \leqslant \infty$ and $n \geqslant 1$ define

$$K_v(n) := \begin{cases} O(n,\mathbb{R}), & \text{if } v = \infty \\ GL(n,\mathbb{Z}_p), & \text{if } v = p, \text{ finite prime.} \end{cases}$$

Let $\chi : A(n,\mathbb{Q}_v)/(K_v(n) \cap A(n,\mathbb{Q}_v)) \to \mathbb{C}^\times$ be a character, i.e., a spherical character of $A(n,\mathbb{Q}_v)$. There exists

$$\ell_v(\chi) = (\ell_{v,1}(\chi), \ldots, \ell_{v,n}(\chi)) \in \mathbb{C}^n$$

such that

$$\chi\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}\right) = \prod_{j=1}^{n} |a_j|_v^{\ell_{v,j}(\chi)}. \tag{3.4}$$

If χ is trivial on the center, i.e., $\chi\left(\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}\right) = 1$ for any $a \in \mathbb{Q}_v^\times$, then $\ell_{v,1}(\chi) + \cdots + \ell_{v,n}(\chi) = 0$ so $\ell_v(\chi) \in \mathfrak{a}_{\mathbb{C}}^*(n)$. If χ is unitary, then $\ell_{v,j}(\chi) \in i\mathbb{R}$ for $j = 1, \ldots, n$.

Let $\ell \in \mathbb{C}^n$ and $\chi_v(\ell)$ be the spherical character of $A(n,\mathbb{Q}_v)$ which is associated to the parameter ℓ as in (3.4). When the representation $\left(\pi^R, \mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi_v(\ell))\right)$ is not irreducible, there exists a unique spherical subconstituent. Denote $\pi_v(\ell)$ as the spherical subconstituent of $\left(\pi^R, \mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi_v(\ell))\right)$. It is called the spherical representation associated to ℓ (or $\chi_v(\ell)$). We abuse notation and denote $\mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi_v(\ell))$ as the vector space of the representation $\pi_v(\ell)$.

For each $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{C}^n$, define the function φ_ℓ with parameter $\ell \in \mathbb{C}^n$ by

$$\varphi_\ell\left(\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} k\right) := \prod_{j=1}^{n} |a_j|_v^{\ell_j} \cdot \delta_v\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}\right)^{\frac{1}{2}} \tag{3.5}$$
$$= \prod_{j=1}^{n} |a_j|_v^{\ell_j + \frac{n-2j+1}{2}}$$

for any $\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \in P(n, \mathbb{Q}_v)$ and $k \in K_v(n)$. Then $\varphi_\ell \in \mathrm{Ind}_{P(n,\mathbb{Q}_v)}^{GL(n,\mathbb{Q}_v)}(\chi_v(\ell))$ for any $\ell \in \mathbb{C}^n$, and it is unique.

Let $n \geqslant 2$ be an integer and $v = \infty$. For $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ we have already defined the eigenfunction I_ν of Casimir operators of type ν in (2.25) and defined the Langlands parameter $\ell_\infty(\nu) \in \mathbb{C}^n$ in Definition 2.10. Then $I_\nu(z) = \varphi_{\ell_\infty(\nu)}(z)$. Conversely, for each $\ell = (\ell_1, \ldots, \ell_n) \in \mathfrak{a}^*_{\mathbb{C}}(n)$, as in (2.33), and for $j = 1, \ldots, n-1$, we have

$$\nu_j(\ell) = \frac{1}{n}(\ell_j - \ell_{j+1} - 1),$$

and

$$\varphi_\ell(z) = (\det(z))^{-\frac{1}{n}\cdot\left(\sum_{k=1}^{n} \ell_k + \frac{n-2k+1}{2}\right)} \cdot \prod_{j=1}^{n-1} (y_1 \cdots y_{n-j})^{\ell_j + \frac{n-2j+1}{2}}$$
$$= \prod_{j=1}^{n-1} y_j^{\sum_{k=1}^{n-j}\left(\ell_k + \frac{n-2k+1}{2}\right)} = I_{\nu(\ell)}(z).$$

Definition 3.8. *Let $n \geqslant 2$ be an integer and $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$. Fix a place $v \leqslant \infty$. Then $\pi_v(\ell)$ is an irreducible spherical representation of $GL(n, \mathbb{Q}_v)$ associated to ℓ with trivial central character. Define:*

- $v = \infty$, *for* $j = 1, \ldots, n-1$,

$$\lambda_{\infty}^{(j)}(\ell) := \lambda_{\nu(\ell)}(\Delta_n^{(j)}) \tag{3.6}$$

where $\nu(\ell) \in \mathbb{C}^{n-1}$ *as in (2.33) and* $\lambda_{\nu(\ell)}$ *is the Harish-Chandra character defined in (2.27);*

- $v = p < \infty$, *for* $j = 1, \ldots, n-1$

$$\lambda_p^{(j)}(\ell) := \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n} p^{-(\ell_{k_1} + \cdots + \ell_{p,k_j})}. \tag{3.7}$$

3.4 Spherical generic unitary representations of $GL(n, \mathbb{Q}_v)$

Definition 3.9. (Additive character) *Fix a prime* $v < \infty$ *or* $v = \infty$. *Let* $e_v : \mathbb{Q}_v \to \mathbb{C}$ *be defined by*

$$e_v(x) := \begin{cases} e^{-2\pi i \{x\}} & \text{if } v < \infty, \\ e^{2\pi i x} & \text{if } v = \infty, \end{cases}$$

where

$$\{x\} = \begin{cases} \sum_{j=-k}^{-1} a_j p^j, & \text{if } x = \sum_{j=-k}^{\infty} a_j p^j \in \mathbb{Q}_p \text{ with } k > 0,\ 0 \leqslant a_j \leqslant p-1, \\ 0, & \text{otherwise,} \end{cases}$$

if $v = p < \infty$.

Definition 3.10. (Whittaker model for a representation of $GL(n, \mathbb{Q}_v)$) *Fix an integer* $n \geqslant 1$ *and* $v = p$ *a finite prime or* $v = \infty$. *Let* $e_v : \mathbb{Q}_v \to \mathbb{C}$ *be the additive character in Definition 3.9. Fix a character* $\psi_v : N(n, \mathbb{Q}_v) \to \mathbb{C}$ *of the form*

$$\psi_v\left(\begin{pmatrix} 1 & u_{1,2} & \ldots & & u_{1,n} \\ & 1 & u_{2,3} & & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}\right) := e_v(a_1 u_{1,2} + \cdots + a_{n-1} u_{n-1,n}) \tag{3.8}$$

for $u_{i,j} \in \mathbb{Q}_v,\ (1 \leqslant i < j \leqslant n)$ *with* $a_i \in \mathbb{Q}_v^{\times},\ (i = 1, \ldots, n-1)$.

(i) *For $v = p$, let (π, V) be a complex representation of $GL(n, \mathbb{Q}_p)$. A Whittaker model for (π, V) relative to ψ_p is the representation $(\pi', \mathcal{W}) \cong (\pi, V)$ where $\mathcal{W}$ is a space of Whittaker functions relative to ψ, i.e., of locally constant functions $W : GL(n, \mathbb{Q}_p) \to \mathbb{C}$ satisfying*

$$W(ug) = \psi_p(u)W(g)$$

for all $u \in N(n, \mathbb{Q}_p)$, $g \in GL(n, \mathbb{Q}_p)$ and π' is given by the right translation.

(ii) *For $v = \infty$, let (π, V) be a $(\mathfrak{g}, K_\infty)$-module where $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $K_\infty = O(n, \mathbb{R})$. Following Theorem 3.1, we refer to (π, V) as a representation of $GL(n, \mathbb{R})$. A Whittaker model for (π, V) relative to ψ_∞ is the representation $(\pi', \mathcal{W}) \cong (\pi, V)$ where $\mathcal{W}$ is a space of Whittaker functions relative to ψ_∞, i.e., of smooth functions of moderate growth satisfying*

$$W(ug) = \psi_\infty(u)W(g)$$

for all $u \in N(n, \mathbb{R})$, $g \in GL(n, \mathbb{R})$ and π' is given as in (3.1).

Remark 3.11. *For $v = \infty$, let $\psi_\infty\left(\begin{pmatrix} 1 & & u_{i,j} \\ & \ddots & \\ & & 1 \end{pmatrix}\right) = \exp(u_{1,2} + \cdots + u_{n-1,n})$. Then Jacquet's Whittaker $W_J(\ ;\nu, 1)$ for some $\nu \in \mathbb{C}^{n-1}$ defined in (2.41) is the Whittaker function relative to ψ_∞. Moreover, for every automorphic cuspidal smooth function f, the Fourier coefficient $W_f(\ ;(m_1, \ldots, m_{n-1}))$ defined in (2.40) is also a Whittaker function relative to an additive character $\psi_\infty\left(\begin{pmatrix} 1 & & u_{i,j} \\ & \ddots & \\ & & 1 \end{pmatrix}\right) = \exp(2\pi i(m_1 u_{n-1,n} + \cdots + m_j u_{n-j,n-j+1} + \cdots + m_{n-1}u_{1,2}))$ for $m_1, \ldots, m_{n-1} \in \mathbb{Z}$.*

Definition 3.12. (Generic representation of $GL(n, \mathbb{Q}_v)$) *Fix an integer $n \geqslant 1$, let v be a finite prime or $v = \infty$, and let ψ_v be an additive character as in (3.8). A representation (π, V) of $GL(n, \mathbb{Q}_v)$ is said to be generic relative to ψ_v if it has a Whittaker model relative to ψ_v as in Definition 3.10.*

Definition 3.13. (Spherical generic character) *Let $n \geqslant 2$ be an integer and $v \leqslant \infty$ be a prime. If there exist*

- *an integer $0 \leqslant r < \frac{n}{2}$ and $t_1, \ldots, t_r \in \mathbb{R}$,*
- *real numbers $\alpha_1, \ldots, \alpha_r \in (0, \frac{1}{2})$,*

such that

$$\begin{aligned}\ell_v &= (\ell_{v,1}, \ldots, \ell_{v,n}) \in \mathfrak{a}^*_{\mathbb{C}}(n) \\ &= (\alpha_1 + it_1, -\alpha_1 + it_1, \ldots, \alpha_r + it_r, -\alpha_r + it_r, it_{r+1}, \ldots, it_{n-r}),\end{aligned} \tag{3.9}$$

then the character $\chi_v(\ell_v) : P(n, \mathbb{Q}_v) \to \mathbb{C}^\times$ is called a spherical generic character.

Theorem 3.14. (Classification of irreducible spherical unitary generic representations) *Let $n \geqslant 2$ be an integer and $v \leqslant \infty$ be a place of $\mathbb{Q}$. Let π be an irreducible spherical unitary generic representaiton of $\mathbb{Q}_v^\times \backslash GL(n, \mathbb{Q}_v)$. Then there exists $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$ which satisfies the condition in Definition 3.13 such that $\pi \cong \pi_v(\ell)$.*

3.5 Quasi-Automorphic parameter and Quasi-Maass form

Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$. Let $n \geqslant 2$ be an integer. Let π be a cuspidal automorphic representation of $GL(n, \mathbb{A})$. The representation π is unramified or spherical if there exists a vector $v^\circ \in V_\pi$ (the complex vector space of π), such that $\pi(k)v^\circ = v^\circ$ for any $k \in K(n, \mathbb{A}) = O(n, \mathbb{R}) \prod_p GL(n, \mathbb{Z}_p)$.

Let π be an unramified cuspidal automorphic representation of $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$. Then by the tensor product theorem ([11], [17], [8]), there exist local generic spherical unitary representations π_v of $\mathbb{Q}_v^\times \backslash GL(n, \mathbb{Q}_v)$ for $v \leqslant \infty$ such that $\pi \cong \bigotimes_{v \leqslant \infty}{}' \pi_v$. Since π_v's are generic, unitary, and spherical, there exist an automorphic parameter $\sigma = \{\sigma_v \in \mathfrak{a}^*_{\mathbb{C}}(n),\ v \leqslant \infty\}$ where σ_v satisfies conditions in Definition 3.13 for any $v \leqslant \infty$, such that $\pi_v \cong \pi_v(\sigma_v)$. So, $\pi \cong \otimes'_v \pi_v(\sigma_v)$ and we may denote

$$\pi(\sigma) := \otimes'_v \pi_v(\sigma_v) \cong \pi. \tag{3.10}$$

Definition 3.15. (Quasi-Automorphic Parameters) *Let $n \geqslant 2$ be an integer and let M be a set of primes including ∞. Let $\ell_M = \{\ell_v \in \mathfrak{a}^*_{\mathbb{C}}(n),\ v \in M\}$ satisfy the conditions in Definition 3.13. Then ℓ_M is called a quasi-automorphic parameter for M.*

By the tensor product theorem combined with the multiplicity one theorem, for any unramified cuspidal automorphic representation π for $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$, there is an automorphic parameter σ for $\{\infty, 2, 3, \ldots,\}$ such that $\pi(\sigma) \cong \pi$ as in (3.10) and σ is also a quasi-automorphic parameter. There exists a unique Hecke-Maass form F_σ of type $\nu(\sigma_\infty)$ such that

- $\Delta_n^{(j)} F_\sigma = \lambda_\infty^{(j)}(\sigma_\infty) F_\sigma$, (for $j = 1, \ldots, n-1$),
- $T_p^{(j)} F_\sigma = \lambda_p^{(j)}(\sigma_p) F_\sigma$, (for $j = 1, \ldots, n-2$), for any finite prime p.

See [13] for more explanation. So, $\ell_v(F_\sigma) = \ell_v(\sigma_v)$ for any $v \leqslant \infty$. Conversely, let F be a Hecke-Maass form of type $\nu \in \mathbb{C}^{n-1}$. Then there exists a unique unramified cuspidal automorphic representation $\pi(\sigma_F)$ for $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$ such that

- $\nu(\sigma_{F,\infty}) = \nu$, so $\ell_\infty(\nu) = \ell_\infty(F) = \ell_\infty(\sigma_{F,\infty})$;
- $\sigma_{F,p} = \ell_p(F)$ and for each $r = 1, \ldots, n-1$, we have

  $$A_F(\underbrace{1, \ldots, 1, p}_{r}, 1, \ldots, 1) = \lambda_p^{(r)}(\sigma_{F,p}),$$

  where $A_F(1, \ldots, 1, p, 1, \ldots, 1)$ is the $(1, \ldots, 1, p, 1 \ldots, 1)$th Fourier coefficient as in Lemma 2.23.

Definition 3.16. **(Quasi-Maass Form)** *Let $n \geqslant 2$ be an integer and M be a set of primes including ∞. Let $\ell_M = \{\ell_v \in \mathfrak{a}^*_{\mathbb{C}}(n),\ v \in M\}$ be a quasi-automorphic parameter for M. Let*

$$\begin{aligned} L &\geqslant \prod_{\substack{q\in M,\\ \text{finite prime}}} q^n, && \textit{(if } M \textit{ is a finite set)} \\ L &= \infty, && \textit{(if } M \textit{ is an infinite set),} \end{aligned} \tag{3.11}$$

and define for $z \in \mathbb{H}^n$,

$$\begin{aligned} F_{\ell_M}(z) = &\sum_{\gamma\in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{L} \cdots \sum_{m_{n-2}=1}^{L} \sum_{\substack{m_{n-1}\neq 0,\\ |m_{n-1}|\leqslant L}} \frac{A_{\ell_M}(m_1,\ldots,m_{n-1})}{\prod_{k=1}^{n-1}|m_k|^{k(n-k)/2}} \\ &\times W_J\left(\begin{pmatrix} m_1\cdots|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1\end{pmatrix}\cdot\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z; \nu(\ell_\infty), \frac{m_{n-1}}{|m_{n-1}|}\right), \end{aligned} \tag{3.12}$$

where $A_{\ell_M}(1,\ldots,1) = 1$ and $A_{\ell_M}(m_1,\ldots,m_{n-1}) \in \mathbb{C}$ satisfy the multiplicative condition in (2.46), if this series is absolutely convergent. For $r = 1,\ldots,n-1$ and any prime $q \in M$,

$$\begin{aligned} A_{\ell_M}(\underbrace{1,\ldots,1,q}_{r},1,\ldots,1) &= \lambda_q^{(j)}(\ell_q) \\ &= \sum_{1\leqslant j_1<\cdots<j_r\leqslant n} p^{-(\ell_{q,j_1}+\cdots+\ell_{q,j_r})}. \end{aligned}$$

Then F_{ℓ_M} is called a quasi-Maass form of ℓ_M of length L.

Remark 3.17. **(i)** *By Theorem 9.4.7, [12], we can rewrite (3.12) as*

$$\begin{aligned} F_{\ell_M}(z) = &\sum_{\gamma\in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{L} \cdots \sum_{m_{n-2}=1}^{L} \sum_{\substack{m_{n-1}\neq 0,\\ |m_{n-1}|\leqslant L}} \frac{A_{\ell_M}(m_1,\ldots,m_{n-1})}{\prod_{k=1}^{n-1}|m_k|^{k(n-k)/2}} \\ &\times e^{2\pi i m_1(a_{n-1,1}x_{1,n}+\cdots+a_{n-1,n-1}x_1)} e^{2\pi i(m_2 x_2^\gamma+\cdots+m_{n-1}x_{n-1}^\gamma)} \\ &\times W_J\left(\begin{pmatrix} m_1\cdots|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1\end{pmatrix}\cdot y^\gamma; \nu(\ell_\infty), 1\right) \end{aligned} \tag{3.13}$$

where

$$\gamma = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} \\ a_{2,1} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{pmatrix} \in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})$$

and $x^\gamma,\ y^\gamma$ *are defined by* $\left(\begin{smallmatrix} \gamma & \\ & 1 \end{smallmatrix}\right) z \equiv x^\gamma \cdot y^\gamma \in \mathbb{H}^n$ *by Iwasawa decomposition, for* $x^\gamma \in N(n,\mathbb{R})$ *as in (2.7) and* $y^\gamma \in A(n,\mathbb{R}^+)$ *as in (2.8).*

(ii) *For any* $\gamma \in P_{n-1,1}(\mathbb{Z}) \cap SL(n,\mathbb{Z})$*, we have*

$$F_{\ell_M}(\gamma z) = F_{\ell_M}(z), \qquad (z \in \mathbb{H}^n). \tag{3.14}$$

(iii) *For* $j = 1, \ldots, n-1$

$$\Delta_n^{(j)} F_{\ell_M} = \lambda_\infty^{(j)}(\ell_\infty) F_{\ell_M},$$

and for any finite prime $q \in M$*,* $j = 1, \ldots, n-1$*,*

$$T_q^{(j)} F_{\ell_M} = \lambda_q^{(j)}(\ell_q) F_{\ell_M},$$

where $\lambda_v^{(j)}(\ell_v)$ *is defined in Definition 3.8 for* $v \in M$*. Moreover,* $\ell_v(F_{\ell_M}) = \ell_v$ *for any* $v \in M$*. For any integer* $1 \leqslant m \leqslant L$*, we have*

$$T_m F_{\ell_M} = A_{\ell_M}(m, 1, \ldots, 1) F_{\ell_M}.$$

Definition 3.18. **(ϵ-closeness)** *Let* $n \geqslant 2$ *and* $\epsilon > 0$*.*

(i) *For* $v \leqslant \infty$*, let* $\pi_v(\ell_v)$ *and* $\pi_v(\sigma_v)$ *be irreducible unramified unitary generic representations of* $GL(n,\mathbb{Q}_v)$ *as in Theorem 3.14 with parameters* $\ell_v, \sigma_v \in \mathfrak{a}^*_{\mathbb{C}}(n)$ *satisfying the condition in Definition 3.13. The representations* $\pi_v(\ell_v)$ *and* $\pi_v(\sigma_v)$ *are* ϵ*-close if*

$$\sum_{j=1}^{m} \left| \lambda_v^{(j)}(\ell_v) - \lambda_v^{(j)}(\sigma_v) \right|^2 < \epsilon \tag{3.15}$$

where $m = n-1$ *for* $v = \infty$ *and* $m = \lfloor \frac{n}{2} \rfloor$ *for* $v < \infty$*.*

(ii) *Let M and M' be sets of primes including ∞ and let ℓ_M and $\sigma_{M'}$ be quasi-automorphic parameters for M and M' respectively as in Definition 3.15. Let $S \subset M \cap M'$ be a finite subset including ∞. The quasi-automorphic parameters ℓ_M and $\sigma_{M'}$ are ϵ-close for S if*

$$\sum_{j=1}^{n-1} \left|\lambda_\infty^{(j)}(\ell_\infty) - \lambda_\infty^{(j)}(\sigma_\infty)\right|^2 + \sum_{\substack{q \in S, \\ \textit{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left|\lambda_q^{(j)}(\ell_q) - \lambda_q^{(j)}(\sigma_q)\right|^2 < \epsilon. \tag{3.16}$$

We obtain a condition for ϵ-closeness with a given quasi-automorphic parameter in the following Lemma. The idea of the lemma and its proof are generalizations of Lemma 1 in [3], 3.1.

Lemma 3.19. *Let $n \geqslant 2$ be an integer and M be a set of places of $\mathbb{Q}$ including ∞. Let ℓ_M be a quasi-automorphic parameter for M as in Definition 3.15. Let $S \subset M$ be a finite subset including ∞. If there exists a smooth function $f \in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$, which is cuspidal, such that*

$$\sum_{j=1}^{n-1} \left|\left| \left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) f \right|\right|_2^2 + \sum_{\substack{q \in S, \\ \textit{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left|\left| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_q)\right) f \right|\right|_2^2 < \epsilon \cdot ||f||_2^2 \tag{3.17}$$

for some $\epsilon > 0$, then there exists an unramified cuspidal automorphic representation $\pi(\sigma)$ as in (3.10) such that the parameters ℓ_M and σ are ϵ-close for S.

Proof. By the spectral decomposition, the space $L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is spanned by Hecke-Maass forms $u_j(z)$ with $||u_j||_2^2 = 1$ for $j = 1, 2, \ldots$. For each u_j there exists an unramified cuspidal automorphic representation $\pi(\sigma_j) = \otimes'_v \pi(\sigma_{j,v})$ such that $\ell_v(u_j) = \ell_v(\sigma_{j,v})$ for any $v \leqslant \infty$.

For any $f \in L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$,

$$f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle \, u_j(z).$$

For $\epsilon > 0$, let

$$\mathcal{U}_\epsilon(\ell_M) := \{u_j \mid \sigma_j \text{ and } \ell_M \text{ are } \epsilon\text{-close for } S\},$$

and define

$$\mathrm{Pr}_\epsilon(f)(z) := \sum_{u_j \in \mathcal{U}_\epsilon(\ell_M)} \langle f, u_j \rangle \, u_j(z) \;\in\; L^2_{\text{cusp}}(SL(n,\mathbb{Z})\backslash\mathbb{H}^n).$$

Assume that f is a smooth automorphic function which satisfies (3.17). Then

$$\begin{aligned}
||\mathrm{Pr}_\epsilon(f)||_2^2 &= ||f||_2^2 - \sum_{u_j \notin \mathcal{U}_\epsilon(\ell_M)} |\langle f, u_j\rangle|^2 \\
&\geqslant ||f||_2^2 \\
&\quad - \sum_{j=1}^{\infty} |\langle f, u_j\rangle|^2 \cdot \frac{1}{\epsilon} \left\{ \sum_{k=1}^{n-1} \left|\lambda_\infty^{(k)}(\sigma_{j,\infty}) - \lambda_\infty^{(k)}(\ell_\infty)\right|^2 + \sum_{\substack{q\in S,\\ \text{finite}}} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left|\lambda_q^{(k)}(\sigma_{j,q}) - \lambda_q^{(j)}(\ell_q)\right|^2 \right\} \\
&= ||f||_2^2 - \frac{1}{\epsilon} \left\{ \sum_{k=1}^{n-1} || \left(\Delta_n^{(k)} - \lambda_\infty^{(k)}(\ell_\infty)\right) f||_2^2 + \sum_{\substack{q\in S,\\ \text{finite}}} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} || \left(T_q^{(k)} - \lambda_q^{(k)}(\ell_q)\right) f||_2^2 \right\} \\
&> 0.
\end{aligned}$$

Therefore, $\mathcal{U}_\epsilon(\ell_M) \neq \varnothing$. □

Definition 3.20. (Automorphic Lifting of Quasi-Maass forms) *Let $n \geqslant 2$ and M be a set of primes including ∞ and let ℓ_M be a quasi-automorphic parameter. Let F_{ℓ_M} be a quasi-Maass form of ℓ_M. Define*

$$\widetilde{F}_{\ell_M}(z) := F_{\ell_M}(\gamma z), \quad (\text{for any } z \in \mathbb{H}^n \text{ and a unique } \gamma \in SL(n,\mathbb{Z}) \text{ such that } \gamma z \in \mathfrak{F}^n). \tag{3.18}$$

We say $\widetilde{F}_{\ell_M}$ is an automorphic lifting of a quasi-Maass form. Here $\mathfrak{F}^n$ is the fundamental domain described in Proposition 2.8.

Remark 3.21. **(i)** *Let $n \geqslant 2$. Define*

$$\widetilde{\mathfrak{F}^n} := \bigcup_{\substack{\gamma \in SL(n-1,\mathbb{Z}),\\ \begin{pmatrix} \gamma & * \\ & 1 \end{pmatrix} \in SL(n,\mathbb{Z})}} \begin{pmatrix} \gamma & * \\ & 1 \end{pmatrix} \mathfrak{F}^n \tag{3.19}$$

where $\mathfrak{F}^n$ *is the fundamental domain described in Proposition 2.8. By (3.14), we have*

$$\widetilde{F}_{\ell_M}(z) = F_{\ell_M}(z), \qquad (z \in \widetilde{\mathfrak{F}^n}). \tag{3.20}$$

(ii) *Since* F_{ℓ_M} *is square-integrable,* $\widetilde{F}_{\ell_M} \in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$. *However* $\widetilde{F}$ *is not continuous and is not cuspidal in general.*

Chapter 4

ANNIHILATING OPERATOR $\natural_p^n$

4.1 Harmonic Analysis for $GL(n,\mathbb{R})/(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$

For vectors $v = (v_1, \ldots, v_n),\ v' = (v'_1, \ldots, v'_n) \in \mathbb{C}^n$, the inner product $\langle\ ,\ \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denotes the usual inner product

$$\langle v, v' \rangle := \sum_{j=1}^{n} v_j \cdot v'_j \in \mathbb{C}.$$

We define the norm $||v|| := \sqrt{\langle v, \overline{v} \rangle}$. For any $w \in W_n$, define

$$w.v := \left(v_{\sigma_w(1)}, \ldots, v_{\sigma_w(n)}\right) \tag{4.1}$$

where σ_w is the permutation on n symbols corresponding to w defined by

$$\begin{pmatrix} v_{\sigma_w(1)} & & \\ & \ddots & \\ & & v_{\sigma_w(n)} \end{pmatrix} = w \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{pmatrix} w^{-1}.$$

Then for any $v, v' \in \mathbb{C}^n$ and $w \in W_n$

$$\langle w.v, w.v' \rangle = \langle v, v' \rangle .$$

For $n \geqslant 2$, $\mathfrak{a}(n)$ is isomorphic to the Lie algebra of $A^1(n, \mathbb{R}^+)$ which is isomorphic to $A^1(n, \mathbb{R}^+) \cong A(n, \mathbb{R})/(\mathbb{R}^\times \cdot (O(n, \mathbb{R}) \cap A(n, \mathbb{R})))$ via the exponential map in (2.17).

Let $\chi : A^1(n, \mathbb{R}^+) \to \mathbb{C}^\times$ be a character. Since $A^1(n, \mathbb{R}^+) \cong A(n, \mathbb{R})/(\mathbb{R}^\times \cdot O(n, \mathbb{R}))$, as discussed in §3.3, there exists $\ell_\infty(\chi) \in \mathbb{C}^n$ such that $\chi\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}\right) = \prod_{j=1}^{n} |a_j|_\infty^{\ell_{\infty,j}(\chi)}$ with $\ell_{\infty,1}(\chi) + \cdots + \ell_{\infty,n}(\chi) = 0$, and it is a one-to-one correspondence. Define

$$\mathfrak{a}^*(n) := \{\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n \mid \ell_1 + \cdots + \ell_n = 0\}, \tag{4.2}$$

$$\mathfrak{a}^*_{\mathbb{C}}(n) := \mathfrak{a}^*(n) + i\mathfrak{a}^*(n) \cong \mathrm{Hom}(\mathfrak{a}_\infty(n), \mathbb{C}^\times).$$

Then $\mathfrak{a}^*_{\mathbb{C}}(n)$ is isomorphic to the group of characters of $A^1(n,\mathbb{R}^+)$ and $i\mathfrak{a}^*(n) \subset \mathfrak{a}^*_{\mathbb{C}}(n)$ is isomorphic to the set of unitary characters of $A^1(n,\mathbb{R}^+)$ and this has the $\mathbb{R}$-vector space structure. For any character $\chi : A^1(n,\mathbb{R}^+) \to \mathbb{C}^\times$, we may write

$$\chi(a) = e^{\langle \ell_\infty(\chi), \ln(a)\rangle}, \qquad (\text{ for } a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in A^1(n,\mathbb{R}^+)).$$

Let

$$\rho = \left(\frac{n-2j+1}{2}\right)_{j=1}^{n} \in \mathfrak{a}^*(n). \tag{4.3}$$

Then for any $a \in A^1(n,\mathbb{R}^+)$, we have

$$e^{\langle \rho, \ln a\rangle} = \delta_\infty(a)^{\frac{1}{2}},$$

where δ_∞ is the modular quasi-character defined in (3.6).

By the Iwasawa decomposition, for any $g \in GL(n,\mathbb{R})$, we have

$$H_{\mathrm{Iwa}}(g) \in \mathfrak{a}(n),\ n(g) \in N(n,\mathbb{R}),\ k(g) \in O(n,\mathbb{R}) \tag{4.4}$$
$$\text{such that } g = |\det g|_\infty^{\frac{1}{n}} \cdot n(g) \cdot \exp(H_{\mathrm{Iwa}}(g)) \cdot k(g).$$

For each $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$ and $g \in GL(n,\mathbb{R})$, we defined the function $\varphi_\ell(g)$ in (3.5). Then

$$\varphi_\ell(g) = e^{\langle \ell+\rho, H_{\mathrm{Iwa}}(g)\rangle} = I_{\nu(\ell)}(g). \tag{4.5}$$

For any $w \in W_n$, the Weyl group, we have

$$\varphi_{\ell-\rho}(wg) = e^{\langle \ell, H_{\mathrm{Iwa}}(wg)\rangle} = e^{\langle \ell, wH_{\mathrm{Iwa}}(g)\rangle} = e^{\left\langle w^{-1}.\ell, H_{\mathrm{Iwa}}(g)\right\rangle} = \varphi_{w.\ell-\rho}(g).$$

This explains the definition of the action of the Weyl group W_n on $\nu \in \mathbb{C}^{n-1}$ in (2.28).

Let $\mathcal{H}\left(\mathfrak{a}_{\mathbb{C}}^{*}(n)\right)^{W_n}$ be the space of holomorphic functions on $\mathfrak{a}_{\mathbb{C}}^{*}(n)$, which are invariant under the action of W_n, the Weyl group of $GL(n,\mathbb{R})$. Define the spherical transform:

$$C_c^{\infty}\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(\mathbb{R}^{\times}\cdot O(n,\mathbb{R}))\right) \hookrightarrow \mathcal{H}\left(\mathfrak{a}_{\mathbb{C}}^{*}(n)\right)^{W_n}$$

Definition 4.1. (Spherical Transform) *For any compactly supported, smooth, bi-$O(n,\mathbb{R})$-invariant function $k \in C_c^{\infty}\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^{\times})\right)$, the spherical transform $\widehat{k}(\ell) \in \mathbb{C}$ is defined as the corresponding eigenvalue of the convolution operator associated to k, i.e.,*

$$\varphi_{\ell} * k(g) = \int_{GL(n,\mathbb{R})/\mathbb{R}^{\times}} \varphi_{\ell}(g\xi^{-1})k(\xi)d\xi = \widehat{k}(\ell)\cdot\varphi_{\ell}(g), \tag{4.6}$$

and

$$\widehat{k}(\ell) = \int_{GL(n,\mathbb{R})/\mathbb{R}^{\times}} e^{\langle \ell+\rho, H_{\mathrm{Iwa}}(\xi)\rangle} k(\xi^{-1})d\xi,$$

where φ_{ℓ} is the eigenfunction of $\mathcal{Z}(\mathcal{D}^n)$ with the parameter ℓ, defined in (4.5).

Definition 4.2. *For each $\ell \in \mathfrak{a}_{\mathbb{C}}^{*}(n)$, define*

$$\beta_{\ell}(g) := \int_{O(n,\mathbb{R})/\mathbb{R}^{\times}} \varphi_{\ell}(\xi g)\, d\xi = \int_{O(n,\mathbb{R})/\mathbb{R}^{\times}} e^{\langle \ell+\rho, H_{\mathrm{Iwa}}(\xi g)\rangle}\, d\xi, \tag{4.7}$$

for any $g \in GL(n,\mathbb{R})$. Then β_{ℓ} is called the spherical function of type ℓ. Moreover, β_{ℓ} is $(\mathbb{R}^{\times}\cdot O(n,\mathbb{R}))$-bi-invariant function. i.e., for any $\xi_1, \xi_2 \in O(n,\mathbb{R})$, we have

$$\beta_{\ell}(\xi_1\cdot g\cdot\xi_2) = \beta_{\ell}(g).$$

The spherical function is again an eigenfunction of the convolution operator whose eigenvalue is the corresponding spherical transform. For any compactly supported, smooth, bi-$O(n,\mathbb{R})$-invariant function $k \in C_c^{\infty}\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(\mathbb{R}^{\times}\cdot O(n,\mathbb{R}))\right)$, we have

$$\beta_{\ell} * k(g) = \widehat{k}(\ell)\beta_{\ell}(g).$$

We recall the following inversion formula for the spherical transform as given in [18]. For any bi-$(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$-invariant, compactly supported smooth function k, we have

$$k(g) = \frac{1}{n!}\int_{i\mathfrak{a}^*(n)} \widehat{k}(\alpha)\beta_\alpha(g)\ \phi_{Planch}(\alpha)d\alpha, \tag{4.8}$$

where

$$\phi_{Planch}(\alpha) = \prod_{1\leqslant k<j\leqslant n}\left|\frac{\Gamma_{\mathbb{R}}(\alpha_k-\alpha_j+1)\Gamma_{\mathbb{R}}(-k+j)}{\Gamma_{\mathbb{R}}(\alpha_k-\alpha_j)\Gamma_{\mathbb{R}}(-k+j+1)}\right|^2, \tag{4.9}$$

for $\alpha = (\alpha_1,\ldots,\alpha_n) \in i\mathfrak{a}^*(n)$, and

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

We recall the Paley-Wiener theorem from [19].

Theorem 4.3. (Paley-Wiener)

(i) *Let $k \in C_c^\infty\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(\mathbb{R}^\times\cdot O(n,\mathbb{R})\right)$, such that $k(g) = 0$ for any $g \in GL(n,\mathbb{R})$ with $||\ln A(g)|| > \delta$ for some $\delta > 0$. Then the spherical transform $\widehat{k} \in \mathcal{H}\left(\mathfrak{a}^*_{\mathbb{C}}(n)\right)^{W_n}$. Moreover, for any integer N, there exists a constant $C_N > 0$ such that*

$$\left|\widehat{k}(\ell)\right| \leqslant C_N\cdot(1+||\ell||)^{-N}\cdot e^{\delta||\Re(\ell)||}$$

for any $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$.

(ii) *Assume that $R_\delta \in \mathcal{H}\left(\mathfrak{a}^*_{\mathbb{C}}(n)\right)^{W_n}$ (for $\delta > 0$) satisfies the following condition. For any integer N there exists a constant $C_N > 0$ such that*

$$|R_\delta(\ell)| \leqslant C_N\cdot(1+||\ell||)^{-N}\cdot e^{\delta||\Re(\ell)||} \tag{4.10}$$

for any $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$. Then there exists $H_\delta \in C_c^\infty\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(\mathbb{R}^\times\cdot O(n,\mathbb{R}))\right)$ with $H_\delta(g) = 0$ for any $g \in GL(n,\mathbb{R})$, $||\ln A(g)|| > \delta$ such that $\widehat{H_\delta} = R_\delta$ and

$$H_\delta(g) = \int_{i\mathfrak{a}^*(n)} R_\delta(\ell)\beta_\ell(g)\ d\mu_{Planch}(\ell).$$

Let $n \geqslant 2$ be an integer and f be a smooth function on $\mathbb{H}^n$. For $D \in \mathcal{Z}(\mathcal{D}^n)$ and any $k \in C_c^\infty\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(\mathbb{R}^\times \cdot O(n,\mathbb{R}))\right)$, since D is invariant under the action of $GL(n,\mathbb{R})$, we have

$$D\left(f * k\right)(z) = (Df) * k(z)$$

if the integral is absolutely convergent. Let S be a Hecke operator and f be a function on $\mathbb{H}^n$. Then

$$S\left(f * k\right)(z) = (Sf) * k(z)$$

when the integral is absolutely convergent. Therefore, the convolution operator associated to the function $k \in C_c^\infty\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/(\mathbb{R}^\times \cdot O(n,\mathbb{R}))\right)$ commutes with the Hecke operators S if the integral is absolutely convergent and also commutes with any $D \in \mathcal{Z}(\mathcal{D}^n)$ if the function is smooth and the integral is absolutely convergent.

Let $D\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/\left(\mathbb{R}^\times \cdot O(n,\mathbb{R})\right)\right)$ be the space of $O(n,\mathbb{R})$-bi-invariant compactly supported distributions on $GL(n,\mathbb{R})/\mathbb{R}^\times$. For any compactly supported distribution $T \in D\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/\left(\mathbb{R}^\times \cdot O(n,\mathbb{R})\right)\right)$, the spherical transform $\widehat{T}(\ell)$ (for any $\ell \in \mathfrak{a}_{\mathbb{C}}^*(n)$) is defined to be the scalar by which T acts on the function φ_ℓ. Furthermore, by [16], for any $R \in \mathcal{H}\left(\mathfrak{a}_\infty^*(n)\right)^{W_n}$ satisfying an inequality

$$|R(\ell)| \leqslant C(1+||\ell||)^N e^{\delta||\Re(\ell)||}, \qquad (\ell \in \mathfrak{a}_{\mathbb{C}}^*(n)) \tag{4.11}$$

for some positive constants C, N and δ, there exists a distribution bi-$(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$-invariant distribution T such that its spherical transform $\widehat{T}(\ell) = R(\ell)$ for any $\ell \in \mathfrak{a}_{\mathbb{C}}^*(n)$.

For any $T \in D\left(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/\left(\mathbb{R}^\times \cdot O(n,\mathbb{R})\right)\right)$ we define the spectral norm

$$||T||_{\text{spec}} := \sup_{\substack{\ell \in \mathfrak{a}_{\mathbb{C}}^*(n), \\ \pi_\infty(\ell(\chi)),\ \text{unitary}}} |\widehat{T}(\ell)|, \qquad \text{(if finite)}. \tag{4.12}$$

4.2 Annihilating operator $\natural_p^n$

The annihilating operator maps $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right) \rightarrow L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$, and has the property that it has a purely cuspidal image.

Lemma 4.4. (Construction of $\natural_p^n$) *Let $n \geqslant 2$ and fix a prime p. For any $\ell_1 = (\ell_{1,1}, \ldots, \ell_{1,n})$, $\ell_2 = (\ell_{2,1}, \ldots, \ell_{2,n}) \in \mathfrak{a}^*_{\mathbb{C}}(n)$, define*

$$\widehat{\natural_p^n}\left(\ell_1, \ell_2\right) := \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \prod_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(1 - p^{-(\ell_{1,i_1} + \cdots + \ell_{1,i_k}) - (\ell_{2,j_1} + \cdots + \ell_{2,j_k})}\right). \quad (4.13)$$

Then there exists an operator denoted $\natural_p^n$, which is a polynomial in convolution operators (associated to some compactly supported bi-$(O(n,\mathbb{R}) \cdot \mathbb{R}^\times)$-distributions), and in Hecke operators at p, satisfies

$$\natural_p^n f(z) = \widehat{\natural_p^n}(\ell_\infty(f), \ell_p(f)) \cdot f(z), \qquad (z \in \mathbb{H}^n).$$

Here f is a smooth function on $\mathbb{H}^n$ which is also an eigenfunction of $\mathcal{Z}(\mathcal{D}^n)$ and the Hecke operators at p. The parameter $\ell_\infty(f) = \ell_\infty(\nu)$, as in (2.30), since f is an eigenfunction of type $\nu \in \mathbb{C}^{n-1}$, and the parameter $\ell_p(f)$ is defined in (2.49).

Remark 4.5. *Before proving this Lemma we give an example of $\natural_p^n$ for the cases $n = 2$ and $n = 3$.*

(i) *For $n = 2$, we have*

$$\natural_p^2 = T_{p^2} + T_p^2 - 2T_p\mathcal{L}_\kappa + 1 \quad (4.14)$$

where $\mathcal{L}_\kappa$ is the convolution operator associated to the distribution κ such that $\widehat{\kappa}(\ell) = p^{\ell_1} + p^{\ell_2}$ for any $\ell = (\ell_1, \ell_2) \in \mathfrak{a}^*_{\mathbb{C}}(2)$. This operator satisfies $\natural_p^2 = \aleph^2$ for the operator $\aleph$ constructed in 2, [19].

(ii) *Let $n = 3$. For $j = 1, 2, 3$, define the compactly supported bi-$\mathbb{R}^\times \cdot O(n, \mathbb{R})$-distributions $\kappa_{\pm j}$ such that*

$$\widehat{\kappa_1}(\ell) = p^{\ell_1} + p^{\ell_2} + p^{\ell_3}, \qquad \widehat{\kappa_{-1}}(\ell) = p^{-\ell_1} + p^{-\ell_2} + p^{-\ell_3},$$

$$\widehat{\kappa_2}(\ell) = -\widehat{\kappa_{-1}}(\ell)^2 + 3\widehat{\kappa_1}(\ell), \qquad \widehat{\kappa_{-2}}(\ell) = \widehat{\kappa_1}(\ell)^2 - 3\widehat{\kappa_{-1}}(\ell),$$

$$\widehat{\kappa_3(\ell)} = -\widehat{\kappa_2}(\ell) \cdot \widehat{\kappa_1}(\ell), \quad \textit{and} \quad \widehat{\kappa_{-3}}(\ell) = -\widehat{\kappa_{-2}}(\ell) \cdot \widehat{\kappa_{-1}}(\ell),$$

for any $\ell = (\ell_1, \ell_2, \ell_3) \in \mathfrak{a}^*_{\mathbb{C}}(n)$. Then

$$\natural_p^3 = T_p \mathcal{L}_{\kappa_3} + T_p^2 \mathcal{L}_{\kappa_2} - T_p^3 - T_p (T_p^{(2)})^2 \mathcal{L}_{\kappa_1} + T_p^2 T_p^{(2)} \mathcal{L}_{\kappa_{-1}} + (T_p^{(2)})^2 \mathcal{L}_{\kappa_{-2}} + (T_p^{(2)})^3 + T_p^{(2)} \mathcal{L}_{\kappa_{-3}}. \tag{4.15}$$

Proof for Lemma 4.4. For any $w_1, w_2 \in W_n$ (the Weyl group of $GL(n, \mathbb{R})$), we have

$$\widehat{\natural_p^n}(w_1.\ell_1, w_2.\ell_2) = \widehat{\natural_p^n}(\ell_1, \ell_2),$$

where $\widehat{\natural_p^n}(\ell_1, \ell_2)$ is holomorphic and satisfies the condition (4.11) for both $\ell_1, \ell_2 \in \mathfrak{a}^*_{\mathbb{C}}(n)$.

For each $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, consider the polynomial

$$\prod_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \left(1 - x p^{-(\ell_{j_1} + \cdots + \ell_{j,k})}\right)$$
$$= 1 - B_{1,k}(\ell) x + \cdots + (-1)^r B_{r,k}(\ell) x^r + \cdots + (-1)^{d_k(n)} x^{d_k(n)}$$

for any $\ell = (\ell_1, \ldots, \ell_n) \in \mathfrak{a}^*_{\mathbb{C}}(n)$, where $d_k(n) = \frac{n!}{k!(n-k)!}$. For each $1 \leqslant r \leqslant d_k(n) - 1$, the coefficients

$$B_{r,k}(\ell) \in \mathcal{H}\left(\mathfrak{a}^*_{\mathbb{C}}(n)\right)^{W_n}$$

satisfy (4.11). For $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, we have,

$$\prod_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(1 - p^{-(\ell_{1,i_1} + \cdots + \ell_{1,i_k}) - (\ell_{2,j_1} + \cdots + \ell_{2,j_k})}\right)$$
$$= \sum_{j=0}^{\tilde{d}_k(n)} a_{j,k}(\ell_1) \cdot b_{j,k}(\ell_2)$$

for $\ell_1 = (\ell_{1,1}, \ldots, \ell_{1,n}),\ \ell_2 = (\ell_{2,1}, \ldots, \ell_{2,n}) \in \mathfrak{a}_{\mathbb{C}}^*(n)$ and some positive integer $\widetilde{d}_k(n)$. So,

$$\begin{aligned}
&\sum_{j=0}^{\widetilde{d}_k(n)} a_{j,k}(\ell_1) \cdot b_{j,k}(\ell_2) \\
&= \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(\sum_{r=0}^{d_k(n)} B_{r,k}(\ell_1) p^{-r(\ell_{2,i_1} + \cdots + \ell_{2,i_k})} \right) \\
&= \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(\sum_{r=0}^{d_k(n)} B_{r,k}(\ell_2) p^{-r(\ell_{1,i_1} + \cdots + \ell_{1,i_k})} \right).
\end{aligned}$$

For $1 \leqslant j \leqslant \widetilde{d}_k(n)$, we have,

$$a_{j,k}(\ell_1),\ b_{j,k}(\ell_2) \in \mathcal{H}\left(\mathfrak{a}_{\mathbb{C}}^*(n)\right)^{W_n}$$

satisfies (4.11) because $a_{j,k}(\ell_1)$ (resp. $b_{j,k}(\ell_2)$) is a polynomial in $B_{r,k}(\ell_1)$ (resp. $B_{r,k}(\ell_2)$) (for $1 \leqslant r \leqslant d_k(n)$). So there exist compactly supported bi-($\mathbb{R}^\times \cdot O(n, \mathbb{R})$)-invariant distributions $\kappa_j^{(k)}$ whose spherical transform is $a_{j,k}(\ell_1)$. For each $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$ and $1 \leqslant j \leqslant \widetilde{d}_k(n)$, let $\mathcal{L}_{\kappa_j^{(k)}}$ be the convolution operator associated to the distribution $\kappa_j^{(k)}$.

We also have the p-adic version of Theorem 4.3 as explained in [19] (also see [9]). So there exist Hecke operators $S_j^{(k)}$ such that

$$S_j^{(k)} f = b_{j,k}(\ell_p(f)) \cdot f$$

where f is an eigenfunction of Hecke operators with parameter $\ell_p(f) \in \mathfrak{a}_{\mathbb{C}}^*(n)$.

Therefore,

$$\natural_p^n = \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{j=0}^{\widetilde{d}_k(n)} S_j^{(k)} \mathcal{L}_{\kappa_j^{(k)}} \right)$$

and

$$\natural_p^n f = \widehat{\natural}_p^n(\ell_\infty(f), \ell_p(f)) \cdot f$$

where f is an eigenfunction of Casimir operators and the Hecke operators. □

Since we use distributions to define $\natural_p^n$, the operator is well defined in the space of smooth functions. For any $\delta > 0$, let

$$U(\delta) := \{g \in GL(n, \mathbb{R}) \mid ||\ln A(g)|| \leqslant \delta\} . \tag{4.16}$$

Let H_δ be a bi-$(\mathbb{R}^\times \cdot O(n, \mathbb{R}))$-invariant compactly supported smooth function with $\mathrm{supp}(H_\delta) \subset U(\delta)$, i.e., $H_\delta(g) = 0$ for any $g \notin U(\delta)$. We define the operator $H_\delta \natural_p^n$ to be

$$H_\delta \natural_p^n F = \natural_p^n (F * H_\delta) \tag{4.17}$$

for a function $F : \mathbb{H}^n \to \mathbb{C}$ which makes the integral convergent. By the Paley-Wiener Theorem 4.3, the operator $H_\delta \natural_p^n$ is a polynomial in convolution operators (associated to the bi-$(\mathbb{R}^\times \cdot O(n, \mathbb{R}))$-invariant, compactly supported smooth functions), and in Hecke operators at the prime p. Then the operator $H_\delta \natural_p^n$ can be defined for the functions in $L^2(\mathbb{H}^n)$ and

$$\widehat{H_\delta \natural_p^n}(\ell_1, \ell_2) = \widehat{H_\delta}(\ell_1) \cdot \widehat{\natural_p^n}(\ell_1, \ell_2) \tag{4.18}$$

where $\ell_j = (\ell_{j,1}, \ldots, \ell_{j,n}) \in \mathfrak{a}_{\mathbb{C}}^*(n)$ for $j = 1, 2$.

Proposition 4.6. *Let $n \geqslant 2$ and p be a prime. Let $E(z)$ be an Eisenstein series as in Definition 2.28. Then*

$$\widehat{\natural_p^n}(\ell_\infty(E), \ell_p(E)) = 0 \quad \textit{and} \quad \natural_p^n E \equiv 0 \tag{4.19}$$

for any prime p. Let ϕ be a self-dual Hecke-Maass form as in Proposition 2.26. Then

$$\widehat{\natural_p^n}(\ell_\infty(\phi), \ell_p(\phi)) = 0 \quad \textit{and} \quad \natural_p^n \phi \equiv 0. \tag{4.20}$$

Moreover, for any constant $C \in \mathbb{C}$,

$$\natural_p^n C \equiv 0. \tag{4.21}$$

Proof. Let $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1, \ldots, n_r < n$ and $r \geqslant 2$. For each $i = 1, \ldots, r$ let ϕ_i be either a Hecke-Maass form for $SL(n_i, \mathbb{Z})\backslash\mathbb{H}^{n_i}$ of type $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,n_i-1}) \in \mathbb{C}^{n_i-1}$ or a constant with $\mu_i = (0, \ldots, 0)$. Let $t = (t_1, \ldots, t_r) \in \mathbb{C}^r$ with $n_1t_1 + \cdots + n_rt_r = 0$. Let $E(z) := E_{P_{n_1,\ldots,n_r}}(z; t; \phi_1, \ldots, \phi_r)$ be an Eisenstein series as in Definition 2.28. Let $\eta_1 = 0$ and $\eta_i = n_1 + \cdots + n_{i-1}$ for $i = 1, \ldots, r$. By Lemma 2.32, we have

$$\sum_{j=\eta_i+1}^{\eta_i+n_i} \ell_{\infty,j}(E) = \left(\frac{n_i - n}{2} + t_i + \eta_i\right) n_i = -\sum_{j=\eta_i+1}^{\eta_i+n_i} \ell_{p,j}(E)$$

for any prime p. Therefore,

$$1 - p^{-(\ell_{\infty,\eta_i+1}(E)+\cdots+\ell_{\infty,\eta_i+n_i}(E))-(\ell_{p,\eta_i+1}(E)+\cdots+\ell_{p,\eta_i+n_i}(E))} = 0$$

and $\widehat{\natural}_p^n(\ell_\infty(E), \ell_p(E)) = 0$.

Let ϕ be a self-dual Maass form for $SL(n, \mathbb{Z})$. Then by Remark 2.27,

$$\ell_v(\phi) = -\ell_v(\phi)$$

up to permutations, for any place $v \leqslant \infty$. So either there exists $1 \leqslant j \leqslant n$ such that $\ell_{v,j}(\phi) = 0$ or there exist $1 \leqslant j \neq j' \leqslant n$ such that $\ell_{v,j}(\phi) + \ell_{v,j'}(\phi) = 0$. Therefore $\widehat{\natural}_p^n(\ell_\infty(\phi), \ell_p(\phi)) = 0$.

Let $C \in \mathbb{C}$ be a constant. Then

$$C = C \cdot I_0(z)$$

for any $z \in \mathbb{H}^n$ and

$$\ell_\infty(C) = \left(-\frac{n - 2j + 1}{2}\right)_{j=1}^{n} \quad \text{and} \quad \ell_p(C) = \left(\frac{n - 2j + 1}{2}\right)_{j=1}^{n}.$$

So $1 - p^{-\ell_{\infty,j}(C)-\ell_{p,j}(C)} = 0$ for any $j = 1, \ldots, n$. Therefore, $\widehat{\natural}_p^n(\ell_\infty(C), \ell_p(C)) = 0$. $\square$

The idea of the following theorem and its proof is in [19].

Theorem 4.7. *Let* $n \geqslant 2$ *be an integer and* p *be a prime. Let* $\delta > 0$ *and* $H_\delta \neq 0$ *be a bi-*$(\mathbb{R}^\times \cdot O(n, \mathbb{R}))$*-invariant compactly supported smooth function with* $\operatorname{supp}(H_\delta) \subset U(\delta)$*. Then the space of the image of* $H_\delta \natural_p^n$ *on* $L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n)$ *is cuspidal and infinite dimensional. So there are infinitely many non self-dual Hecke-Maass forms.*

Proof. The Langlands spectral decomposition states that

$$\begin{aligned} & L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n) \\ & = L^2_{\text{cont}}(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n) \oplus L^2_{\text{residue}}(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n) \oplus L^2_{\text{cusp}}(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n) \end{aligned}$$

where L^2_{cusp} denote the space of Maass forms, L^2_{residue} consists of iterated residues of Eisenstein series and L^2_{cont} is the space spanned by integrals of Eisenstein series. The Eisenstein series are studied in §2.6. So, for any $f \in L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n)$ there exists $f_{\text{cont}}(z) \in L^2_{\text{cont}}$, $f_{\text{residue}}(z) \in L^2_{\text{residue}}$ and $f_{\text{cusp}}(z) \in L^2_{\text{cusp}}$ such that

$$f(z) = f_{\text{cont}}(z) + f_{\text{residue}}(z) + f_{\text{cusp}}(z).$$

By Proposition 4.6, for any Eisenstein series E and constant C, we have $\natural_p^n E = \natural_p^n C \equiv 0$. Since the invariant integral operators and Hecke operators preserve the space of cuspidal functions,

$$H_\delta \natural_p^n f = H_\delta \natural_p^n f_{\text{cusp}} \in L^2_{\text{cusp}}(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n).$$

Therefore the image of $H_\delta \natural_p^n$ on $L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n)$ is cuspidal.

We will show that the image of $H_\delta \natural_p^n$ on $L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n)$ is infinite dimensional. First we show that it is non-zero. Take $\alpha_\infty = (\alpha_{\infty,1}, \ldots, \alpha_{\infty,n})$, $\alpha_p = (\alpha_{p,1}, \ldots, \alpha_{p,n}) \in \mathfrak{a}^*_{\mathbb{C}}(n)$ such that $\widehat{H_\delta}(\alpha_\infty) \cdot \widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \neq 0$ and α_∞ and α_p satisfies the condition in Definition 3.13.

As in Definition 3.16, we construct a quasi-Maass form F of type $\nu(\alpha_\infty)$ for $\{\infty, p\}$ and of length $L = \infty$ such that

$$F(z) = \sum_{\gamma \in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{k_1,\ldots,k_{n-1} \geqslant 0} \frac{A_F\left(p^{k_1}, \ldots, p^{k_{n-1}}\right)}{p^{\frac{1}{2}\sum_{j=1}^{n-1} k_j(n-j)j}} \cdot W_J\left(\begin{pmatrix} p^{k_1+\cdots+k_{n-1}} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z; \nu(\alpha_\infty), 1\right).$$

where

$$A_F(\underbrace{1, \ldots, 1, p}_{j}, 1, \ldots, 1) = \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n} p^{-(\alpha_{p,k_1} + \cdots + \alpha_{p,k_j})}, \text{ (for } j = 1, \ldots, n-1)$$

and $A_F(p^{k_1}, \ldots, p^{k_{n-1}})$ satisfies the multiplicative condition (2.46) and (2.47). Then

$$H_\delta \natural_p^n F(z) = \widehat{H_\delta}(\alpha_\infty) \cdot \widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \cdot F(z)$$

for $z \in \mathbb{H}^n$.

Let $\widetilde{F}$ be the automorphic lifting of F as in Definition 3.20. Then $\widetilde{F} \in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ and $H_\delta \natural_p^n \widetilde{F} \in C^\infty \cap L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is cuspidal as we show above. To show that $H_\delta \natural_p^n \widetilde{F} \neq 0$, we need Lemma below.

Let

$$\Sigma_T := \left\{ \begin{pmatrix} 1 & x_{1,2} & \ldots & x_{1,n} \\ & 1 & \ldots & x_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \in \mathbb{H}^n \;\middle|\; \begin{matrix} y_i > T, \\ \text{for } i = 1, \ldots, n-1 \end{matrix} \right\} \tag{4.22}$$

then

$$\Sigma_T \subset \bigcup_{\gamma \in P_{n-1,1}(\mathbb{R}) \cap SL(n,\mathbb{Z})} \gamma \mathfrak{F}^n \qquad (T > 1).$$

Lemma 4.8. *Let*

$$T > \max\left\{\exp(4\delta),\ \exp\left(\frac{n!\ln p}{2\left(\lfloor\frac{n}{2}\rfloor - 1\right)!\left(n - \lfloor\frac{n}{2}\rfloor\right)!}\right)\right\} \tag{4.23}$$

then for any $z \in \Sigma_T$,

$$H_\delta \natural_p^n \widetilde{F}(z) = \widehat{H_\delta}(\alpha_\infty)\widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \cdot F(z).$$

By Lemma 4.8, for any $z \in \Sigma_T$, and for any T as in (4.23), we have

$$H_\delta \natural_p^n \widetilde{F}(z) = H_\delta \natural_p^n F(z) = \widehat{H_\delta}(\alpha_\infty) \cdot \widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \cdot F(z) \neq 0.$$

So $H_\delta \natural_p^n \widetilde{F} \neq 0$. Therefore, the image of $H_\delta \natural_p^n$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is not empty.

Assume that the space of image of $\natural_p^n$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is finite dimensional. Let

$$\natural_p^n\mathcal{U} := \left\{u_j,\ \text{a Hecke-Maass form of type } \mu_j \in \mathbb{C}^{n-1} \ \middle|\ \begin{array}{c}\natural_p^n u_j \neq 0, \\ \text{and } ||u||_2^2 = 1\end{array}\right\}$$
$$\neq \varnothing.$$

Then it is the basis of the space of image of $H_\delta \natural_p^n$. Since we assume that it is finite dimensional, it follows that $\natural_p^n\mathcal{U}$ is a finite set. Suppose that the number of elements of $\natural_p^n\mathcal{U}$ is $B < \infty$, where B is the positive integer and

$$\natural_p^n\mathcal{U} = \{u_1, \ldots, u_B\}.$$

Then there are $c_1, \ldots, c_B \in \mathbb{C}$ such that

$$H_\delta \natural_p^n \widetilde{F}(z) = \sum_{j=1}^{B} c_j u_j(z). \tag{4.24}$$

Compare Fourier coefficients on both sides. For nonnegative integers $k_1, \ldots, k_{n-1}$, the $(p^{k_1}, \ldots, p^{k_{n-1}})$th Whittaker-Fourier coefficient for $H_\delta \natural_p^n \widetilde{F}$ is

$$W_{H_\delta \natural_p^n \widetilde{F}} \left(z; p^{k_1}, \ldots, p^{k_{n-1}}\right) = \int_0^1 \cdots \int_0^1 H_\delta \natural_p^n \widetilde{F} \left(\begin{pmatrix} 1 & v_{1,2} & \ldots & v_{1,n} \\ & 1 & \ldots & v_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} z \right) \cdot e^{-2\pi i (p^{k_1} v_{n-1,n} + p^{k_2} v_{n-2,n-1} + \cdots + p^{k_1} v_{1,2})} \, dv_{n-1,n} \cdots dv_{1,2}.$$

For each $j = 1, \ldots, B$, the Hecke-Maass form u_j is of type $\mu_j \in \mathbb{C}^{n-1}$, and let $A_j(p^{k_1}, \ldots, p^{k_{n-1}}) \in \mathbb{C}$ be the $(p^{k_1}, \ldots, p^{k_{n-1}})$th Fourier coefficient of u_j. By (4.24) we have

$$W_{H_\delta \natural_p^n \widetilde{F}} \left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; p^{k_1}, \ldots, p^{k_{n-1}} \right) = \sum_{j=1}^{B} \frac{c_j \cdot A_j\left(p^{k_1}, \ldots, p^{k_{n-1}}\right)}{p^{\frac{1}{2}\sum_{i=1}^{n-1} k_i (n-i) i}} \cdot W_J \left(\begin{pmatrix} p^{k_1 + \cdots + k_{n-1}} & & \\ & \ddots & \\ & & 1 \end{pmatrix} z; \mu_j, 1 \right)$$

for any $z \in \mathbb{H}^n$. For $z \in \Sigma_T$, we have

$$\begin{aligned} W_{H_\delta \natural_p^n \widetilde{F}} &\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; p^{k_1}, \ldots, p^{k_{n-1}} \right) \\ &= \widehat{H_\delta}(\alpha_\infty) \cdot \widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \cdot W_J \left(\begin{pmatrix} p^{k_1 + \cdots + k_{n-1}} & & \\ & \ddots & \\ & & 1 \end{pmatrix} z; \nu(\alpha_\infty), 1 \right) \\ &= \sum_{j=1}^{B} \frac{c_j \cdot A_j\left(p^{k_1}, \ldots, p^{k_{n-1}}\right)}{p^{\frac{1}{2}\sum_{i=1}^{n-1} k_i (n-i) i}} \cdot W_J \left(\begin{pmatrix} p^{k_1 + \cdots + k_{n-1}} & & \\ & \ddots & \\ & & 1 \end{pmatrix} z; \mu_j, 1 \right), \end{aligned}$$

and $\widehat{H_\delta}(\alpha_\infty) \cdot \widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \neq 0$ by our assumption. Fix $k_1 = \cdots = k_{n-1} = 0$. Since B is a finite positive integer, it is possible to assume that $\nu(\alpha_\infty) \neq \mu_j$ for $j = 1, \ldots, B$. Then there are $c'_1, \ldots, c'_B \in \mathbb{C}$ such that for $y_1, \ldots, y_{n-1} > T$, and

$$W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \nu, 1\right) = \sum_{j=1}^{B} c'_j \cdot W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \mu_j, 1\right)$$

for at least one $c'_j \neq 0$ (for $j = 1, \ldots, B$). Assume that $c'_1 \neq 0$. Since W_J is an eigenfunction of $\Delta_n^{(i)}$, for $y_1, \ldots, y_{n-1} > T$ and for any $i = 1, \ldots, n-2$, we have

$$\left(\Delta_n^{(i)} - \lambda_\infty^{(i)}(\alpha_\infty)\right) W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \nu(\alpha_\infty), 1\right) \equiv 0,$$

so

$$\sum_{j=1}^{B} c'_j \cdot \left(\lambda_\infty^{(i)}(\ell_\infty(\mu_j)) - \lambda_\infty^{(i)}(\alpha_\infty)\right) \cdot W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \mu_j\right) \equiv 0,$$

where $\lambda_\infty^{(i)}(\alpha_\infty)$ and $\lambda_\infty^{(j)}(\ell_\infty(\mu_j))$ (for $j = 1, \ldots, B$ and $i = 1, \ldots, n-1$) are eigenvalues of $\Delta_n^{(i)}$ as in (2.34). Since we assume that $\nu(\alpha_\infty) \neq \mu_1, \ldots, \mu_B$, there exists $i = 1, \ldots, n-1$ such that

$$\lambda_\infty^{(i)}(\ell_\infty(\mu_j)) - \lambda_\infty^{(i)}(\alpha_\infty) \neq 0, \qquad (\text{ for } j = 1, \ldots, B).$$

Again, there exist $c''_2, \ldots, c''_M \in \mathbb{C}$ such that

$$W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \mu_1, 1\right) = \sum_{j=2}^{B} c''_j \cdot W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \mu_j, 1\right)$$

for $y_1, \ldots, y_{n-1} > T$ and $c''_j \neq 0$ for at least one $j = 2, \ldots, B$. So in a similar manner, we deduce that there exists $\mu \in \{\mu_1, \ldots, \mu_B\}$ such that

$$W_J\left(\begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}; \mu, 1\right) \equiv 0$$

for any $y_1, \ldots, y_{n-1} > T$. This gives a contradiction. Therefore, $\natural_p^n \mathcal{U}$ should be an infinite set. It follows that the image of $H_\delta \natural_p^n$ on $L^2\left(SL(n, \mathbb{Z})\backslash \mathbb{H}^n\right)$ is infinite dimensional. $\square$

To complete the proof of the Theorem, we give a proof of Lemma 4.8.

Proof for Lemma 4.8. Let κ be a compactly supported function with support in $U(\delta)$. Let $t > \exp(4\delta)$. For any $z \in \Sigma_t$, assume that $||\ln A(zh^{-1})|| \leqslant \delta$ for some $h \in GL(n, \mathbb{R})$. By Iwasawa decomposition,

$$z = x \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

for $x \in N(n, \mathbb{R}), y_1, \ldots, y_{n-1} > 0$ and

$$h = \left(\frac{|\det(h)|}{\prod_{j=1}^{n-1} v_j^{n-j}} \right)^{\frac{1}{n}} \cdot u \begin{pmatrix} v_1 \cdots v_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix} k,$$

for $u \in N(n, \mathbb{R}), v_1, \ldots, v_{n-1} > 0$ and $k \in O(n, \mathbb{R})$. Then by Lemma 2.5, for $j = 1, \ldots, n-1$, we have

$$\exp(-4\delta) \leqslant \frac{y_j}{v_j} \leqslant \exp(4\delta).$$

So,

$$v_j \geqslant y_j \cdot \exp(-4\delta) \geqslant t \cdot \exp(-4\delta) > 1.$$

Then $\widetilde{F}(h) = F(h)$ because $\Sigma_1 \subset \bigcup_{\gamma \in P_{n-1,1}(\mathbb{R}) \cap SL(n,\mathbb{Z})} \gamma \mathfrak{F}^n$. So for $z \in \Sigma_t$, we have

$$\begin{aligned} \widetilde{F} * \kappa(z) &= \int_{GL(n,\mathbb{R})/\mathbb{R}^\times} \widetilde{F}(h) \kappa(zh^{-1}) \, dh \\ &= \int_{GL(n,\mathbb{R})/\mathbb{R}^\times} F(h) \kappa(zh^{-1}) \, dh = F * \kappa(z) = \widehat{\kappa}(\alpha_\infty) \cdot F(z). \end{aligned}$$

Let $T \in \mathbb{R}$ satisfies (4.23). For a non-negative integer $B \leqslant \exp\left(\frac{n! \ln p}{2\left(\lfloor \frac{n}{2} \rfloor - 1\right)!\left(n - \lfloor \frac{n}{2} \rfloor\right)!} \right)$, and for any $z \in \Sigma_T$,

$$T_{p^B} \widetilde{F}(z) = A_F(p^B, 1, \ldots, 1) \cdot F(z).$$

The operator $H_\delta \natural_p^n$ is a polynomial in Hecke operators and convolution operators associated with compactly supported functions which have support in $U(\delta)$. By combining the above computations, for any $z \in \Sigma_T$, we obtain

$$H_\delta \natural_p^n \widetilde{F}(z) = \widehat{H_\delta}(\alpha_\infty) \cdot \widehat{\natural_p^n}(\alpha_\infty, \alpha_p) \cdot F(z).$$

$\square$

Lemma 4.9. *Let $n \geqslant 2$ and p be a prime. Then*

$$||\natural_p^n f||_2^2 \leqslant \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}} \right)^{4d_k(n)} \cdot ||f||_2^2 \tag{4.25}$$

for any $f \in C^\infty \cap L^2 \left(SL(n,\mathbb{Z}) \backslash \mathbb{H}^n \right)$. Here $d_k(n) = \frac{n!}{k!(n-k)!}$ for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Moreover, for any $H_\delta \in C_c^\infty \left(O(n,\mathbb{R}) \backslash GL(n,\mathbb{R}) / (\mathbb{R}^\times \cdot O(n,\mathbb{R})) \right)$ and $\delta > 0$, there exists a positive real number $C_{H_\delta} < \infty$ such that

$$||H_\delta \natural_p^n f||_2^2 \leqslant C_{H_\delta}^2 \cdot \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}} \right)^{4d_k(n)} \cdot ||f||_2^2 \quad , \tag{4.26}$$

for any $f \in L^2 \left(SL(n,\mathbb{Z}) \backslash \mathbb{H}^n \right)$.

Proof. Since $\natural_p^n$ kills the continuous part, we only need to consider cuspidal functions. For any cuspidal function $f \in C^\infty \cap L^2 \left(SL(n,\mathbb{Z}) \backslash \mathbb{H}^n \right)$, we have

$$f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle \, u_j(z) \quad ,$$

where $u_j(z)$'s are Hecke-Maass forms for $SL(n,\mathbb{Z})$ of type μ_j with $||u_j||_2^2 = 1$. So

$$||\natural_p^n f||_2^2 \leqslant \sum_{n=1}^{\infty} \left| \widehat{\natural_p^n}(\ell_\infty(u_j), \ell_p(u_j)) \right|^2 |\langle f, u_j \rangle|^2 .$$

If there exists a constant $A > 0$ such that $\left| \widehat{\natural_p^n}(\ell_\infty(u_j), \ell_p(u_j)) \right| \leqslant A$ for any u_j, then

$$||\natural_p^n f||_2^2 \leqslant A^2 \cdot ||f||_2^2.$$

By (4.13),

$$\left|\widehat{\natural}_p^n(\ell_1,\ell_2)\right| = \left|\prod_{k=1}^{\lfloor\frac{n}{2}\rfloor}\prod_{1\leqslant j_1<\cdots<j_k\leqslant n}\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(1-p^{-(\ell_{1,i_1}+\cdots+\ell_{1,i_k})-(\ell_{2,j_1}+\cdots+\ell_{2,j_k})}\right)\right| \tag{4.27}$$

$$\leqslant \prod_{k=1}^{\lfloor\frac{n}{2}\rfloor}\prod_{1\leqslant j_1<\cdots<j_k\leqslant n}\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\left|1-p^{-(\ell_{1,i_1}+\cdots+\ell_{1,i_k}+\ell_{2,j_1}+\cdots+\ell_{2,j_k})}\right|$$

$$\leqslant \prod_{k=1}^{\lfloor\frac{n}{2}\rfloor}\prod_{1\leqslant j_1<\cdots<j_k\leqslant n}\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(1+p^{-\Re(\ell_{1,i_1}+\cdots+\ell_{1,i_k}+\ell_{2,j_1}+\cdots+\ell_{2,j_k})}\right)$$

for any $\ell_1,\ell_2 \in \mathfrak{a}_{\mathbb{C}}^*(n)$. Recall the following theorem from [12].

Theorem 4.10. (Luo-Rudnick-Sarnak) *Fix an integer $n \geqslant 2$. Let f be a Hecke-Maass form for $SL(n,\mathbb{Z})$. Then for $j = 1,\ldots,n$ and any prime $v \leqslant \infty$, including ∞,*

$$\Re\left(\ell_{v,j}(f)\right) \leqslant \frac{1}{2} - \frac{1}{n^2+1}. \tag{4.28}$$

For $\ell_1 = \ell_\infty(u)$ and $\ell_2 = \ell_p(u)$ for any Hecke-Maass forms u, the last line of the (4.27) is less than or equal to

$$\prod_{k=1}^{\lfloor\frac{n}{2}\rfloor}\prod_{j=1}^{2\cdot d_k(n)}\left(1+p^{x_j^{(k)}}\right),$$

where $x_j^{(k)} \leqslant \frac{k(n^2-1)}{n^2+1}$ and $\sum_{j=1}^{2\cdot d_k(n)} x_j^{(k)} = 0$. So for each $k = 1,\ldots,\lfloor\frac{n}{2}\rfloor$,

$$\prod_{j=1}^{2\cdot d_k(n)}\left(1+p^{x_j^{(k)}}\right) = p^{-(\frac{x_1^{(k)}}{2}+\cdots+\frac{x_{2\cdot d_k(n)}^{(k)}}{2})}\cdot\prod_{j=1}^{2\cdot d_k(n)}\left(1+p^{x_j^{(k)}}\right)$$

$$= \prod_{j=1}^{2\cdot d_k(n)}\left(p^{-\frac{x_j^{(k)}}{2}}+p^{\frac{x_j^{(k)}}{2}}\right)$$

$$\leqslant \prod_{j=1}^{2\cdot d_k(n)}\left(p^{-\frac{k(n^2-1)}{n^2+1}}+p^{\frac{k(n^2-1)}{n^2+1}}\right),$$

since $p^{\frac{k(n^2-1)}{n^2+1}} > 1$. Therefore, for any Hecke-Maass form u,

$$\left|\widehat{\natural}_p^n(\ell_\infty(u),\ell_p(u))\right| \leqslant \prod_{k=1}^{n}\left(p^{-\frac{k(n^2-1)}{n^2+1}}+p^{\frac{k(n^2-1)}{n^2+1}}\right)^{2\cdot d_k(n)}.$$

By Theorem 4.3, for any Hecke-Maass form u, we have

$$\left|\widehat{H_\delta}(\ell_\infty(u))\right| \ll (1+||\ell_\infty(u)||)^{-1} e^{\delta||\Re(\ell_\infty(u))||}.$$

By Theorem 4.10, $||\Re(\ell_\infty(u))||$ is bounded. So there exists a real positive constant C_{H_δ} such that

$$\left|\widehat{H_\delta}(\ell_\infty(u))\right| \leqslant C_{H_\delta}$$

for any Hecke-Maass form u. Since $H_\delta \natural_p^n f$ is cuspidal for any $f \in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ it follows that

$$||H_\delta \natural_p^n f||_2^2 \leqslant C_{H_\delta}^2 \cdot \prod_{k=1}^{n} \left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}}\right)^{4\cdot d_k(n)} \cdot ||f||_2^2 \;\;.$$

□

4.3 Example for H_δ

For any $g \in GL(n,\mathbb{R})$, by (2.13), we have

$$g = |\det g|^{\frac{1}{n}}\, k_1 \cdot A(g) \cdot k_2, \qquad (\text{ for } k_1,\ k_2 \in O(n,\mathbb{R})),$$

then define

$$u(g) := \frac{1}{n}\mathrm{tr}(A(g)^2) - 1 = \frac{1}{n}\left(e^{2a_1} + \cdots + e^{2a_n}\right) - 1, \tag{4.29}$$

where $A(g) = \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n}\end{pmatrix}$ such that $a_1, \ldots, a_n \in \mathbb{R}$ and $a_1 + \cdots + a_n = 0$. Then since $e^{2a_1} \cdot \cdots \cdot e^{2a_n} = 1$,

$$0 \leqslant u(g), \qquad (\text{ for } g \in GL(n,\mathbb{R})),$$

and

$$u(g) + 1 \leqslant \exp(2||\ln A(g)||), \qquad (\text{ for any } g \in GL(n,\mathbb{R})).$$

We generalize the function used in [3]. Let $\phi : [0,\infty) \to [0,\infty)$ be a smooth function with $\mathrm{supp}(\phi) \subset [0,1]$ and

$$\int_0^\infty \phi(x)\, dx = 1.$$

For any $\delta > 0$, let $Y = \frac{1}{e^{2\delta}-1}$. Define

$$H_\delta(g) := \phi\left(Y \cdot u(g)\right), \qquad (\text{ for } g \in GL(n,\mathbb{R})). \tag{4.30}$$

Then H_δ is a compactly supported smooth bi-$(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$-invariant function. Since $\phi(Y \cdot u(g)) = 0$ for $u(g) > \frac{1}{Y}$, we have

$$\mathrm{supp}(H_\delta) \subset \{g \in GL(n,\mathbb{R}) \ \mid ||\ln A(g)|| \leqslant \delta\}.$$

For example, for $x \in [0,\infty)$, let

$$\phi(x) := \begin{cases} \frac{1}{c}\exp\left(-\frac{1}{x(1-x)}\right), & \text{for } 0 < x < 1; \\ 0, & \text{otherwise,} \end{cases} \tag{4.31}$$

where $c = \int_0^1 \exp\left(-\frac{1}{t(1-t)}\right)\, dt$. Then ϕ is always non-negative and it is a smooth function with a support $(0,1)$.

For any $g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & \cdots & \vdots \\ g_{n,1} & \cdots & g_{n,n} \end{pmatrix} \in GL(n,\mathbb{R})$, we have

$$\mathrm{tr}\left({}^t g \cdot g\right) = \sum_{1 \leqslant i,j \leqslant n} g_{i,j}^2 =: |\det g|^{\frac{2}{n}}\, ||g||^2 = |\det g|^{\frac{2}{n}}\, \mathrm{tr}\left(A(g)^2\right).$$

So,

$$u(g) = \frac{1}{n}||g||^2 - 1, \qquad (\text{ for } g \in GL(n,\mathbb{R})).$$

Lemma 4.11. *Take* $\delta > 0$ *such that* $\left(e^{2\delta} - 1\right) \leqslant 1$ *and*

$$\left(e^{2\delta} - 1\right) \cdot \int_{\substack{\mathbb{H}^n, \\ u(z)=t}} 1\, d^*z \ \leqslant \ 1, \qquad (\textit{for } 0 \leqslant t \leqslant 1),$$

and let H_δ be a function defined in (4.30). Then we have

$$\int_{\mathbb{H}^n} H_\delta(z)\ d^*z \ \leqslant\ 1.$$

Proof. Let $Y = \frac{1}{e^{2\delta}-1}$. Then

$$\int_{\mathbb{H}^n} H_\delta(z)\ d^*z = \int_{\mathbb{H}^n} \phi(Y \cdot u(z))\ d^*z = \int_0^\infty \int_{\substack{\mathbb{H}^n, \\ u(z)=t}} \phi(Y \cdot t)\ d^*z\ dt$$
$$= \frac{1}{Y} \int_0^\infty \phi(t) \int_{\substack{\mathbb{H}^n, \\ u(z)=\frac{t}{Y}}} 1\ d^*z\ dt.$$

Since $\phi(t) = 0$ for $t > 1$, we have $0 \leqslant \frac{t}{Y} \leqslant \frac{1}{Y} \leqslant 1$. So

$$\int_{\mathbb{H}^n} H_\delta(z)\ d^*z \leqslant \int_0^\infty \phi(t)\ dt = 1.$$

□

Chapter 5

APPROXIMATE CONVERSE THEOREM

5.1 Approximate converse theorem

Let S be a finite set of primes including ∞. Let $q_S := \max\{v \in S \mid v < \infty\}$ for $S \neq \{\infty\}$ and $q_S := 1$ for $S = \{\infty\}$. For $\delta > 0$, define

$$
\begin{aligned}
B^n(S;\delta) := &\left\{ z \notin \widetilde{\mathfrak{F}^n} \;\middle|\; ||\ln A(z^{-1}\tau)|| \leqslant \delta \text{ for some } \tau \in \mathfrak{F}^n \right\} \\
&\bigcup \left\{ z \in \mathfrak{F}^n \;\middle|\; ||\ln A(z^{-1}\tau)|| \leqslant \delta \text{ for some } \tau \notin \widetilde{\mathfrak{F}^n} \right\} \\
&\bigcup \left\{ z \in \mathfrak{F}^n \;\middle|\; \begin{matrix} \begin{pmatrix} q_S^{\alpha_1} & & \\ & \ddots & \\ & & q_S^{\alpha_n} \end{pmatrix} z \notin \widetilde{\mathfrak{F}^n} \text{ for some} \\ \text{non-negative integers } \alpha_1 + \cdots + \alpha_n = \lfloor \tfrac{n}{2} \rfloor \end{matrix} \right\} \\
&\bigcup \left\{ \begin{pmatrix} q_S^{\alpha_1} & & \\ & \ddots & \\ & & q_S^{\alpha_n} \end{pmatrix} z \notin \widetilde{\mathfrak{F}^n} \;\middle|\; \begin{matrix} \text{for some } z \in \mathfrak{F}^n, \text{ for some} \\ \text{non-negative integers } \alpha_1 + \cdots + \alpha_n = \lfloor \tfrac{n}{2} \rfloor \end{matrix} \right\}
\end{aligned}
\tag{5.1}
$$

where $\widetilde{\mathfrak{F}^n}$ is the extended fundamental domain defined in (3.19).

We state the main theorem.

Theorem 5.1. (Approximate Converse Theorem) *Let $n \geqslant 2$ be an integer and M be a set of primes including ∞ and at least one finite prime. Let $\ell_M = \{\ell_v \in \mathfrak{a}^*_{\mathbb{C}}(n),\ v \in M\}$ be a quasi-automorphic parameter for M and F_{ℓ_M} be a quasi-Maass form of ℓ_M of length L as in Definition 3.16. Let $\widetilde{F}_{\ell_M}$ be the automorphic lifting of F_{ℓ_M} in (3.18). Assume that there exists a prime $p \in M$ such that $\widehat{\natural^n_p}(\ell_\infty, \ell_p) \neq 0$. Let $S \subset M$ be a finite subset including ∞. Choose arbitrary $\delta > 0$ and an arbitrary bi-$(\mathbb{R}^\times \cdot O(n,\mathbb{R}))$-invariant compactly supported* *smooth function* H_δ *with* $\operatorname{supp}(H_\delta) \subset U(\delta)$*, satisfying* $\widehat{H}_\delta(\ell_\infty) \neq 0$*.*

Then there exists an unramified cuspidal automorphic representation $\pi(\sigma) = \otimes_v' \pi_v(\sigma_v)$ *with an automorphic parameter* σ *as in (3.10) such that* ℓ_M *and* σ *are* ϵ*-close for* S *where*

$$\epsilon := \frac{\sup\limits_{B^n(S;\delta)} \left| \widetilde{F}_{\ell_M} - F_{\ell_M} \right|^2 \cdot C_p(n, S, H_\delta; \ell_\infty)}{\left| \widehat{\natural}_p^n(\ell_\infty, \ell_p) \right|^2 \cdot \left| \widehat{H}_\delta(\ell_\infty) \right|^2 \cdot L(F_{\ell_M})^2}. \tag{5.2}$$

Here $C_p(n, S, H_\delta; \ell_\infty)$ *is a positive constant (which is determined by* ℓ_∞*, the prime* p *and* H_δ*) given explicitly as*

$$\begin{aligned} C_p(n, S, H_\delta; \ell_\infty) :=& \operatorname{Vol}\left(B^n(S;\delta) \cap \mathfrak{F}^n\right) \cdot \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}} \right)^{4 d_k(n)} \\ & \times \left\{ \sum_{j=1}^{n-1} \left(\int_{\mathbb{H}^n} \left| \left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty) \right) H_\delta(\tau) \right| \, d^*\tau \right)^2 \right. \\ & \left. + C_{H_\delta}^2 \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{q \in S, \\ \textit{finite prime}}} \left(q^{-\frac{j(n+1)}{2}} \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n} q^{k_1 + \cdots + k_j} \right)^2 \right\}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} L(F_{\ell_M})^2 :=& \sum_{m_1=1}^{L} \cdots \sum_{m_{n-2}=1}^{L} \sum_{0 \neq |m_{n-1}| \leqslant L} \left| \frac{A_{\ell_M}(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} \right|^2 \\ & \cdot \int_T^\infty \cdots \int_T^\infty \left| W_J \left(\begin{pmatrix} m_1 \cdots |m_{n-1}| & & \\ & \ddots & \\ & & 1 \end{pmatrix} y; \nu(\ell_\infty), 1 \right) \right|^2 d^*y, \end{aligned}$$

where $d_k(n) = \frac{n!}{k!(n-k)!}$ *for* $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ *and* $C_{H_\delta} > 0$ *is a constant defined in Lemma 4.9 for* H_δ*. For* $r = 1, \ldots, n-1$*, we have*

$$A_{\ell_M}(\underbrace{1, \ldots, 1, q}_{r}, 1, \ldots, 1) = \begin{cases} \sum_{1 \leqslant k_1 < \cdots < k_r \leqslant n} q^{-(\ell_{q,k_1} + \cdots + \ell_{q,k_r})}, & \textit{if } q \in M, \\ 0, & \textit{otherwise,} \end{cases}$$

and $A_{\ell_M}(1,\ldots,1) = 1$ *while* $A_{\ell_M}(m_1,\ldots,m_{n-1})$ *is determined by the multiplicative relations in (2.46) for* $(m_1,\ldots,m_{n-1}) \in \mathbb{Z}^{n-1}$*. Here* T *is a constant such that*

$$T \geqslant \max\left\{\exp(4\delta),\ \exp\left(\frac{n!\ln p}{2\left(\lfloor\frac{n}{2}\rfloor - 1\right)!\left(n-\lfloor\frac{n}{2}\rfloor\right)!}\right)\right\}.$$

Remark 5.2. **(i)** *If* ϵ *in (5.2) is sufficiently small, then by Remark 8 [4],* $\pi(\sigma)$ *is uniquely determined.*

(ii) *The constant* ϵ *in (5.2) mainly depends on* $\sup_{B^n(\delta,S)}\left|\widetilde{F}_{\ell_M}(z) - F_{\ell_M}(z)\right|^2$*. It is an interesting problem to choose* H_δ *so that* ϵ *is as small as possible.*

(iii) *Taking* δ *and* H_δ *is also important to get a good* ϵ*. We give an example for* H_δ *in §4.3.*

(iv) *For any finite set* S*, since the space* $B^n(S;\delta)$ *is bounded, there are finitely many* $\gamma_1,\ldots,\gamma_r \in SL(n,\mathbb{Z})$ *such that*

$$\gamma_1\mathfrak{F}^n \cup \cdots \cup \gamma_r\mathfrak{F}^n \supset B^n(\delta,S)$$

and

$$\sup_{B^n(\delta,S)}\left|\widetilde{F}_{\ell_M}(z) - F_{\ell_M}(z)\right|^2 = \sup_{B^n(\delta,S)\cap\mathfrak{F}^n}\left\{\left|F_{\ell_M}(\gamma_j z) - F_{\ell_M}(z)\right|^2 \mid j = 1,\ldots,r\right\}.$$

(v) *For an unramified cuspidal representation* $\pi \cong \otimes_v \pi_v(\sigma_v)$ *of* $\mathbb{A}^\times\backslash GL(n,\mathbb{A})$*, define an analytic conductor*

$$\mathcal{C}(\pi) := \prod_{j=1}^{n}\left(1 + |\sigma_{\infty,j}|\right)$$

as in [4], where $\sigma_\infty = (\sigma_{\infty,1},\ldots,\sigma_{\infty,n}) \in \mathfrak{a}^*_{\mathbb{C}}(n)$*. Fix* $Q \geqslant 2$*. By [4], for any unramified cuspidal representation* $\pi \cong \otimes_v(\sigma_v)$ *of* $\mathbb{A}^\times\backslash GL(n,\mathbb{A})$ *with* $\mathcal{C}(\pi) \leqslant Q$*, there exists a prime* $p \ll \log Q$ *such that* $\left|\widehat{\mathfrak{h}}^n_p(\sigma_\infty,\sigma_p)\right|$ *is sufficiently large.*

5.2 Proof of Theorem 5.1

Proof of Theorem 5.1. Take H_δ such that $\widehat{H_\delta}(\ell_\infty) \neq 0$. By Theorem 4.7, since $\widehat{\natural_p^n}(\ell_\infty, \ell_p) \neq 0$, the automorphic function $H_\delta \natural_p^n \widetilde{F}_{\ell_M} \neq 0$ and

$$\natural_p^n \left(\widetilde{F}_{\ell_M} * H_\delta \right) = H_\delta \natural_p^n \widetilde{F}_{\ell_M} \in L^2_{\text{cusp}} \left(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n \right).$$

If $\natural_p^n \left(\widetilde{F}_{\ell_M} * H_\delta \right)$ satisfies (3.17) for $\epsilon > 0$ then, by Lemma 3.19, there exists an unramified cuspidal representation with an automorphic parameter σ, which is ϵ-close to ℓ_M. Let $\nu := \nu(\ell_\infty)$ as in (2.33).

To get the lower bound for $||H_\delta \natural_p^n \widetilde{F}_{\ell_M}||_2^2$, we use the following Lemma.

Lemma 5.3. *For an integer $n \geqslant 2$, let f be a square-integrable, cuspidal, automorphic, smooth function for $\mathbb{H}^n$. For $T \geqslant 1$,*

$$||f||_2^2 > \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \int_T^\infty \cdots \int_T^\infty \left| W_f \left(y; m_1, \ldots, m_{n-2}, m_{n-1} \right) \right|^2 \, d^*y$$

where $y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$, $y_1, \ldots, y_{n-1} > 0$ *and* $d^*y = \prod_{j=1}^{n-1} y_j^{-j(n-j)-1} dy_j$. *Here* $W_f(z; m_1, \ldots, m_{n-1})$ *is the Fourier coefficient of f for $m_1, \ldots, m_{n-1} \in \mathbb{Z}$, defined by*

$$\begin{aligned} &W_f(z; m_1, \ldots, m_{n-1}) \\ &\quad = \int_{\mathbb{Z}\backslash\mathbb{R}} \cdots \int_{\mathbb{Z}\backslash\mathbb{R}} f(uz) e^{-2\pi i (m_1 u_{n-1,n} + \cdots + m_{n-2} u_{2,3} + m_{n-1} u_{1,2})} \, d^*u \end{aligned}$$

where $u = \begin{pmatrix} 1 & & u_{i,j} \\ & \ddots & \\ & & 1 \end{pmatrix}$, $u_{i,j} \in \mathbb{R}$ *for* $1 \leqslant i < j \leqslant n$ *and* $d^*u = \prod_{1 \leqslant i < j \leqslant n} du_{i,j}$.

Let $T \geqslant \max \left\{ \exp(4\delta), \ \exp \left(\frac{n! \ln p}{2 \left(\lfloor \frac{n}{2} \rfloor - 1 \right)! \left(n - \lfloor \frac{n}{2} \rfloor \right)!} \right) \right\} > 1$. By Lemma 4.8, for any $z \in \Sigma_{T, \frac{1}{2}} \subset \mathfrak{F}^n$,

$$\natural_p^n \left(\widetilde{F}_{\ell_M} * H_\delta \right)(z) = \widehat{H_\delta}(\ell_\infty) \widehat{\natural_p^n}(\ell_\infty, \ell_p) \cdot F_{\ell_M}(z).$$

Then for $z \in \Sigma_{T,\frac{1}{2}}$, and for integers $1 \leqslant m_1, \ldots, m_{n-2}, |m_{n-1}| \leqslant L$, the $(m_1, \ldots, m_{n-1})$th Fourier coefficient for $H_\delta \natural_p^n \widetilde{F}_{\ell_M}$ is

$$\begin{aligned}
&W_{H_\delta \natural_p^n \widetilde{F}_{\ell_M}}(z; m_1, \ldots, m_{n-1}) \\
&= \widehat{H_\delta}(\ell_\infty) \widehat{\natural_p^n}(\ell_\infty, \ell_p) \cdot \frac{A_{\ell_M}(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} \\
&\times W_J\left(\begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & \\ & \ddots & \\ & & 1 \end{pmatrix} y; \nu(\ell_\infty), 1\right) e^{2\pi i(m_1 x_{n-1,n} + m_2 x_{n-2,n-1} + \cdots + m_{n-1} x_{1,2})}.
\end{aligned}$$

Therefore, by Lemma 5.3,

$$\begin{aligned}
||H_\delta \natural_p^n \widetilde{F}_{\ell_M}||_2^2 \geqslant \sum_{m_1=1}^{L} \cdots \sum_{m_{n-2}=1}^{L} \sum_{0 \neq |m_{n-1}| \leqslant L} \left| \widehat{H_\delta}(\ell_\infty) \widehat{\natural_p^n}(\ell_\infty, \ell_p) \cdot \frac{A_{\ell_M}(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} \right|^2 \\
\times \int_T^\infty \cdots \int_T^\infty \left| W_J\left(\begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & \\ & \ddots & \\ & & 1 \end{pmatrix} y; \nu(\ell_\infty), 1\right) \right|^2 d^*y.
\end{aligned} \tag{5.4}$$

Consider the case when $v = \infty$. For $j = 1, \ldots, n-1$, there exist $\lambda_\infty^{(j)}(\ell_\infty) \in \mathbb{C}$ as in Definition 3.8 for the corresponding character associated to the parameter ℓ_∞. So for $j = 1, \ldots, n-1$, we have

$$\begin{aligned}
&|| \left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) \natural_p^n \left(\widetilde{F}_{\ell_M} * H_\delta\right) ||_2^2 \\
&\leqslant \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}}\right)^{4d_k(n)} \cdot || \left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) \widetilde{F}_{\ell_M} * H_\delta||_2^2 \ ,
\end{aligned}$$

since the operator $\natural_p^n$ commutes with the invariant differential operators $\Delta_n^{(j)}$. Since

$$\left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) F_{\ell_M} * H_\delta(z) = 0$$

for any $z \in \mathbb{H}^n$, it follows that

$$\begin{aligned}
|| \left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) \widetilde{F}_{\ell_M} * H_\delta||_2^2 &= || \left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right) * H_\delta||_2^2 \\
&= || \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right) * \left((\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)) H_\delta\right) ||_2^2
\end{aligned}$$

and

$$
\begin{aligned}
&|| \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right) * \left((\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty))H_\delta\right) ||_2^2 \\
&\quad = \int_{\mathfrak{F}^n} \left| \int_{GL(n,\mathbb{R})/\mathbb{R}^\times} \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right)(\xi) \cdot \left((\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty))H_\delta(\xi^{-1}z\right) \ d\xi \right|^2 \ d^*z \\
&\quad \leqslant \int_{B^n(S;\delta)\cap\mathfrak{F}^n} \left(\int_{B^n(S;\delta)} \left| \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right)(\xi)\right| \cdot \left|\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty))H_\delta\right| \ d\xi \right)^2 \ d^*z \\
&\leqslant \sup_{B^n(S;\delta)} \left|\widetilde{F}_{\ell_M} - F_{\ell_M}\right|^2 \cdot \mathrm{Vol}\left(B^n(S;\delta) \cap \mathfrak{F}^n\right) \cdot \left(\int_{GL(n,\mathbb{R})/\mathbb{R}^\times} \left|(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty))H_\delta\right| \ d\xi \right)^2 .
\end{aligned}
$$

Consider the case when $v = q < \infty$ and $q \in S$. For $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ there exists $\lambda_q^{(j)}(\ell_q) \in \mathbb{C}$, as in Definition 3.8, for the corresponding character associated to the parameter ℓ_q. Since $H_\delta \natural_p^n$ commutes with Hecke operators, it follows that

$$
\begin{aligned}
&|| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_q)\right) H_\delta \natural_p^n \widetilde{F}_{\ell_M} ||_2^2 \\
&\quad \leqslant \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}} \right)^{4d_k(n)} \cdot C_{H_\delta}^2 \cdot || \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_q)\right) \widetilde{F}_{\ell_M} ||_2^2 \ ,
\end{aligned}
$$

for $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Since $\left(T_q^{(j)} - \lambda_q^{(j)}(\ell_q)\right) F_{\ell_M} \equiv 0$ and $\widetilde{F}_{\ell_M}(z) = F_{\ell_M}(z)$ for $z \in \mathfrak{F}^n$, we have

$$
\begin{aligned}
|| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_q)\right) \widetilde{F}_{\ell_M} ||_2^2 &= \int_{\mathfrak{F}^n} \left| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_q)\right) \widetilde{F}_{\ell_M}(z) \right|^2 \ d^*z \\
&= \int_{\mathfrak{F}^n} \left| T_q^{(j)} \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right)(z) \right|^2 \ d^*z.
\end{aligned}
$$

By the definition of $T_q^{(j)}$ in (2.48) for each $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, there exists a positive integer $\sharp\left(T_q^{(j)}\right)$ such that

$$
\begin{aligned}
T_q^{(j)} \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right)(z) &= \frac{1}{q^{\frac{j(n-1)}{2}}} \sum_{0 \leqslant k_1 \leqslant \cdots \leqslant k_j \leqslant j} c_{(k_1,\ldots,k_j)} \cdot T_{q^{k_1}} \cdots T_{q^{k_j}} \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right)(z) \\
&= \frac{1}{q^{\frac{j(n-1)}{2}}} \sum_{k=1}^{\sharp\left(T_q^{(j)}\right)} \left(\widetilde{F}_{\ell_M} - F_{\ell_M}\right)(C_k z),
\end{aligned}
$$

where $c_{(k_1,\ldots,k_j)} \in \mathbb{Z}$ and C_k's are upper triangular matrices with integer coefficients which are determined by Hecke operators in the first line. So

$$\int_{\mathfrak{F}^n} \left| T_q^{(j)} \left(\widetilde{F}_{\ell_M} - F_{\ell_M} \right)(z) \right|^2 \, d^*z \leqslant \int_{\mathfrak{F}^n} \left(\frac{1}{q^{\frac{j(n-1)}{2}}} \sum_{k=1}^{\sharp\left(T_q^{(j)}\right)} \left| \left(\widetilde{F}_{\ell_M} - F_{\ell_M} \right)(C_k z) \right| \right)^2 \, d^*z$$

$$\leqslant \mathrm{Vol}\left(B^n(S;\delta) \cap \mathfrak{F}^n\right) \cdot \left(T_q^{(j)} 1\right)^2 \cdot \sup_{B^n(S;\delta)} \left| \widetilde{F}_{\ell_M} - F_{\ell_M} \right|^2$$

$$= \left(q^{-\frac{j(n+1)}{2}} \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n} q^{k_1 + \cdots + k_j} \right)^2 \cdot \mathrm{Vol}\left(B^n(S;\delta) \cap \mathfrak{F}^n\right) \cdot \sup_{B^n(S;\delta)} \left| \widetilde{F}_{\ell_M} - F_{\ell_M} \right|^2.$$

□

To complete the proof of the main theorem, we give the proof of Lemma 5.3.

Proof of Lemma 5.3. Let $f : SL(n,\mathbb{Z})\backslash\mathbb{H}^n \to \mathbb{C}$ be a cuspidal automorphic function, which is smooth and square integrable. For $j = 1, \ldots, n-1$, let

$$u_{n-j+1} := \begin{pmatrix} & u_{1,n-j+1} & \\ I_{n-j} & \vdots & 0_{n-j \times j-1} \\ & u_{n-j,n-j+1} & \\ & & \\ 0_{j \times n-j} & & I_j \end{pmatrix} \in N(n,\mathbb{R})$$

where $u_{1,n-j+1}, \ldots, u_{n-j,n-j+1} \in \mathbb{R}$ and $0_{a \times b}$ is an $a \times b$ matrix with 0 for every entry. Here $N(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ is the set of $n \times n$ unitary upper triangular matrices. We follow the argument in 5.3, [12]. Let $n \geqslant 2$ be an integer. Fix $j = 1, \ldots, n-1$. For $m_1, \ldots, m_j \in \mathbb{Z}$, define

$$f_j(z; m_1, \ldots, m_j) := \int_{\mathbb{Z}\backslash\mathbb{R}} \cdots \int_{\mathbb{Z}\backslash\mathbb{R}} f\left(u_n \cdot u_{n-1} \cdots u_{n-j+1} z\right) \times e^{-2\pi i (m_1 u_{n-1,n} + \cdots + m_j u_{n-j,n-j+1})} \, d^*u_n \cdots d^*u_{n-j+1},$$

where

$$d^*u_{n-j+1} = \prod_{k=1}^{n-j} du_{k,n-j+1}.$$

Then for $m_1, \ldots, m_{n-1} \in \mathbb{Z}$,

$$f_{n-1}(z; m_1, \ldots, m_{n-1}) = W_f(z; m_1, \ldots, m_{n-1}).$$

Let $f_0(z) := f(z)$ with $z \in \mathbb{H}^n$. By following the proof of Theorem 5.3.2, [12], we can also prove the following.

(i) For $j = 1, \ldots, n-1$, we have

$$f_j(z; m_1, \ldots, m_j) \\ = \int_{\mathbb{Z}\backslash\mathbb{R}} \cdots \int_{\mathbb{Z}\backslash\mathbb{R}} f_{j-1}(u_{n-j+1}z; m_1, \ldots, m_{j-1}) e^{-2\pi i m_j u_{n-j,n-j+1}} \; d^* u_{n-j+1}.$$

(ii) Fix $j = 1, \ldots, n-2$. For positive $m_1, \ldots, m_{j-1} \in \mathbb{Z}$, we have

$$f_{j-1}(z; m_1, \ldots, m_{j-1}) \\ = \sum_{m_j=1}^{\infty} \sum_{\gamma_{n-j} \in P_{n-j-1,1}(\mathbb{Z})\backslash SL(n-j,\mathbb{Z})} f_j\left(\begin{pmatrix} \gamma_{n-j} & \\ & I_j \end{pmatrix} z; m_1, \ldots, m_{j-1}, m_j\right).$$

(iii) For positive integers $m_1, \ldots, m_{n-2}$, we have

$$\begin{aligned} f_{n-2}(z; m_1, \ldots, m_{n-2}) &= \sum_{0 \neq m_{n-1} \in \mathbb{Z}} f_{n-1}(z; m_1, \ldots, m_{n-2}, m_{n-1}) \\ &= \sum_{0 \neq m_{n-1} \in \mathbb{Z}} W_f\left(z; m_1, \ldots, m_{n-2}, m_{n-1}\right). \end{aligned}$$

Since the Siegel set $\Sigma_{1,\frac{1}{2}} \subset \mathfrak{F}^n$,

$$||f||_2^2 = \int_{\mathfrak{F}^n} |f(z)|^2 \; d^* z \; > \; \int_1^{\infty} \cdots \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^2 \; d^* z.$$

Then

$$\int_1^\infty \cdots \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^2 \; d^*z$$
$$= \int_1^\infty \cdots \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1=1}^{\infty} \sum_{\gamma_{n-1} \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \overline{f(z)} e^{2\pi i m_1 (\gamma_{n-1,1} x_{1,n} + \cdots \gamma_{n-1,n-1} x_{n-1,n})}$$
$$\times f_1\left(\begin{pmatrix} \gamma_{n-1} & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}; m_1\right) d^*z,$$

where $\gamma_{n-1} = \left(\begin{smallmatrix} & * & \\ \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-1} \end{smallmatrix} \right) \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})$. For a positive integer m_1 and $\gamma_{n-1} = \left(\begin{smallmatrix} & * & \\ \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-1} \end{smallmatrix} \right) \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})$, it follows that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{f(z)} e^{2\pi i m_1 (\gamma_{n-1,1} x_{1,n} + \cdots \gamma_{n-1,n-1} x_{n-1,n})} \prod_{k=1}^{n-1} dx_k$$
$$= \overline{f_1\left(\begin{pmatrix} \gamma_{n-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}\right)}$$

So,

$$\int_1^\infty \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^2 \; d^*z$$
$$= \int_1^\infty \cdots \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1=1}^{\infty}$$
$$\sum_{\gamma_{n-1} \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \left| f_1\left(\begin{pmatrix} \gamma_{n-1} & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}; m_1\right) \right|^2$$
$$\prod_{1 \leqslant i < j \leqslant n-1} dx_{i,j} \; d^*y$$
$$\geqslant \sum_{m_1=1}^{\infty} \int_1^\infty \cdots \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| f_1\left(\left(\begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}; m_1\right)\right) \right|^2 \prod_{1 \leqslant i < j \leqslant n-1} dx_{i,j} \; d^*y.$$

Then using

$$z' = \begin{pmatrix} & & & x_{1,n-1} \\ & I_{n-2} & & \vdots \\ & & & x_{n-2,n-1} \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} & & & 0 \\ & y_2 z'' & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

with $y_2 > 0$, $z'' \in \mathbb{H}^{n-2}$, for $0 < m_1 \in \mathbb{Z}$, we obtain

$$f_1\left(\begin{pmatrix} & & & x_{1,n-1} & 0 \\ & I_{n-2} & & \vdots & \vdots \\ & & & x_{n-2,n-1} & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & & 0 & 1 \end{pmatrix} \begin{pmatrix} & & & 0 & 0 \\ & y_1 y_2 z'' & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \dots & 0 & y_1 & 0 \\ 0 & \dots & & 0 & 1 \end{pmatrix}; m_1\right)$$

$$= \sum_{m_2=1}^{\infty} \sum_{\gamma_{n-2} \in P_{n-3,1} \backslash SL(n-2,\mathbb{Z})} f_2\left(\begin{pmatrix} \gamma_{n-2} & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \begin{pmatrix} & & & 0 & 0 \\ & y_1 y_2 z'' & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \dots & 0 & y_1 & 0 \\ 0 & \dots & & 0 & 1 \end{pmatrix}; m_1, m_2\right) \times e^{2\pi i m_2 (\gamma_{n-2,1} x_{1,n-1} + \cdots + \gamma_{n-2,n-2} x_{n-2,n-1})},$$

where $\gamma_{n-2} = \left(\begin{smallmatrix} & * & \\ \gamma_{n-2,1} & \cdots & \gamma_{n-2,n-2} \end{smallmatrix} \right)$. We get again,

$$||f||_2^2 \geqslant \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \int_1^{\infty} \cdots \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| f_2\left(\begin{pmatrix} & & & 0 & 0 \\ & y_1 y_2 z'' & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \dots & 0 & y_1 & 0 \\ 0 & \dots & & 0 & 1 \end{pmatrix}; m_1, m_2\right)\right|^2 \prod_{1 \leqslant i < j \leqslant n-2} dx_{i,j}\, d^*y.$$

After continuing this process inductively for $n-1$ steps, we finally obtain

$$||f||_2^2 > \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \int_1^{\infty} \cdots \int_1^{\infty} \left| W_f\left(y; m_1, \ldots, m_{n-2}, m_{n-1}\right)\right|^2 \, d^*y$$

where $y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$, $y_1, \ldots, y_{n-1} > 0$. $\square$

5.3 Proof of Theorem1.1

Let $H_\delta(z) = \phi(Y \cdot u(z))$ where ϕ is a function defined in (4.31) and $Y = \frac{1}{e^{2\delta}-1}$. Choose $\delta > 0$ such that

$$\max\left\{\left|\lambda_\infty^{(j)}(\ell_\infty)\right|\right\}_{j=1,\dots,n-1} \cdot \max\left\{\int\limits_{\substack{\mathbb{H}_n,\\ u(z)=t}} 1\ d^*z\right\}_{0\leqslant t\leqslant 1} \cdot (e^{2\delta}-1) \leqslant 1.$$

Then for $j = 1, \dots, n-1$,

$$\begin{aligned}&\int_{\mathbb{H}^n} \left|\left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) H_\delta(\tau)\right|\ d^*\tau \\ &\qquad \leqslant \int_{\mathbb{H}^n} \left|\Delta_n^{(j)} H_\delta(\tau)\right|\ d^*\tau\ +\ \left|\lambda_\infty^{(j)}(\ell_\infty)\right| \int_{\mathbb{H}^n} \phi(Y\cdot u(\tau))\ d^*\tau\end{aligned}$$

As in Lemma 4.11, we have

$$\begin{aligned}\left|\lambda_\infty^{(j)}(\ell_\infty)\right| \cdot \int_{\mathbb{H}^n} \phi(Y\cdot u(\tau))\ d^*\tau &= \int_0^\infty \phi(t) \left(\frac{\left|\lambda_\infty^{(j)}(\ell_\infty)\right|}{Y} \int\limits_{\substack{\mathbb{H}^n,\\ u(\tau)=\frac{t}{Y}}} 1\ d^*\tau\right) dt \\ &\leqslant \int_0^\infty \phi(t)\ dt = 1.\end{aligned}$$

So,

$$\int_{\mathbb{H}^n} \left|\left(\Delta_n^{(j)} - \lambda_\infty^{(j)}(\ell_\infty)\right) H_\delta(\tau)\right|\ d^*\tau\ \leqslant\ \int_{\mathbb{H}^n} \left|\Delta_n^{(j)} H_\delta(\tau)\right|\ d^*\tau\ +1\ ,$$

then

$$\begin{aligned}C_p(n,\delta;S) :=&\mathrm{Vol}(B^n(S;\delta)\cap\mathfrak{F}^n)\cdot \prod_{k=1}^{\lfloor\frac{n}{2}\rfloor}\left(p^{-\frac{k(n^2-1)}{n^2+1}} + p^{\frac{k(n^2-1)}{n^2+1}}\right)^{4d_k(n)} \\ &\times\left[\sum_{j=1}^{n-1}\left(\int_{\mathbb{H}^n}\left|\Delta_n^{(j)}\phi(Y\cdot u(\tau))\right|\ d^*\tau + 1\right)^2\right. \\ &\left. + C_{H_\delta}^2 \sum_{j=1}^{\lfloor\frac{n}{2}\rfloor} \sum_{\substack{q\in S,\\ \text{finite prime}}} \left(q^{-\frac{j(n+1)}{2}} \sum_{1\leqslant k_1<\cdots<k_j\leqslant n} q^{k_1+\cdots+k_j}\right)^2\right].\end{aligned} \tag{5.5}$$

By (5.2) and (5.3), we have Theorem 1.1. $\square$

BIBLIOGRAPHY

[1] Ce Bian. Computing $GL(3)$ automorphic forms. *Bull. London Math. Soc. (2010)*, 42(5):827–842, 2010.

[2] Andrew Booker. Uncovering a new L-function. *Notices of the AMS*, 55:1088–1094, 2008.

[3] Andrew Booker, Andreas Strömbergsson, and Akshay Venkatesh. Effective Computation of Maass cusp forms. *IMRN*, 2006.

[4] Farrell Brumley. Effective multiplicity on on GL_N and narrow zero-free regions for Rankin-Selberg L-functions. *American Journal of Mathematics*, 128:1455–1474, 2006.

[5] W. Casselman. On some results of Atkin and Lehner. *Math.Ann.*, 201:301–314, 1973.

[6] J. W. Cogdell and I.I. Piatetski-Shapiro. Converse theorems for GL_n. *Inst. Hautes Études Sci. Publ. Math.*, 79:157–214, 1994.

[7] J. W. Cogdell and I.I. Piatetski-Shapiro. Converse theorems for GL_n. II. *J. Reine Angew. Math.*, 507:165–188, 1999.

[8] D. Flath. Decomposition of representations into tensor products. In *Automorphic Forms, Representations, and L-functions*, volume 33 of *Proceedings of Symposia in Pure Mathematics*, pages 170–183. American Mathematical Society, Providence, RI, 1979.

[9] R. Gangolli. On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups. *Ann. of Math.*, 93:150–165, 1971.

[10] I. Gelfand and D. Kazhdan. *Representations of the group $GL(n, K)$ whre K is a local field*. Lie groups and their Representations. John Wiley and Sons, New York, 1975.

[11] I. M. Gelfand, M. Graev, and I.I. Piatetski-Shapiro. *Representation theory and automorphic functions*. Academic Press, Inc., Boston, MA, 1990.

[12] Dorian Goldfeld. *Automorphic Forms and L-functions for the group $GL(n, \mathbb{R})$*. Number 99 in Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, 2006.

[13] Dorian Goldfeld and Joseph Hundley. *Automorphic representations and L-functions for the general linear group*. Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, prepare.

[14] Douglas Grenier. On the shape of fundamental domains in $GL(n, \mathbb{R})$. *Pacific Journal of Math.*, 160:53–65, 1993.

[15] Dennis A. Hejhal. On eigenfunctions of the Laplacian for Hecke triangle groups. In *Emerging applications of number theory (Minneapolis, MN, 1996)*, volume 109 of *IMA Vol. Math. Appl.*, pages 291–315. Springer, New York, 1999.

[16] Sigurdur Helgasson. *Groups and Geometric Analysis: Integral geometry, invariant differential operators, and spherical functions*. Number 83 in Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000.

[17] Hervé Jacquet and Robert P. Langlands. *Automorphic forms on $GL(2)$*, volume 114 of *Lecture Notest in Mathematics*. Spring-Verlag, Berlin-New York, 1970.

[18] Erez Lapid and Werner Müller. Spectral asymptotics for arithmetic quotients of $SL(n, \mathbb{R})/SO(n)$. *Duke Mathematical Journal*, 149:117–155, 2009.

[19] Elon Lindenstrauss and Akshay Venkatesh. Existence and Weyl's law for Spherical Cusp forms. *GAFA*, 17:220–251, 2007.

[20] Boris Mezhericher. *Evaluating Jacquet's Whittaker functions and Maass forms for $SL(3, \mathbb{Z})$*. PhD thesis, Columbia University, 2008.

[21] Stephen D. Miller. On the existence and temperedness of cusp forms for $SL_3(\mathbb{Z})$. *J. Reine Angew. Math.*, 533:127–169, 2001.

[22] Werner Müller. Weyl's law for the cuspidal spectrum of SL_n. *Ann. of Math.*, 165:275–333, 2007.

[23] I.I. Piatetski-Shapiro. Multiplicity One Theorems. In *Automorphic Forms, Representations, and L-functions*, volume 33 of *Proceedings of Symposia in Pure Mathematics*, pages 209–212. American Mathematical Society, Providence, RI, 1979.

[24] Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956.

[25] J. Shalika. The Multiplicity One Theorem on $GL(n)$. *Ann. Math.*, 100:171–193, 1974.

[26] Harold Stark. Fourier coefficients of Maass waveforms. In *Modular forms (Durham, 1983)*, pages 263–269. Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984.

[27] Holger Then. Maass cusp forms for large eigenvalues. *Math. Comp. 74*, 249:363–381, 2005.