Soergel Diagrammatics for Dihedral Groups

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Abstract

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We give a diagrammatic presentation for the category of Soergel bimodules for the dihedral group W, finite or infinite. The (two-colored) Temperley-Lieb category is embedded inside this category as the degree 0 morphisms between color-alternating objects. The indecomposable Soergel bimodules are the images of Jones-Wenzl projectors. When W is finite, the Temperley-Lieb category must be taken at an appropriate root of unity, and the negligible Jones-Wenzl projector yields the Soergel bimodule for the longest element of W.

Contents

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To my parents

Chapter 1

Introduction

1.1 Soergel Bimodules

1.1.1 The construction

Given an additive graded monoidal category $\mathcal{C}$, its additive (i.e. split) Grothendieck group $[\mathcal{C}]$ has the natural structure of a $\mathbb{Z}[v, v^{-1}]$-algebra. Multiplication by v corresponds to the grading shift $\{1\}$. We say that $\mathcal{C}$ is a *categorification* of $[\mathcal{C}]$. When $\mathcal{C}$ has the Krull-Schmidt property, the ring $[\mathcal{C}]$ will have a $\mathbb{Z}[v, v^{-1}]$-basis given by the classes of indecomposable objects (up to grading shift). We will use *indecomposable* as a noun, to refer to an indecomposable object.

Let $(W, \mathcal{S})$ be any Coxeter group, finite or infinite, equipped with a natural reflection representation $\mathfrak{h}$. We are interested in categorifications of $\mathcal{H}$, the associated Hecke algebra of W. When W is a Weyl group, geometric representation theory provides us with a natural categorification of $\mathcal{H}$, using equivariant perverse sheaves on the flag variety. This construction does not generalize to an arbitrary Coxeter group, though it does have analogs for affine Weyl groups and other crystallographic Coxeter groups. In the early 1990s, Soergel explored what happens when one takes the hypercohomology of a semisimple equivariant perverse sheaf on the flag variety. This will naturally be a graded bimodule over the polynomial ring $R = \mathbb{C}[\mathfrak{h}]$

(with linear terms graded in degree 2). Examining the properties of the bimodules which appear, Soergel defined a class of R-bimodules, now called *Soergel bimodules*. These can be defined for any Coxeter group W (agreeing with the hypercohomology bimodules in the Weyl group case), and they categorify $\mathcal{H}$. In other words, Soergel bimodules are an algebraic replacement for flag varieties, in situations with no ambient geometry. In a similar fashion, Soergel bimodules are an algebraic replacement for Harish-Chandra bimodules acting on the BGG category $\mathcal{O}$. We refer the reader to [27] for a purely algebraic account of Soergel bimodules, and to numerous other papers [23, 24, 25, 26] for the complete story.

Defining Soergel bimodules is a simple matter. Let us call a subset $J \subset \mathcal{S}$ *finitary* if the corresponding parabolic subgroup $W_J \subset W$ is finite. The ring R is naturally equipped with a W-action, and for any finitary $J \subset \mathcal{S}$ one may take the subring $R^J \subset R$ of invariants under W_J. When $s \in \mathcal{S}$ is a simple reflection, we define the bimodule $B_s \stackrel{\text{def}}{=} R \otimes_{R^s} R\{-1\}$. Tensor products of various bimodules B_s and their direct sums and grading shifts will form the (additive) category $\mathcal{B}_{\text{BS}}$ of *Bott-Samelson bimodules*, so named because they are obtained geometrically from "Bott-Samelson resolutions." Including all direct summands, we get the category $\mathcal{B}$ of Soergel bimodules. One may also wish to consider the category $\mathcal{B}_{\text{gBS}}$ of *generalized Bott-Samelson bimodules*, which is generated by the bimodules $B_J \stackrel{\text{def}}{=} R \otimes_{R^J} R\{-l(J)\}$. Here, $l(J)$ indicates the length of the longest element w_J of the finite subgroup W_J. Though not immediately obvious, it is true that $\mathcal{B}_{\text{gBS}} \subset \mathcal{B}$.

When W is a finite group, every subset of $\mathcal{S}$ is finitary, and the set of all rings R^J for all subsets forms a *Frobenius hypercube*, in the sense of [9]. This is to say that whenever $J \subset I \subset \mathcal{S}$, every ring extension $R^I \subset R^J$ is actually a Frobenius extension, and that these extensions are mutually compatible in some sense. This has immediate applications to understanding the morphisms in $\mathcal{B}$, and implies that the category can be depicted diagrammatically. Using diagrammatics to study categorification was pioneered by Khovanov and Lauda [18, 19].

We will be examining the case of the dihedral group in this paper. In this case, $\mathcal{S} = \{s, t\}$ consists of two elements, and the group $W = W_m$ is determined by a single number $m = m_{s,t}$, which is either ∞ or a natural number ≥ 2.

1.1.2 General subtlety; dihedral simplicity

Soergel proves in [27] that there is one indecomposable Soergel bimodule B_w for each element $w \in W$, characterized by some condition depending on w. What exactly these modules are is a difficult question. One might expect that the image of B_w in the Grothendieck group will be the Kazhdan-Lusztig basis element b_w, but this is simply false in general. The *Soergel Conjecture* (see Conjecture 1.13 in [27] and following) states that $[B_w] = b_w$ for all $w \in W$ in an arbitrary Coxeter group, when we define R over $\mathbb{C}$ as above, or any field $\Bbbk$ of characteristic 0. On the other hand, there are known cases where one can choose a base field $\Bbbk$ of finite characteristic for which some $[B_w] \neq b_w$. Passing to finite characteristic may remove idempotents, changing which objects are indecomposable.

The case of a dihedral group is trivial in this regard: Soergel explicitly constructed the indecomposables B_w and proved that they descend to the Kazhdan-Lusztig basis, when working over any infinite field $\Bbbk$ of characteristic $\neq 2$ (see chapter 4 in [27]). It seems to the author that Soergel's techniques could be easily generalized to a broader class of base ring $\Bbbk$.

The reason that dihedral groups are easy is that b_w is *smooth* for all w. This is an algebraic condition on the Kazhdan-Lusztig basis element, stating that $b_w = v^{l(w)} \sum_{x \leq w} T_x$ where $\{T_w\}$ is the standard basis. For Weyl groups, this corresponds to a geometric condition: that the closure of the Bruhat orbit $\mathcal{O}_w$ in the flag variety G/B is smooth (or rationally smooth, in the sense of intersection cohomology). In this case, the indecomposable B_w is (up to a shift) none other than the equivariant cohomology of the orbit closure. When $w = w_J$ is a longest element of a parabolic subgroup, the closure of $\mathcal{O}_w$ is P/B for the appropriate parabolic P; this is smooth, and one can deduce that $B_w = B_J$. While there is no geometry for general dihedral groups, Soergel performs an analogous algebraic construction to produce B_w for smooth elements. This constructive method, while useful for smooth elements, does not seem to generalize well.

Another way to produce indecomposable Soergel bimodules would be to find the idempotents which express them as direct summands of tensor products $B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_d}$ in $\mathcal{B}_{\text{BS}}$. Soergel's results imply that B_w occurs with multiplicity one in such a tensor product, when $w = s_1 s_2 \cdots s_d$ is a reduced expression. Finding these idempotents hinges upon a thorough understanding of morphism spaces in $\mathcal{B}_{\text{BS}}$. Explicitly constructing these idempotents and understanding the coefficients which appear would also give a direct understanding of how the Soergel Conjecture depends on the choice of base field $\Bbbk$. This is the method we pursue here.

1.1.3 The Main Results

The primary result of this paper is a presentation of the morphisms in the category $\mathcal{B}_{\text{BS}}$ by generators and relations, when W is a dihedral group. The presentation will be given in terms of planar diagrams. As part of this, we give an explicit description of the idempotents which pick out each indecomposable B_w.

The same presentation has been given before by Libedinsky [20] for the "right-angled" cases $m = 2, \infty$. His work is complimentary, as he does not discuss idempotents, or connections to the Temperley-Lieb algebra, and his proofs are entirely different.

A morphism will be represented by a graph with boundary properly embedded in the planar strip $\mathbb{R} \times [0, 1]$. The edges of this graph are labelled by elements of $\mathcal{S}$, which we call "colors." The only vertices appearing are univalent vertices, trivalent vertices joining three edges of the same color, and if m is finite, vertices of valence $2m$ whose edge labels alternate between the two colors. A number of relations are placed on these graphs, which (after some abstraction) can be represented in a way independent of the finite value of m.

A more significant goal would be to find a diagrammatic presentation for $\mathcal{B}_{\text{BS}}$ in the case of an arbitrary Coxeter group. For type A, this was done by the author and M. Khovanov in [7]. The form of the presentation in type A is revealing. The objects of $\mathcal{B}_{\text{BS}}$, as we know, are generated by objects B_s associated to a single color $s \in \mathcal{S}$. The morphisms are generated by one-color morphisms for each color, and two-color morphisms for each pair $s, t \in \mathcal{S}$, corresponding to the $2m$-valent vertex for the corresponding dihedral group. The relations are generated by one-color, two-color, and three-color relations. In this paper, for an arbitrary 2-color Coxeter group, we find the new 2-color generator and formulae for all the 2-color relations, for which the 2-color relations of [7] are the special cases $m_{s,t} = 2, 3$. However, there is no 2-color generator for $m_{s,t} = \infty$! In fact, the general story follows a similar pattern: objects are determined by size 1 finitary subsets of $\mathcal{S}$, generating morphisms by size 2 finitary subsets, and generating relations by size 3 finitary subsets. This is work in progress with Geordie Williamson.

Having a diagrammatic presentation in type A has led to numerous other results:

- Categorifications of induced trivial modules [5].
- Diagrammatics for $\mathcal{B}_{\mathrm{gBS}}$ [5].
- A categorification of the Temperley-Lieb quotient of $\mathcal{H}$, and an identification of Hom spaces therein with coordinate rings of subvarieties in $\mathfrak{h}$ [6].
- Diagrammatics for the entire 2-category of Singular Soergel bimodules [8] (work in progress with Geordie Williamson).

We provide the dihedral analogs of all these results in this paper. We do not prove that the Temperley-Lieb categorfication works, but do provide a sketch of the proof. It should not be necessary for the reader to read those papers to understand the arguments or results herein.

1.1.4 Coefficients

Instead of defining $\mathcal{B}$ over the field $\mathbb{C}$, we try to define it over a ring $\Bbbk$ which is as general as possible. There are several motivations for this pedantry. Studying Soergel bimodules over fields of arbitrary characteristic is of considerable interest for questions in modular geometric representation theory. Studying Soergel bimodules over $\mathbb{Z}$ (or as close as one may come) may be of interest to knot theorists or topological quantum field theorists, in order to define invariants which may have torsion. While no applications are given in this paper, we attempt to be responsible for the future. For each result (outside of the introduction) we specify what properties $\Bbbk$ must satisfy in order for the result to hold.

We never change the base ring of the Hecke algebra $\mathcal{H}$, which is always $\mathbb{Z}[v, v^{-1}]$. We only change the base ring of potential categorifications. In general, this base ring will be an extension of the ring $\mathbb{Z}[x, y]$, where x and y are two parameters determined by a choice of basis for the linear terms in R.

1.2 Temperley-Lieb

1.2.1 The basics

The *Temperley-Lieb category* $\mathcal{TL}$ is a monoidal algebroid governing the category of representations of quantum $\mathfrak{sl}_2$. Let us first consider the semi-simple version of the story, where the category is defined to be $\mathbb{Q}(q)$-linear. The objects are given by $n \in \mathbb{N}$, and $\mathrm{Hom}_{\mathcal{TL}}(n, m) = \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V^{\otimes n}, V^{\otimes m})$, where V is the standard representation. In particular, within $\mathrm{End}_{\mathcal{TL}}(n)$ one can find idempotents corresponding to the projections of $V^{\otimes n}$ to its indecomposable summands. In the Karoubi envelope of $\mathcal{TL}$ there is one indecomposable V_n for each $n \in \mathbb{N}$, first appearing as an idempotent inside $V^{\otimes n}$. This idempotent is called the *Jones-Wenzl projector* JW_n ([16, 30]). The endomorphism ring $TL_n = \mathrm{End}_{\mathcal{TL}}(n)$ is commonly known as the *Temperley-Lieb algebra* on n strands. It is a quotient of $\mathcal{H}_{S_n}$, the Hecke algebra in type A.

The Temperley-Lieb algebra first appeared in [28], and was used for the study of subfactors by Jones in [15]. Most useful is the diagrammatic description of the Temperley-Lieb category given by Kauffman [17], using crossingless matchings. A crossingless matching is essentially a 1-manifold with boundary properly embedded in the planar disk, separating that disk into regions. A closed 1-manifold evaluates to a polynomial in $\mathbb{Q}(q)$. For example, a circle evaluates to $-[2] = -(q+q^{-1})$. We will mostly discuss $\mathcal{TL}$ in terms of crossingless matchings, but we will often use the description as the intertwiner category for $U_q(\mathfrak{sl}_2)$ in order to deduce certain facts. Jones-Wenzl projectors can be understood diagrammatically as linear combinations of crossingless matchings which satisfy certain conditions, and can be calculated using recurrence relations.

There is a minor generalization of the Temperley-Lieb category known as the *two-colored Temperley-Lieb 2-category* $2\mathcal{TL}$. Consider a crossingless matching, and color each region of the planar disk with one of two colors (say, red and blue) so that adjacent regions alternate colors. This is a 2-morphism in $2\mathcal{TL}$. Each crossingless matching can be colored in precisely 2 ways, giving two different (though isomorphic) 2-morphism spaces, so the difference between $\mathcal{TL}$ and $2\mathcal{TL}$ is mostly bookkeeping. However, one may evaluate a circle in two different ways based on the color on the interior, so that the category is now $\mathbb{Q}(x,y)$-linear. The product $\xi = xy$ of these two circles is known as the *index*. If one fixes the color which appears on the far left, one almost has a copy of $\mathcal{TL}$ embedded inside $2\mathcal{TL}$ except that there is an additional parameter which complicates calculations. Nonetheless, one still has Jones-Wenzl projectors in the two-color case.

In the literature, people typically work with one of two specializations of $\mathbb{Q}(x,y)$: the *spherical* case, where $x = y$, and the *lopsided* case, where $x = 1$ and $y = \xi$ (see section 2.5 of [22]). In fact, the general case $\mathbb{Q}(x,y)$ is a perturbation of the spherical case, in the sense of [4]. Because the general case is not often used, it may be difficult to look up formulae for Jones-Wenzl projectors with coefficients in $\mathbb{Q}(x,y)$. We will often state a formula for $2\mathcal{TL}$ and only give a reference for the analogous formula in $\mathcal{TL}$; we leave the modification to the reader.

1.2.2 Roots of Unity

The Temperley-Lieb category can be defined over $\mathbb{Z}[q+q^{-1}]$. Every quantum number in q can be expressed as a polynomial in $[2]=q+q^{-1}$. We write ζ_m for an arbitrary primitive m-th root of unity. The statement that $q^2=\zeta_m$ is equivalent to the statement that $[m]=0$ and $[n]\neq 0$ for $n<m$. So we can specialize $\mathbb{Z}[q+q^{-1}]$ algebraically to the case where $q^2=\zeta_m$ by setting the appropriate polynomial in [2] equal to zero. In the case m odd, $q^2=\zeta_m$ allows q itself to be either ζ_m or ζ_{2m}, and one can distinguish these two cases with a further specialization.

Passing to this quotient drastically affects the representation theory of $\mathcal{TL}$, and the indecomposables in its Karoubi envelope. The representations of this algebroid are no longer semi-simple. The Jones-Wenzl projector on n strands has $[n]!$ in the denominator of its coefficients, so that it is no longer defined when $n\geq m$. The Jones-Wenzl projector on $m-1$ strands is well-defined, however it becomes *negligible*. Diagrammatically, a linear combination of crossingless matchings is negligible if all ways to glue it into a closed 1-manifold evaluate to 0. Algebraically, this says that it is in the kernel of any reasonably adjoint bilinear form on TL_{m-1}. In fact, JW_{m-1} generates the monoidal ideal of negligible morphisms in $\mathcal{TL}$. It is common to study the category $\mathcal{TL}_{\text{negl}}$ obtained by killing all negligible maps, because this directly relates to topological quantum field theories. In fact, Jones' original application of Temperley-Lieb to subfactors in [15] used the negligible quotient. For more on killing negligible morphisms in general, see chapter 2 of [2], which references [29].

The two-colored version can be defined over $\mathbb{Z}[x,y]$, and there are also specializations of $\mathbb{Z}[x,y]$ corresponding to roots of unity. The behavior of colored Jones-Wenzl projectors at a root of unity is the same as the uncolored case.

Working modulo $[m]=0$, the negligible Jones-Wenzl projector JW_{m-1} is actually rotation-invariant. This fact, though fairly trivial, seems not to be commonly known, and is crucial in this paper. Rotation in the Temperley-Lieb algebra has been studied before (see [14, 12]), but typically in the negligible quotient.

Let us be more precise. Rotating JW_{m-1} by two strands does nothing. Rotating JW_{m-1} by one strand will multiply the map by 1 if $q = \zeta_{2m}$ and by -1 if $q = \zeta_m$. The two-colored Jones-Wenzl projector can only be rotated by two strands (to preserve colors), and this does nothing.

1.2.3 Connections to Soergel Bimodules

We now work in the infinite dihedral group (so that $m_{s,t} = \infty$), and we also let s and t label our two colors. Consider the Soergel bimodule $M_d = \underbrace{B_s \otimes B_t \otimes B_s \otimes B_t \otimes \cdots}_{d+1}$ whose indices alternate. Let $i \in \{s,t\}$ be the index of the final tensor; it is s if d is even and t otherwise. Inside the 2-category $2\mathcal{TL}$ we have the 1-category $\mathrm{Hom}(i,s)$ consisting of all diagrams whose leftmost color is s and rightmost is i. Inside this 1-category there is an object d corresponding to the picture with d strands, and an object d' for any d' with the same parity as d.

Proposition 1.2.1. *(For $m_{s,t} = \infty$) Suppose that $d, d' \geq 0$ have the same parity. The graded $\mathbb{Z}[x,y]$-module $\mathrm{Hom}_{\mathcal{B}}(M_d, M_{d'})$ is concentrated in non-negative degree. The degree 0 part is isomorphic to $\mathrm{Hom}_{\mathrm{Hom}_{2\mathcal{TL}}(i,s)}(d,d')$, and this isomorphism is compatible with composition.*

Remark 1.2.2. The same statement holds with the colors switched. By contrast, if $\widetilde{M_d}$ is defined the same way except starting with B_t instead of B_s, then the Hom spaces $\mathrm{Hom}_{\mathcal{B}}(M_d, \widetilde{M_{d'}})$ and $\mathrm{Hom}_{\mathcal{B}}(\widetilde{M_{d'}}, M_d)$ are concentrated in strictly positive degrees, for any d, d'. There is no corresponding morphism space in $2\mathcal{TL}$, because the object on the left does not even match up.

Other Hom spaces in $\mathcal{B}$ reduce to these. There will be an isomorphism $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$, so we can remove duplications in a tensor product and assume that the sequence alternates.

We will interpret this proposition to imply that the two-colored Temperley-Lieb 2-category essentially controls the morphisms of minimal degree between Soergel bimodules for W_∞.

When looking at a graded category, all idempotents will be in degree 0. Therefore, the idempotent endomorphisms of M_d can be understood in terms of idempotents in TL_d (or rather, its two-colored version with parameters x and y). In particular, over $\mathbb{Q}(x,y)$ we have Jones-Wenzl projectors which give indecomposable objects in the Karoubi envelope. These will pick out the indecomposable Soergel bimodules B_w, when $\underbrace{stst\cdots}_{d+1}$ is a reduced expression for w. Of course, the full category $\mathcal{B}$ is more complicated than its degree 0 part, but the degree 0 part is sufficient for understanding the Grothendieck group and the Karoubi envelope.

Now fix $2 \le m < \infty$ and let $W = W_m$ be the finite dihedral group. Simultaneously, let $\mathbb{Z}[x,y]$ be specialized appropriately so that q^2 is an m-th root of unity. The proposition above still holds whenever $d + d' \le 2(m-1)$. However, we now have a new map of degree 0 from $M = \underbrace{B_s \otimes B_t \otimes \cdots}_{m}$ to $\widetilde{M} = \underbrace{B_t \otimes B_s \otimes \cdots}_{m}$, where these are the two reduced expressions for the longest element w_0. This map is the projection from M to the common summand B_{w_0} followed by the inclusion into $\widetilde{M}$. Similarly, there is a map $\widetilde{M} \to M$. As noted above, these new degree 0 maps can not correspond to anything in $\mathcal{2TL}$ because the boundaries do not even match up correctly. However, following the new maps twice $M \to \widetilde{M} \to M$ yields an endomorphism of M which gives the projection to the indecomposable B_{w_0}, and is therefore equal to the negligible Jones-Wenzl projector JW_{m-1}! Roughly speaking, we've added "square roots" of the negligible Jones-Wenzl projector (or rather, two new maps α and β with $\alpha\beta$ equal to JW_{m-1} with one color on the left, and $\beta\alpha$ equal to JW_{m-1} with the other color on the left).

Proposition 1.2.3. *(For $m_{s,t} < \infty$) The degree 0 morphisms in $\mathcal{B}$ form some non-trivial "extension" of the 2-category $\mathcal{2TL}$ when $q^2 = \zeta_m$, generated by "square roots" of the negligible Jones-Wenzl projector.*

Of course, what we mean by an "extension" of the 2-category is not entirely clear, since this new map can not exist in a 2-category, and the 2-category must be demoted to some kind of monoidal category before any sense can be made. We will not be more precise than this.

1.2.4 Terminological Disasters

Technically, for any Coxeter group W there is a *Temperley-Lieb algebra* TL_W for W, the quotient of $\mathcal{H}_W$ by the elements b_{w_J} where J ranges over all size 2 finitary parabolic subsets *except* those where $m_{s,t} = 2$. The algebra commonly known as the Temperley-Lieb algebra is the case for type A.

However, we will also need to use the Temperley-Lieb algebra TL_{W_m} for the dihedral group of size $2m$. This is the quotient of $\mathcal{H}$ by b_{w_0}. To categorify this, we will take the quotient of $\mathcal{B}$ by the ideal generated by all morphisms factoring through B_{w_0}. This kills the negligible Jones-Wenzl projector, and it kills the new square roots thereof, meaning that what remains of the degree 0 part is generated by the images of Temperley-Lieb elements.

Proposition 1.2.4. *There is a categorification of TL_{W_m} by a graded additive monoidal category. Within this category, certain degree 0 Hom spaces are given precisely by $2\mathcal{TL}_{\text{negl}}$.*

We will only provide a sketch of this result.

So we use Temperley-Lieb algebras in type A, at $2m$-th roots of unity, to category the Temperley-Lieb algebra in dihedral type! Not only are there two different Temperley-Lieb algebras, but there are two different sets of parameters. The Temperley-Lieb algebra TL_{W_m} is an algebra over $\mathbb{Z}[v, v^{-1}]$, and multiplication by v corresponds to grading shift upstairs. The categorification is $\mathbb{Z}[x, y]$-linear (or if we specialize, $\mathbb{Z}[q, q^{-1}]$-linear). Polynomials and quantum numbers in q should not be confused with those in v, though they will both appear! Yuck! The author denies any responsibility for this overloaded terminology - it is not his fault.

1.3 Structure of the paper

This paper is intended to be an omnibus of all things dihedral: a Dihedral Cathedral. It seems to the author that the dihedral group has been (perhaps rightly) ignored in the basic study of Kazhdan-Lusztig theory, because it is trivial algebraically and unconnected to geometry. As a result, there is not much literature describing the case of the dihedral group in detail, so a large quantity of background information is warranted.

This paper assumes very little outside knowledge. In introduction to diagrammatics for 2-categories can be found in [19], chapter 4. An introduction to Karoubi envelopes can be found in [1]. References to the author's earlier work occur only when the computation is simple enough to be left as an exercise.

In section 2.1 we discuss the Kazhdan-Lusztig presentations of the Hecke algebra and the Hecke algebroid, and traces on these objects. We also fix some basic notation. In section 2.2 we describe the polynomial ring on which W acts, its invariant subrings, and the Frobenius extension structure between them. We define Soergel bimodules and their 2-categorical analog Singular Soergel bimodules. Section 2.3 contains an introduction to Jones-Wenzl idempotents and their analogs for the two-colored Temperley-Lieb category. Counting colored regions in a Jones-Wenzl projector will yield a polynomial which cuts out all the reflection lines in $\mathfrak{h}$. In section 2.4 we discuss the standard trick played in categorification, whereby calculating certain Hom spaces in a category will be sufficient to prove strong categorification results.

In section 3.1 (resp. section 3.2) we provide diagrammatics for (resp. Singular) Soergel bimodules for the case $m = \infty$. In sections 4.1 and 4.2 we provide the additional generators and relations for the case $m < \infty$. The remaining sections 4.3 and 4.4 are simple consequences, giving a diagrammatic presentation for generalized Bott-Samelson bimodules and a categorification of the Temperley-Lieb algebra of W.

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Chapter 2

Background

2.1 The dihedral group and its Hecke algebra

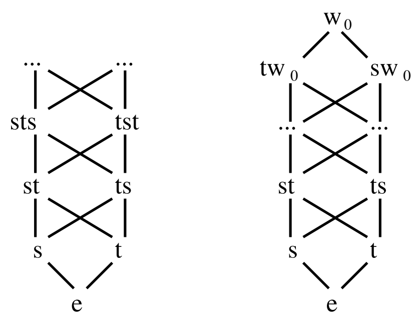
2.1.1 The dihedral group

In this paper, m will always represent either an integer ≥ 2 or ∞. We will be viewing the dihedral group as a Coxeter group with 2 generators, and we refer the reader to numerous easily-found sources for the basics of Coxeter groups, such as [13].

The *infinite dihedral group* W_∞ is freely generated by two involutions $s = s_i$ and $t = s_2$; in other words, the only relations are $s_i^2 = e$ for $i = 1, 2$, where e is the identity element. For any integer $m \geq 2$, the *finite dihedral group* W_m is the quotient of W_∞ by the relation $\underbrace{sts\ldots}_{m} = \underbrace{tst\ldots}_{m}$. An equivalent relation is $(st)^m = e$.

Notation 2.1.1. In Coxeter theory many things are labelled by the vertices of the Coxeter graph. In this case, there are only two vertices and a lot of text, so it will be simpler to write $stst\ldots$ rather than $s_1s_2s_1s_2\ldots$. We will be drawing pictures and using colors to represent the vertices as well. From now on, the ordered sets $\{1, 2\}$, $\{s, t\}$, and $\{b, r\}$ will all be identified with the vertices of the Coxeter graph, and elements will be called *colors* or *indices*. The letters b and r are short for "blue" and "red", and these colors will be used consistently. The letters i and j will always refer to indices, and we will write $\underline{\boldsymbol{i}} = i_1 i_2 \ldots i_d$ and $\underline{\boldsymbol{j}}$ for

Figure 2.1: The Hasse diagrams of W



sequences of arbitrary indices, with "start" i_1 and "end" i_d, and *length* d. The length zero sequence is denoted $\emptyset$. We call $\underline{\boldsymbol{i}}$ *alternating* if the indices alternate between red and blue. Given a sequence $\underline{\boldsymbol{i}}$, $s_{\underline{\boldsymbol{i}}}$ will be the product $s_{i_1}s_{i_2}\ldots$. We say that $\underline{\boldsymbol{i}}$ is a reduced expression for $w \in W$ when $s_{\underline{\boldsymbol{i}}}$ is.

We use capital letters like I, J, K, L to denote subsets of $\{1, 2\}$, which we call *parabolic subsets*. For a parabolic subset J, the corresponding Coxeter subgroup is written W_J. We often use shorthand: $\{1, 2\}$ may be denoted $\underline{W}$, $\{s\}$ may be denoted $\underline{s}$, and $\emptyset$ may be denoted $\underline{e}$. We may also use $\underline{\boldsymbol{i}}$ for a sequence of parabolic subsets.

N.B. When we give a result, *the equivalent statement with s and t reversed will always be assumed.* The same is true with right multiplication vis a vis left multiplication.

The elements of W can be easily enumerated. When $m = \infty$, the elements are $\{e, s, t, st, ts, sts, tst, \ldots\}$. When $m < \infty$, this list terminates at $\underbrace{sts\ldots}_{m} = \underbrace{tst\ldots}_{m} = w_0$, where the longest element w_0 is the only element with multiple reduced expressions. There is a length function l on W, and the Bruhat order agrees entirely with the length function: if $l(x) < l(w)$ then $x < w$, and if $l(x) = l(w)$ and $x \neq w$ then x and w are incomparable. The length function and Bruhat order are encapsulated in Figure 2.1, for W_∞ and W_m respectively. This diagram is not the *Bruhat graph*, which is something different, but merely the Hasse diagram of the Bruhat order; we will still refer to it as the Bruhat chart.

The *Poincare polynomial* $\widetilde{\pi}(W)$ of a Coxeter group W is $\sum_{w \in W} v^{2l(w)}$, an element of $\mathbb{Z}[[v]]$.

For finite Coxeter groups, the *balanced Poincare polynomial* $\pi(W)$ is $\frac{\widetilde{\pi}(W)}{v^{l(w_0)}}$, an element of $\mathbb{Z}[v, v^{-1}]$ which is invariant under flipping v and v^{-1}. We also write this as $[W]$, "quantum W". When J is a parabolic subset, we write $[J]$ for the balanced Poincare polynomial of W_J. Note that $(v + v^{-1})$ is the balanced Poincare polynomial of any singleton J. We will always write out $(v + v^{-1})$, preserving [2] exclusively for polynomials in the variable q!

2.1.2 The Hecke Algebra

The Hecke algebra $\mathcal{H} = \mathcal{H}_m$ is a $\mathbb{Z}[v, v^{-1}]$-algebra with several useful presentations. The *standard presentation* has generators T_i, $i = 1, 2$ with relations

$$T_i^2 = (v^{-2} - 1)T_i + v^{-2}1 \tag{2.1.1}$$

$$\underbrace{T_1T_2T_1\dots}_{m} = \underbrace{T_2T_1T_2\dots}_{m} \tag{2.1.2}$$

This second relation is suppressed when $m = \infty$. We write $T_{\underline{i}}$ for $T_{i_1}T_{i_2}\cdots T_{i_d}$. We define the *standard basis* by $T_w \overset{\text{def}}{=} T_{\underline{i}}$ whenever $\underline{\boldsymbol{i}}$ is a reduced expression for $w \in W$. The identity of $\mathcal{H}$ is T_e. These T_w, for $w \in W$, form a basis of $\mathcal{H}$ as a free $\mathbb{Z}[v, v^{-1}]$-module.

In general, a $\mathbb{Z}[v, v^{-1}]$-linear map $\mu\colon \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ satisfying $\mu(ab) = \mu(ba)$ is called a *trace*. We also allow traces to take values in $\mathbb{Z}[[v, v^{-1}]]$. One can show that the map ε given by $\varepsilon(T_w) = \delta_{w,1}$ is a trace, called the *standard trace*. Later, we will be identifying this trace as the graded rank of certain free modules over R, the polynomial ring in two variables. Therefore, if we wanted the graded dimension of these modules over the ground ring we would have to multiply by the graded dimension of R, which is $\frac{1}{(1-v^2)^2}$. We call this renormalized trace $\widetilde{\varepsilon}$.

The Hecke algebra also has a *Kazhdan-Lusztig basis* $\{b_w\}$. This basis can be defined implicitly as the unique basis which satisfies certain criteria (see [13] for a basic introduction). However, for the case of dihedral groups, the solution to these criteria is particularly easy.

Claim 2.1.2. *For all* $w \in W$, $b_w = v^{l(w)} \sum_{x \leq w} T_x$. *This holds for* m *finite or infinite.*

For instance, $b_i \overset{\text{def}}{=} b_{s_i} = v(T_i + 1)$ for $i = 1, 2$, and when W is finite $b_W \overset{\text{def}}{=} b_{w_0} = v^m \sum_{w \in W} T_w$. We are using e to represent the identity of W, so there is no ambiguity between $b_1 = b_{s_1}$ and $b_e = 1 \in \mathcal{H}$. When $\underline{\boldsymbol{i}}$ is a sequence, $b_{\underline{i}}$ denotes the product of each b_{i_l} as usual. Note that this is quite different from $b_{s_{\underline{i}}}$.

Clearly $\varepsilon(b_w) = v^{l(w)}$. Another useful structure on $\mathcal{H}$ is the v-antilinear antiinvolution ω defined by $\omega(b_i) = b_i$. This allows one to put a semi-linear product on $\mathcal{H}$ via $(x, y) \overset{\text{def}}{=} \varepsilon(\omega(x)y)$. Conversely, $\varepsilon(x) = (1, x)$. Arbitrary traces μ are in bijection with semi-linear products for which b_i is self-biadjoint, by replacing ε with μ in these formulas. Since a trace is determined by its values at each b_w, the corresponding semi-linear product is determined by the values $(1, b_w)$.

As $\{T_1, T_2\}$ generates $\mathcal{H}$, so does $\{b_1, b_2\}$. The presentation involving generators b_i is called the *Kazhdan-Lusztig presentation*, and it will take a little work to derive. The quadratic relation is easy:

$$b_i^2 = (v + v^{-1})b_i \tag{2.1.3}$$

When $m = \infty$ this relation clearly suffices. In the finite case there is one more relation, which relates b_1 and b_2 using the fact that both can be used to express the longest element b_{w_0}. For pedagogical reasons, however, we shall discuss the case $m = \infty$ in more depth before giving this additional relation (2.1.4).

2.1.3 Three related recursions

In this section, $W = W_\infty$. We are interested in expressing the Kazhdan-Lusztig basis elements b_w in terms of the generators b_i. First we go the other way, expressing products $b_{\underline{i}}$ in terms of b_w. The following two claims demonstrate what happens when b_i is multiplied by b_w.

Claim 2.1.3. *Let $w \in W$ be an element whose reduced expression starts with t. Then $b_t b_w = (v + v^{-1})b_w$.*

Claim 2.1.4. *Let $w \in W$ be an element whose reduced expression starts with st. Then $b_t b_w = b_{tw} + b_{sw}$. In the Bruhat chart of Figure 2.1, this sends an element to the sum of the two elements diagonally attached to it.*

Proof. Given Claim 2.1.2, these are simple exercises for the reader. ☐

Example 2.1.5. This implies that $b_t b_{st} = b_{tst} + b_t$, $b_s b_{tst} = b_{stst} + b_{st}$, and so forth. However, $b_s b_t = b_{st}$ and $b_s b_e = b_s$.

The second claim gives us an iterative way of understanding the decomposition of the product $b_{\underline{i}}$ when $\underline{\boldsymbol{i}}$ is alternating into sums of b_w. In fact, the same recursive formula appears in several other places, and this is no accident. We let $[n]$ denote the n-th *quantum number* in q, which is $\frac{q^n - q^{-n}}{q - q^{-1}}$.

Claim 2.1.6. *We have $[2][n] = [n+1] + [n-1]$ for $n \geq 2$.*

The correspondence between this formula and Claim 2.1.4 is clear, when we send $[n]$ to the length n alternating sequence (ending, say, with s) and we think of multiplication by $[2]$ as multiplying by the next b_s or b_t in line. Note that $[2][1] = [2]$, just as $b_t b_s = b_{ts}$.

Now let $V = V_1$ be the standard two-dimensional representation of $\mathfrak{sl}_2$ (or its quantum analog), and let V_l denote the $l+1$-dimensional irreducible representation.

Claim 2.1.7. *We have $V \otimes V_l \cong V_{l+1} \oplus V_{l-1}$ for $l \geq 1$. Meanwhile, $V \otimes V_0 \cong V_1$.*

By taking the q-character of these representations we obtain Claim 2.1.6, since the character of V_l is $[l+1]$. Because of this, we see that the same combinatorics should govern the decomposition of $V^{\otimes n}$ into irreducibles as should govern the decomposition of $b_{\underline{i}}$ alternating into a sum of b_w. Using the inverse matrix, the same combinatorics governs writing $[n]$ as a polynomial in $[2]$ as governs writing b_w as a linear combination of monomials in b_s and b_t.

One can encode these familiar decomposition numbers in "truncated Pascal triangles."

Definition 2.1.8. Let the integer c_p^q be determined from the following table, which is populated by letting each entry be the sum of the one or two entries diagonally below.

5		9		5		1
	5		4		1	
2		3		1		
	2		1			
1		1				
	1					
1						

We number the rows starting with 1 and number the columns so that each row has only every other column. We let c_p^q be the entry in the p-th row and q-th column, and we say that $q \in c(p)$ if q is a column with an entry on the p-th row. For example, $c_1^1 = 1$, $c_2^2 = 1$, $c_3^1 = c_3^3 = 1$, and $c_4^2 = 2$.

The following claim is obvious from the inductive formulae. The indexing is slightly annoying, but we have chosen it so that it is most convenient for the Hecke algebra.

Claim 2.1.9.

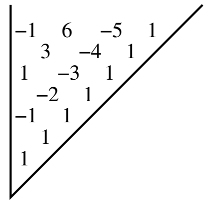
- *For $p \geq 1$ we have that $V^{\otimes(p-1)} \cong \oplus_{q \in c(p)} V_{q-1}^{\oplus c_p^q}$.*
- *For $p \geq 1$ we have that $[2]^{p-1} = \sum_{q \in c(p)} c_p^q [q]$.*
- *Let us only consider elements of W whose reduced expressions end in s. Let $\underline{\boldsymbol{i}}_p = \underbrace{\ldots sts}_{p}$ for $p \geq 1$ be an alternating sequence, and let $w_p \in W$ be the element with that reduced expression. Then for $p \geq 1$, $b_{\underline{\boldsymbol{i}}_p} = \sum_{q \in c(p)} c_p^q b_{w_q}$.*

Example 2.1.10. $b_s b_t b_s b_t b_s b_t = b_{ststst} + 4b_{stst} + 5b_{st}$.

Note that this claim covers one zigzag path up the Bruhat chart, and the same claim with s and t switched covers the complimentary zigzag path. The element e is not addressed by this claim at all, and b_e never appears in the decomposition of any $b_{\underline{\boldsymbol{i}}}$ where $\underline{\boldsymbol{i}} \neq \emptyset$. After all, b_s and b_t generate a non-trivial ideal.

Now we invert matrices to solve our original question.

Definition 2.1.11. Let the integer d_p^q be determined from the following table (with the same conventions as before), which is populated by letting $d_p^q = d_{p-1}^{q-1} - d_{p-2}^q$.



Claim 2.1.12. • *For $p \geq 1$, in the Grothendieck group of $\mathfrak{sl}_2$ representations, we have that $[V_{p-1}] = \sum_{q \in c(p)} d_p^q [V_{q-1}]$.*

- *For $p \geq 1$ we have that $[p] = \sum_{q \in c(p)} d_p^q [2]^{q-1}$.*

- *With the same conventions as in Claim 2.1.9, for $p \geq 1$, $b_{w_p} = \sum_{q \in c(p)} d_p^q b_{\underline{\boldsymbol{i}}_q}$.*

These recursion formulae give us all we need to know about $\mathcal{H}_\infty$.

2.1.4 The finite case

In this section, $W = W_m$ for $m < \infty$.

Because of Claim 2.1.2 and because the Bruhat charts of W_∞ and W_m agree for elements of length $< m$, we see that the formulas of Claims 2.1.9 and 2.1.12 for writing $b_{\underline{\boldsymbol{i}}}$ in terms of b_w and vice versa will still hold for all $p < m$. A moment's thought will confirm that they hold for $p = m$ as well. Now we see that the missing relation for the Kazhdan-Lusztig presentation, analogous to (2.1.2), can be obtained from the two expressions for $b_W = b_{w_0} = b_{sts\ldots} = b_{tst\ldots}$. For the following relation, let $\underline{\boldsymbol{i}}_q^s$ be the length q alternating sequence ending with s, and let $\underline{\boldsymbol{i}}_q^t$ end with t, for $q \geq 1$.

$$\sum_{q \in c(m)} d_m^q b_{\underline{\boldsymbol{i}}_q^s} = b_{w_0} = \sum_{q \in c(m)} d_m^q b_{\underline{\boldsymbol{i}}_q^t} \tag{2.1.4}$$

Note that the recursive formula of Claim 2.1.4 will terminate at w_0, because $w_{\underline{\boldsymbol{i}}_m^s} = w_{\underline{\boldsymbol{i}}_m^t}$ so that $b_t b_{\underline{\boldsymbol{i}}_m^s} = (v + v^{-1}) b_{\underline{\boldsymbol{i}}_m^s}$. If one wished to find the coefficients of $b_{\underline{\boldsymbol{i}}_p}$ for $p > m$, it would not be difficult to modify the triangle of Definition 2.1.8 into a "Pascal trapezoid" with at most m columns.

It is easy to deduce the following equalities from the facts above. Remember that $[W] = v^{-m}(1 + 2v^2 + 2v^4 + \ldots + 2v^{2(m-1)} + v^{2m})$.

$$b_i b_W = b_W b_i = (v + v^{-1}) b_W \tag{2.1.5}$$

$$b_W b_W = [W]\, b_W \tag{2.1.6}$$

In particular, the right or left ideal of b_W is rank 1 as a $\mathbb{Z}[v, v^{-1}]$-module, spanned by b_W. An element of the ideal is thus determined by its image under ε. Moreover, if x is any element in a $\mathcal{H}$-module for which $b_i x = (v + v^{-1})x$ for any $i = 1, 2$, it must be the case that $b_W x = [W]\, x$ (this is more obvious in the standard basis; see [31], Lemma 2.2.3).

2.1.5 The Hecke Algebroid

The Hecke algebroid is a general construction for Coxeter groups, a key reference for which being [31]. We give the simple description here, in such a way that the reader could guess how to generalize it to other Coxeter groups.

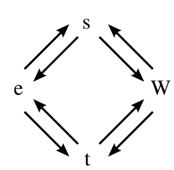
There are 4 parabolic subsets, $\underline{W}$, $\underline{s}$, $\underline{t}$, and $\underline{e}$. If J represents a finitary parabolic subset (which only ignores $\underline{W}$ when $m = \infty$), we write b_J for the Kazhdan-Lusztig basis element for the longest element of W_J, which with our notation is precisely b_e, b_s, b_t, or b_W.

The Hecke algebroid $\mathfrak{H}$ is a $\mathbb{Z}[v, v^{-1}]$-linear category with objects labelled by *finitary* parabolic subsets. The $\mathbb{Z}[v, v^{-1}]$-module $\mathrm{Hom}_{\mathfrak{H}}(J, K)$ is the *intersection* in $\mathcal{H}$ of the left ideal $\mathcal{H} b_J$ with the right ideal $b_K \mathcal{H}$. Composition from $\mathrm{Hom}(J, K) \times \mathrm{Hom}(L, J) \to \mathrm{Hom}(L, K)$ is given by renormalized multiplication in the Hecke algebra: we multiply the two elements and divide by $[J]$. From this, it is clear that b_J is the identity element of $\mathrm{End}(J)$. It is also clear that $\mathrm{End}(\underline{e}) = \mathcal{H}$ as an algebra. Whenever $J \subset K$, b_K is in both the right and left ideal of b_J, and therefore there is an inclusion of ideals yielding $\mathrm{Hom}(K, L) \subset \mathrm{Hom}(J, L)$. This inclusion is realized by precomposition with $b_K \in \mathrm{Hom}(J, K)$. A similar statement can be made about $\mathrm{Hom}(L, K) \subset \mathrm{Hom}(L, J)$ and postcomposition with $b_K \in \mathrm{Hom}(K, J)$.

This construction of the Hecke algebroid should be reminiscent of idempotization. When A is an algebra with a family of idempotents e_α, the idempotization is a category with objects parametrized by α and Hom spaces given by $e_\beta A e_\alpha$. This category will be equivalent to a subcategory of projective A-modules. Instead, we have a modification, where A has a family of "almost-idempotents" satisfying $e_\alpha^2 = c_\alpha e_\alpha$ for some (not necessarily invertible) scalars c_α. One must use the intersection $e_\beta A \cap A e_\alpha$ and must renormalize the multiplication in order to have similar properties.

2.1.6 Presenting the Hecke algebroid as a quiver algebroid

Whenever $J \subset K$ we may view b_K as an element both of $\mathrm{Hom}(J, K)$ and $\mathrm{Hom}(K, J)$. It turns out that these maps generate the algebroid. Moreover, whenever $J \subset K \subset L$ it is clear that $b_L \star b_K = b_L$, where $\star$ denotes composition in $\mathfrak{H}$, in this case from $\mathrm{Hom}(K, L) \times \mathrm{Hom}(J, K) \to \mathrm{Hom}(J, L)$. Similarly $b_K \star b_L = b_L \in \mathrm{Hom}(L, J)$. Therefore, we need only consider these maps when $K \setminus J$ is a single index. We take these generators and view them as arrows in a path algebroid.



When $m = \infty$, there is no object $\underline{W}$ so there are only 3 vertices and 2 doubled edges in this quiver.

We may describe morphisms using paths between parabolic subsets. By $b_{\underline{esWtWs}}$ we mean the morphism which follows this path, from $\underline{s}$ up to $\underline{W}$ and eventually to $\underline{e}$. For instance, $b_{\underline{s}}$ would be the identity morphism of $\underline{s}$. What relations should be imposed between these arrows? In $\mathcal{H} = \mathrm{End}(\underline{e})$ one knows that $b_i^2 = [2]\, b_i$, so that $b_{\underline{esese}} = [2]\, b_{\underline{ese}}$. However, this equality will follow from a simpler relation: $b_{\underline{ses}} = (v + v^{-1}) b_{\underline{s}}$.

Consider the map $b_{\underline{tesetes}}$. It is easy to observe that this is precisely $b_{\underline{i}}$ for $\underline{\boldsymbol{i}} = tsts$, seen as an element of $b_t \mathcal{H} \cap \mathcal{H} b_s = \mathrm{Hom}(\underline{s}, \underline{t})$. On the other hand, $b_{\underline{sWt}} = b_{w_0} \in \mathrm{Hom}(\underline{t}, \underline{s})$.

Therefore, we may rephrase the first equality of relation (2.1.4) as a statement in $\mathrm{Hom}(\underline{s}, \underline{t})$, and the second equality of (2.1.4) in $\mathrm{Hom}(\underline{t}, \underline{s})$, using these paths to relate both sides. This is the only interesting relation in the Hecke algebroid. In general, let us temporarily denote by a_p the path corresponding to $\underline{\boldsymbol{i}}_p^s$ in $\mathrm{Hom}(\underline{s}, \underline{i})$ for $p \geq 1$ (and i determined by the parity).

Proposition 2.1.13. *The following relations on paths define the Hecke algebroid $\mathfrak{H}$. Recall that i can represent an arbitrary index s or t.*

$$b_{\underline{iei}} = (v + v^{-1}) b_{\underline{i}} \tag{2.1.7}$$

In fact, this equation is sufficient for the case $m = \infty$.

$$b_{\underline{WiW}} = \frac{[W]}{v + v^{-1}} b_{\underline{W}} \tag{2.1.8}$$

$$b_{\underline{esW}} = b_{\underline{etW}} \tag{2.1.9}$$

$$b_{\underline{Wse}} = b_{\underline{Wte}} \tag{2.1.10}$$

$$b_{\underline{sWi}} = \sum_{q \in c(m)} d_m^q b_{a_q} \tag{2.1.11}$$

A similar relation holds for t.

Proof. It is clear that all these relations do hold in the Hecke algebroid, so that there is a map from this quiver algebroid with relations to $\mathfrak{H}$. It is easy to see that this map is surjective. We know what the ranks of all Hom spaces should be as free $\mathbb{Z}[v, v^{-1}]$-modules. It remains to show that these relations reduce all paths to a certain set of paths which has the correct size. This is a simple exercise for the reader. □

2.1.7 Traces on the Hecke algebroid

A *trace* on an algebroid is a $\mathbb{Z}[v, v^{-1}]$-linear map $\varepsilon_X \colon \mathrm{End}(X) \to \mathbb{Z}[v, v^{-1}]$ for each object X, such that $\varepsilon_X(ab) = \varepsilon_Y(ba)$ whenever $a \in \mathrm{Hom}(Y, X)$ and $b \in \mathrm{Hom}(X, Y)$.

Claim 2.1.14. *Any trace on the Hecke algebroid is determined by $\varepsilon_{\underline{e}}$.*

Proof. Because of the defining property of a trace, any endomorphism of any object which can be expressed as a path going through $\underline{e}$ has its value determined by $\varepsilon_{\underline{e}}$. The identity of every object in $\mathfrak{H}$ is given (up to a scalar) by a path through $\underline{e}$, since we have the relation $B_{J\underline{e}J} = [J]b_J$. Therefore, every path can be rewritten (up to a scalar) as a path through $\underline{e}$. ☐

It is not hard to show that the standard trace on $\mathcal{H}$ extends to a trace on $\mathfrak{H}$, also called the *standard trace*.

2.1.8 Induced trivial representations

Consider $\mathrm{Hom}(J, \underline{e})$ (resp. $\mathrm{Hom}(\underline{e}, J)$) for some J. This is none other than the left (resp. right) ideal generated by b_J inside $\mathcal{H}$, and the left (resp. right) action of $\mathrm{End}(\underline{e})$ on this space by composition is precisely the action of $\mathcal{H}$ on the ideal. The Hecke algebra for has a *trivial representation* $\mathbb{T}$, a left $\mathcal{H}$-module. It is free of rank 1 as a $\mathbb{Z}[v, v^{-1}]$-module, and b_i acts by multiplication by $v + v^{-1}$. When $m < \infty$ the trivial representation can be embedded inside the regular representation of $\mathcal{H}$ by looking at the left ideal of b_W. For a finitary parabolic subset $J \subset I$, we may induce the trivial representation of $\mathcal{H}_J$ up to $\mathcal{H}$ to get a representation $\mathbb{T}_J$, and this can also (less obviously) be embedded inside the regular representation of $\mathcal{H}$ by looking at the left ideal of b_J. Therefore, this induced trivial module is none other than $\mathrm{Hom}(J, \underline{e})$.

2.2 The Soergel categorification

2.2.1 The reflection representation

Let $V = \mathbb{R}^2$, equipped with the standard euclidean product. Choose two distinct lines in V which form angle θ. We let $q = e^{i\theta}$. Define an action of W_∞ on V by letting s be reflection across the first line, and t be reflection across the second. The product st will be rotation by 2θ, and the order of this rotation will determine which quotient W_m acts on V faithfully.

Let R be the polynomial ring of V, equipped with the induced action of W. The ring R is graded, with linear functionals placed in degree 2. Choose two linear functionals f_s and f_t, such that $s(f_s) = -f_s$ and $t(f_t) = -f_t$ (we also denote these f_1 and f_2, or f_r and f_b). Clearly $R \cong \mathbb{R}[f_s, f_t]$.

Let R^i denote the subring invariant under s_i. There is a map $\partial_i \colon R \to R^i$, known as the *Demazure operator* or *divided difference operator*, where $\partial_i(f) = \frac{f - s_i f}{f_i}$. It is well-defined since the numerator must divide the denominator, being s_i-anti-invariant. The map is clearly R^i-linear, of degree -2, and $\partial_i(f_i) = 2$. Also, $\partial_i(f) = 0$ if $f \in R^i$, and $\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g)$.

Claim 2.2.1. *There is an isomorphism of graded R^i-modules $R \cong R^i \oplus R^i\{2\}$.*

Proof. The map from left to right is $g \mapsto (\frac{1}{2}\partial_i(gf_i), \partial_i(g))$ and the map from right to left is $(g, h) \mapsto g + \frac{1}{2} f_i h$. We leave the calculation to the reader. □

In fact, we can say more: the ring R is actually a Frobenius extension of R^i. More details are found in the next section.

The Cartan matrix is $a_{i,j} \stackrel{\text{def}}{=} \partial_i(f_j)$. As usual, $a_{i,i} = 2$. Rescaling the generators f_s and f_t may alter $a_{i,j}$ but not the product $\xi = a_{s,t}a_{t,s} = 4\cos^2(\theta) = [2]^2$. Note that $\xi = 4$ is impossible because $\theta \neq 0$. We will not care about the action of W on V, but only on the algebraic story, the action of W on R. As such, it is unnecessary to work over the base field $\mathbb{R}$. We may work over any base ring $\Bbbk$ which contains the values $a_{i,j}$. If 2 is not invertible, we need to find another proof of Claim 2.2.1 (see section 2.2.4). Trigonometric identities will transform into algebraic statements about the values of $a_{i,j}$ and ξ, which we discuss in section 2.2.2. Henceforth, we will define everything algebraically, replacing $a_{s,t}$ and $a_{t,s}$ with variables x and y respectively.

Definition 2.2.2. We let the *universal ring* be $R_u = \mathbb{Z}[x, y, (xy - 4)^{-1}, f_s, f_t]$, graded with

f_s, f_t in degree 2 and x, y in degree 0. We place an action of the infinite dihedral group W_∞ on R_u via $s(f_s) = -f_s$, $s(f_t) = f_t - xf_s$, $t(f_t) = -f_t$ and $t(f_s) = f_s - yf_t$. The W_∞ action on the degree 0 part is trivial. We define the subrings R_u^i and the Demazure operators as before.

Notation 2.2.3. We let $\mathbb{K}$ denote the ring $\mathbb{Z}[x, y, (xy-4)^{-1}]$. We will always use $\Bbbk$ to represent a $\mathbb{K}$-algebra to which we might want to specialize. We let $R_\Bbbk = R_u \otimes_\mathbb{K} \Bbbk$. We will still denote the linear terms of $R_\Bbbk$ by V^*. When it is irrelevant which $\Bbbk$ is chosen, or when $\Bbbk$ is understood, we simply denote the ring by R.

We see that $x = \partial_s(f_t)$ and $y = \partial_t(f_s)$. Whenever we switch s and t, we should also switch x and y. We write $\xi = xy$. Because x and y are algebraically independent in R_u, the action of W_∞ is faithful.

2.2.2 Trigonometry and Two-colored Quantum Numbers

We now investigate algebraically the condition that $R_\Bbbk$ admits an action of W_m for $m < \infty$. The case when $x = y = -[2]$ for some q should be familiar to anyone well-versed in quantum numbers. In fact, what occurs below is a two-color variation on quantum numbers, appearing in the two-colored Temperley-Lieb algebra.

Using the basis $\{f_s, f_t\}$ for the degree 2 part of R_u, we see that the matrix of st is precisely $A = \begin{pmatrix} xy-1 & x \\ -y & -1 \end{pmatrix}$.

Definition 2.2.4. We define a family of polynomials $Q_k \in \mathbb{Z}[\xi]$ inductively, for $k \geq 1$. Start with $Q_0 = 0$ and $Q_1 = Q_2 = 1$. When k is odd, we let $Q_k = \xi Q_{k-1} - Q_{k-2}$. When k is even, we let $Q_k = Q_{k-1} - Q_{k-2}$.

Example 2.2.5. $Q_3 = \xi - 1$, $Q_4 = \xi - 2$, $Q_5 = \xi^2 - 3\xi + 1$.

Claim 2.2.6. *Consider the coefficients d_p^q from Definition 2.1.11. Then for k odd, $Q_k = \sum_{q \in c(k)} d_k^q \xi^{\frac{q-1}{2}}$. For k even, $Q_k = \sum_{q \in c(k)} d_k^q \xi^{\frac{q-2}{2}}$.*

Claim 2.2.7. *We have* $A^k = \begin{pmatrix} Q_{2k+1}(\xi) & xQ_{2k}(\xi) \\ -yQ_{2k}(\xi) & -Q_{2k-1}(\xi) \end{pmatrix}$.

Claim 2.2.8. *The following are equivalent (even in characteristic* 2*):*

- $Q_{2m} = 0$ *and* $Q_{2m-1} = -1$
- $Q_{2m-k} = -Q_k$ *for* $0 \leq k \leq 2m$
- $Q_m = 0$.

The proofs are simple exercises.

A specialization $R_{\Bbbk}$ admits an action of W_m for $2 \leq m < \infty$ if and only if A^m is the identity matrix, which is equivalent to $Q_{2m+1} = 1$, $Q_{2m-1} = -1$, and $xQ_{2m} = yQ_{2m} = 0$ (and one of the first two equations is redundant). Thus $\mathbb{Z}[x,y]/(xQ_{2m}, yQ_{2m}, Q_{2m-1}+1)$ is the universal ring admitting an action of W_m. We don't really wish to work in this much generality (although if you do, be my guest!). Let us make the further assumption that x, y are either zero or non-zero-divisors, and the common assumption in Cartan matrices that $x = 0 \iff y = 0$. If $x = y = 0$ then A has order 2. Ignoring $m = 2$ we know that $x, y \neq 0$ so that $Q_{2m} = 0$ is our replacement relation. By the above claim, $Q_{2m} = 0$ and $Q_{2m-1} = -1$ is equivalent to $Q_m = 0$.

Definition 2.2.9. We say that a $\mathbb{Z}[x,y]$-algebra $\Bbbk$ is *m-admissible* if it factors through the following quotient. When $m = 2$, we have $x = y = 0$. When $3 \leq m < \infty$, we have $Q_m(xy) = 0$.

Let us change our notation to place everything in a more familiar context.

Claim 2.2.10. *Inside* $\mathbb{Z}[q, q^{-1}]$, *we have* $Q_k([2]^2) = [k]$ *when* k *is odd, and* $Q_k([2]^2) = \frac{[k]}{[2]}$ *when* k *is even.*

The typical specialization is $x = y = -[2]$, for which the matrix $A^k = \begin{pmatrix} [2k+1] & -[2k] \\ [2k] & -[2k-1] \end{pmatrix}$. (Whether one uses $x = [2]$ or $x = -[2]$, such as for the value of a circle in the Temperley-Lieb algebra, is a matter of convention; we always use $-[2]$.) Instead of specializing this way, we merely use the symbol $[k]_x$ to denote $Q_k(xy) \in \mathbb{Z}[x,y]$ when k is odd, and $-xQ_k(xy)$ when k is even. We use $[k]_y$ accordingly, and call these the *two-colored quantum numbers*. When k is odd, $[k]_x = [k]_y$, and we may denote it by $[k]_{x,y}$ for emphasis. The two-colored quantum numbers replace the usual quantum numbers when doing any computations with the two-color Temperley-Lieb algebras (such as calculating coefficients of Jones-Wenzl projectors). One should think of two-colored quantum numbers as though they were actual quantum numbers.

Any identity involving quantum numbers has an analogous identity for two-colored quantum numbers. For instance, $[2]_x[k]_y = [k+1]_x + [k-1]_x$.

Claim 2.2.11. *When k divides m, Q_k divides Q_m in $\mathbb{Z}[\xi]$, and $[k]_x$ divides $[m]_x$ in $\mathbb{Z}[x,y]$.*

Proof. Consider the remainder term when dividing Q_m by Q_k. This polynomial vanishes when evaluated at any $\xi = [2]^2 \in \mathbb{R}$, and therefore is zero everywhere. □

We are interested in a *faithful* action of W_m on R, not merely an action. We need $Q_m = 0$ and $Q_k \neq 0$ for any $k < m$. This can be established by looking at the analog of the cyclotomic polynomials.

Definition 2.2.12. Let P_k be defined inductively by letting $P_k = Q_k/(\prod_{d|k} P_d)$.

Note that the P_k, the analogs of cyclotomic polynomials, also have integer coefficients (one can use the same arguments as for cyclotomic polynomials themselves). For complex numbers, $P_k(\xi) = 0$ if and only if $\xi = [2]^2$ where $q^2 = \zeta_k$.

Remark 2.2.13. For reasons which will become obvious in section 2.2.5, we also want to assume that m is invertible.

Definition 2.2.14. Let R_m be the *universal "nice" ring admitting a faithful action of W_m*. When $m = 2$, $R_m = \mathbb{Z}[x,y,\frac{1}{2}]/(x,y)$. When $m \geq 3$, $R_m = \mathbb{Z}[x,y,(xy-4)^{-1},m^{-1}]/(P_m(xy))$. We say $\Bbbk$ is *m-faithful* if the specialization factors through R_m, and write it as $\Bbbk_m$. Whenever we say a ring is m-faithful we imply that $m < \infty$.

Remark 2.2.15. Sometimes $P_m(xy) = 0$ implies that $xy - 4$ is invertible, and sometimes it does not. For instance, when $m = 2, 4$ we must invert 2, while for $m = 3, 5, 6$ we need not invert anything. It never happens that $P_m(xy) = 0 \implies xy - 4 = 0$. Usually (or always?) the conditions that $P_m(xy) = 0$ and $xy - 4$ is invertible imply that m is invertible.

Before we pass on to other matters, let us address one further specialization. In order to have complete color symmetry, setting $x = y$ is not enough. When m is odd, we must also require the algebraic analog of the fact that $q = \zeta_{2m}$, not ζ_m. See section 2.3.3 for the significance of this. This distinction is invisible in terms of algebraic conditions on ξ. Assume that $x = y$ below, and $P_m(x^2) = 0$. For normal quantum numbers, $[m+1] = q^{-1}[m] + q^m$ so that when $[m] = 0$, $[m+1] = q^m = \pm 1$. Therefore, ignoring q, our analogous condition will be $[m+1]_x = -1$, or $[m-1]_x = 1$. This is already implied by $P_m = 0$ when m is even.

Definition 2.2.16. We say that an m-faithful ring $\Bbbk$ is *m-symmetric* if it factors through $x - y = 0$ and $xQ_{m-1}(x^2) - 1 = 0$, and if m is invertible.

Most of the results in this paper will be stated for m-symmetric rings, but could be extended to work for m-faithful rings after some painful tweaking.

2.2.3 Invariant Subrings

Let $\mathcal{L}$ denote the set of $L \in V^*$ (up to scalar) such that $L = 0$ cuts out a reflection line. Then $\mathcal{L} = \mathcal{L}_s \cup \mathcal{L}_t$, where $\mathcal{L}_s = \{f_s, t(f_s), st(f_s), \ldots\}$ and $\mathcal{L}_t = \{f_t, s(f_t), ts(f_t), \ldots\}$. When $m = \infty$ these two subsets are both infinite and disjoint. Over an m-faithful ring, both subsets are finite, and we terminate them at the appropriate place to avoid redundancy. When m is even the two sets are disjoint, and when m is odd the two sets are equal (with the ordering flipped). In this finite case, we denote by $\mathbb{L}$ the product of all $L \in \mathcal{L}$, a polynomial of degree m (which, in the doubled grading on R, is in graded degree $2m$). It is not hard to show that $s(\mathbb{L}) = t(\mathbb{L}) = -\mathbb{L}$.

Let $\mathcal{L}^{\perp}$ denote the set of $L \in V^*$ (up to scalar) such that $L = 0$ cuts out a line “perpendicular" to a reflection line. For instance, L is perpendicular to f_s if $s(L) = L$, so that the perpendicular line is $L_s = xf_s - 2f_t$. The perpendicular line to f_t is $L_t = yf_t - 2f_s$. One can easily check that (for the right choice of scalars) $\mathcal{L}^\perp \subset R_u$. Over an m-faithful ring, $\mathcal{L}^\perp$ is finite, and we denote by Z the product of all $L \in \mathcal{L}^\perp$, a polynomial of degree m. It is not hard to show that $s(Z) = t(Z) = Z$.

There is also a polynomial $z = xf_s^2 - xyf_sf_t + yf_t^2$ of degree 2, for which $s(z) = t(z) = z$. When $x = y$ we may instead use $z = f_s^2 + f_t^2 - xf_sf_t$. Over an m-faithful ring, z and Z generate all the W-invariants $R_\Bbbk^W$, and $\mathbb{L}$ generates the W-antiinvariants as a module over $R_\Bbbk^W$. When W_∞ acts faithfully, there are no antiinvariants, and the invariants are generated by z.

We summarize. For any $\Bbbk$, $R = R_\Bbbk$, we have

$$R^s = \Bbbk[f_s^2, L_s] \tag{2.2.1}$$

$$R^t = \Bbbk[f_t^2, L_t]. \tag{2.2.2}$$

When x is invertible we may write

$$R^s = \Bbbk[z, L_s] \tag{2.2.3}$$

and similarly for y invertible and R^t.

When $\Bbbk$ is m-faithful we have

$$R^W = \Bbbk[z, Z]. \tag{2.2.4}$$

When $\Bbbk$ is not m-faithful for any $m < \infty$ we have

$$R^{W_\infty} = \Bbbk[z]. \tag{2.2.5}$$

In particular, R_u and R_u^i are both rings of dimension 2 over $\mathbb{K}$, while $R_u^{W_\infty}$ has dimension 1, so that the extension $R_u^i \subset R_u$ is finite while the extension $R_u^{W_\infty} \subset R_u^i$ or R_u is infinite.

This is a general phenomenon related to whether or not the parabolic subgroup is finitary. We only care about finite extensions of rings, so that we will never want to deal with $R_u^{W_\infty}$.

R is a free module of rank 2 over R^s, generated by 1 and L_t. For instance, $f_s = \frac{yL_s+2L_t}{xy-4}$. Moreover, $\partial_s(L_t) = xy-4$ is invertible. Therefore, R is generated by 1 as a $R^s - R^t$-module, and there is an element of R^t which maps by ∂_s to 1.

2.2.4 Frobenius Extensions and R^i

Definition 2.2.17. Let $A \subset B$ is an extension of commutative rings such that B is free and finitely generated as an A-module, equipped with an A-linear map $\partial = \partial_A^B \colon B \to A$. We say that two bases $\{a_\alpha\}$ and $\{b_\alpha\}$ for B over A are *dual bases* if $\partial(a_\alpha b_\beta) = \delta_{\alpha\beta}$. If dual bases exist, B is called a *Frobenius extension* of A.

For a Frobenius extension, one is equipped with a comultiplication map $B \to B \otimes_A B$, which sends $1 \mapsto \Delta_A^B = \sum_\alpha a_\alpha \otimes b_\alpha$ for some dual basis. This element of $B \otimes_A B$ did not depend on the choice of dual basis, and neither does $\mu_A^B = \mu(\Delta_A^B) = \sum_\alpha a_\alpha b_\alpha \in A$. Frobenius extensions appear often in geometric representation theory, and are useful because the functors of induction and restriction between A-mod and B-mod are biadjoint.

We will prove that, under certain circumstances, the invariant subrings will form a square of Frobenius extensions, in the sense of [9]. In the notation Δ_A^B or ∂_A^B we replace A and B by either W, s, t or emptiness, as in ∂_W^s and Δ_i. The smaller ring is always placed on bottom. We have already defined ∂_i. We will also use Sweedler notation for coproducts, so that $\mu_s = \Delta_{s(1)}\Delta_{s(2)}$.

The proof of Claim 2.2.1 relied on the presence of $\frac{1}{2} \in \mathbb{R}$. In fact, it only relied on the fact that $\partial_s(\frac{f_s}{2}) = 1$, but any other element f with $\partial_s(f) = 1$ would work, such as $\frac{L_t}{xy-4}$.

Claim 2.2.18. *Let $f \in V^*$ for which $\partial_i(f) = 1$. Then $\{1, f\}$ forms a basis for R as a free R^i-module, and $\{-s(f), 1\}$ forms a dual basis over the map ∂_i. Moreover, any pair of dual bases can be formed as above by choosing such an f, up to multiplication by invertible scalars. Therefore, $R^i \subset R$ is a Frobenius extension, $R \cong R^i \oplus R^i\{2\}$, and $\mu_i = f_i$.*

Proof. That $\{1, f\}$ and $\{-s(f), 1\}$ are dual is clear. The isomorphism $R \cong R^i \oplus R^i\{2\}$ is given by sending $g \in R$ to $(\partial_i(gf), \partial_i(g))$, and conversely by sending $(g, h) \mapsto g - s(f)h$. We leave the rest to the reader. □

Here is an obvious corollary.

Claim 2.2.19. *Letting* $B_i \overset{def}{=} R \otimes_{R^i} R\{-1\}$, *we have an isomorphism of graded* R*-bimodules*

$$B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}. \tag{2.2.6}$$

Proof. This is clear from the R^i-bimodule isomorphism $R \cong R^i \oplus R^i\{2\}$. Explicitly, choosing f with $\partial_i(f) = 1$, the map from left to right sends $1 \otimes g \otimes 1 \mapsto (\partial_i(fg) \otimes 1, \partial_i(g) \otimes 1)$ and the map from right to left sends $(1 \otimes 1, 0) \mapsto 1 \otimes -1 \otimes 1$ and $(0, 1 \otimes 1) \mapsto 1 \otimes -s(f) \otimes 1$. □

Equation (2.2.6) is a categorification of relation (2.1.3), as shall be seen.

Note that we may choose a pair of dual bases of R over R^s such that one bases lies within R^t, and vice versa. This is a technical statement, used in [9].

2.2.5 Frobenius Extensions and R^W

Now we wish to discuss the extensions $R^W \subset R^i$ and $R^W \subset R$, which are only finite extensions when $W = W_m$ is a finite dihedral group. We assume for the rest of the chapter that $\Bbbk$ is m-symmetric for some $m \geq 2$. There is a more general Demazure operator $\partial_W : R \to R^W$, whose degree is $-2m$. It is defined by letting $\underline{\boldsymbol{i}}$ be a reduced expression for w_0, and letting $\partial_W = \partial_{\underline{\boldsymbol{i}}} \overset{\text{def}}{=} \partial_{i_1}\partial_{i_2}\dots\partial_{i_m}$. This does not depend on the choice of reduced expression if $\Bbbk$ is m-symmetric. There are also relative Demazure operators $\partial_W^i : R^i \to R^W$, defined by letting $\underline{\boldsymbol{i}}$ be a relative reduced expression for w_0. For example, $\partial_W^s = \underbrace{\partial_t\partial_s\partial_t\cdots\partial_t}_{m-1}$, because $tst\cdots t$ is a reduced expression for w_0 after multiplying by s. These Demazure operators make each inclusion a Frobenius extension.

These definitions imply that $\partial_W^s \partial_s = \partial_W^t \partial_t$, which states that this square of Frobenius extensions is compatible, in the sense of [9]. It is possible but annoying to alter the results for the m-faithful case, where the Frobenius square is only compatible up to a scalar. One must keep track of this scalar everywhere below (and even in isotopy of diagrams, see [9]), which we have not done.

Theorem 1. *Suppose that $\Bbbk$ is m-symmetric. Then the extensions $R^W \subset R$ and $R^W \subset R^i$ are Frobenius extensions. Therefore, R is a free R^W-module of graded rank $[W]$, and R^i is a free R^W-module of graded rank $\frac{[W]}{(v+v^{-1})}$. Any dual bases satisfy the following properties. The latter ones involve considering an element of R^s as though it were in R and taking ∂_t of it.*

$$\mu_W = \mathbb{L} \tag{2.2.7}$$

$$\mu_W^s = \frac{\mathbb{L}}{f_s} \tag{2.2.8}$$

$$\Delta_{W(1)}^s \partial_t(\Delta_{W(2)}^s) = \partial_t(\Delta_{W(1)}^s)\Delta_{W(2)}^s = \frac{\mathbb{L}}{f_s f_t} \tag{2.2.9}$$

$$\Delta_{W(1)}^s \otimes \partial_t(f\Delta_{W(2)}^s) = \partial_s(f\Delta_{W(1)}^t) \otimes \Delta_{W(2)}^t \quad \in R^s \otimes_{R^W} R^t \text{ for any } f \in R \tag{2.2.10}$$

In particular, the map $R \to R^s \otimes_{R^W} R^t$ sending $f \mapsto \Delta_{W(1)}^s \otimes \partial_t(f\Delta_{W(2)}^s) = \partial_s(f\Delta_{W(1)}^t) \otimes \Delta_{W(2)}^t$ is well-defined, R^s-linear on the left, and R^t-linear on the right.

Sketch of Proof. First, we need to show that R^s is a Frobenius extension of R^W. One need only find dual bases. Computing these bases explicitly is a calculation, but showing that dual bases exist is not difficult via a constructive argument. Once one chooses a basis (say, powers of L_s) then one need only find a dual basis, which one can show exists inductively. The tricky part is that the coefficients in these bases lie within $\Bbbk_m$. Essentially, however, this will follow from the fact that m is invertible, that $\frac{\mathbb{L}}{f_s} \in R^s$, and that $\partial_W^s(\frac{\mathbb{L}}{f_S}) = m$. This can be checked by hand.

The equations above hold in general for any square of Frobenius extensions, as shown in [9]. The only interesting piece of data is that $\mu_W = \mathbb{L}$. This even implies that $\partial_W^s(\frac{\mathbb{L}}{f_S}) = m$, since for any Frobenius extension, $\partial_A^B(\mu_A^B)$ is equal to the degree of B over A. To show that $\mu_W = \mathbb{L}$, note first that $\partial_s(\mu_W) = \partial_s(\mu_W^s \mu_s) = 2\mu_W^s = 2\frac{\mu_W}{f_s}$. By the definition of ∂_s, we see that $s(\mu_W) = -\mu_W$. Also, $t(\mu_W) = -\mu_W$. So μ_W is antiinvariant, and for degree reasons must be equal to $\mathbb{L}$. □

Corollary 2.2.20. *Letting* $B_W \overset{def}{=} R \otimes_{R^W} R\{-m\}$ *we have the following isomorphisms:*

$$B_i \otimes B_W \cong B_W \otimes B_i \cong (v + v^{-1}) B_W \tag{2.2.11}$$

$$B_W \otimes B_W \cong [W]\, B_W \tag{2.2.12}$$

We mention briefly that, choosing any basis $\{a_\alpha\}$ with dual basis $\{b_\alpha\}$ that the $[W]$-many projection maps from R to R^W are $f \mapsto \partial_W(f a_\alpha)$ and the inclusion maps are $g \mapsto g b_\alpha$. These maps, applied to the middle factor in $R \otimes_{R^W} R \otimes_{R^W} R$, give you the projections and inclusions in (2.2.12) as well. To deduce (2.2.11) we write $R \otimes_{R^i} R \otimes_{R^W} R$ as $R \otimes_{R^i} R \otimes_{R^i} R^i \otimes_{R^W} R$ and reduce the second factor of R as in Claim 2.2.18.

2.2.6 Soergel Bimodules

Definition 2.2.21. As in the introduction, let $\mathcal{B}_{\mathrm{BS}}$ denote the subcategory of R-modules generated (over $\otimes$, $\oplus$, $\{1\}$) by B_s and B_t, and let $\mathcal{B}$ denote its Karoubi envelope. When $R = R_m$, let $\mathcal{B}_{\mathrm{gBS}}$ denote the subcategory generated by B_s, B_t, and B_W. When $R = R_u$ we write $\mathcal{B}_{\mathrm{BS},u}$ and $\mathcal{B}_u$. When $\Bbbk = \Bbbk_m$ is m-faithful, we may write $\mathcal{B}_m$ instead of $\mathcal{B}_\Bbbk$.

Remark 2.2.22. We will show that, under suitable conditions on $\Bbbk_m$, the bimodule B_W is a summand of $\underbrace{B_s \otimes B_t \otimes B_s \otimes \cdots}_{m}$, so that it lives inside $\mathcal{B}_m$ already, and $\mathcal{B}_m$ is also the Karoubi envelope of $\mathcal{B}_{\mathrm{gBS},m}$.

Remark 2.2.23. The category $\mathcal{B}_{\mathrm{BS},\Bbbk}$ is *not* simply the base change of the category $\mathcal{B}_{\mathrm{BS},u}$. The modules have had their base changed, but in the new ring there may be additional morphisms which were not present before, and the graded rank of the Hom spaces may change. In fact, we will show that the new morphisms in $\mathcal{B}_m$ that are not in $\mathcal{B}_u$ are generated by a single new morphism of degree 0.

Theorem 2. *(See Soergel [27] Theorems 1.10, 4.2, 5.5, 6.16) Let $m \geq 2$ or $m = \infty$. Let $\Bbbk$ be an infinite field of characteristic $\neq 2$, and let $W = W_m$ be the dihedral group acting faithfully on $R = R_{\Bbbk}$. There is a map from $\mathcal{H}$ to the Grothendieck ring of $\mathcal{B}$ sending v to $[R\{1\}]$, b_i to $[B_i]$, and b_W to $[B_W]$ when $m < \infty$. This map is an isomorphism. There is one indecomposable bimodule $B_w \in \mathcal{B}$ for each $w \in W$, and if $\underline{\boldsymbol{i}}$ is a sequence giving a reduced expression of w then B_w is a summand of $B_{\underline{i}}$ of multiplicity 1. For dihedral groups, we know that this isomorphism also sends b_w to $[B_w]$. For two objects $X, Y \in \mathcal{B}$, the space* $\mathrm{Hom}(X,Y)$ *is a free left (or right) R-module of graded rank $([X],[Y]) \overset{def}{=} \varepsilon(\omega([X])[Y])$.*

Remark 2.2.24. Soergel's results apply in great generality to other Coxeter groups. In the general case, one can still say that the indecomposables B_w are in bijection with W, but can not assert that they descend to the Kazhdan-Lusztig basis.

Remark 2.2.25. We believe that Soergel's techniques could have been generalized to other base rings, although it is somewhat subtle. We present a different method towards proving these results for dihedral groups over more general rings $\Bbbk$. We will call $\Bbbk$ a *Soergel ring* when we want to emphasize that the full force of Soergel's results may be brought to bear. For now, this just means an infinite field of characteristic $\neq 2$.

2.2.7 Singular Soergel Bimodules

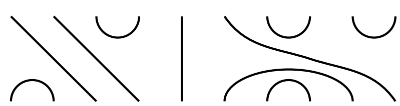
Definition 2.2.26. Let **Bim** denote the 2-category whose objects are rings, and for which $\mathrm{Hom}(B,A)$ is the category of $A-B$-bimodules. Let $\mathfrak{B}_{\mathrm{BS}}$ denote the full (additive, graded) sub-2-category whose objects are R^J for J finitary, and whose 1-morphisms are generated by R^J as an $R^J - R^K$-bimodule and as a $R^K - R^J$-bimodule, whenever $J \subset K$. These are known as *induction* and *restriction* bimodules, respectively. Its Karoubi envelope is the 2-category of *singular Soergel bimodules*, denoted $\mathfrak{B}$. As usual, we use the subscript u when this is done over $R = R_u$, the subscript $\Bbbk$ for $R = R_{\Bbbk}$, and the subscript m when $R = R_m$.

Theorem 3. *(See Williamson [31] Theorems 7.5.1, 7.4.1 and others) Let $\Bbbk$ be a Soergel ring. There is a functor from $\mathfrak{H}$ to the Grothendieck category of $\mathfrak{B}$, sending v to the grading shift and $b_J \in \mathrm{Hom}(J,K)$ to $R^J \in \mathrm{Hom}(R^J, R^K)$ for $J \subset K$, and similarly for $\mathrm{Hom}(K,J)$. This map is an isomorphism, and over the empty parabolic it restricts to the isomorphism from $\mathcal{H}$ to $[\mathcal{B}]$. There is a formula for calculating the size of 2-morphism spaces in $\mathfrak{B}$ by using the standard trace map on $\mathfrak{H}$.*

2.3 Temperley-Lieb categories

2.3.1 The Uncolored Temperley-Lieb category

The (uncolored) *Temperley-Lieb algebra* on n strands TL_n is an algebra over $\mathbb{Z}[d]$ which can be realized pictorially. It has a basis given by crossingless matchings with n points on bottom and n on top (see the example below). Multiplication is given by concatenation of diagrams, and by replacing any closed component (i.e. circle) with the scalar d. We denote the crossingless matching representing the identity element by $\mathbb{1}_n$.

Example 2.3.1. An element of TL_{10}: 

The Temperley-Lieb algebra is part of the *Temperley-Lieb category* $\mathcal{TL}$, a monoidal category whose objects are $n \in \mathbb{N}$, thought of as n points on a line, and where the morphisms from n to m are spanned by crossingless matchings with n points on bottom and m on top. Composition is given by concatenation and resolving circles, as usual, so that $\mathrm{End}(n) = TL_n$. As a monoidal category we should really view it as a 2-category with a single object, the uncolored region.

Let $U = U_q(\mathfrak{sl}_2)$ be the quantum group of $\mathfrak{sl}_2$. Let V_k be the irreducible representation with highest weight $-q^k$, and let $V = V_1$. We have already remarked that the Temperley-Lieb category governs the intertwiners between tensor products of V, so that $\mathrm{Hom}_{\mathcal{TL}}(n,m) \otimes_{\mathbb{Z}[d]}$

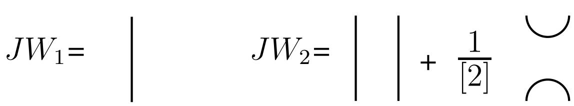
$\mathbb{Q}(q) = \mathrm{Hom}_U(V^{\otimes n}, V^{\otimes m})$. Under this base change, $d \mapsto -[2]$, and we use quantum numbers interchangeably with the polynomials in d that express them.

Proposition 2.3.2. *The Temperley-Lieb algebra TL_n, after extension of scalars, contains* canonical *idempotents which project $V^{\otimes n}$ to each isotypic component. It contains primitive idempotents refining the isotypic idempotents, which project to each individual irreducible component (although this splitting is non-canonical). Given a choice of primitive idempotents, TL_n contains maps which realize the isomorphisms between the different irreducible summands of the same isotypic component. These maps can be defined in any extension of $\mathbb{Z}[d]$ where the quantum numbers $[2], [3], \ldots, [n]$ are invertible.*

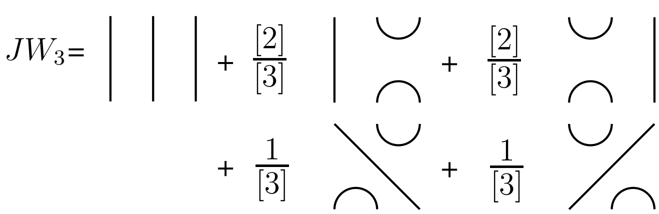
Proof. The only part of this proposition which is not tautological is the statement about coefficients. This follows from recursion formulas for the idempotents, some of which can be found below. For more on recursion formulas and coefficients see [10]. □

We say an extension of $\mathbb{Z}[d]$ is *TL_n-sufficient* if the first n quantum numbers are invertible, and is *TL-sufficient* if it is TL_n-sufficient for all n.

The highest non-zero projection, from $V^{\otimes n}$ to V_n, is known as the *Jones-Wenzl projector* $JW_n \in TL_n$. Here are some examples of Jones-Wenzl projectors.



Example 2.3.3.



Claim 2.3.4. *Jones-Wenzl projectors satisfy the following relevant properties:*

- *JW_n is the unique map which is killed when any cap is applied on top or any cup on bottom, and for which the coefficient of $\mathbb{1}_n$ is 1.*

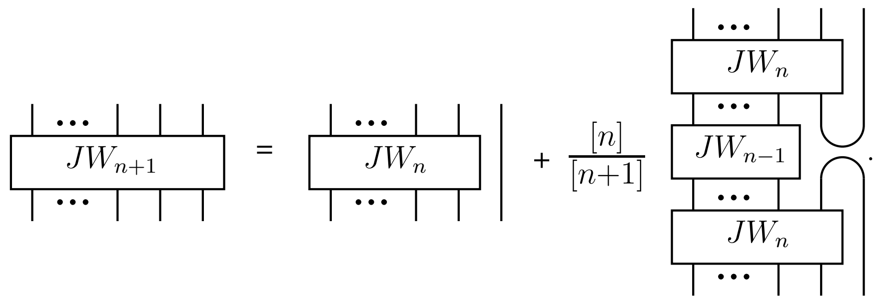
- *The ideal generated by JW_n in TL_n is one-dimensional over $\mathbb{Q}(d)$, since any other element $x \in TL_n$ acts on JW_n by the coefficient of $\mathbb{1}_n$ in x.*

- *JW_n is invariant under left-right and top-bottom reflection.*

- *The coefficient of every crossingless matching in JW_n is non-zero.*

- *JW_n can be defined so long as $[n]$ is invertible.*

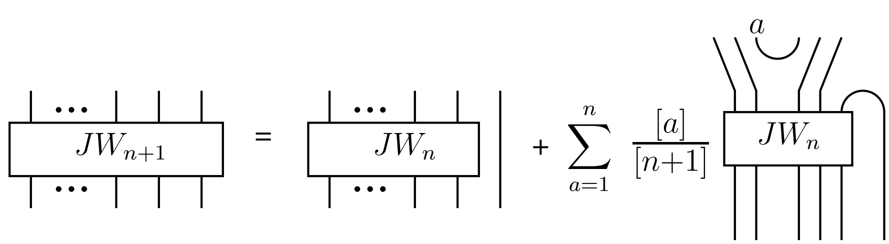
- *There is a recursive formula (see [30]) which is:*



(2.3.1)

It is more typical to see this formula with JW_{n-1} replaced by $\mathbb{1}_{n-1}$.

- *There is an alternate recursive formula ([10], Theorem 3.5), which sums over the possible positions of cups, and follows quickly from (2.3.1):*

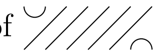


(2.3.2)

In $\mathbf{Kar}(\mathcal{TL})$ we let V_n denote the image of JW_n and $(\cdot) \otimes V$ denote the functor of adding a new line on the right. The recursive formula (2.3.1) gives a diagrammatic proof of the following obvious proposition.

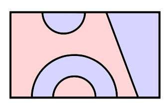
Proposition 2.3.5. *The Karoubi envelope $\textbf{\textit{Kar}}(\mathcal{TL})$ of $\mathcal{TL}$ has indecomposables V_n, $n \in \mathbb{N}$. These satisfy $V \otimes V_0 \cong V_0 \otimes V \cong V_1$ and $V \otimes V_n \cong V_n \otimes V \cong V_{n+1} \oplus V_{n-1}$ for $n \geq 1$.*

The proofs of the above facts are standard. Some references on the Temperley-Lieb algebra and planar algebras include [11, 3, 10, 21]. In particular, formulae for Jones-Wenzl projectors were produced in [10]; the unpublished paper [21] has a more detailed version.

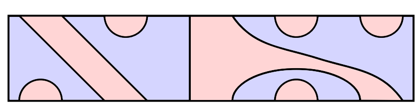
Let us pause to calculate the coefficient ρ_k of  in JW_k, using (2.3.2). The first term on the right hand side contributes nothing to this coefficient. The sum only contributes when $a = 1$, and it contributes $\frac{[1]}{[k]}$ times the coefficient of $\mathbb{1}_{k-2}$ in JW_{k-2}, which is 1. Therefore $\rho_k = \frac{1}{[k]}$. When $q = \zeta_{2(k+1)}$, $[k] = 1$ and $\rho_k = 1$.

2.3.2 The Two-colored Temperley-Lieb Category

Any 1-manifold will divide the plane into regions which can be colored alternately with 2 colors. Let us assume these two colors are red and blue. We may construct a variation on the Temperley-Lieb algebra by coloring the regions, and specifying different values for a circle based on its interior color. The *two-color Temperley-Lieb 2-category* $2\mathcal{TL}$ has two objects, red and blue, and its 1-morphisms are generated by a point on a line either switching blue to red or vice versa. The 2-morphisms are the $\mathbb{Z}[x, y]$-module spanned by appropriately-colored crossingless matchings. Multiplication is defined as in $\mathcal{TL}$ except that a circle with red (resp. blue) interior evaluates to x (resp. y).

Example 2.3.6. An element of $\mathrm{Hom}(rbrbrb, rbrb)$: 

Note that there is no 1-morphisms from red to red, or from blue to blue, so that the colors must alternate. A 1-morphism (and its source and target objects) can be specified by the alternating sequence of colors it passes through (e.g. $rbrbrb$). Fixing a number of strands and a color (say, the rightmost color), we get an algebra which we might call the *two-color Temperley-Lieb algebra* $2TL_n$.

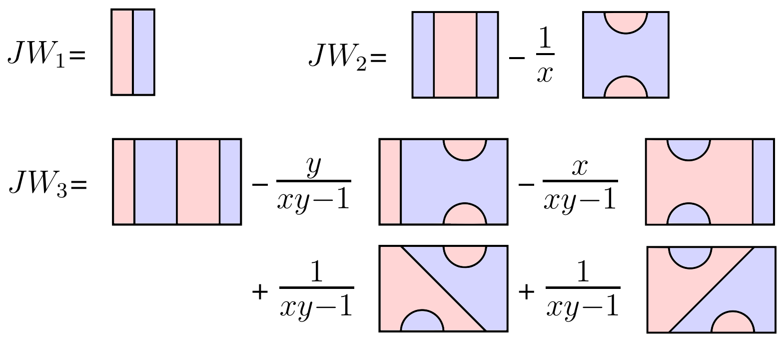
Example 2.3.7. An element of $2TL_{10}$: 

If we specialize to $x = y = d$, we may forget the colors and obtain the usual *uncolored* Temperley-Lieb category. The difference between TL and $2TL$ is not very large, consisting of a minor generalization of coefficients. However, adding colors does restrict which 1-morphisms and 2-morphisms can be concatenated. In $2\mathcal{TL}$ there are 4 kinds of diagrams (depending on whether blue or red appears on the left and right), corresponding to the 4

Hom categories between the 2 objects. In other words, a 2-morphism which has blue on the right and red on the left in the source also has blue on the right and red on the left in the target.

Jones-Wenzl projectors JW_n exist in $2\mathcal{TL}$ as well, one for each number of strands and choice of rightmost color. Its coefficients will involve inverting two-colored quantum numbers $[n]_x$ and $[n]_y$. The recursion formulae (2.3.1) and (2.3.2) can be generalized, using two-color quantum numbers. We give examples of the first few projectors, when blue is the rightmost color.

Example 2.3.8.



As mentioned in the introduction, people tend to use the spherical or lopsided specialization of $2\mathcal{TL}$, so that the references for the facts below may be more difficult to find. The proofs, however, are completely analogous to the uncolored case.

Proposition 2.3.9. *The canonical isotypic idempotents, the non-canonical primitive idempotents, and the intra-isotypic isomorphisms of Proposition 2.3.2 all have analogs in $2TL_n$ after localization. More precisely, there are two versions of each idempotent, one with red on the right and one with blue. These maps are defined over any extension of $\mathbb{Z}[x,y]$ for which the two-color quantum numbers up to $[n]$ are invertible.*

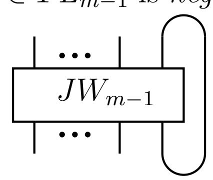
We say that a specialization $\Bbbk$ of $\mathbb{Z}[x,y]$ is $2TL_n$-*sufficient* if primitive idempotents and intra-isotypic isomorphisms can be constructed in $2TL_n$ using coefficients in $\Bbbk$. We say it is 2TL-*sufficient* if it is $2TL_n$-sufficient for all n.

Every indecomposable in the Karoubi envelope of $\mathcal{TL}$ is doubled in the Karoubi envelope of $2\mathcal{TL}$ because of the two choices of color. For instance, V_0 is replaced by two distinct indecomposables, V_b and V_r, represented by the empty diagram with the region colored blue or red respectively. Similarly, V_1 is replaced by V_{rb} and V_{br}, and so forth.

Proposition 2.3.10. *The Karoubi envelope* $\boldsymbol{Kar}(2\mathcal{TL})$ *of* $2\mathcal{TL}_{\Bbbk}$ *(for a 2TL-sufficient ring* $\Bbbk$*) has indecomposables* $V_{\underline{i}}$ *where* $\underline{\boldsymbol{i}}$ *is an nonempty alternating sequence of red and blue. Therefore, there are two sequences of each length* $n \geq 1$*. We denote by* $r \otimes \cdot$ *the functor of placing a line on the left with the lefthand region colored red; this operation can only be performed on a 1-morphism which currently has blue in the lefthand region. This satisfies* $r \otimes V_b \cong V_{rb}$ *and* $r \otimes V_{br\ldots} \cong V_{rbr\ldots} \oplus V_{r\ldots}$ *for any appropriate sequence* $\underline{\boldsymbol{i}} = br\ldots$ *of length* ≥ 2*. Similar statements are made switching* r *and* b*, or switching left and right.*

Note that we can label these indecomposables by the elements of $W_\infty \setminus \{e\}$. The multiplication rule given here is a categorification of the rule given in Claim 2.1.4.

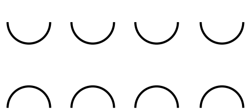
2.3.3 Roots of Unity and Rotation

It is well known that when $[m] = 0$, the Jones-Wenzl projector $JW_{m-1} \in TL_{m-1}$ is *negligible*. This is equivalent, for Jones-Wenzl projectors, to the statement that 

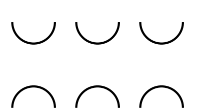
$= 0$.

In fact, JW_{m-1} is negligible (when it is defined) if and only if $[m] = 0$. See [11] for more details.

In particular, rotating JW_{m-1} by any number of strands yields another morphism which is killed on the top and bottom by all cups and caps, and is therefore a scalar multiple of JW_{m-1}. We say that JW_{m-1} has a "rotational eigenvalue." Conversely, if JW_{m-1} has a rotational eigenvalue, then it must be negligible. To compute this eigenvalue one may compare the coefficients of two pictures which are related by rotation. Rotation by 2 strands always has eigenvalue 1, since the following picture is unchanged after rotation by 2 strands.



or



Therefore, rotation by 1 strand has eigenvalue ± 1. When $m-1$ is odd, again the eigenvalue is 1 since the above picture rotates by 1 strand to be the left-right flip of itself. In general, the coefficient of $\mathbb{1}_{m-1}$ after rotation is $\rho_{m-1} = \frac{1}{[m-1]}$, which is 1 when $q = \zeta_{2m}$ and -1 when $q = \zeta_m$.

For the two-color case, we may only rotate by an even number of strands to preserve colors. However, if we set $x = y$ and desire the coefficients of JW_{m-1} to be invariant under color swap, then we require analogously that $[m-1]_x = 1$. Therefore:

Claim 2.3.11. *The two-color projector JW_{m-1} is rotation-invariant (and m is minimal with this property) $\iff$ our base ring is m-faithful. It is also invariant under color swap $\iff$ our base ring is m-symmetric.*

2.3.4 Coxeter Lines

In section 2.2.3, we have already described the set $\mathcal{L} = \mathcal{L}_s \cup \mathcal{L}_t$ of linear polynomials which cut out the reflection lines for W_∞ or W_m. We consider both sets $\mathcal{L}_s$ and $\mathcal{L}_t$ as being ordered. We work below in R_u, although the implications for R_m are clear.

Definition 2.3.12. Let $\mathbb{L}_{k,l} \in R$ be the product of the first k elements of $\mathcal{L}_s$ and the first l elements of $\mathcal{L}_t$.

Definition 2.3.13. Given a crossingless matching in $2\mathcal{TL}$, its *associated monomial* will be $f_s^a f_t^b \in R$, where a is the number of blue regions and b the number of red regions. Given an arbitrary 2-morphism in $2\mathcal{TL}$, its *associated polynomial* will be obtained by writing it in the basis of crossingless matchings and taking the appropriate linear combination of monomials.

Proposition 2.3.14. *When $m = 2k$, the associated polynomial of JW_{m-1} with either blue or red on the right is equal to $\mathbb{L}_{k,k}$. When $m = 2k+1$, the associated polynomial of JW_{m-1} with blue (resp. red) on the right is equal to $\mathbb{L}_{k,k+1}$ (resp. $\mathbb{L}_{k+1,k}$).*

Remark 2.3.15. Note that, in either case, the polynomial in question will descend to $\mathbb{L}$ in R_m (up to a scalar). Therefore, this polynomial cuts out in $\mathfrak{h}$ the union of all the reflection-fixed lines.

This proposition is the analog of Section 3.7 in [6]. In that paper we examine Soergel bimodules in general type A, not in dihedral type, and instead of a single polynomial we obtain an ideal in $\Bbbk[f_1, f_2, \ldots, f_n]$. This ideal cuts out the *Coxeter lines* in $\mathfrak{h}_{\mathfrak{sl}_n}$ (there called the "Weyl lines"), which are the lines given by transverse intersections of reflection hyperplanes in $\mathfrak{h}$. While it is not obvious in this dihedral setup for reasons of dimension, what we are doing is creating an ideal which cuts out *lines*, not one that cuts out codimension one reflection hyperplanes.

Proof. We prove this statement inductively. Let $Y_{m,b}$ denote the associated polynomial of JW_m with blue on the right. Let L denote the $k+1$-st line in $\mathcal{L}_t$. We will be done if we can show that $Y_{m+1,b} = LY_{m,r}$ for either $m = 2k$ or $m = 2k+1$, and the same for the color swap.

We need the 2-colored version of (2.3.2). The differences between the one-colored and two-colored versions are

1. If $n+1$ is even, then $[n+1]$ becomes $[n+1]_{x,y}$.
2. If $n+1$ is odd, then the outsides are the same color. If this color is blue, $[n+1]$ becomes $[n+1]_x$, and otherwise becomes $[n+1]_y$.
3. In the sum, $[a]$ becomes $[a]_x$ if the interior of the new cup is blue, and $[a]_y$ if the interior is red.

The recursive formula says that $Y_{m+1,b} = \frac{1}{[m+1]_x} Y_{m,r}(\sum_{a=1, n+1-a \text{ even}}^{m+1} [a]_y f_t + \sum_{a=1, n+1-a \text{ odd}}^{m+1} [a]_x f_s)$. The term $a = n+1$ comes from the first term of (2.3.2). Thus it is enough to show that $(\sum_{a=1, n+1-a \text{ even}}^{m+1} [a]_y f_t + \sum_{a=1, n+1-a \text{ odd}}^{m+1} [a]_x f_s)$ is an expression for L (up to a scalar). This, too, can be proven by induction, dealing with the cases of even and odd separately. We leave this exercise to the intrepid reader. $\square$

Remark 2.3.16. Every coefficient in JW_m is nonzero. There is only a single crossingless matching with precisely one blue (resp. red) connected component, giving a nonzero leading term $cf_s f_t^m$. Therefore the associated polynomial of JW_m is nonzero when $\Bbbk$ is $2TL_m$-sufficient.

2.4 Main Techniques

2.4.1 Diagrammatics and Rotation

There are numerous excellent introductions to diagrammatics for cyclic (i.e. pivotal) monoidal categories and 2-categories, such as chapter 4 of [19]. For an example of a diagrammatic category which is self-biadjoint, see [7].

We make only one remark, also made in [7]. *Cyclicity* states that taking any morphism and using adjunction maps to rotate it by 360 degrees will not change the morphism. Cyclicity is required to draw morphisms on a plane, because any symbol we use to depict the morphism is evidently invariant under 360 degree rotation. However, consider a morphism with boundary $B_s \otimes B_s \otimes B_s$, reading around the circle. It is possible to rotate this morphism by 120 degrees, and cyclicity is no guarantee that this will not change the morphism. If it is the case that 120 degree rotation does not change the morphism, then one may depict the morphism using a diagram which is 120 degree rotation invariant, such as a trivalent vertex. One may pose the same question for any 2-morphism whose boundary 1-morphism has rotational symmetry. In this paper, we will also have morphisms whose boundary is $\underbrace{B_s \otimes B_t \otimes \cdots}_{2m}$, and it will be invariant under rotation by $\frac{360}{m}$ degrees, so that we may draw it as a two-colored $2m$-valent vertex (see section 4.1).

When one gives a diagrammatic category by generators and relations, and generators are drawn to have some non-trivial rotational invariance, then there is a hidden relation (called the *isotopy relation*) which states that rotating that 2-morphism does nothing. This relation will go unstated, but will need to be checked when applying functors into non-diagrammatic categories.

2.4.2 Hom spaces and Grothendieck rings

Suppose that $\mathcal{C}$ is a $\Bbbk$-linear graded additive monoidal category with objects B_i for $i = 1, 2$, satisfying relation (2.2.6). Let $B_\emptyset$ denote the monoidal identity. There is an obvious $\mathbb{Z}[v, v^{-1}]$-linear map from $\mathcal{H}_\infty$ to the Grothendieck ring $[\mathcal{C}]$, sending $b_i \mapsto [B_i]$. If in addition $\mathcal{C}$ satisfies the categorified version of (2.1.4) for some $m < \infty$, then this map factors through the finite Hecke algebra $\mathcal{H}_m$. We need not assume the existence of an object which categorifies b_{w_0}, only that tensor products of B_i satisfy this isomorphism. If $\mathcal{C}$ satisfies the Krull-Schmidt property, one can deduce that such an object B_{w_0} does exist; we will not use this fact, and leave the proof to the reader. Let W be the Coxeter group acting faithfully and $\mathcal{H}$ its Hecke algebra.

The category $\mathcal{C}$ induces a semi-linear pairing on $\mathcal{H}$, via $(b_{\underline{i}}, b_{\underline{j}}) \mapsto \mathrm{grk}_{\Bbbk}\mathrm{Hom}_{\mathcal{C}}(B_{\underline{i}}, B_{\underline{j}})$. If, in addition, each object B_i is self-biadjoint, then biadjointness descends to the map $\omega\colon \mathcal{H} \to \mathcal{H}$, and the semi-linear pairing is determined by the map $\varepsilon(b_{\underline{i}}) = \mathrm{grkHom}(B_\emptyset, B_{\underline{i}})$. We use the graded rank, which is well-defined even though we do not assume that Hom spaces are free.

Lemma 2.4.1. *Let $\mathcal{C}$ be a category as above (with B_i self-biadjoint), and let ε be any trace map on $\mathcal{H}$. Suppose that*

- *For each $w \in W$ there is an object B_w for which $b_w \mapsto [B_w]$. The biadjoint of B_w is $B_{w^{-1}}$.*
- *The categorified version of the relation in Claim 2.1.9 holds, decomposing $B_{\underline{i}}$ for a* reduced expression *$\underline{i}$ into direct sums of B_w.*
- *The Hom spaces $\mathrm{Hom}_{\mathcal{C}}(B_\emptyset, B_w)$ are free $\Bbbk$-modules for all $w \in W$. (More generally, we may assume they are free R-modules for some $\Bbbk$-algebra R for which composition in $\mathcal{C}$ is R-linear.)*

- *The graded rank of* $\mathrm{Hom}_{\mathcal{C}}(B_{\emptyset}, B_w)$ *over* $\Bbbk$ *(resp. over R) is equal to* $\varepsilon(b_w)$*.*

Then we may deduce that

- *All Hom spaces between various objects* B_w *and* $B_{\underline{i}}$ *in* $\mathcal{C}$ *are free as* $\Bbbk$*-modules (resp. as R-modules).*

- *For any sequences* $\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}$ *we know that* $\mathrm{Hom}_{\mathcal{C}}(B_{\underline{i}}, B_{\underline{j}})$ *has graded rank over* $\Bbbk$ *(resp. R) equal to* $(b_{\underline{i}}, b_{\underline{j}})$*, the semi-linear product induced by* ε*. The same is true replacing* $\underline{\boldsymbol{i}}$ *or* $\underline{\boldsymbol{j}}$ *with* w *for any* $w \in W$*.*

Proof. We note first that we may replace the category $\mathcal{C}$ with its subcategory consisting of direct sums of summands of various B_w. This is the Karoubi envelope of the monoidal category generated by B_1 and B_2, and it is itself closed under monoidal product.

Using biadjointness and direct sum decompositions, we see that any Hom space between various $B_{\underline{i}}$ or B_w is isomorphic to a direct sum of Hom spaces $\mathrm{Hom}(B_{\emptyset}, B_w)$ for various B_w. Therefore the freeness of $\mathrm{Hom}(B_{\emptyset}, B_w)$ implies the freeness of all Hom spaces. The combinatorics of biadjointness and decomposition in $\mathcal{C}$ are the same as the combinatorics of ω and the additive relations in $\mathcal{H}$ when determining (x, y), so that the final statements are obvious. □

Corollary 2.4.2. *Suppose that* $\mathcal{C}$ *satisfies the conditions of Lemma 2.4.1 for the standard trace* ε*, as the graded rank of Hom spaces over a graded ring R. We assume that R is concentrated in non-negative degree, and has no nontrivial idempotents. Suppose that* $\mathcal{C}$ *consists only of direct sums of summands of various* $B_{\underline{i}}$*. Then each object* B_w *is indecomposable and the category has the Krull-Schmidt property. The objects* $\{B_w\}$ *for* $w \in W$ *form a complete list of non-isomorphic indecomposables up to grading shift. The map* $\mathcal{H} \to \boldsymbol{Kar}[\mathcal{C}]$ *is an isomorphism.*

Proof. Calculations with the bilinear form imply that the graded rank of $\mathrm{Hom}_{\mathcal{C}}(B_w, B_x)$ is either in $1 + v\mathbb{Z}[v]$ for $w = x$, or in $v\mathbb{Z}[v]$ for $w \neq x$. This is sufficient to imply that $\{B_w\}$

form a list of pairwise non-isomorphic indecomposables. Since every $B_{\underline{i}}$ splits into these indecomposables, our list must be complete. ☐

This observation was made by Soergel in [27] and elsewhere; that one need only show the existence of objects B_w descending to b_w to show that $\mathcal{B}$ categorifies the Kazhdan-Lusztig basis, because once one has those objects abstract nonsense implies that they form a complete list of non-isomorphic indecomposables.

Corollary 2.4.3. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two such categories as in Lemma 2.4.1. Assume that $\mathcal{C}$ is generated by direct sums of summands of $B_{\underline{i}}$, and that both categories have the Krull-Schmidt property. Suppose $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is an additive R-linear monoidal functor sending B_i to B_i (which implies that B_w is sent to B_w). Suppose that $\mathcal{F}$ induces isomorphisms of Hom spaces* $\mathrm{Hom}(B_\emptyset, B_w)$ *for all w, or alternatively of* $\mathrm{Hom}(B_\emptyset, B_{\underline{i}})$ *for every reduced expression $\underline{i}$. Then $\mathcal{F}$ is fully faithful.*

Proof. Left to the reader. ☐

The upshot of these statements is that one need only investigate very few Hom spaces to understand or identify the entire category.

To see these proofs done in more detail for a completely analogous case, see section 3.3 of [6].

Chapter 3

Dihedral Diagrammatics: $m = \infty$

In this chapter we give a presentation of an R_u-linear diagrammatic category $\mathcal{D}(\infty)$, and a functor $\mathcal{F} \colon \mathcal{D}(\infty) \to \mathcal{B}_u$. This functor will be an equivalence of categories after base change. We will show, for sufficiently nice rings $\Bbbk$ over $\mathbb{Z}[x, y]$, that the category after base change $\mathcal{D}(\infty)_{\Bbbk}$ is a categorification of the Hecke algebra. In the Karoubi envelope $\mathbf{Kar}(\mathcal{D}(\infty)_{\Bbbk})$, the isomorphism classes of indecomposable objects are in bijection with the elements of W_∞, so we may label them B_w for $w \in W_\infty$. There is an isomorphism $\mathcal{H}_\infty \to [\mathbf{Kar}(\mathcal{D}(\infty)_{\Bbbk})]$ for which $b_w \mapsto [B_w]$.

The key to the proof is the observation that certain morphism spaces in $\mathcal{D}(\infty)$ can be described by the two-colored (type A) Temperley-Lieb category. In fact, the idempotents in $\mathcal{D}(\infty)$ which correspond to the indecomposable objects B_w (for $w \neq e$) are given by Jones-Wenzl projectors. The theory of modules over the Temperley-Lieb algebra will provide us with the information we need to check all the desired direct sum decompositions.

Once we have the existence of objects B_w, we will use Lemma 2.4.1 and its corollaries to finish the proof. In order to do this, we need to investigate certain specific Hom spaces. To show that these Hom spaces are not "too big" we use diagrammatic arguments to reduce all pictures to certain forms. To show that they are not "too small" we evaluate the pictures using the functor $\mathcal{F}$ to bimodules, and use this to demonstrate linear independence. Putting this together, we pinpoint exactly the size of Hom spaces, and show that $\mathcal{F}$ is an equivalence. In previous papers, we proved that $\mathcal{H} \to [\mathbf{Kar}(\mathcal{D})]$ was an isomorphism by constructing an isomorphism $\mathcal{D} \to \mathcal{B}$ and using Soergel's results, while here we prove it directly using Corollary 2.4.2.

3.1 The category $\mathcal{D}(\infty)$

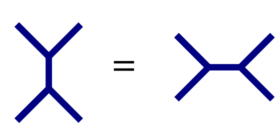
3.1.1 Definitions

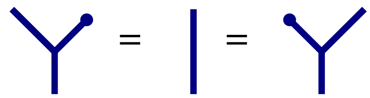
Definition 3.1.1. A *Soergel graph* for $m = \infty$ is an isotopy class of a particular kind of graph with boundary, properly embedded in the planar strip (so that the boundary of the graph is always embedded in the boundary of the strip). The edges in this graph are colored by either s or t. The vertices in this graph are either univalent (dots) or trivalent, with all three adjoining edges having the same color. The boundary of the graph gives two sequences of colors, the *top* and *bottom boundary.* Soergel graphs have a degree, where trivalent vertices have degree -1 and dots have degree 1.

When there is no ambiguity we refer to a Soergel graph merely as a "graph," even though it is an isotopy class of embedded graph.

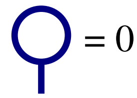
Definition 3.1.2. Let $\mathcal{D}(\infty)$ be the $\mathbb{Z}[x, y, (xy - 4)^{-1}]$-linear monoidal category defined herein. The objects will be finite sequences $\underline{\boldsymbol{i}} = i_1 i_2 \dots i_d$ of indices s and t, with a monoidal structure given by concatenation. The space $\mathrm{Hom}_{\mathcal{D}(\infty)}(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}})$ will be the free $\mathbb{Z}[x, y, (xy-4)^{-1}]$-module generated by Soergel graphs with bottom boundary $\underline{\boldsymbol{i}}$ and top boundary $\underline{\boldsymbol{j}}$, modulo the relations below. Hom spaces will be graded by the degree of the Soergel graphs.

In giving relations, we will always use the convention that blue represents s and red t. The first four relations hold for both red and blue:

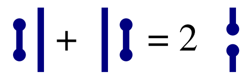
 (3.1.1)



(3.1.2)

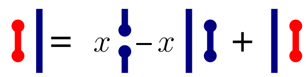


(3.1.3)

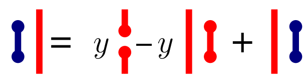


(3.1.4)

The remaining relations, along with (3.1.4) above, are called *forcing relations*. They are:

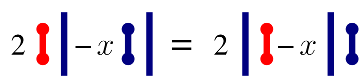


(3.1.5)

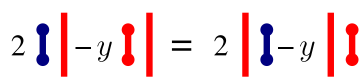


(3.1.6)

The forcing relations immediately imply the *sliding relations*:



(3.1.7)



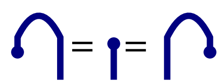
(3.1.8)

Remark 3.1.3. Technically, we have defined a non-graded, non-additive category whose Hom spaces are enriched in graded $\Bbbk$-modules. One can take its graded additive closure, which formally adds direct sums and grading shifts, and restricts to degree 0 maps. This is a graded additive category with exactly the same information. We always assume we are working with the graded additive closure.

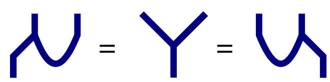
Definition 3.1.4. Let $\mathcal{F} = \mathcal{F}_\infty$ be the $\mathbb{Z}[x, y, (xy-4)^{-1}]$-linear monoidal functor from $\mathcal{D}(\infty)$ to $\mathcal{B}_{\mathrm{BS},u}$ defined as follows. The object $\underline{\boldsymbol{i}}$ is sent to $B_{\underline{i}}$. A dot colored i can be oriented as a map from i to $\emptyset$, or as a map from $\emptyset$ to i. The former is sent by $\mathcal{F}$ to the multiplication map from $R \otimes_{R^i} R \to R$. The latter sends $1 \in R$ to $\Delta_i \in R \otimes_{R^i} R$ (see section 2.2.4). The trivalent vertex colored i can be oriented as a map from ii to i or vice versa. The former is sent by $\mathcal{F}$ to the map $f \otimes g \otimes h \mapsto f\partial_i(g) \otimes h$ from $R \otimes_{R^i} R \otimes_{R^i} R \to R \otimes_{R^i} R$, and the latter is sent to the map $f \otimes g \mapsto f \otimes 1 \otimes g$ in the opposite direction.

Claim 3.1.5. *The previous definition gives a well-defined functor.*

Proof. Not only must we show that the relations on $\mathcal{D}(\infty)$ given above are satisfied by their images in $\mathcal{B}$, but we must show that the morphism designated by a graph is independent of the graph's embedding into the plane. This latter statement corresponds to some additional "isotopy relations," as pictured here.



(3.1.9)



(3.1.10)

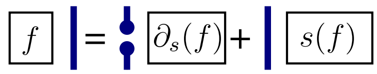
For the first four relations and the isotopy relations, this was proven in [7], and we leave the simple computation to the reader. We will discuss the sliding and forcing relations presently. □

3.1.2 Double Dots

Under the functor $\mathcal{F}$, a *double dot* colored i (such as those appearing in the forcing and sliding relations) will be sent to the polynomial $\mu_i = f_i \in R_u$. Placing double dots in the leftmost and rightmost regions will yield an R_u-bimodule structure on Hom spaces.

One should think of the vertical line colored i as being the identity map of B_i. The bimodule B_i is designed so that left and right multiplication are the same for polynomials invariant in s_i. The sliding relations say precisely that $L_s = 2f_t - xf_s$ slides through a blue line, and L_t slides through a red line. Using (3.1.4) one can show that f_s^2 slides through a blue line as well. Since L_s and f_s^2 generate R_u^s, and polynomial in R_u^s will slide across a blue line. Moving a symmetric polynomial across a line we refer to as "dot sliding."

The dot forcing relations (which are implied by the sliding relations if 2 is invertible) describe how to take any polynomial and "force" it across a line, rewriting it as a sum of terms where either some polynomial appears on the other side, or where the line itself is "broken." The reader can check that the dot forcing relations imply the following general dot forcing rule:

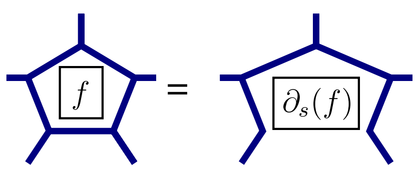


(3.1.11)

Here, f represents an arbitrary polynomial (a sum of products of double dots). Since $s(f_t) = f_t - x f_s$, we see that this agrees with (3.1.5), for instance. The reader can now verify that the functor $\mathcal{F}$ preserves the dot sliding and forcing relations.

It is an immediate conclusion from (3.1.1) and (3.1.3) that any cycle of one color with an empty interior evaluates to 0. Combining this with the dot forcing rule, a cycle surrounding a polynomial f can be replaced with the "broken" cycle with $\partial_i(f)$ outside (where $\partial_i(f)$ is sometimes 0, as in (3.1.3)). The broken cycle is just a tree, and which tree is irrelevant by (3.1.1), while the polynomial $\partial_i(f)$ is s_i-invariant and slides across all the edges, meaning that it is irrelevant which edge of the cycle was broken in order to force f out.

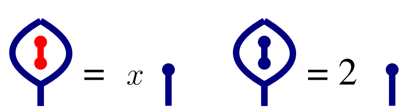
Example 3.1.6.



As an example, we have the following relations, saying that a double dot colored j inside a opening labelled i gives $a_{i,j}$ times the identity, with $a_{i,j}$ replaced by x or y or 2 as appropriate.

(3.1.12)

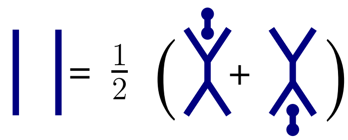
We call this the *Cartan relation*, because it so quickly yields the entries of the Cartan matrix. We give the same name to the "needle" version of this relation.



(3.1.13)

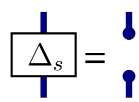
3.1.3 Splitting ii

We have the following implication of (3.1.4), if we allow division by 2:



(3.1.14)

We obtain this by stretching dots from the two lines towards the center, and applying (3.1.4). This gives a decomposition of $\mathbb{1}_{ss}$ into two orthogonal idempotents, each factoring through the object s, and therefore induces the isomorphism $ss \cong s\{1\} \oplus s\{-1\}$. As to be expected from Section 2.2.4, we need not divide by 2, by choosing any other representative of Δ_i. In particular, we may replace (3.1.4) with



(3.1.15)

and accordingly we have a different idempotent decomposition. It should not be hard to determine what we mean by Δ_s in the picture above, because we have shown that $R \otimes_{R^s} R$ acts on the object s of $\mathcal{D}(\infty)$ via left and right multiplication by double dots. This is exactly analogous to Claim 2.2.18 and the isomorphism (2.2.6).

Thus $\mathcal{D}(\infty)$ satisfies

$$ii \cong i\{1\} \oplus i\{-1\}. \tag{3.1.16}$$

This relation is sufficient to give a map from $\mathcal{H}$ to the Grothendieck group of $\mathbf{Kar}(\mathcal{D}(\infty))$, because it categorifies (2.1.3).

3.1.4 Spanning sets and minimal degrees

Given a graph, each connected component of the graph will have a single color. By (3.1.2) any monocolor component with no cycles will reduce to a tree with no more dots than necessary: a double dot, a single dot attached to the boundary, or a tree with no dots connecting ≥ 2

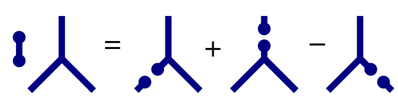
boundary points. We call the latter two *simple trees.* Any choice of simple tree for a given boundary represents the same morphism by (3.1.1). Any cycles in a component can be broken by inductively reducing the interior of the cycle to a polynomial, and then forcing it out. This gives the induction step for the following proposition, whose proof is now obvious.

Proposition 3.1.7. *Any morphism in $\mathcal{D}(\infty)$ can be written as a linear combination of graphs where each component is either a simple tree or a double dot; moreover, all double dots are in the left-most (alternatively, right-most) region. Therefore, any morphism with no boundary reduces to double dots.*

Corollary 3.1.8. *The endomorphism ring of $\emptyset$ in $\mathcal{D}(\infty)$ is precisely R.*

Proof. The endomorphism ring is spanned by double dots, so that R surjects on to it. After applying $\mathcal{F}$, we get an isomorphism $R \to \mathrm{End}(R)$. Therefore $R \cong \mathrm{End}(\emptyset)$. □

Morphisms of this form do not constitute a basis in general, however, since we have equalities like



(3.1.17)

Now we endeavor to find a basis at least for the morphisms of minimal degree.

Definition 3.1.9. Consider a diagram, each component of which is a simple tree (so there are no double dots). The plane minus the red subgraph is split into connected components, each of which contains a (possibly empty) blue subgraph. We call this diagram *maximally connected* if the blue subgraph is connected within every component of the plane minus the red subgraph, and the same with the colors switched. We consider these diagrams up to (3.1.1).

Clearly, a maximally connected diagram is determined by its boundary and the blue subgraph (resp. the red subgraph) since the other color is determined by this. Many pictures of maximaly connected diagrams can be found in Section 3.1.6.

Claim 3.1.10. *Up to scalars, all morphisms are generated over maximally connected diagrams by placing double dots in* any *of the regions.*

Proof. Using (3.1.4) or (3.1.15), we may break any line. This is sufficient to reach any collection of simple trees by disconnecting some maximally connected diagram. By Proposition 3.1.7 we are done. □

Claim 3.1.11. *Fix a sequence of colors along the boundary of a planar disk. Morphisms represented by maximally connected diagrams with this boundary all have the same degree, and are the minimal degree attainable for morphisms in $\mathcal{D}(\infty)$ with that boundary. If the colors on the boundary alternate, this degree is* 2*; for every repetition on the boundary this degree is lowered by* 1*.*

Example 3.1.12. Given boundary *brbrrrbbrb*, the minimal degree would be -2 because there are 4 repetitions (this sequence lies on a circle so the end is adjacent to the beginning).

Proof. Removing a double dot lowers the degree by 2, so we assume there are no double dots, only simple trees. Given two blue trees with no red in between, one can connect them with a blue edge, meeting each tree in a new trivalent vertex. This operation reduces the degree by 2, and sends two simple trees to a simple tree (or in the case of single dots, to something which immediately reduces to a simple tree). Therefore a minimal degree map must be maximally connected.

It remains to show that all maximally connected graphs have the appropriate degree. Whenever there is repetition, the repeated indices must be in the same component in a maximally connected graph. Precomposing this morphism with the trivalent vertex which fuses the two adjacent boundary lines in the repetition, we reduce to the case with one fewer repetition. Therefore we may assume that the colors on the boundary alternate. We leave it as an exercise to show, in this case, that all maximally connected diagrams have degree 2 (one way is to use the Euler characteristic). □

Claim 3.1.13. *The number of maximally connected diagrams with an alternating boundary rbrb... of length $2m$ is the m-th Catalan number, $\frac{1}{m+1}\binom{2m}{m}$.*

The bijection between maximally connected diagrams and crossingless matchings will be made explicit soon enough.

Proposition 3.1.14. *Maximally connected graphs (considered up to (3.1.1)) form a* basis *for the minimal degree part of the Hom space in $\mathcal{D}(\infty)$.*

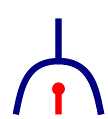
The responsible thing to do would be to prove this proposition now. It is enough to show that the images of maximally connected diagrams after applying the functor $\mathcal{F}$ are linearly independent. The proof should be no more than a clever calculation, finding combinatorics for bimodule elements which would distinguish these maps. However, Theorem 4 will also imply this result, and we are too lazy to do the calculation when an elegant proof can replace it. We will prove a few propositions using this one, but we do give a proof of Theorem 4 which does not require anything dependent on this. We state the result now because it is the real motivating reason why everything works.

Remark 3.1.15. One can calculate the degree 2 part of the renormalized trace $\widetilde{\varepsilon}(b_{\underline{i}})$ for $\underline{\boldsymbol{i}}$ alternating of even length $2k$ (this is c^1_{2m-1}), and show that it is precisely the k-th Catalan number. We leave this as an exercise. Thus $\widetilde{\varepsilon}$ agrees with the trace induced on $\mathcal{H}_\infty$ by $\mathcal{D}(\infty)$ in their minimal degree terms. We will show soon that they are equal on the nose.

3.1.5 Pitchforks and Alldots

We make two comments about higher degree morphisms.

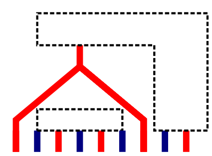
Let us call the following map (occurring in either color) a *pitchfork*.



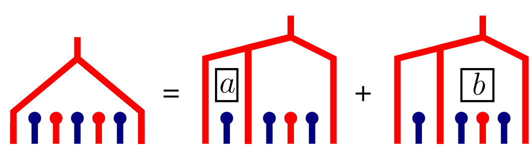
Let us call the map from $B_{\underline{i}}$ to $\emptyset$ which immediately terminates every boundary line in a dot by the name *all-dot*.

Claim 3.1.16. *Let $\underline{\boldsymbol{i}} = rbrbrb\dots$ be an alternating sequence of length $m \geq 1$ (though we do not assume that m is even). Then $\mathrm{Hom}(\underline{\boldsymbol{i}}, \emptyset)$ is generated (over double dots in the exterior region) by maps which begin with pitchforks, and by the all-dot.*

Proof. Without loss of generality, we may assume that our morphism is represented by a collection of simple trees. We will use induction on m, where the statement is obviously true for $m = 1, 2$ (and there are no pitchforks). Consider the first red boundary. If it ends in a dot, then we may use the inductive hypothesis for $m - 1$ on the remainder of the diagram. Therefore, suppose that it connects to some other red boundary. Consider the first other boundary that it is connected to, and how this divides the graph into two regions.



By induction, the morphism in the inner region either begins with a pitchfork (and thus satisfies our criterion) or is the all-dot. But this latter possibility is within the span of pitchforks as well, as can be seen by the following application of (3.1.15).



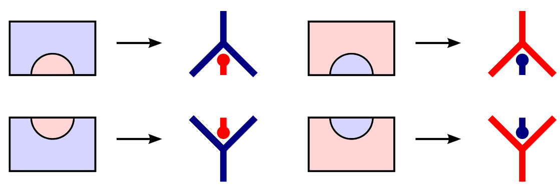
Here, $\{1, a\}$ and $\{b, 1\}$ are dual bases. The map with b begins with a pitchfork, while the map with a does not yet begin with a pitchfork. However, in the map with a, the region where b is absent looks like what we began with, so by induction it is within the span of pitchforks. □

We will be using this claim in the following way: if there is an idempotent in $\mathrm{End}(B_{\underline{\boldsymbol{i}}})$ which is killed by all pitchforks, then the only maps in the Karoubi envelope from the image of that idempotent to $\emptyset$ will factor through the all-dot.

3.1.6 Temperley-Lieb and Soergel

Fix two colors i and j (among red and blue), and consider the Hom category $\mathrm{Hom}_{2\mathcal{TL}}(i,j)$. In particular, this is a *non-monoidal* category consisting of all diagrams where the rightmost and leftmost colors are fixed beforehand, and its objects may be thought of as alternating sequences $\underline{\boldsymbol{i}}$ of red and blue with fixed extremal colors.

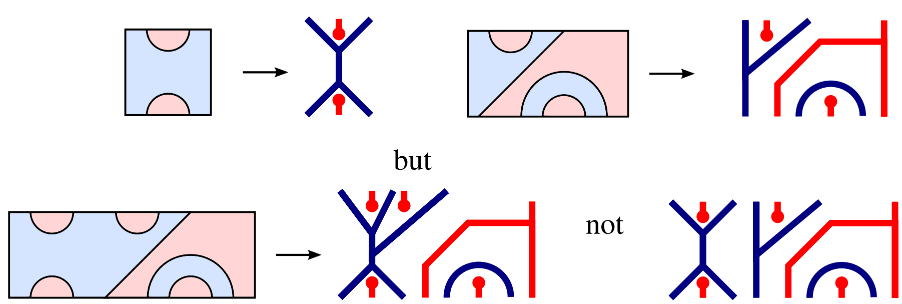
Definition 3.1.17. We now provide a functor from $\mathrm{Hom}_{2\mathcal{TL}}(i,j)$ to $\mathcal{D}(\infty)$. On objects it sends an alternating sequence $\underline{\boldsymbol{i}}$ to the object $\underline{\boldsymbol{i}} \in \mathcal{D}(\infty)$. We map the generating morphisms as follows:



All the morphisms in the image of this functor are degree 0.

Using isotopy, (3.1.2) and the Cartan relation (3.1.12), it is immediate to see that this functor is well-defined and $\mathbb{Z}[x,y]$-linear. Note that $\mathcal{D}(\infty)$ contains many non-alternating sequences $\underline{\boldsymbol{i}}$, but only the alternating sequences, and certain degree 0 maps between them, are in the image of this functor. This functor does *not* lift to any sort of 2-functor from $2\mathcal{TL}$ to $\mathcal{D}(\infty)$, as can be seen in the following example.

Example 3.1.18.



A genuine 2-functor which explains this one will be produced soon, once we discuss singular Soergel bimodules.

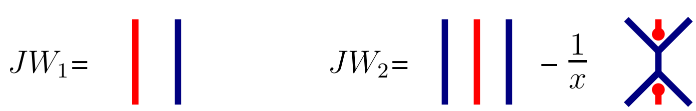
Starting with a diagram in $2\mathcal{TL}$ we can "squeeze" each region into a tree and get the corresponding Soergel graph, which is maximally connected. Given any maximally connected graph, we may draw lines in the complement of the graph and color each region according to the graph, to get a crossingless matching in $2\mathcal{TL}$. This gives a clear bijection between maximally connected graphs (with the appropriate boundary) and diagrams in $2\mathcal{TL}$.

Proposition 3.1.19. *Let* $\underline{\boldsymbol{i}}$ *and* $\underline{\boldsymbol{j}}$ *be objects in* $\mathrm{Hom}_{2\mathcal{TL}}(i,j)$ *for some colors* i,j. *Then the map* $\mathrm{Hom}_{2\mathcal{TL}}(\underline{\boldsymbol{i}},\underline{\boldsymbol{j}}) \to \mathrm{Hom}_{\mathcal{D}(\infty)}(\underline{\boldsymbol{i}},\underline{\boldsymbol{j}})_0$*, the degree 0 part, is an isomorphism. Moreover,* $\mathrm{Hom}_{\mathcal{D}(\infty)}(\underline{\boldsymbol{i}},\underline{\boldsymbol{j}})$ *has no morphisms of negative degree.*

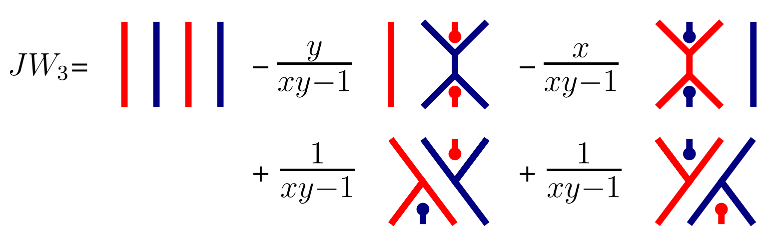
Warning: this proof uses the (not-yet-proven) Proposition 3.1.14. Without this Proposition, the following only proves that the map is surjective.

Proof. The boundary of the Hom space we're looking at (reading all the way around the circle) is almost an alternating sequence, except for two repetitions, because both $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$ start with i and end with j. It is clear that this functor gives a bijection from crossingless matchings to maximally connected diagrams. That we get precisely the morphisms of minimal degree 0 is given by Claim 3.1.11, so the Hom space map is surjective. It is also injective, thanks to Proposition 3.1.14. ☐

The most important consequence of the functor above is that we end up with idempotents and isomorphisms in (localized) $\mathcal{D}(\infty)$ and its Karoubi envelope, given by the images of the projectors and idempotents guaranteed by Proposition 2.3.9. In order to construct these idempotents we need to be in a 2TL-sufficient ring. For instance, here are the idempotents which are the images of the first few Jones-Wenzl projectors. These come in two variants: pictured is the variant which ends with blue.

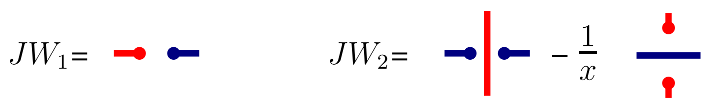


Example 3.1.20.

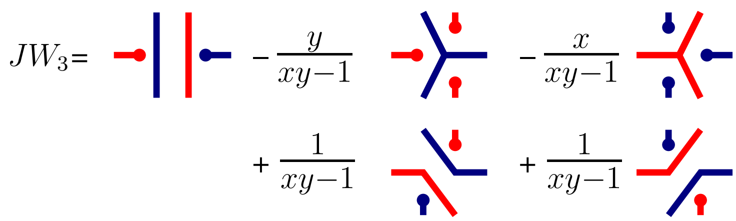


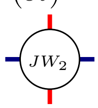
Remark 3.1.21. Thanks to Claim 2.3.4, we can describe this map as the unique degree 0 map killed by any pitchfork on top or bottom.

We may also construct Soergel graphs which correspond to crossingless matchings but have degree 2 by ignoring the two repetitions from earlier. Here are a few Jones-Wenzl projectors in this context.



Example 3.1.22.



Notation 3.1.23. When we view the Jones-Wenzl projector as a map of degree 2 with boundary $(st)^{m-1}$ reading around the circle, we will draw it as a circle labelled by JW. For instance,  . This map is *not* rotation-invariant for arbitrary $\Bbbk$, so we can not draw it in a rotation-invariant way.

3.1.7 The Grothendieck Group

Because the Temperley-Lieb algebra subsumes all degree 0 maps, there are no additional idempotents than those we already know about. All the idempotents which are primitive (i.e. indecomposable) in the Temperley-Lieb algebra are primitive in $\mathcal{D}(\infty)$ as well. We may now transfer everything we know about $2\mathcal{TL}$ (namely Proposition 2.3.10) into a statement about $\mathcal{D}(\infty)$. (If we don't assume Proposition 3.1.14, we can only provide the idempotents we do know about; we can not claim that there are no additional idempotents, or that these idempotents are non-zero.)

Theorem 4. *Let $\Bbbk$ be a 2TL-sufficient ring. Then the indecomposables $\{B_w\}$ in $\boldsymbol{Kar}(\mathcal{D}(\infty)_{\Bbbk})$ can be labelled by elements $w \in W_\infty$. The indecomposable B_e corresponds to the empty diagram. Each other indecomposable corresponds to the image of the appropriate Jones-Wenzl* *projector in* $B_s \otimes B_t \otimes \ldots$. *There is an isomorphism from* $\mathcal{H}_\infty$ *to the Grothendieck ring of* $\boldsymbol{Kar}(\mathcal{D}(\infty)_{\Bbbk})$ *sending* b_w *to* $[B_w]$. *For any Soergel ring* $\Bbbk$, *the functor* $\mathcal{F}$ *is an equivalence, and the indecomposables* B_w *go to the indecomposable Soergel bimodules that he also labels* B_w.

Proof. We know that $\mathbf{Kar}(\mathcal{D}(\infty)_{\Bbbk})$ has objects B_w corresponding to the appropriate Jones-Wenzl projectors, although we do not know that these objects are non-zero. As before, any $B_{\underline{\boldsymbol{i}}}$ can be decomposed into direct sums of B_w, in the appropriate way, so that we have a $\mathbb{Z}[v, v^{-1}]$-algebra map $\mathcal{H}_\infty \to [\mathbf{Kar}(\mathcal{D}(\infty)_{\Bbbk})]$ sending $b_w \to [B_w]$. We now attempt to pin down the trace map on $\mathcal{H}$ induced by $\mathcal{D}(\infty)_{\Bbbk}$. This trace is determined by the graded rank of $\mathrm{Hom}(B_w, \emptyset)$. Suppose that we can show that $\mathrm{Hom}(B_w, \emptyset)$ is a free $R_{\Bbbk}$-module of rank 1, generated in degree $l(w)$, so that $\varepsilon_{\mathcal{D}(\infty)}(b_w) = v^{l(w)}\mathrm{grk}(R)$. This would imply that $\varepsilon_{\mathcal{D}(\infty)} = \widetilde{\varepsilon}_{\mathrm{std}}$. Now Lemma 2.4.1 and its corollaries will finish the proof. Also, this allows us to prove Proposition 3.1.14.

It remains to calculate $\mathrm{Hom}(B_w, \emptyset)$. If $w = \underline{\boldsymbol{i}}$ is a reduced expression, then this Hom space can be described as $\mathrm{Hom}(B_{\underline{\boldsymbol{i}}}, \emptyset)$ precomposed with the Jones-Wenzl idempotent. However, by Claim 3.1.16, almost all the maps from $B_{\underline{\boldsymbol{i}}}$ to $\emptyset$ begin with a pitchfork, which kills the Jones-Wenzl projector. Therefore, $\mathrm{Hom}(B_w, \emptyset)$ is generated inside $\mathrm{Hom}(B_{\underline{\boldsymbol{i}}}, \emptyset)$ by the map which composes the idempotent with the all-dot. It is at most rank 1 over R (acting by double dots) with its generator in degree $l(w)$, the degree of the all-dot. Suppose that we can show that this map is nonzero after applying the functor $\mathcal{F}$. Since R acts freely on Hom spaces in $\mathcal{B}$, this would imply that $\mathrm{Hom}(B_w, \emptyset)$ must be a free R-module as well.

If one takes the Jones-Wenzl projector JW_m and applies dots to *every* strand, top and bottom, then one obtains a polynomial. It is immediate that this is precisely the associated polynomial of the element $JW_m \in 2\mathcal{TL}$, which is equal to some $\mathbb{L}_{k,l}$, as discussed in section 2.3.4. This polynomial is nonzero in $R_{\Bbbk}$. Therefore, one also gets a nonzero answer when applying $\mathcal{F}$ to the Jones-Wenzl projector with dots just on the top. This concludes the proof. $\square$

3.2 Singular Soergel Bimodules: $m = \infty$

3.2.1 Definitions

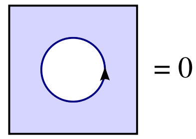
Definition 3.2.1. A *singular Soergel graph* or *Soergel 1-manifold diagram* for $m = \infty$ is (almost) an isotopy class of a particular kind of 1-manifold with boundary, properly embedded in the planar strip (so that the boundary of the graph is always embedded in the boundary of the strip). The regions cut out by this 1-manifold are labelled by finitary parabolic subsets $\{\underline{e}, \underline{s}, \underline{t}\}$, in such a way that $\underline{s}$ never abuts $\underline{t}$. Soergel 1-manifold diagrams are graded, where the degree of a clockwise cup or cap is $+1$, and the degree of a counterclockwise cup or cap is -1.

In addition to labeling regions, one can color and orient the 1-manifold itself. A line can be colored s or t, and separates two regions whose label differs by that element. The orientation is such that the larger parabolic subset is on the right hand of the 1-manifold. The boundary of the graph gives two sequences of colored oriented points, the *top* and *bottom boundary*. Not every oriented colored 1-manifold gives rise to a consistent labeling of regions.

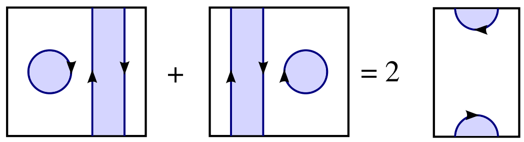
If there is no ambiguity, we shorten the name to "Soergel diagram." We think of a Soergel diagram as being the data of two oriented 1-manifolds, one blue and one red, which are not allowed to overlap. This is different from the $m < \infty$ case, where they are allowed to intersect transversely. Note that the degree of a diagram is *not* isotopy invariant, so that rotating a relation will give a relation in a different degree. This is the only sense in which we do not consider Soergel diagrams as isotopy classes.

Definition 3.2.2. Let $\mathfrak{D}(\infty)$ be the 2-category defined as follows. The objects are $\{\underline{e}, \underline{s}, \underline{t}\}$. The 1-morphisms are generated by maps from $\underline{e}$ to $\underline{s}$ and back, and maps $\underline{e}$ to $\underline{t}$ and back. A path in the object space (like $\underline{sesetes}$) uniquely specifies a 1-morphism. The 2-morphism space between 1-morphisms is the free $\mathbb{Z}[x, y, (xy{-}4)^{-1}]$-module spanned by Soergel diagrams with the appropriate boundary, modulo the relations below. Hom spaces will be graded by the degree of the Soergel diagrams.

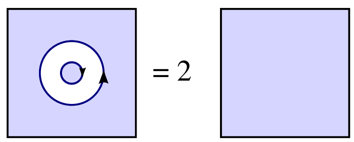
The first two relations hold for both colors:



(3.2.1)

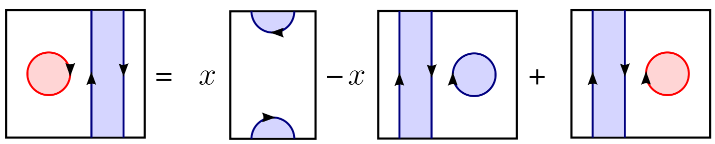


(3.2.2)

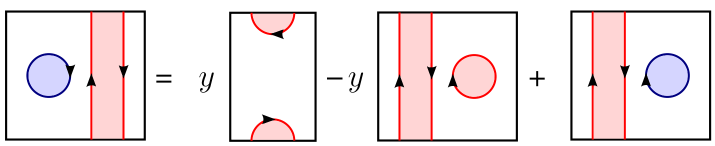


(3.2.3)

The next relations, along with (3.2.2) above, are called the *circle forcing relations*.

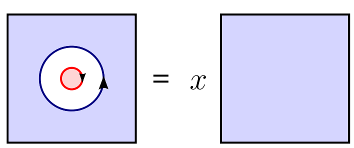


(3.2.4)

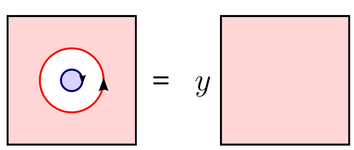


(3.2.5)

Finally, along with (3.2.3) we have the *Cartan relations*:

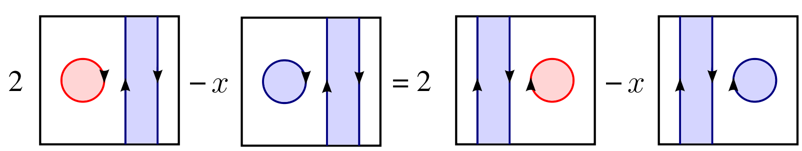


(3.2.6)

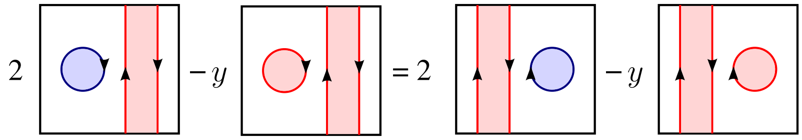


(3.2.7)

The forcing relations immediately imply the *circle sliding relations*.



(3.2.8)

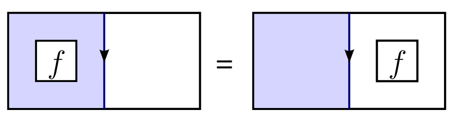


(3.2.9)

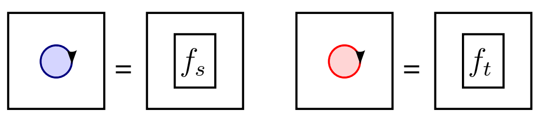
Given (3.2.3) and the circle forcing relations, the remaining Cartan relations are redundant if we invert 2. Also, one can demonstrate using (3.2.2) that the left side of (3.2.3) squares to twice itself, so that (3.2.3) is already implied if we desire our endomorphism ring to be a domain.

One should think of clockwise circles as the new double dots. That is, they give a way for R_u to act in any $\underline{e}$ region. Placing a polynomial inside a counterclockwise circle surrounded by region $\underline{i}$ is effectively taking ∂_i of that polynomial. The map ∂_i is surjective, so this gives an action of R^i in a region labelled $\underline{i}$. We will still use a box labelled f to represent the polynomial f, and if it appears inside a blue region then it must be in R^s.

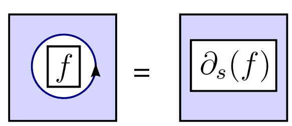
Given this notation (which is only consistent because of the relations above) we may rewrite the circle sliding, circle forcing, and Cartan relations as follows:



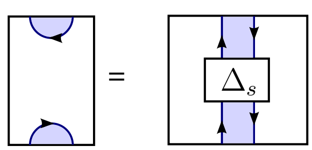
when $f \in R^s$ (3.2.10)



(3.2.11)



(3.2.12)

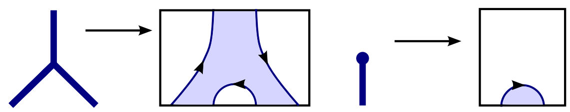


(3.2.13)

The map Δ_s in (3.2.13) is the comultiplication element in B_s, as described in section 2.2.4.

3.2.2 Functors

Definition 3.2.3. We give a functor $\iota\colon \mathcal{D}(\infty) \to \mathrm{Hom}_{\mathfrak{D}(\infty)}(\underline{e}, \underline{e})$. On objects, it sends s to the path $\underline{ese}$ and t to the path $\underline{ete}$. We define the functor on generators:



Claim 3.2.4. *The above definition gives a well-defined functor.*

Proof. Both categories only consider pictures up to isotopy, so we may ignore questions of isotopy invariance. Relations (3.1.2) and (3.1.1) also correspond to mere isotopies in $\mathfrak{D}(\infty)$, and relation (3.1.3) corresponds to relation (3.2.1). A double dot in $\mathcal{D}(\infty)$ goes to a clockwise circle in $\mathfrak{D}(\infty)$. The correspondence between the dot sliding and forcing relations of $\mathcal{D}(\infty)$ and the circle sliding and forcing relations of $\mathfrak{D}(\infty)$ is clear. $\square$

Proposition 3.2.5. *The functor ι is an* isomorphism *of categories.*

Proof. Clearly any path from $\underline{e}$ to itself will be composed out of the smaller loops $\underline{ese}$ and $\underline{ete}$, so that we have a bijection of objects between sequences of indices i and sequences of paths $\underline{eie}$. To give a functor in the reverse direction, take any morphism in $\mathrm{Hom}(\underline{e}, \underline{e})$ and deformation retract the shaded regions to some graph with the appropriate boundaries. Which graph you choose is irrelevant because of relations (3.1.2) and (3.1.1). The same correspondence of relations in the previous proof shows that all of the relations in $\mathfrak{D}(\infty)$ go to zero in $\mathcal{D}(\infty)$. Perhaps the only subtlety is relation (3.2.1), which is not phrased as within $\mathrm{Hom}(\underline{e}, \underline{e})$. However, any instance of a counterclockwise circle within $\mathrm{Hom}(\underline{e}, \underline{e})$ will clearly be sent to an empty cycle within a Soergel graph, and this is zero. The same can be said about the Cartan relations and their previous counterparts. Therefore this functor is well-defined, and clearly yields the inverse functor. $\square$

Definition 3.2.6. We give a (strict) 2-functor $\mathfrak{F}\colon \mathfrak{D}(\infty) \to \mathfrak{B}_u$ to the 2-category of singular Soergel bimodules. This 2-functor is the identity on objects, so that we think of *ule* as

$R = R_u$, $\underline{s}$ as R^s and $\underline{t}$ as R^t. The maps from $\underline{e}$ to $\underline{i}$ and back correspond to restriction and induction bimodules, respectively. To define the 2-functor on 2-morphisms we need only give the image of the clockwise and counterclockwise cups and caps. In each case, this corresponds to the appropriate map for the Frobenius extension $R^s \subset R$: that is, the (blue) clockwise cap is sent to multiplication $R \otimes_{R^s} R \to R$, the clockwise cup is sent to comultiplication $1 \mapsto \Delta_s$ in the other direction, the counterclockwise cap is sent to the Demazure operator $\partial_s \colon R \to R^s$, and the counterclockwise cup is sent to the inclusion $R^s \subset R$.

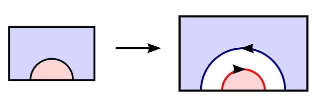
Claim 3.2.7. *The above definition gives a well-defined 2-functor. The functor $\mathcal{F} \colon \mathcal{D}(\infty) \to \mathcal{B}$ factors through this 2-functor via ι.*

Proof. The isotopy relations follow by properties of Frobenius extensions. Relation (3.2.1) follows because $\partial_s(1) = 0$. Finally, the circle sliding and forcing relations and the Cartan relations follow for exactly the same reasons that the dot sliding and forcing relations did previously. The factoring of functors is clear. □

3.2.3 Temperley-Lieb and the Grothendieck algebroid

Now let us take the map from $2\mathcal{TL}$ to $\mathcal{D}(\infty)$ and make it into a bona fide 2-functor.

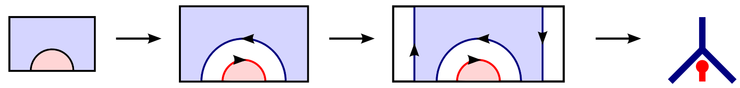
Definition 3.2.8. We give 2-functor from $2\mathcal{TL}$ to $\mathfrak{D}(\infty)$ as follows. The line in $2\mathcal{TL}$ from blue to red is sent to the 1-morphism $\underline{tes}$, and the line from red to blue is sent to $\underline{set}$. Visually, the map on 2-morphisms takes a crossingless matching and widens each strand into a region labelled $\underline{e}$, with its boundary oriented counter-clockwise.



Claim 3.2.9. *The functor above is well-defined, and its image consists of degree* 0 *maps.*

Proof. That the isotopy relations of $2\mathcal{TL}$ are satisfied is obvious. That reduction of circles works follows from the Cartan relations. □

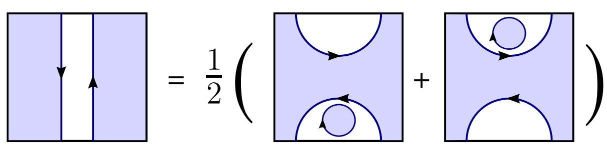
To obtain the morphism in $\mathcal{D}(\infty)$ previously obtained from a diagram in $2\mathcal{TL}$, one applies this functor and then composes on the right and left with the path to $\underline{e}$, to obtain an endomorphism of $\underline{e}$.



That the functor from $2\mathcal{TL}$ surjects onto the space of all degree 0 maps follows immediately from the following theorem.

Theorem 5. *Let $\Bbbk$ be a 2TL-sufficient ring. The 2-category $\boldsymbol{Kar}(\mathfrak{D}(\infty)_{\Bbbk})$ is a categorification of the Hecke category $\mathfrak{H}$, and categorifies the standard trace. The 2-functor $\mathfrak{F}_{\Bbbk}$ is an equivalence.*

Proof. The only relations in the Hecke category are $\underline{iei} \cong [2]\,\underline{i}$. This is categorified by a rotation of (3.2.14), such as



(3.2.14)

We now proceed as in Lemma 2.4.1. We have a functor from the Hecke category to the Grothendieck category of $\mathfrak{D}(\infty)$, and Hom spaces induce a semi-linear pairing on $\mathfrak{H}$. Because a blue up arrow is clearly biadjoint to a blue down arrow, this pairing is determined by a trace on the category. Any trace on $\mathfrak{H}$ is determined by its values on $\mathrm{End}_{\mathfrak{H}}(\underline{e})$ (see section 2.1.7). Moreover, we know the graded rank of all Hom spaces in $\mathrm{Hom}_{\mathfrak{D}(\infty)}(\underline{e}, \underline{e})$ because it is equivalent to $\mathcal{D}(\infty)$. Therefore, the trace on $\mathrm{End}_{\mathfrak{H}}(\underline{e}) = \mathcal{H}$ agrees with the standard trace, and the trace on all of $\mathfrak{H}$ agrees with the standard trace. Moreover, $\mathfrak{F}$ induces isomorphisms on Hom spaces in the category $\mathrm{End}_{\mathfrak{D}(\infty)}(\underline{e})$ since $\mathcal{F}$ does.

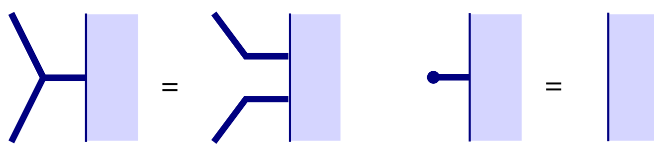
The idempotents in $2\mathcal{TL}$ give rise to idempotents in $\mathfrak{D}(\infty)$ which give a number of direct sum decompositions. In particular, we can find objects in the Karoubi envelope which descend to the Kazhdan-Lusztig basis of $\mathfrak{H}$, and are therefore indecomposable and pairwise non-isomorphic. Therefore the map from $\mathfrak{H}$ to the Grothendieck category is an equivalence of categories. Similar arguments show that $\mathfrak{F}$ is an equivalence. □

3.2.4 Induced Modules

Corollary 3.2.10. *Over a 2TL-sufficient ring* $\Bbbk$*, the category* $\mathrm{Hom}_{\mathfrak{D}(\infty)}(\underline{e}, \underline{s})$ *categorifies the left ideal of* b_s*, which is the induced module from the trivial representation of* $\mathcal{H}_s$*. The module action is categorified by the monoidal action of* $\mathrm{End}_{\mathfrak{D}(\infty)}(\underline{e}) = \mathcal{D}(\infty)$*.*

Note that this Hom category can easily be described using a slight modification of $\mathcal{D}(\infty)$, in a precise analogy to Chapter 4 in [5]. One may draw the usual Soergel graphs but require that they end in a "blue region". One adds a new morphism corresponding to a trivalent vertex with the blue region, and imposes relations corresponding to (3.1.1) and (3.1.2).

Explicitly, the new generator is . The new relations are


(3.2.15)

Any usual Soergel graph, with the usual Soergel relations, may be drawn to the left of the blue region.

It is quite easy to provide the equivalence of categories between this diagrammatic category and $\mathrm{Hom}_{\mathfrak{D}(\infty)}(\underline{e}, \underline{s})$, in analogy to the functor ι above.

Chapter 4

Dihedral Diagrammatics: $m < \infty$

When $m < \infty$ we know that that $b_{sts\dots} = b_{tst\dots}$. In $\mathbf{Kar}(\mathcal{D}(\infty))$ we have found two idempotents corresponding to $B_{sts\dots}$ and $B_{tst\dots}$, and requiring the images of these idempotents to be isomorphic will give a category equipped with map from $\mathcal{H}_m$ to its Grothendieck algebra. Therefore, to define $\mathbf{Kar}(\mathcal{D}_m)$ we could formally add this isomorphism as a new generator. Instead we do this same thing before taking the Karoubi envelope, modifying $\mathcal{D}(\infty)$ into a category $\mathcal{D}_m$ by adding maps from $B_s \otimes B_t \otimes B_s \otimes \dots \to B_t \otimes B_s \otimes B_t \otimes \dots$ and back which interact in a precise way with the various idempotents. In order for this category to be consistently defined, we must change base to a ring $\Bbbk_m$ where the action of W_∞ on $\mathfrak{h}$ descends to W_m. Note that changing base for $\mathcal{D}(\infty)$ would not be sufficient: one must still add this new morphism.

Let $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$ be two sequences which are reduced expressions for elements of W_m and live as objects in some $\mathrm{Hom}_{2\mathcal{TL}}(i, j)$ (so that they both end with i and start with j). Then it is still true that $\mathrm{Hom}_{2\mathcal{TL}}(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}})$ surjects onto the degree 0 part of $\mathrm{Hom}_{\mathcal{D}_m}(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}})$. However, while in $\mathcal{D}(\infty)$ these are the only degree 0 maps between reduced expressions in the entire category, in $\mathcal{D}_m$ there is now an additional degree 0 map from $sts\dots$ to $tst\dots$. This map could not correspond to anything in $2\mathcal{TL}$ because the source and target objects do not match up! Neither is this some sort of 2-categorical extension of $2\mathcal{TL}$, because objects are inconsistent.

What we have is an extension of $2\mathcal{TL}$ but not as a 2-category. We have not encountered this sort of categorical extension before and without further examples we are unsure what general structure to define to encapsulate it.

What is remarkable is that the relations in $\mathcal{D}_m$ are truly uniform over all m. That is, abstracting the Jones-Wenzl projector into a map JW_m depending on m, the relations have a very simple form which does not depend on m. Of course, the morphism JW_m and its behavior does depend strongly on m.

4.1 The category $\mathcal{D}_m$

4.1.1 Definitions

Definition 4.1.1. A *Soergel graph* for $m < \infty$ is an isotopy class of a particular kind of graph with boundary, properly embedded in the planar strip (so that the boundary of the graph is always embedded in the boundary of the strip). The edges in this graph are colored by either s or t. The vertices in this graph are either univalent (dots), trivalent with all three adjoining edges having the same color, or $2m$-valent with alternating edge colors. The boundary of the graph gives two sequences of colors, the *top* and *bottom boundary.* Soergel graphs have a degree, where trivalent vertices have degree -1, dots have degree 1, and $2m$-valent vertices have degree 0.

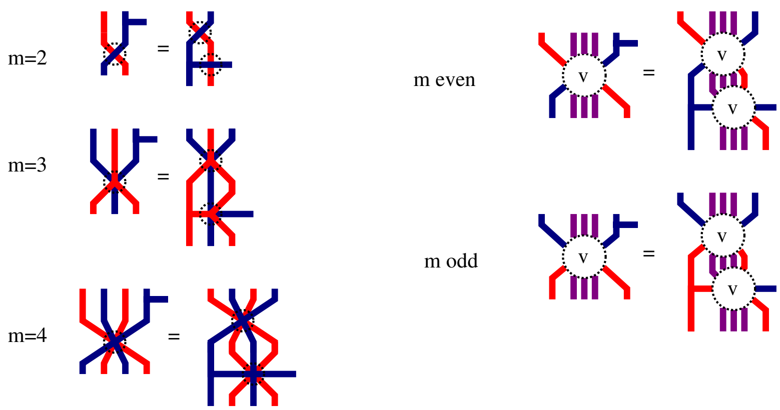
When there is no ambiguity we refer to a Soergel graph for m merely as a "graph." If we want to talk about Soergel graphs for $m = \infty$ again, we will say so explicitly.

Fix a base ring $\Bbbk_m$ which is m-symmetric and $2TL_{m-1}$-sufficient.

Definition 4.1.2. Let $\mathcal{D}_m$ be the $\Bbbk_m$-linear monoidal category defined herein. The objects will be finite sequences $\underline{\boldsymbol{i}} = i_1 i_2 \dots i_d$ of indices s and t, with a monoidal structure given by concatenation. The space $\mathrm{Hom}_{\mathcal{D}_m}(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}})$ will be the free $\Bbbk_m$-module generated by Soergel graphs with bottom boundary $\underline{\boldsymbol{i}}$ and top boundary $\underline{\boldsymbol{j}}$, modulo the relations below. Hom spaces will be graded by the degree of the Soergel graphs.

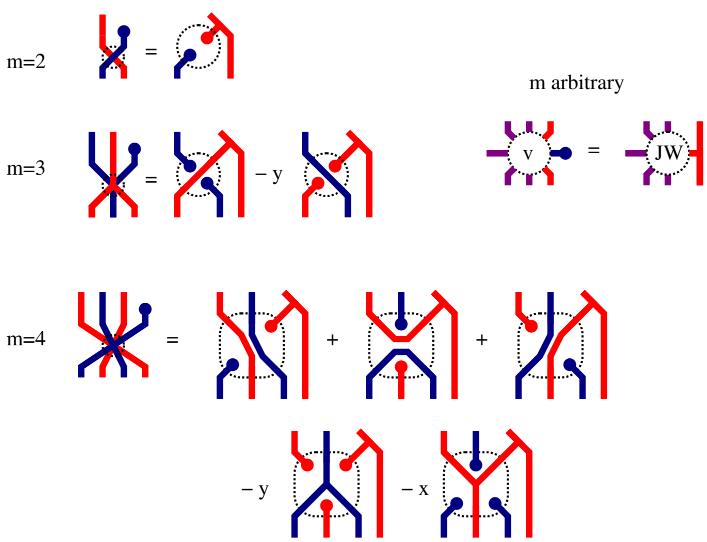
We have all the relations that define $\mathcal{D}(\infty)$ (see section 3.1.1) as well as two new relations, called the *two-color relations*, found in equations (4.1.1) and (4.1.2). They hold with the colors switched as well. It is difficult to draw these relations for all m at once, since the number of strands entering a vertex changes. A circle labelled JW contains the Jones-Wenzl projector JW_{m-1} as a degree 2 map (see Notation 3.1.23), and a circle labelled v contains the $2m$-valent vertex. A sequence of a few purple lines will indicate an alternating sequence of red and blue lines of the appropriate length (depending on m).

The new relations are *two-color associativity*:



(4.1.1)

and *dotting the vertex*:

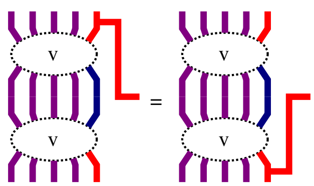


(4.1.2)

Each figure has some examples, with the $2m$-valent vertex in these relations circled. Note that the expansion of the Jones-Wenzl projector in these examples has already incorporated the additional relations coming from the fact that our base ring is m-faithful.

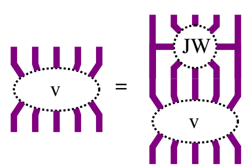
Remark 4.1.3. Thanks to the results of section 2.3.3, these relations are invariant under switching colors when $\Bbbk$ is m-symmetric. We write it in this form, without imposing $x = y$, to make it clear that one could define this category for the m-faithful case too. The category will still be cyclic, so that diagrammatics make sense. However, there will be no color-switching symmetry. We could not draw the new morphism so that its 180-degree rotation is equal to its color-switch; instead, the two are equal up to a scalar which needs to be kept track of. This is far too much of a headache.

Now we derive some other relations in $\mathcal{D}_m$. The following two pictures become the same after an application of (4.1.1).



(4.1.3)

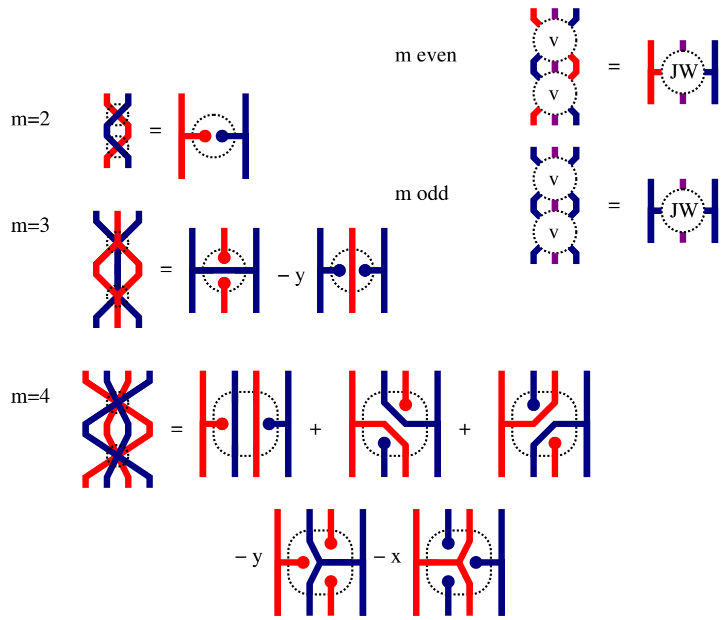
Relation (4.1.2) implies that the $2m$-valent vertex is killed by any pitchfork. Thus we may deduce that



(4.1.4)

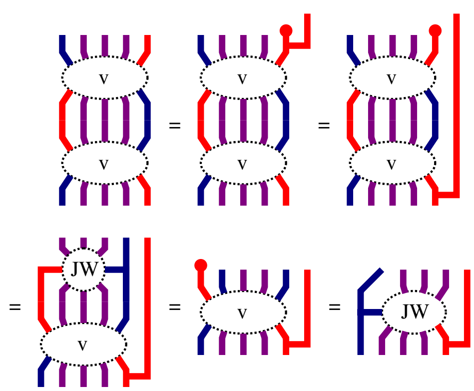
Only one term in JW survives: the term yielding the identity map. The other terms produce pitchforks.

Claim 4.1.4. *The following relation holds. (Note: this will be the idempotent which projects onto B_{w_0}.)*



(4.1.5)

Proof. Using in order: (3.1.2), (4.1.3), (4.1.2), (4.1.4), and (4.1.2). The coloration is as though m is even, although if m is odd one only need change the color on the leftmost strands.



□

We invite the reader to compare relation (4.1.1) with relation (3.7) in [5].

4.1.2 The Grothendieck group

Now let us assume that $\Bbbk_m$ is m-symmetric and $2TL_{m-1}$-sufficient.

Claim 4.1.5. *There is a functor $\mathcal{F}_m \colon \mathcal{D}_m \to \mathcal{B}_m$, such that the composition $\mathcal{D}(\infty)_{\Bbbk} \to \mathcal{D}_m \to \mathcal{B}_m$ agrees with our earlier functor $\mathcal{F}_\infty$.*

Proposition 4.1.6. *Any morphism in $\mathcal{D}_m$ with an alternating boundary of length $< 2m$ reduces to a sum of diagrams with no $2m$-valent vertices.*

Both of these should be results that we are equipped to demonstrate now. The former consists in defining the image of the $2m$-valent vertex and checking the new relations. The latter should be no more than a clever feat of planar graph theory. However, both results will be easier to prove once we discuss singular Soergel bimodules, so we postpone the proofs until then. The functor is defined in section 4.2.4, and the proposition proven thereabouts. We will not use the following results until the previous ones have been proven.

Lemma 4.1.7. *The functor $\mathcal{F}_m$ induces an isomorphism $\mathrm{End}_{\mathcal{D}_m}(\emptyset) \cong R$.*

Proof. Proposition 4.1.6 shows that all maps reduce to double dots. The functor to bimodules gives us a surjective map from a rank 1 R-module to $\mathrm{End}_{\mathcal{B}}(B_e) = R$, which must be an isomorphism. $\square$

The upshot is that the new relations (4.1.1) and (4.1.2) do not impose any new relations on polynomials.

There is clearly still a map from $\mathcal{H}_\infty \to [\mathbf{Kar}(\mathcal{D}_m)]$, because $ii \cong i\{1\} \oplus i\{-1\}$ still holds. Our ring $\Bbbk$ is definitely *not* $2TL_m$ sufficient, since $[m]_x = 0$, but one can still define all the idempotents in $\mathcal{D}_m$ coming from $\mathit{2}\mathcal{TL}$ for elements $w \in W_\infty$ of length $\leq m$. We call the image of the corresponding idempotent B_w. We make no claim yet that these are indecomposable or non-zero. We still have $b_w \mapsto [B_w]$, for $l(w) \leq m$. Relation (4.1.5) and its color switch implies that, in the Karoubi envelope, the $2m$-valent vertex is precisely an isomorphism from $B_{sts\ldots} \to B_{tst\ldots}$, whose inverse isomorphism is its own rotation. Therefore, the element $b_{sts\ldots} - b_{tst\ldots}$ is in the kernel, so that there is an induced map $\mathcal{H}_m \to [\mathbf{Kar}(\mathcal{D}_m)]$. For an element $w \in W_m$, we write $B_w \stackrel{\text{def}}{=} B_{\widetilde{w}}$ where $\widetilde{w} \in W_\infty$ is a reduced lift for w, and for $w = w_0$ the choice of $\widetilde{w}$ is irrelevant up to isomorphism.

Theorem 6. *Suppose that* $\Bbbk$ *is* m*-symmetric and* $2TL_{m-1}$*-sufficient. There is an isomorphism* $\mathcal{H}_m \to [\boldsymbol{Kar}(\mathcal{D}_m)]$ *which sends* $b_s \to [B_s]$*,* $b_{w_0} \to [B_{w_0}]$*, and in general* $b_w \to [B_w]$*. This categorifies the standard trace map. For any Soergel ring,* $\mathcal{F}_m$ *is an equivalence of categories, sending indecomposables* B_w *to* B_w*.*

Proof. That a map $\mathcal{H}_m \to [\mathbf{Kar}(\mathcal{D}_m)]$ exists and sends the elements to their images as stated is now clear. We have not shown that the map is injective or surjective (or that the right hand side is nonzero) but nonetheless there is induced on $\mathcal{H}_m$ a trace map. It remains to show that this trace map is the standard trace map, for this will imply that all the objects mentioned are nonzero, pairwise non-isomorphic, and indecomposable. Again, the trace map can be computed by calculating $\mathrm{Hom}(B_w, \emptyset)$ for all $w \in W_m$, which is computed by looking at maps $B_{\underline{i}} \to \emptyset$ for reduced expressions which are unchanged by composition with the appropriate idempotent.

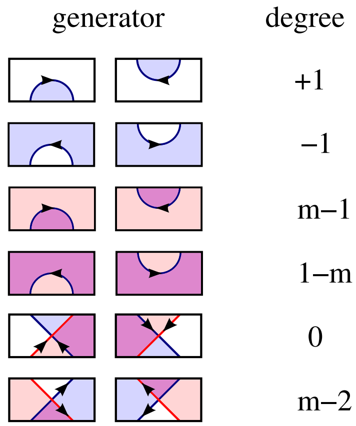
Let $\underline{\boldsymbol{i}}$ be an alternating sequence of length $\leq m$, representing a reduced expression. Proposition 4.1.6 implies that every diagram representing a map from $\mathrm{Hom}(B_{\underline{i}}, \emptyset)$ can be reduced to a linear combination of diagrams involving only dots and trivalent vertices. We can use the exact same arguments as in Theorem 4 to show that the only possible map which wouldn't kill the idempotent is the all-dot. We have already calculated the polynomial which appears when you apply the all-dot on both sides to a Jones-Wenzl projector, and computed that this polynomial is nonzero (thanks to Lemma 4.1.7). The remainder of the proof continues exactly as in Theorem 4, using the appropriate analog of Lemma 2.4.1. ☐

4.2 Singular Soergel Bimodules: $m < \infty$

4.2.1 Definitions

Definition 4.2.1. A *singular Soergel graph* or *Soergel 1-manifold diagram* for m is (almost) an isotopy class of two oriented 1-manifolds with boundary properly embedded in the planar strip, one of each color, which can only intersect transversely. Moreover, there must be a

Figure 4.1: Degrees of Soergel diagrams



consistent labeling of the regions between these edges. Regions may be labelled by parabolic subsets $\{\underline{e}, \underline{s}, \underline{t}, \underline{W}\}$. A line can be colored s or t, and separates two regions whose label differs by that element. The orientation is such that the larger parabolic subset is on the right hand of the 1-manifold. The boundary of the graph gives two sequences of colored oriented points, the *top* and *bottom boundary.* Not every oriented colored 1-manifold gives rise to a consistent labeling of regions. Soergel 1-manifold diagrams are graded as in figure 4.1.

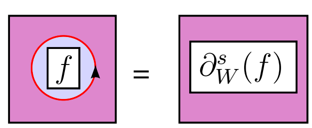
If there is no ambiguity, we shorten the name to "Soergel diagram." One can remember that the degree of a cup or cap is always "in minus out," regardless of orientation, where we associate to a region the length of the longest element of the corresponding parabolic subgroup (which is 0, 1, or m). The degree of a sideways crossing is "big minus middle minus middle plus small," which in this case is always $m - 1 - 1 + 0 = m - 2$. Note that the degree of a diagram is *not* an isotopy invariant. This is the only sense in which we do not consider Soergel diagrams as isotopy classes. The degree of morphisms is also the only place where m enters into the definition.

A picture is often worth a thousand words. We will try to be very clear about what we refer to in a picture. "Lines" or "strands" will refer to sections of the red or the blue 1-manifold. A "blue circle" will denote a blue line in the shape of a circle (which can separate either a red region from a purple one, or blue from white), while a “blue circular region” will refer a blue region enclosed by a circle of indefinite color. We might refer to a 1-morphism by the sequence of lines comprising it (i.e. “red up blue down”) or by the colors of the regions (i.e. “red purple blue”). Purple designates $\underline{W}$. We refer to a diagram without any purple regions as an ∞-*diagram*.

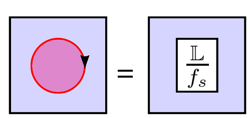
Let $\Bbbk_m$ be m-symmetric and $2TL_{m-1}$-sufficient.

Definition 4.2.2. Let $\mathfrak{D}_m$ be the 2-category defined as follows. The objects are $\{\underline{e}, \underline{s}, \underline{t}, \underline{W}\}$, thought of as parabolic subsets. The 1-morphisms are generated by maps from I to J and whenever their difference is a single element. A path in the object space (like $\underline{sesWtWse}$) uniquely specifies a 1-morphism. The 2-morphism space between 1-morphisms is the free $\Bbbk$-module spanned by Soergel diagrams with the appropriate boundary, modulo the relations below. Hom spaces will be graded by the degree of Soergel diagrams.

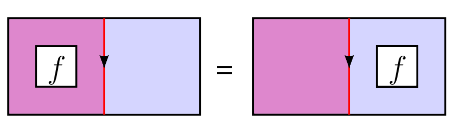
We have all the relations present in $\mathfrak{D}(\infty)$ (see section 3.2.1), which give all the relations between ∞-diagrams. This means that we have a functor $\mathfrak{D}(\infty) \to \mathfrak{D}_m$, and we already know how to place an (appropriately invariant) polynomial inside a region labelled $\underline{e}$, $\underline{s}$, or $\underline{t}$. We will write some of the remaining relations in terms of polynomials, instead of expressing them in terms of circles. The first relation gives us a way of writing elements of R^W inside a region labelled $\underline{W}$, since ∂_W^s is surjective.



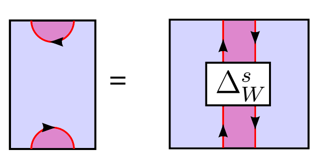
(4.2.1)



(4.2.2)



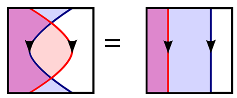
when $f \in R^W$ (4.2.3)



(4.2.4)

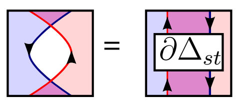
These relations hold with colors switched, and represent standard relations for a Frobenius extension. The element $\Delta^s_W \in R^s \otimes_{R^W} R^s$ is described in section 2.2.5. Because of the relations, there is a map from $R^s \otimes_{R^W} R^s \to \mathrm{End}(\underline{sWs})$ given by placing boxes in the right and left regions. Similar actions are represented by boxes which cross multiple lines: they represent linear combinations of diagrams with boxes on each side, not in the middle.

The next relation, an analog of Reidemeister II, says that like-oriented strands can be pulled apart.

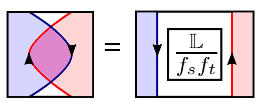


(4.2.5)

We also have non-oriented Reidemeister II relations. The element $\partial\Delta_{st}$ represents the element $\Delta^s_{W(1)} \otimes \partial_t(\Delta^s_{W(2)}) = \partial_s(\Delta^t_{W(1)}) \otimes \Delta^t_{W(2)} \in R^s \otimes_W R^t$, as described in section 2.2.5.



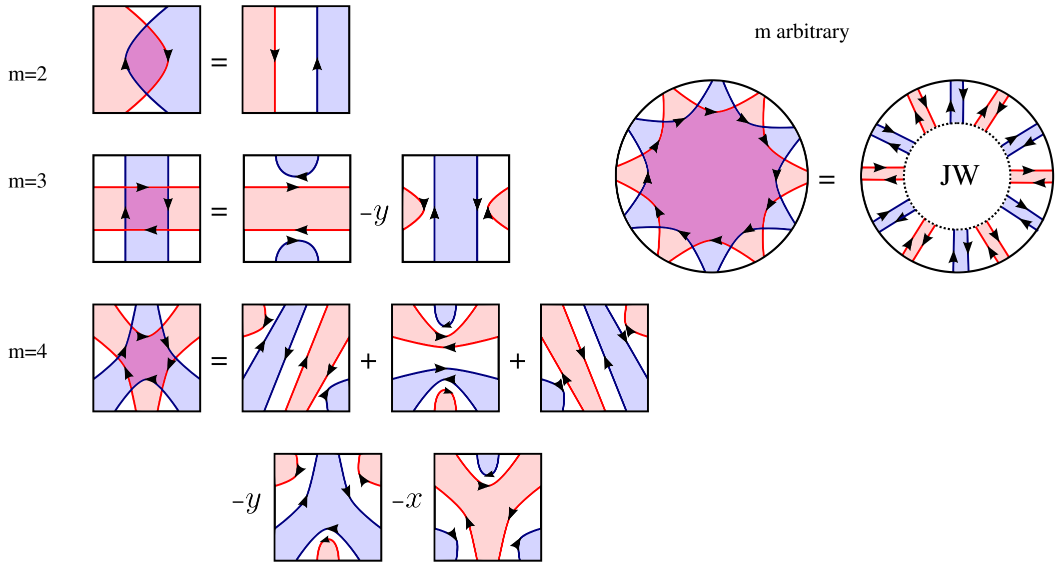
(4.2.6)



(4.2.7)

All of the above relations hold in generality for squares of Frobenius extensions. See [9] for more details.

There is only one truly interesting relation, unique to the dihedral group. This relation starts with $m-1$ alternating arcs of each color, oriented around an inner purple region. It replaces this with an ∞-diagram, the image of JW_{m-1} under the functor $\mathit{2TL} \to \mathfrak{D}(\infty) \to \mathfrak{D}_m$.



(4.2.8)

4.2.2 Graph simplifications

Remember that the word "polynomial" in this context refers to a linear combination of certain configurations of circles, which we know gives a map from some invariant ring R^I to Hom spaces in $\mathfrak{D}_m$. We have not yet shown that this map is injective, or even that polynomials are non-zero.

Lemma 4.2.3. (The Circle Removal Lemma) *Any morphism in $\mathfrak{D}_m$ can be represented as a linear combination of diagrams with no closed components of either color, but with polynomials in arbitrary regions. Any Soergel diagram with empty boundary reduces to a polynomial.*

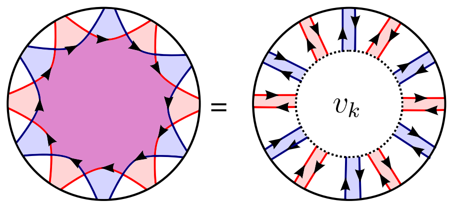
Proof. This is proven in [9], and is a general statement about squares of Frobenius algebras.

□

In other words, a collection of nested circles easily evaluates to a polynomial using the relations above, and a more complicated system of overlapping circles may be pulled apart using Reidemeister II moves, and reduced to a polynomial as well. Note that this lemma holds regardless of the color of the region on the boundary, so that a closed Soergel diagram in a white region evaluates to a polynomial in R, and a closed Soergel diagram in a blue region becomes a polynomial in R^s.

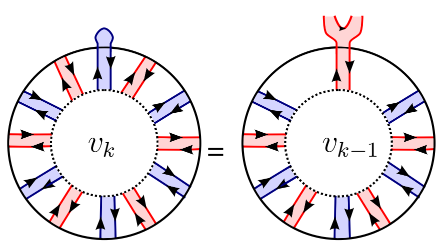
Now we attempt to simplify more complicated graphs with boundary.

Notation 4.2.4. When given k alternating arcs of each color oriented around an inner purple region as in (4.2.8), we will denote the map by v_k. Our interesting relation says that $v_k = JW_k$ only for $k = m-1$, *not* for arbitrary k. In the example below, $k = 6$.



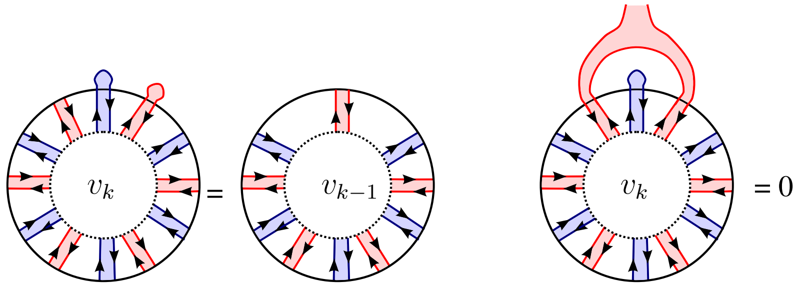
The map v_k can be oriented in many ways, and its orientation determines its degree. However, when oriented as a map from <u>*esetese...*</u> to <u>*etesete...*</u>, it will have degree $2(m-k)$. We let v_0 denote a purple circle in a blue annulus in a white region (or purple in red in white, these are equal). Using the relations above, v_0 is equal to $\mathbb{L}$. It is obvious that any purple region which does not meet the boundary must have a neighborhood equal to v_k for some $k \geq 1$, or simply be a purple circular region.

If we place a colored cap on one of the colored sections of the boundary of v_k for $k \geq 1$, we can use (4.2.5) to pull two strands apart, and obtain v_{k-1} with an added "trivalent vertex."



(4.2.9)

From this, it is easy to show that "pitchforks" kill v_k, and double capping v_k will yield v_{k-1}.

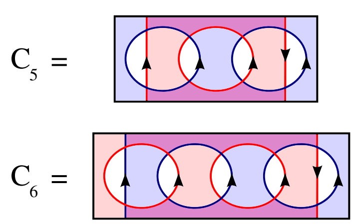


Relation (4.2.8) says that v_{m-1} can be de-purplified, i.e. rewritten as a sum of ∞-diagrams. Therefore, any v_k for $k \leq m-1$ can also be de-purplified. After all, v_k equals v_{m-1} with a number of cups attached, and will be equal to JW_{m-1} with a number of cups attached. Note that relation (4.2.7) is actually a statement about v_1; in fact, this relation is redundant, merely stating that the RHS is what one obtains when one caps JW_{m-1} almost everywhere.

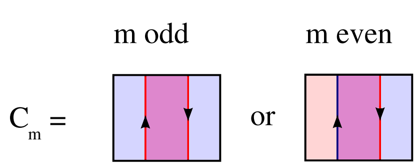
Warning: Remember that caps do not kill JW_{m-1} in this context, only the analogs of pitchforks kill JW_{m-1}. Also, capping off JW_{m-1} certainly does not yield JW_k for $k < m-1$.

For $k > m-1$, one can not use (4.2.8) to de-purplify v_k. In fact, v_m is the smallest diagram which is not in the span of ∞-diagrams, and will be the image of the $2m$-valent vertex under the functor from $\mathcal{D}_m$ to $\mathfrak{D}_m$ to be defined. Thankfully, all v_k for $k > m$ can be expressed using only v_m. In order to show this, we introduce an auxiliary map..

For $k \geq 2$, consider the following maps C_k of degree $2(m-k)$, where the number of circles is $k-2$:

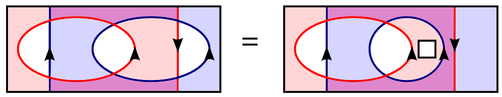


Claim 4.2.5. *When $m = k$ (so that this map is degree 0), we have*



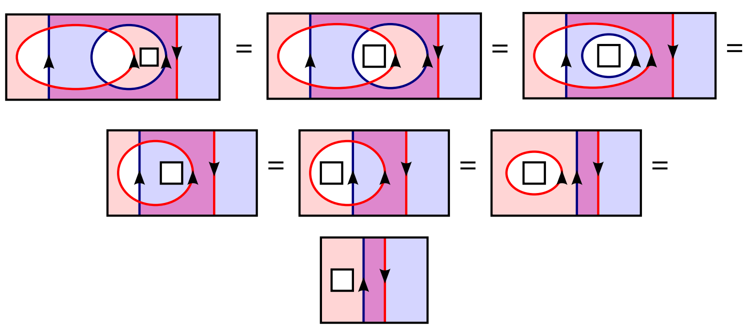
(4.2.10)

Proof. Using the circle elimination lemma as well as polynomial manipulation rules, we see that this map must reduce to some polynomial in $R^s \otimes_{R^W} R^t$. For degree reasons this polynomial is a scalar, so we must only check that this scalar is 1. In fact, the derivation goes as follows, for the case $m = 4$:



The first step is to apply (4.2.6) to obtain a sum of diagrams with boxes in the rightmost blue region, and a box in the red region as pictured. However, only one term in this sum survives: the term where the box in the rightmost region is 1. Were the box in the rightmost region of degree > 0, then the diagram to the left of that region would have to reduce to a polynomial of negative degree, equal to 0. Therefore c_m is equal to a single diagram as pictured, where if f is the polynomial dual to 1 under ∂^s_W, then the element in the box is $\partial_t(f)$.

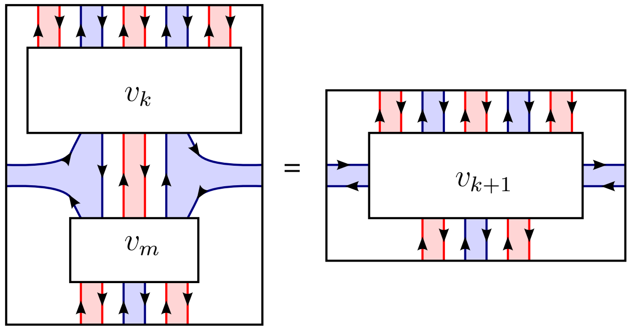
Now we apply (3.2.10), (4.2.5), and (3.2.12) until the diagram only has a box in the left region.



We place $\partial_t(f)$ in the box in every diagram on the second row. But then $\partial_s(\partial_t(f))$ appears in the box on the third row, and $\partial_t(\partial_s(\partial_t(f)))$ on the fourth row (and so forth, for $m > 4$). Therefore, the final polynomial appearing is $\partial^s_W(f)$, which is 1. □

Now we use C_m to help understand v_k.

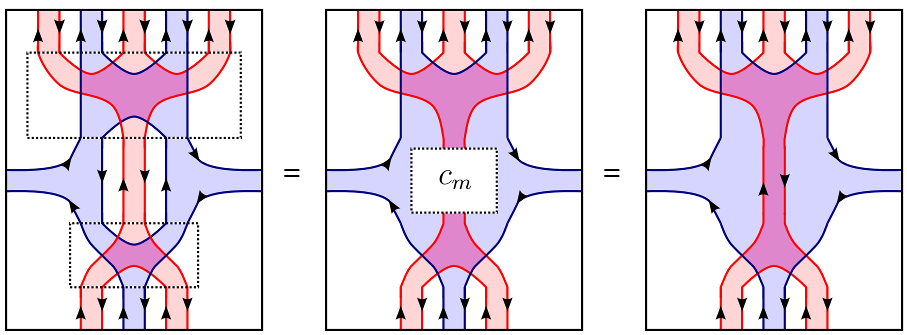
Claim 4.2.6. *For every $k \geq m$ we have*



(4.2.11)

The example here has $k = 4$ and $m = 3$, although the typical example is similar. When m is even, the new side colors are different rather than the same.

Proof. The example below should make the general proof clear.



□

Remark 4.2.7. In fact, $c_k = 0$ for $k > m$, since it reduces to a polynomial of negative degree. A similar proof shows that replacing v_m with v_l for any $l > m$ in the above diagram would yield 0. In other words, if any two purple regions v_k and v_l are connected by more than m "colored bands," then the result is 0.

Lemma 4.2.8. *Suppose that a diagram in $\mathfrak{D}_m$ has no purple appearing on the boundary. Then it reduces to linear combinations of diagrams generated by ∞-diagrams and v_m.*

Proof. Let us note that any polynomial may always be slid to a lighter region (i.e. purple to blue to white), and polynomials in blue or red or white regions can be expressed in terms of ∞-diagrams. Now suppose that there is a purple region in the diagram. If this purple region is a circular region, it reduces to a polynomial by (4.2.2). If this purple region is v_k for $1 \leq k < m$ then we may de-purplify it, as previously discussed. If this purple region is v_k for $k \geq m$ then we may express it using only copies of v_m, by using (4.2.11) iteratively. This procedure will strictly decrease the number of purple regions labelled by $k \neq m$, and therefore we may eliminate all such purple regions. □

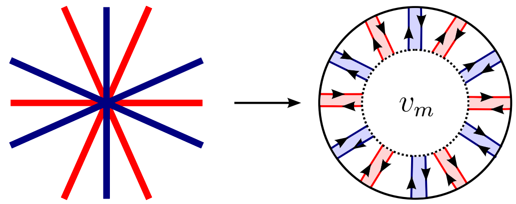
Note that if there is no purple on the boundary, then we are dealing with objects that may as well live inside $\mathfrak{D}(\infty)$. In particular, suppose that there is some white region on the boundary. Then, reading around the boundary, the colors of regions must be of the form $\underline{ei_1ei_2ei_3e\ldots}$ for some sequence $\underline{\boldsymbol{i}}$, and corresponds to some Bott-Samelson bimodule. Using the relations of $\mathfrak{D}(\infty)$, we may as well remove repetitions in the sequence $\underline{\boldsymbol{i}}$ using the standard isomorphisms.

Corollary 4.2.9. *Suppose that a diagram has boundary $\underline{ei_1ei_2e\ldots ei_de}$ when reading around the boundary of the disk, where $\underline{\boldsymbol{i}}$ is an alternating sequence of length d. If $d < 2m$ then every diagram with that boundary can be de-purplified.*

Proof. This is a simple consequence of the circle removal lemma. We may reduce the diagram to a diagram with no closed 1-manifold components of either color, so the only components come from those on the boundary. Clearly then, there are not enough components for v_m to appear in the diagram (or v_k for $k \geq m$), unless different strands in the boundary of v_m are connected to each other. However, if any two strands in v_m are connected to each other, then by simple planar arguments there must be a cup somewhere, implying that v_m reduces to v_{m-1} and eventually to ∞-diagrams. □

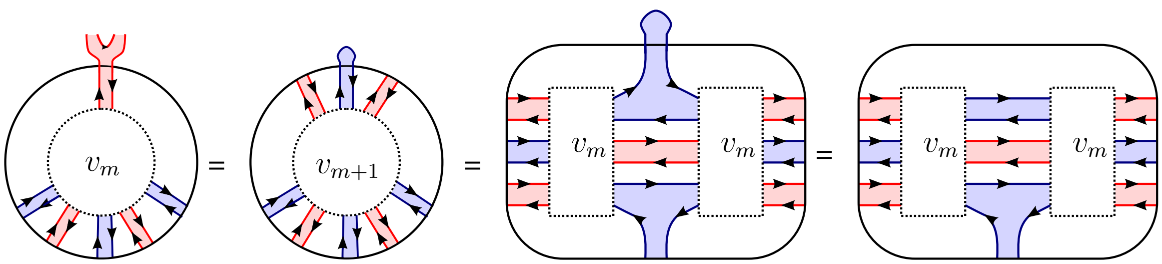
4.2.3 Comparison to the non-singular case

Definition 4.2.10. We give a functor $\iota_m \colon \mathcal{D}_m \to \mathrm{Hom}_{\mathfrak{D}_m}(\underline{e}, \underline{e})$. On objects, it sends s to the path $\underline{ese}$ and t to the path $\underline{ete}$. We define the functor on dots and trivalent vertices as in Definition 3.2.3. We define the functor on the $2m$-valent vertex here.



Claim 4.2.11. *The above definition gives a well-defined functor.*

Proof. Because this was true for $\iota\colon \mathcal{D}(\infty) \to \mathrm{Hom}_{\mathfrak{D}(\infty)}(1,1)$, we need only check the relations involving 6-valent vertices, (4.1.1) and (4.1.2). Both categories only consider pictures up to isotopy, so we may ignore questions of isotopy invariance. Relation (4.1.2) follows from (4.2.9) and (4.2.8). Relation (4.1.1) follows from



□

Claim 4.2.12. *This functor is full.*

Proof. This is precisely the statement of Lemma 4.2.8. □

We will also show that this functor is faithful, although it is not essentially surjective, because there are loops based at $\underline{e}$ which pass through $\underline{W}$. The functor is an equivalence after passing to the Karoubi envelope. Once we have shown that the functor is faithful, Corollary 4.2.9 will give an immediate proof of Proposition 4.1.6.

4.2.4 The functor to bimodules

Definition 4.2.13. Let $\Bbbk_m$ be m-symmetric and $2TL_{m-1}$-sufficient. We define a 2-functor $\mathfrak{F}_m\colon \mathfrak{D}_m \to \mathfrak{B}_m$ as follows. On objects it is the identity, and on 1-morphisms it sends the map from I to J for $I \subset J$ to the restriction bimodule R^I (as an $R^J - R^I$-bimodule), and it sends the map from J to I to the induction bimodule R^I (as an $R^I - R^J$-bimodule). Cups and caps are sent to the appropriate maps of Frobenius algebras, as discussed in sections 2.2.4 and 2.2.5. The upwards-pointing crossing goes to the canonical isomorphism $R \otimes_{R^s} R^s \otimes_{R^W} R^W \cong R \otimes_{R^t} R^t \otimes_{R^W} R^W$, which are both R as an $R - R^W$-bimodule. Similarly, the downwards-pointing crossing is a canonical isomorphism between R and itself as an $R^W - R$-bimodule. The sideways crossings are maps between R and $R^s \otimes_{R^W} R^t$ as $R^s - R^t$-bimodules, which are either multiplication or the map $f \to \Delta^s_{W,(1)} \otimes \partial_t(f\Delta^s_{W,(2)})$ discussed in section 2.2.5.

Proposition 4.2.14. *This 2-functor is well-defined.*

Proof. We must check the relations of the category, as well as isotopy relations. Almost all these relations (including the isotopy relations) hold in more generality for squares of Frobenius extensions, and they are proven in [9]. The only relation that needs to be checked is (4.2.8), expressing v_{m-1} as JW_{m-1}. Thankfully, we understand how Hom spaces work in $\mathfrak{B}_m$. We may view the map v_{m-1} as a map from $\underbrace{B_s \otimes B_t \otimes \dots}_{2(m-1)} \to R$, which is killed by every pitchfork and has minimal degree. We have already seen that the space of such maps is 1-dimensional, and that placing a dot on every B_i (which corresponds in the singular world to placing caps on every colored band) gives an *injective* map from this 1-dimensional space into the endomorphisms of R (i.e. R itself). Therefore, we need only show that v_{m-1} and JW_{m-1} produce the same polynomial when capped off everywhere. But we have already shown that v_{m-1} capped off everywhere is $v_0 = \mu_W = \mathbb{L}$, while JW_{m-1} capped off everywhere is its associated polynomial, which is also $\mathbb{L}$ (see section 2.3.4). ☐

Corollary 4.2.15. *For any parabolic subset,* $\mathrm{End}_{\mathfrak{D}_m}(I) \cong R^I$.

Proof. We have already seen that all such diagrams reduce to polynomials, and the existence of the functor implies that there can be no additional relations between polynomials. ☐

In fact, we can make this corollary even stronger, in light of Corollary 4.2.9.

Corollary 4.2.16. *For $\underline{\boldsymbol{i}}$ an alternating sequence of length $< 2m$ and X the corresponding 1-morphism $\underline{ei_1 ei_2 e \dots ei_d e}$ in $\mathfrak{D}_m$ or $\mathfrak{D}(\infty)$, the natural 2-functor from $\mathfrak{D}(\infty)_{\Bbbk} \to \mathfrak{D}_m$ induces isomorphisms on* $\mathrm{Hom}(X, \underline{e})$.

Proof. The map has already been shown to be surjective. It must be injective, since the functor from $\mathfrak{D}(\infty)_{\Bbbk} \to \mathfrak{B}_{\Bbbk}$ is an equivalence, and factors through $\mathfrak{D}_m$. □

This saves us from having to calculate explicitly the bimodule morphism corresponding to the $2m$-valent vertex.

Definition 4.2.17. We define the functor $\mathcal{F}_m \colon \mathcal{D}_m \to \mathcal{B}_m$ as the composition of ι_m and $\mathfrak{F}_m$.

Clearly this functor agrees with $\mathcal{F}_\infty$ on the image of $\mathcal{D}(\infty)$ in $\mathcal{D}_m$, since $\mathfrak{F}_m$ agrees with $\mathfrak{F}_\infty$ on ∞-diagrams.

4.2.5 The Grothendieck Algebroid

Theorem 7. *The 2-category* ***Kar***$(\mathfrak{D}_m)$ *categorifies the Hecke algebroid $\mathfrak{H}_m$. If $\Bbbk_m$ is a Soergel ring then $\mathfrak{F}_m$ is an equivalence after passage to the Karoubi envelope. In addition, ι_m is fully faithful and $\mathcal{F}_m$ is an equivalence of categories.*

Proof. First we show that there is a map from $\mathfrak{H}_m$ to the Grothendieck category of $\mathfrak{D}_m$. We have presented $\mathfrak{H}_m$ by generators and relations in section 2.1.6, and it is clear what happens to the generators, so we need only check the relations. Equation (2.1.7) follows from the modification (3.2.14) of (3.2.13) as in $\mathfrak{D}(\infty)$, and equation (2.1.8) follows from the equivalent modification of (4.2.4). The isomorphism $\underline{esW} \cong \underline{etW}$ required by equation (2.1.9) is realized by the upwards-pointing crossing, and the same for (2.1.10) and the downwards-pointing crossing. Finally, the most interesting relation is (2.1.11). This is implied by the decompositions in $\mathcal{2TL}$, after applying the functor to $\mathfrak{D}(\infty)$.

Therefore, $\mathfrak{D}_m$ induces a trace on $\mathfrak{H}_m$, which can be identified by its values on $\mathrm{End}_{\mathfrak{H}_m}(\underline{e}) = \mathcal{H}_m$. These in turn are determined by the graded dimensions of the Hom spaces specified in

Corollary 4.2.16, which agree with the graded ranks of Hom spaces in $\mathfrak{D}(\infty)$. Therefore, the trace induced by $\mathfrak{D}_m$ is equal to the standard trace. The remaining arguments proceed as usual. □

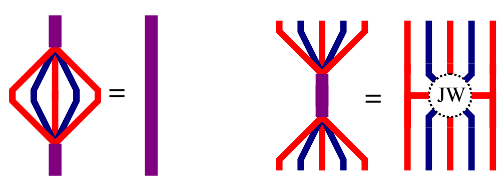
4.3 Thickening

4.3.1 Diagrams for $\mathcal{B}_{\text{gBS}}$

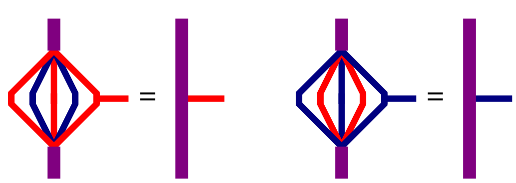
Recall that $\mathcal{B}_{\text{gBS}}$ is the full subcategory of $\mathcal{B}$ monoidally generated by B_s, B_t, and B_W. We present this category diagrammatically, in precise analogy with [5], chapter 3.5.

Definition 4.3.1. A *thick Soergel graph* has edges labelled either s, t, or W (purple). The new vertices (compared to a Soergel graph) are: trivalent with 3 purple edges (degree $-m$); trivalent with two purple and one other (degree -1); univalent with one purple (degree m); and $m+1$-valent with one purple edge and the remainder alternating between s and t (degree 0).

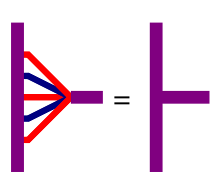
Definition 4.3.2. The category $\mathcal{D}_{m,\text{gBS}}$ has morphisms given by thick Soergel graphs, with the relations of $\mathcal{D}_m$ as well as the following relations.



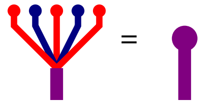
(4.3.1)



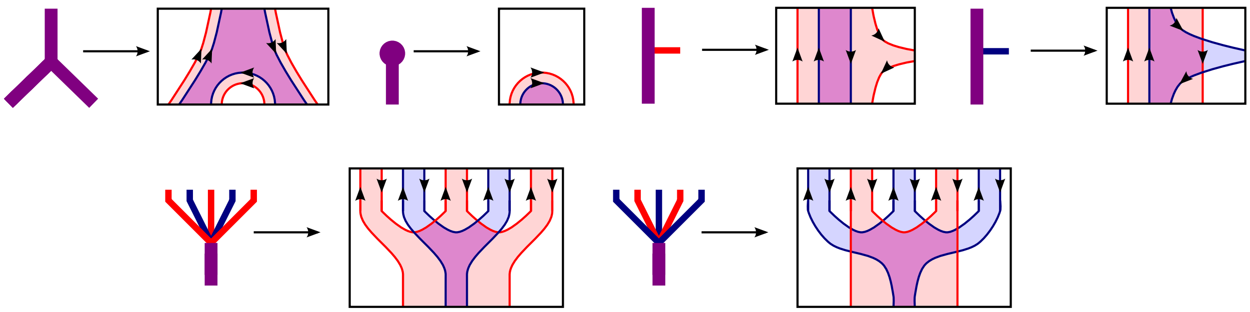
(4.3.2)



(4.3.3)

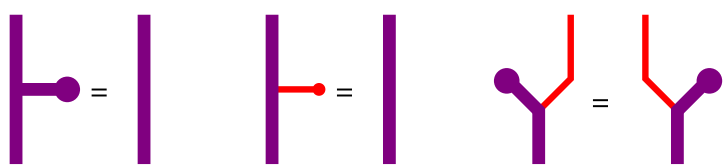
 (4.3.4)

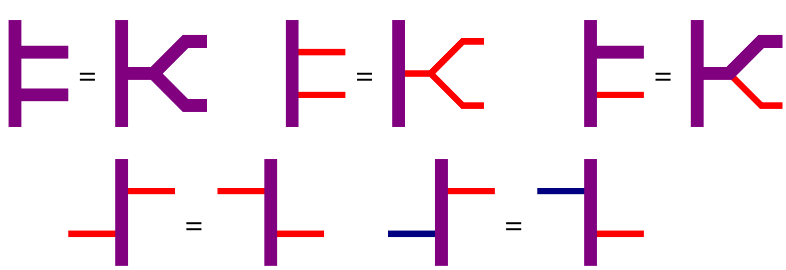
Relation (4.3.1) identifies the purple line as the image of the idempotent which picks out B_W inside $B_s \otimes B_t \otimes \ldots$. The remaining equalities identify the new generators as pre-existing maps in $\mathcal{B}$. Therefore, the fact that this category is equivalent to $\mathcal{B}_{\mathrm{gBS}}$ is entirely obvious. We can also describe these morphisms within $\mathfrak{D}_m$ as follows.



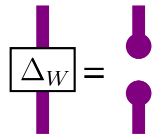
This defines a functor from $\mathcal{D}_{m,\mathrm{gBS}}$ to $\mathrm{End}_{\mathfrak{D}_m}(\underline{e})$, which on objects sends the purple object to $\underline{etWte}$. There is an isomorphic functor sending purple to $\underline{esWse}$, passing through blue instead of red.

Proposition 4.3.3. *We also have the following equalities.*

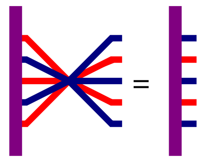
 (4.3.5)

 (4.3.6)

(4.3.7)



(4.3.8)



(4.3.9)

Proof. These are each very easy to show within $\mathfrak{D}_m$. The only interesting part is to show that the purple trivalent vertex is equal to what one might expect it to be in $\mathfrak{D}_m$: the analogous picture to what the red and blue trivalent vertices are. This check requires (4.2.10), as does the check of (4.3.9). The remaining relations then follow from isotopy or Frobenius extension relations. □

Remark 4.3.4. These relations were much more difficult to show in [5] for the case of type A, because at the time of writing the diagrammatics for singular Soergel bimodules had not yet been developed.

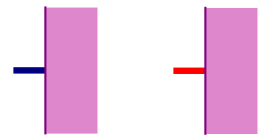
Remark 4.3.5. Note also that neither the $2m$-valent vertex nor the $m+1$-valent vertex are actually required, in the presence of the other maps. For instance:

There are perhaps many more interesting equalities to find.

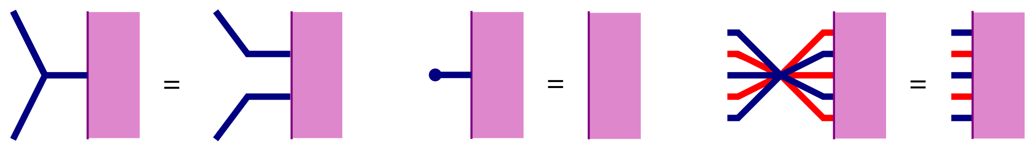
4.3.2 Induced modules

As in section 3.2.4, we may represent $\mathrm{Hom}_{\mathfrak{D}_m}(\underline{e}, I)$ for $I \subset \{s,t\}$ simply using Soergel graphs with a shaded region,in precise analogy with Chapter 4 of [5]. When $I = \{s\}$ or $I = \{t\}$, we have already described the answer for $\mathfrak{D}(\infty)$ in section 3.2.4, and the result for $\mathfrak{D}_m$ is identical, except that we have additional relations between Soergel graphs. When $I = \{s,t\}$, we require that graphs end in a "purple region," and add new morphisms corresponding to trivalent vertices with the purple region.

Explicitly, the new generators are



The new relations are



(4.3.10)

Any usual Soergel graph, with the usual Soergel relations, may be drawn to the left of the purple region.

Again, the equivalence between these diagrams and singular Soergel diagrams is easy.

4.4 Temperley-Lieb categorifies Temperley-Lieb

Suppose that $2 < m < \infty$. In this section we state the following result, without proof (due to time/space constraints):

Theorem 8. *Consider the quotient of $\mathcal{D}_m$ by the $2m$-valent vertex. This categorifies TL_W, the Temperley-Lieb algebra of W_m. The degree 0 Hom spaces are given precisely by $2\mathcal{TL}_{\mathrm{negl}}$. Similarly, the quotient of $\mathfrak{D}_m$ by the purple region categorifies the Temperley-Lieb algebroid of W_m.*

Here is a sketch of the proof. There is clearly a map from TL_W to the Grothendieck ring of this quotient, because the map from $\mathcal{H}_m$ factors through the ideal $b_{w_0} = 0$. Therefore, this quotient induces a trace map on TL_W. We still have idempotents which yield objects B_w, $w \in W \setminus \{w_0\}$, and though we have not yet shown that these objects are nonzero, we do know that all other objects can be expressed as direct sums of these. If they remain indecomposable and pairwise non-isomorphic in the quotient, then they will descend to a basis of TL_W, and the map from TL_W will be an isomorphism. This, in turn, will follow from the calculation of the trace on TL_W, because the graded rank of $\mathrm{End}(B_w)$ will be in $1 + v\mathbb{Z}[v]$, and the graded rank of $\mathrm{Hom}(B_w, B_x)$ for $w \neq x$ will be in $v\mathbb{Z}[v]$.

In order to calculate the trace map, we must determine what elements of $\mathrm{Hom}(B_{\underline{\boldsymbol{i}}}, B_\emptyset)$ survive in the quotient, for each reduced expression $\underline{\boldsymbol{i}}$. This is a diagrammatic calculation.

For instance, what will be $\mathrm{End}(B_\emptyset)$ in the quotient category? We rephrase this in singular language: consider a Soergel 1-manifold diagram with a purple region and with empty white boundary. We know that this reduces to a polynomial in R, but which polynomials can appear? We claim that the polynomials which appear are precisely the ideal generated by $\mathbb{L}$, and that therefore $\mathrm{End}(B_\emptyset) \cong R/\mathbb{L}$.

Let us excise a neighborhood of v_m from our diagram: the remainder of the picture is a Soergel diagram on the *punctured plane* whose boundary consists of $2m$ colored bands entering the puncture. The first step will be to show that any diagram on the punctured plane with that boundary which entirely wraps the puncture (so that it can't be contained in a simply-connected subregion) will reduce to a polynomial in this ideal. We know what all the diagrams on the plane with that boundary reduce to: either ∞-diagrams or a single $2m$-valent vertex. Two $2m$-valent vertices which are connected by all $2m$ edges will be zero, however, by Remark 4.2.7. The only ∞-diagram which can be placed safely next to a $2m$-valent vertex is generated over R by the all-dot, and the result is generated over R by $\mathbb{L}$. This calculation is done for the case $m = 3$ in [6].

Similarly, $\mathrm{Hom}(B_s, B_\emptyset)$ should be isomorphic to $R/\frac{\mathbb{L}}{f_s}\{1\}$, where each polynomial is placed next to the blue dot. This kernel is generated by the $2m$-valent vertex with all but one dot attached. $\mathrm{Hom}(B_s \otimes B_t, B_\emptyset)$ should be isomorphic to $R/\frac{\mathbb{L}}{f_s f_t}\{2\}$, generated by a dot of each color. However, $\mathrm{Hom}(B_s \otimes B_t \otimes B_s)$ has two generators, and one will need to be more precise to describe the Hom spaces. Nonetheless, a simple diagrammatic argument should still suffice to calculate this Hom space.

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