The discrete Dirac operator and the discrete generalized Weierstrass representation in pseudo-Euclidean spaces

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ABSTRACT

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In this thesis we consider the problem of finding a integrable discretization of the Dirac operator. We show that an appropriate deformation of the spectral properties of the eigenfunction of the smooth Dirac operator leads to a discrete integrable Dirac operator. We use this discrete Dirac operator to construct a discrete analogue of the modified Novikov–Veselov hierarchy and a discrete analogue of the generalized Weierstrass representation of isotropically embedded surfaces in pseudo-Euclidean spaces.

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To my parents

Chapter 1

Introduction

An important recent development in soliton theory is the growing interest in the study of *discrete integrable systems*—nonlinear difference equations possesing integrability properties similar to their continuous counterparts.

There are many reasons why discrete integrable systems are interesting. Discrete systems are more fundamental from a mathematical viewpoint, because a discrete equation admits a unique continuous limit, while constructing an integrable discretization of a differential equation is a non-trivial problem. Discrete integrable systems are also interesting in their own right and have numerous applications to other areas of mathematics. For example, the recent works of Krichever and others on the characterization of Jacobian varieties [16; 17] and Prym varieties [18] are based on the study of discrete integrable systems.

Another application of the theory of discrete integrable systems is the recent emergence of a field known as *discrete differential geometry* [3]. Classical differential geometry studies smooth geometric shapes, many of which are known to be described by integrable equations. Discrete differential geometry aims to find lattice analogues of the methods and constructions of the smooth theory which are described by integrable discretizations of the corresponding equations.

In this thesis, we consider the problem of finding an integrable discretization of the *Dirac operator*:

$$D = \begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar{z}} & 0 \end{pmatrix} - \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad u = \bar{u}. \tag{1.1}$$

The Dirac operator is related to a classical construction in differential geometry known as the *Weierstrass representation*. It is also the auxiliary operator for the integrable hierarchy known as the *modified Novikov–Veselov hierarchy*.

Based on an appropriate deformation of the spectral data of the Dirac operator, we construct the following integrable discretization of the Dirac operator:

$$D = \begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad \alpha^2 - \beta^2 = 1. \tag{1.2}$$

We then use this operator to construct a difference analogue of the modified Novikov–Veselov hierarchy and a discrete analogue of a certain generalization of the Weierstrass representation.

The thesis is organized as follows. In Chapter 2, we describe the classical Weierstrass representation of conformally embedded surfaces in $\mathbb{R}^3$, its relationship to the Dirac equation, and several generalizations of this construction that have recently been found. In Chapter 3, we describe an integrable discretization of the generalized Dirac operator and of the modified Novikov–Veselov hierarchy. In Chapter 4, we use this discretization to construct a discrete analogue of the generalized Weierstrass representation of isotropic surfaces in pseudo-Euclidean spaces.

The author's results have been published in the two papers [27] and [28], which correspond to Chapters 3 and 4, respectively.

Chapter 2

The Weierstrass representation

The Weierstrass representation is a classical construction of differential geometry that has received significant attention in recent years due to its relationship with the theory of integrable systems. In a nutshell, the Weierstrass representation is a convenient way of parameterizing an arbitrary smoothly embedded surface in $\mathbb{R}^3$.

2.1 The classical Weierstrass representation

We begin with a definition.

Definition. Let (M, g) be a Riemannian manifold of dimension n, let $U \subset S$ be an open subset, and let $x_1, \ldots, x_n$ be coordinates on U. We say that the coordinates x_i are *conformal* or *isothermal* if in terms of these coordinates the metric tensor has the form

$$ds^2 = e^{\varphi}(dx_1^2 + \cdots + dx_n^2)$$

for some function φ, i.e. if the metric tensor is a scalar multiple of the Euclidean metric tensor.

On a Riemannian manifold of dimension two, conformal coordinates can always be found in the neighborhood of any point. This is a consequence of the existence theorem for the Beltrami equation $\partial_{\bar{z}} w = \mu \partial_z w$. In particular, any embedded surface in a Riemannian manifold locally admits conformal coordinates. In dimensions $n \geq 3$, a necessary and sufficient condition for the existence of conformal coordinates is the vanishing of the Cotton tensor (for $n = 3$) or the Weyl tensor (for $n \geq 4$).

The Weierstrass representation is a natural way of parameterizing a surface embedded in $\mathbb{R}^3$ with conformal coordinates. By the existence theorem stated above, conformal coordinates can always be locally found on any embedded surface, so this parametrization can describe an arbitrary embedded surface.

Let $\vec{X} : S \to \mathbb{R}^3$ be a smooth immersion. Let $U \subset S$ be an open set, and let x, y be conformal coordinates on U with respect to the metric on S induced by the embedding. The condition that the coordinates x and y are conformal is equivalent to the following condition:

$$\left\langle \frac{\partial \vec{X}}{\partial x}, \frac{\partial \vec{X}}{\partial x} \right\rangle = \left\langle \frac{\partial \vec{X}}{\partial y}, \frac{\partial \vec{X}}{\partial y} \right\rangle, \quad \left\langle \frac{\partial \vec{X}}{\partial x}, \frac{\partial \vec{X}}{\partial y} \right\rangle = 0, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^3$. Introducing the complex coordinate $z = x + iy$, we can write this condition in an equivalent form:

$$\left\langle \frac{\partial \vec{X}}{\partial z}, \frac{\partial \vec{X}}{\partial z} \right\rangle = \left(\frac{\partial X_1}{\partial z} \right)^2 + \left(\frac{\partial X_2}{\partial z} \right)^2 + \left(\frac{\partial X_3}{\partial z} \right)^2 = 0. \tag{2.2}$$

We parametrize solutions of this equation by Pythagorean triples as follows:

$$\frac{\partial X_1}{\partial z} = \frac{\psi_1^2 - \bar{\psi}_2^2}{2}, \quad \frac{\partial X_2}{\partial z} = i\frac{\psi_1^2 + \bar{\psi}_2^2}{2}, \quad \frac{\partial X_3}{\partial z} = \psi_1 \bar{\psi}_2, \tag{2.3}$$

where ψ_1, ψ_2 are some complex-valued functions on U, not necessarily holomorphic. In terms of these functions, the embedding $\vec{X}$ can then be expressed in the following way:

$$\begin{aligned} X_1 &= \tfrac{1}{2} \int (\psi_1^2 - \bar{\psi}_2^2) dz + (\bar{\psi}_1^2 - \psi_2^2) d\bar{z}, \\ X_2 &= \tfrac{i}{2} \int (\psi_1^2 + \bar{\psi}_2^2) dz - (\bar{\psi}_1^2 + \psi_2^2) d\bar{z}, \\ X_3 &= \int \psi_1 \bar{\psi}_2 dz + \bar{\psi}_1 \psi_2 d\bar{z}. \end{aligned} \tag{2.4}$$

The consistency condition for these expressions has the form

$$\partial_{\bar{z}} \psi_1 = u \psi_2, \quad \partial_z \psi_2 = -u \psi_1, \quad \bar{u} = u, \tag{2.5}$$

where u is some real-valued function on U. Conversely, given a solution of equation (2.5) on a simply-connected domain U, we can use formulas (2.4) to construct a conformal embedding of U into $\mathbb{R}^3$. The metric and the mean curvature on U have the form

$$ds^2 = e^{2\alpha}(dx^2 + dy^2), \quad H = \frac{2u}{e^{\alpha}}, \quad e^{\alpha} = |\psi_1^2| + |\psi_2^2|. \tag{2.6}$$

This construction is known as the *Weierstrass representation.* It was originally introduced by Weierstrass to describe minimal surfaces (corresponding to $u = 0$) and later extended by arbitrary surfaces by Eisenhart in [8]. It was first written in the above form by Konopelchenko in [10].

The Weierstrass representation has received significant attention in recent years due to its relationship with the theory of integrable systems and its application to the Willmore conjecture. The consistency condition (2.5) above is known as the *Dirac equation.* It can be written in the form $D\psi = 0$, where $\psi = (\psi_1, \psi_2)^T$ and D is the *Dirac operator*:

$$D = \begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar{z}} & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad u = \bar{u}. \tag{2.7}$$

The Dirac operator is the auxiliary linear operator for the modified Novikov–Veselov hierarchy, which will be described in detail in Chapter 3. It is an infinite hierarchy of commuting flows on the space of Dirac operators (2.7). The flows of the hierarchy define deformations of the corresponding embedded surface S that preserve the value of the integral of the squared mean curvature over the surface, known as the *Willmore functional*:

$$W(S) = \int_S H^2 d\mu. \tag{2.8}$$

2.2 Generalized Weierstrass representations

There exist numerous ways to generalize the classical Weierstrass representation. In [2; 23] Taimanov constructed a Weierstrass representation of surfaces in three-dimensional Lie groups. Konopelchenko extended the Weierstrass representation to conformally embedded surfaces in $\mathbb{R}^4$:

Theorem 2.2.1 *[13] Suppose that the vector functions $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$ are* *defined on some simply-connected domain U and satisfy the following Dirac equations*

$$\left[\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix} + \begin{pmatrix} \bar u & 0 \\ 0 & u \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad \left[\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & \bar u \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{2.9}$$

where the potential u is complex-valued. Then the formulas

$$\begin{aligned} X_1 &= \tfrac{1}{2}\int(-\varphi_1\psi_1 + \bar\varphi_2\bar\psi_2)dz + (-\bar\varphi_1\bar\psi_1 + \varphi_2\psi_2)d\bar z, \\ X_2 &= \tfrac{i}{2}\int(\varphi_1\psi_1 + \bar\varphi_2\bar\psi_2)dz - (\bar\varphi_1\bar\psi_1 + \varphi_2\psi_2)d\bar z, \\ X_3 &= \tfrac{1}{2}\int(\varphi_1\bar\psi_2 + \bar\varphi_2\psi_1)dz + (\bar\varphi_1\psi_2 + \varphi_2\bar\psi_1)d\bar z, \\ X_4 &= \tfrac{i}{2}\int(-\varphi_1\bar\psi_2 + \bar\varphi_2\psi_1)dz + (\bar\varphi_1\psi_2 - \varphi_2\bar\psi_1)d\bar z, \end{aligned} \tag{2.10}$$

define an embedding $\vec X : U \to \mathbb{R}^4$ such that the coordinates x and y on U are conformal.

In this thesis we study a generalization of the Weierstrass representation that describes surfaces embedded in *pseudo-Euclidean spaces.* This construction was described by Konopelchenko in [10; 11; 12].

Let S be a surface embedded in a pseudo-Riemannian manifold. At every point of S the restriction of the metric tensor to S is either positive (or negative) definite, or it is indefinite. In the first case, it is possible to locally introduce conformal coordinates on S, multiplying the metric tensor by -1 if it is negative definite. If the metric tensor has signature $(1,1)$, the natural analogue of conformal coordinates are the coordinates in which the metric tensor is off-diagonal:

Definition. Let (M,g) be a pseudo-Euclidean manifold of dimension two, where the metric g has signature $(1,1)$, let $U \subset M$ be an open subset, let x and y be coordinates on U. We say that the coordinates x and y are *isotropic* if the metric tensor has the form

$$ds^2 = e^{\varphi}dxdy$$

for some function $\varphi(x,y)$. The geometric meaning of isotropic coordinates is that the tangent vectors to the coordinate lines $x = const$ and $y = const$ are isotropic vectors, i.e. have zero length.

The Weierstrass representation can be extended to surfaces embedded with isotropic coordinates in the pseudo-Euclidean spaces $\mathbb{R}^{2,1}$, $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$. The corresponding formulas are obtained by a Wick rotation of the representations (2.4) and (2.10) in $\mathbb{R}^3$ and $\mathbb{R}^4$. The operators in the consistency conditions for these representations are various reductions of the generalized Dirac operator [11]:

$$D = \begin{pmatrix} 0 & \partial_z \\ -\partial_w & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}. \tag{2.11}$$

where z and w are complex variables. We describe these three cases individually and give explicit formulas. All of these formulas are taken directly from the paper [11].

2.2.1 The $\mathbb{R}^{2,1}$ case

Suppose that the complex-valued functions ψ_1, ψ_2 are defined on some simply-connected domain $U \subset \mathbb{R}^2$ and satisfy the equations

$$\left[\begin{pmatrix} 0 & \partial_x \\ \partial_y & 0 \end{pmatrix} - \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \bar{p} = p. \tag{2.12}$$

Then the formulas

$$\begin{aligned} X_1 &= \tfrac{1}{2}\int \left[(\psi_1^2 + \bar{\psi}_1^2)dx + (\psi_2^2 + \bar{\psi}_2^2)dy\right], \\ X_2 &= \tfrac{i}{2}\int \left[(\psi_1^2 - \bar{\psi}_1^2)dx + (\psi_2^2 - \bar{\psi}_2^2)dy\right], \\ X_3 &= \int \left[\psi_1\bar{\psi}_1 dx + \psi_2\bar{\psi}_2 dy\right], \end{aligned} \tag{2.13}$$

define an embedding $\vec{X} : U \to \mathbb{R}^{2,1}$, such that the induced metric on U has signature $(1,1)$ and the coordinates x and y are isotropic. Conversely, any surface embedded in $\mathbb{R}^{2,1}$ locally admits such a representation if the restriction of the metric tensor has signature $(1,1)$ everywhere.

2.2.2 The $\mathbb{R}^{3,1}$ case

Suppose that the complex-valued functions φ_i, ψ_i, $i = 1, 2$ are defined on some simply-connected domain $U \subset \mathbb{R}^2$ and satisfy the equations

$$\left[\begin{pmatrix} 0 & \partial_x \\ \partial_y & 0 \end{pmatrix} - \begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad \left[\begin{pmatrix} 0 & \partial_x \\ \partial_y & 0 \end{pmatrix} - \begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{2.14}$$

Then the formulas

$$\begin{aligned} X_1 &= \tfrac{1}{2}\int \left[(\varphi_1\bar{\psi}_1 + \bar{\varphi}_1\psi_1)dx + (\varphi_2\bar{\psi}_2 + \bar{\varphi}_2\psi_2)dy\right], \\ X_2 &= \tfrac{i}{2}\int \left[(\varphi_1\bar{\psi}_1 - \bar{\varphi}_1\psi_1)dx + (\varphi_2\bar{\psi}_2 - \bar{\varphi}_2\psi_2)dy\right], \\ X_3 &= \tfrac{1}{2}\int \left[(\varphi_1\bar{\varphi}_1 - \psi_1\bar{\psi}_1)dx + (\varphi_2\bar{\varphi}_2 - \psi_2\bar{\psi}_2)dy\right], \\ X_4 &= \tfrac{1}{2}\int \left[(\varphi_1\bar{\varphi}_1 + \psi_1\bar{\psi}_1)dx + (\varphi_2\bar{\varphi}_2 + \psi_2\bar{\psi}_2)dy\right], \end{aligned} \tag{2.15}$$

define an embedding $\vec{X} : U \to \mathbb{R}^{3,1}$, such that the induced metric on U has signature $(1, 1)$ and the coordinates x and y are isotropic. Conversely, any surface embedded in $\mathbb{R}^{3,1}$ locally admits such a representation if the restriction of the metric tensor has signature $(1, 1)$ everywhere.

2.2.3 The $\mathbb{R}^{2,2}$ case

Suppose that the functions φ_i, ψ_i, $i = 1, 2$ are defined on some simply-connected domain $U \subset \mathbb{R}^2$ and satisfy the equations

$$\left[\begin{pmatrix} 0 & \partial_x \\ \partial_y & 0 \end{pmatrix} - \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad \left[\begin{pmatrix} 0 & \partial_x \\ \partial_y & 0 \end{pmatrix} - \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{2.16}$$

where $\bar{p} = p$ and $\bar{q} = q$. Then the formulas

$$\begin{aligned} X_1 &= \tfrac{1}{2}\int \left[(\varphi_1\psi_1 + \bar{\varphi}_1\bar{\psi}_1)dx + (\varphi_2\psi_2 + \bar{\varphi}_2\bar{\psi}_2)dy\right], \\ X_2 &= \tfrac{i}{2}\int \left[(\varphi_1\psi_1 - \bar{\varphi}_1\bar{\psi}_1)dx + (\varphi_2\psi_2 - \bar{\varphi}_2\bar{\psi}_2)dy\right], \\ X_3 &= \tfrac{1}{2}\int \left[(\varphi_1\bar{\psi}_1 + \bar{\varphi}_1\psi_1)dx + (\varphi_2\bar{\psi}_2 + \bar{\varphi}_2\psi_2 dy\right], \\ X_4 &= \tfrac{i}{2}\int \left[(\varphi_1\bar{\psi}_1 - \bar{\varphi}_1\psi_1)dx + (\varphi_2\bar{\psi}_2 - \bar{\varphi}_2\psi_2 dy\right], \end{aligned} \tag{2.17}$$

define an embedding $\vec{X} : U \to \mathbb{R}^{2,2}$, such that the induced metric on U has signature $(1,1)$ and the coordinates x and y are isotropic. Conversely, any surface embedded in $\mathbb{R}^{2,2}$ locally admits such a representation if the restriction of the metric tensor has signature $(1,1)$ everywhere.

2.3 The author's results

The principal result of this thesis is an integrable discretization of the generalized Dirac operator (2.11) and an integrable reduction of this discretization. We will see that several versions of this reduction can be used to construct discrete analogues of the generalized Weierstrass representations (2.13), (2.15), and (2.17). These discrete representations describe $\mathbb{Z}^2$ lattices in pseudo-Euclidean spaces with the geometric property that every edge is an isotropic vector.

Chapter 3

The discrete Dirac operator

In this section, we construct a discretization of the generalized Dirac operator (2.11) and an integrable reduction of this discretization. We also construct an integrable hierarchy of equations based on this discrete Dirac operator.

3.1 Discretization of finite-gap operators

The problem of constructing an integrable discretization of an integrable differential equation is not mathematically well-posed and does not have a universal solution. Several methods for constructing integrable discretizations have been developed in soliton theory. They are generally based on constructing a discrete analogue of the auxiliary linear problems, which involves an appropriate deformation of the analytic properties of the solutions of these linear problems.

One of the most interesting aspects of the theory of integrable systems is its relationship to the algebraic geometry of curves. This theory is known as the *theory of finite-gap integration* and was developed in [21; 7; 19]. A *finite-gap* operator is an integrable linear operator whose eigenfunctions are defined on a compact Riemann surface known as the *spectral curve*. The eigenfunction has singularities of a prescribed type on the spectral curve and can be uniquely characterized by these singularities. In the framework of the theory of finite-gap integration, there exists a natural approach to the discretization problem, based on deforming the singularities of the eigenfunctions.

To illustrate this approach, let us consider the following simple example. Consider the linear equation

$$\partial_x \psi(x,\lambda) = \lambda\psi(x,\lambda), \quad \psi(0,\lambda) = 1. \tag{3.1}$$

where x is a complex variable and the spectral parameter λ is defined on the Riemann sphere $\mathbb{CP}^1$. The solution to this problem is the exponential function

$$\psi(x,\lambda) = e^{\lambda x}. \tag{3.2}$$

The function $\psi(x,\lambda)$ is the unique function on $\mathbb{CP}^1$ satisfying the following conditions:

- $\psi(x,\lambda)$ is holomorphic in λ on $\mathbb{CP}^1\backslash\{\infty\}$.
- At the infinite point, $\psi(x,\lambda)$ has an essential singularity of the form

$$\psi(x,\lambda) = e^{\lambda x}(1 + O(\lambda^{-1})). \tag{3.3}$$

It is also easy to show that any function satisfying the above conditions also satisfies equation (3.1), in other words equation (3.1) can be reconstructed from the spectral properties of its solution.

We now consider a discretization of equation (3.1). Let $n \in \mathbb{Z}$ be a discrete variable, and consider the equation

$$\psi(n+1,\lambda) - \psi(n,\lambda) = \lambda\psi(n,\lambda), \quad \psi(0,\lambda) = 1. \tag{3.4}$$

The solution to this equation is the meromorphic function

$$\psi(n,\lambda) = (1+\lambda)^n. \tag{3.5}$$

The function $\psi(n,\lambda)$ can be uniquely specified by the following conditions:

- $\psi(n,\lambda)$ is meromorphic in λ on $\mathbb{CP}^1$.
- $\psi(n,\lambda)$ has a zero of order n at $\lambda = -1$ and a pole of order n at ∞ of the form

$$\psi(n,\lambda) = \lambda^n + O(\lambda^{n-1}). \tag{3.6}$$

Conversely, a function satisfying these conditions also satisfies the discrete equation (3.4), so we can reconstruct equation (3.4) from the spectral properties.

To obtain the differential equation (3.1) as a continuous limit of the discrete equation (3.4), introduce a small parameter h and consider the function

$$\psi_h(n,\lambda) = (1+h\lambda)^n. \tag{3.7}$$

This function has a zero of order n at $\lambda = -1/h$ and satisfies the difference equation

$$\psi_h(n+1,\lambda) - \psi_h(n,\lambda) = h\lambda\psi_h(n,\lambda), \quad \psi_h(0,\lambda) = 1, \tag{3.8}$$

which is a rescaling of (3.4). Let $n \to \infty$ and $h \to 0$ so that $x = nh$ remains constant. We have that

$$\psi_h(n,\lambda) = \exp(n\log(1+h\lambda)) = \exp(n(h\lambda + O(h^2))) \to e^{\lambda x} = \psi(x,\lambda), \tag{3.9}$$

while

$$\psi_h(n+1,\lambda) - \psi_h(n,\lambda) - h\lambda\psi(x,\lambda) \to h(\partial_x\psi(x,\lambda) - \lambda\psi(x,\lambda)), \tag{3.10}$$

so in the limit we obtain equation (3.1).

In general, the eigenfunction of a finite-gap linear differential operator, known as the *Baker–Akhiezer function*, is defined on an algebraic Riemann surface and has exponential singularities controlled by the continuous variables at one or more marked points of the surface. To construct a discrete analogue of the operator, we replace each exponential singularity with a pair of meromorphic singularities consisting of a pole and a zero of the same order, which we view as the discrete variable. This deformed eigenfunction then satisfies a difference equation that is a discretization of the original differential equation. The continuous limit in a discrete variable is obtained by merging the pole and zero corresponding to that variable as described in the example above.

This method was first used for constructing algebro-geometric solutions of the Ablowitz–Ladik equation [1], [20], which is a discretization of the nonlinear Schrödinger equation. In [15], Krichever used this approach to construct a discretization of the Schrödinger operator in a magnetic field

$$H = \partial_z\partial_{\bar z} + v\partial_{\bar z} + u, \tag{3.11}$$

which is equivalent (by excluding a component of the eigenfunction) to the generalized Dirac operator (2.11).

In this section we give a matrix variant of Krichever's construction. We will see that using this approach to discretize the generalized Dirac operator (2.11) leads to the following discrete equation, which we call the *generalized discrete Dirac equation*:

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{3.12}$$

Here $\psi = (\psi_1(n,m), \psi_2(n,m))^T$ is a vector function of two discrete variables $n, m \in \mathbb{Z}$,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha(n,m) & \beta(n,m) \\ \gamma(n,m) & \delta(n,m) \end{pmatrix} \tag{3.13}$$

is a (2×2)-matrix function of the discrete variables, and T_1 and T_2 denote the translation operators in the discrete variables

$$T_1 f(n,m) = f(n+1,m), \quad T_2 f(n,m) = f(n,m+1). \tag{3.14}$$

We also obtain the following integrable reduction of this generalized Dirac equation:

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \alpha^2 - \beta^2 = 1. \tag{3.15}$$

We will then use this reduction to construct discretizations of the generalized Weierstrass representations (2.12)-(2.17).

3.2 The finite-gap Dirac operator and the modified Novikov–Veselov hierarchy

To construct an integrable discretization of the generalized Dirac operator (2.11) we need to know the spectral properties of its finite-gap eigenfunctions. This theory has been developed by Taimanov in [24; 25].

The spectral curve of the generalized Dirac operator (2.11) can be arbitrary, and the eigenfunction has two essential singularities on the curve. The reduction to the elliptic

Dirac operator (2.7) corresponds to considering spectral curves which admit a pair of involutions, one holomorphic and one anti-holomorphic, such that the essential singularities satisfy certain symmetry conditions with respect to the involutions. The exact statement is the following:

Theorem 3.2.1 *[24] A. Let X be a compact Riemann surface with the following data:*

- *A pair of distinct marked points $P_\pm$.*
- *Local parameters $z_\pm = k_\pm^{-1}$ defined in some neighborhoods of these points.*
- *A nonspecial effective divisor $\mathcal{D}$ of degree $g+1$ on X supported away from the marked points.*

Then

1. *There exists a unique vector-function $\psi(z, w, P) = (\psi_1, \psi_2)$ which is meromorphic in P on $X \backslash \{P_\pm\}$ with poles only at the divisor $\mathcal{D}$, and which has the following expansion near the marked points:*

$$\psi = \exp(k_+ z)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(k_+^{-1})\right] \quad \text{near } P_+, \tag{3.16}$$

$$\psi = \exp(k_- w)\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(k_-^{-1})\right] \quad \text{near } P_-. \tag{3.17}$$

2. *There exist functions $u(z, w)$ and $v(z, w)$ such that the function ψ satisfies the generalized Dirac equation*

$$\left[\begin{pmatrix} 0 & \partial_z \\ -\partial_w & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{3.18}$$

B. *Suppose that the following additional data are given:*

- *A holomorphic involution $\sigma : X \to X$ such that $\sigma(P_\pm) = P_\pm$ and $\sigma(k_\pm) = -k_\pm$.*

- *A meromorphic differential ω on X having two poles of degree two at the points $P_\pm$ with principal parts $(\pm k_\pm^2 + O(k_\pm^{-1}))dk_\pm^{-1}$ and zeroes in $\mathcal{D} + \sigma(\mathcal{D})$.*

Then the potentials of the Dirac equation (3.18) satisfy the condition $v = -u$.

C. *Set $w = \bar{z}$, and suppose that the following additional data are given:*

- *An anti-holomorphic involution $\tau : X \to X$ such that $\tau(P_\pm) = P_\mp$ and $\tau(k_\pm) = \bar{k}_\mp$.*
- *A meromorphic function f on X with divisor $\sigma(\mathcal{D}) - \tau(\mathcal{D})$ such that $f(P_\pm) = \pm 1$.*

Then the potential of the Dirac equation (3.18) is real-valued: $\bar{u} = u$.

The Dirac operator (2.7) is an auxiliary linear operator for the *modified Novikov–Veselov hierarchy*, introduced by Bogdanov in [4; 5]. This is an integrable hierarchy of equations that have the form of Manakov triples:

$$\frac{\partial D}{\partial t_n} = DA_n + B_n D, \tag{3.19}$$

where D is the Dirac operator, t_n is an infinite sequence of times, and A_n and B_n are matrix differential operators, where the operator A_n has order $2n+1$ and has a leading term of the form

$$A_n = \begin{pmatrix} \partial^{2n+1} + \bar{\partial}^{2n+1} & 0 \\ 0 & \partial^{2n+1} + \bar{\partial}^{2n+1} \end{pmatrix} + \cdots \tag{3.20}$$

These equations are the consistency conditions for the overdetermined linear system

$$D\psi = 0, \tag{3.21}$$

$$\frac{\partial \psi}{\partial t_n} + A_n \psi = 0, \tag{3.22}$$

where $\psi = (\psi_1, \psi_2)^T$. The first equation ($n = 1$) has the following form:

$$u_t = \left(u_{zzz} + 3u_z v + \frac{3}{2} u v_z\right) + \left(u_{\bar{z}\bar{z}\bar{z}} + 3u_{\bar{z}} \bar{v} + \frac{3}{2} u \bar{v}_{\bar{z}}\right), \tag{3.23}$$

where

$$v_{\bar{z}} = (u^2)_z \tag{3.24}$$

To construct finite-gap solutions to the modified Novikov–Veselov hierarchy, Taimanov introduces the time-dependent Baker–Akhiezer function $\psi(z, \bar{z}, t_1, t_2, \dots, P)$ on the spectral curve.

Theorem 3.2.2 *[24] Suppose that X is a compact Riemann surface with the data $P_\pm$, $k_\pm$, $\mathcal{D}$, σ, ω, τ and f as described above above. Then*

- *There exists a unique vector function $\psi(z, \bar{z}, t_1, \dots, P) = (\psi_1, \psi_2)$ on X depending on the variables z, $\bar{z}$ and on the time variables $t_1, \dots$, which is meromorphic in P on $X \setminus \{P_\pm\}$ and has poles only in the divisor $\mathcal{D}$, and has the following expansions at the marked points:*

$$\psi = \exp\left(k_+ z + k_+^3 t_1 + \dots + k_+^{2n+1} t_n + \dots\right) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(k_+^{-1}) \right] \text{ near } P_+, \tag{3.25}$$

$$\psi = \exp\left(k_- \bar{z} + k_-^3 t_1 + \dots + k_-^{2n+1} t_n + \dots\right) \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(k_-^{-1}) \right] \text{ near } P_-. \tag{3.26}$$

- *There is a unique operator D of the form (2.7) and unique operators A_n of degrees $2n+1$ with principal parts (3.20) such that equations (3.21)-(3.22) are satisfied.*

- *The potential $u(z, \bar{z}, t_1, \dots)$ of the Dirac operator D satisfies the modified Novikov–Veselov equations (3.19).*

Later in this chapter, we will see that it is possible to construct an integrable hierarchy of differential-difference equations for which the discrete Dirac operator (3.15) is the auxiliary linear operator. We call this hierarchy of equations the *discrete modified Novikov–Veselov hierarchy.*

3.3 Discretization of the Dirac equation

We now construct a discretization of the Dirac equation, taking Th.3.2.1 as our starting point. As we have seen, the eigenfunctions of the Dirac equation have two essential singularities on the spectral curve and in addition poles at a certain divisor. We replace each of the two exponential singularities of the eigenfunction by a pole and a zero of the same order, and construct a discrete equation satisfied by such a function. We then impose reductions on this equation by introducing symmetries on the curve, as in parts B and C of Th.3.2.1. For convenience, we provide a short summary of the theory of divisors on Riemann surfaces.

Let X be a smooth compact Riemann surface of genus g. We denote the ring of meromorphic functions on X by $\mathrm{Mer}(X)$. A *divisor* on X is an element of the free abelian group generated by the points of X, that is to say, a finite set of points of X counted with integer coefficients. The *degree* of a divisor D, denoted $\deg D$, is the sum of its coefficients. A divisor is called *effective* if all of its coefficients are non-negative; if D and D' are two divisors, we write $D \geq D'$ if $D - D'$ if an effective divisor. If $f \in \mathrm{Mer}(X)$ is a meromorphic function on X, its *associated divisor* (f) is the set of zeroes of f minus the set of poles of f, each point being counted with the appropriate multiplicity (so for example the divisor of the function $x^2 - x$ on $\mathbb{C}P^1$ is $1 \cdot 0 + 1 \cdot 1 - 2 \cdot \infty$); in the same way we define the associated divisor (ω) of a meromorphic 1-form ω. A divisor associated to a meromorphic function is called *principal*; two divisors D and D' are called *linearly equivalent* if their difference is principal. The *canonical class* K of X is the divisor associated to a non-trivial meromorphic 1-form on X (any two such divisors are linearly equivalent).

It is natural to formulate the following question: given a finite set of points on a Riemann surface X, does there exist a meromorphic function having poles only at those points, with some maximum specified orders, and no other singularities? What if we also require the function to have zeroes at some other points of the surface, with some minimum specified orders? In terms of divisors, we can formulate this question as follows. Given a divisor D, what is the dimension of the vector space $H^0(D) = \{f \in \mathrm{Mer}(X) | (f) + D \geq 0\}$? This dimension only depends on the linear equivalence class of D, and can be found using the *Riemann–Roch theorem*. Denote $h^0(D) = \dim H^0(D)$ and $h^1(D) = \dim H^0(K - D)$. Then the theorem states that

$$h^0(D) = 1 - g + \deg D + h^1(D). \tag{3.27}$$

We now proceed with our construction. Let X be a smooth compact Riemann surface of genus g. We consider the following data on X:

Data A.

- Four distinct marked points $P_1^\pm, P_2^\pm$ on X.
- Local parameters $z_i^\pm = (k_i^\pm)^{-1}$ defined in some neighborhoods of these points.
- An effective divisor $\mathcal{D} = \gamma_1 + \cdots + \gamma_{g+1}$ of degree $g+1$ on X, supported away from the marked points, which satisfies the following condition of general position:

$$h^1(\mathcal{D} + (n-1)P_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) = 0 \text{ for all } n, m \in \mathbb{Z}. \tag{3.28}$$

To construct solutions of equation (2.7), we consider spaces of meromorphic functions on X with singularities controlled by the discrete variables (functions of this type were first introduced by Krichever in [15]):

$$\Psi_{n,m} = H^0(\mathcal{D} + nP_1^+ - nP_1^- + mP_2^+ - mP_2^-) \subset \mathrm{Mer}(X), \quad n, m \in \mathbb{Z}.$$

The Riemann-Roch theorem implies the following

Proposition 3.3.1 *Suppose that X is a Riemann surface with data A defined above. Then each of the spaces $\Psi_{n,m}$ is two-dimensional:*

$$\dim \Psi_{n,m} = h^0(\mathcal{D} + nP_1^+ - nP_1^- + mP_2^+ - mP_2^-) = 2 \textit{ for all } n, m \in \mathbb{Z},$$

the intersection of two of these spaces at adjacent lattice points is one-dimensional:

$$\dim \Psi_{n,m} \cap \Psi_{n,m-1} = h^0(\mathcal{D} + nP_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) = 1 \textit{ for all } n, m \in \mathbb{Z},$$

$$\dim \Psi_{n,m} \cap \Psi_{n-1,m} = h^0(\mathcal{D} + (n-1)P_1^+ - nP_1^- + mP_2^+ - mP_2^-) = 1 \textit{ for all } n, m \in \mathbb{Z},$$

and these two one-dimensional subspaces of $\Psi_{n,m}$ span the entire space, i.e. their intersection is trivial:

$$\dim \Psi_{n,m} \cap \Psi_{n,m-1} \cap \Psi_{n-1,m} = h^0(\mathcal{D} + (n-1)P_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) = 0.$$

Therefore, we can fix a basis $\psi_1(n,m,P), \psi_2(n,m,P)$ in each of the spaces $\Psi_{n,m}$ by letting $\psi_1(n,m,P)$ be any non-zero element of $\Psi_{n,m} \cap \Psi_{n,m-1}$, and letting $\psi_2(n,m,P)$ to be any non-zero element of $\Psi_{n,m} \cap \Psi_{n-1,m}$:

$$\psi_1(n,m,P) \in H^0(\mathcal{D} + nP_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) - \{0\}, \tag{3.29}$$

$$\psi_2(n,m,P) \in H^0(\mathcal{D} + (n-1)P_1^+ - nP_1^- + mP_2^+ - mP_2^-) - \{0\}. \tag{3.30}$$

The principal observation concerning these functions can be summarized in the following statement:

Proposition 3.3.2 *Suppose that X is a Riemann surface with data A as defined above. Then there exist functions $\alpha(n,m)$, $\beta(n,m)$, $\gamma(n,m)$, $\delta(n,m)$ such that the functions $\psi_1(P)$ and $\psi_2(P)$ defined by (3.29)-(3.30) satisfy the Dirac equation:*

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{3.31}$$

Proof. Indeed, by construction, both $\psi_1(n,m+1,P)$ and $\psi_2(n+1,m,P)$ actually lie in the space $\Psi_{n,m}$, hence they can be expressed as linear combinations of the basis functions $\psi_1(n,m,P)$ and $\psi_2(n,m,P)$, which is equivalent to saying that they satisfy the Dirac equation (3.31).

Therefore, a Riemann surface X together with the additional data given above allows us to construct a family of solutions $(\psi_1(n,m,P), \psi_2(n,m,P))^T$ of the Dirac equation (3.31), parametrized by the points P of X.

In order to construct reductions on the Dirac equation (3.31), we first express the coefficients $\alpha(n,m)$, $\beta(n,m)$, $\gamma(n,m)$ and $\delta(n,m)$ in terms of the principal parts of the basis functions at the marked points. In terms of the chosen local coordinates, the basis functions $\psi_1(n,m,P)$ and $\psi_2(n,m,P)$ have the following expansions at the marked points, where k denotes the appropriate local coordinate $k_i^{\pm}$:

$$\psi_1(n,m,P) = \begin{cases} a_1^+(n,m)k^n + O(k^{n-1}), & \text{as } P \to P_1^+ \\ a_1^-(n,m)k^{-n} + O(k^{-n-1}), & \text{as } P \to P_1^- \\ O(k^{m-1}), & \text{as } P \to P_2^+ \\ a_2^-(n,m)k^{-m} + O(k^{-m-1}), & \text{as } P \to P_2^- \end{cases} \tag{3.32}$$

$$\psi_2(n,m,P) = \begin{cases} O(k^{n-1}), & \text{as } P \to P_+^1 \\ b_1^-(n,m)k^{-n} + O(k^{-n-1}), & \text{as } P \to P_-^1 \\ b_2^+(n,m)k^{m} + O(k^{m-1}), & \text{as } P \to P_+^2 \\ b_2^-(n,m)k^{-m} + O(k^{-m-1}), & \text{as } P \to P_-^2 \end{cases} \tag{3.33}$$

where the $a_i^\pm(n,m)$ and $b_i^\pm(n,m)$ are functions of the discrete variables n and m. Considering the Dirac equation (3.31) near the marked points $P_1^\pm, P_2^\pm$ gives us the following system of equations (in what follows, we usually suppress the indices n and m and replace them with the translation operators T_1 and T_2):

$$\begin{array}{rclrcl} T_2 a_1^+ &=& \alpha a_1^+, & 0 &=& \gamma a_1^- + \delta b_1^-, \\ T_2 a_1^- &=& \alpha a_1^- + \beta b_1^-, & T_1 b_2^+ &=& \delta b_2^+, \\ 0 &=& \alpha a_2^- + \beta a_2^+, & T_1 b_2^- &=& \gamma a_2^- + \delta b_2^-. \end{array} \tag{3.34}$$

The functions ψ_1 and ψ_2 have so far been defined up to multiplication by a constant factor dependent on n and m. We impose the following additional conditions on the functions ψ_1 and ψ_2:

$$a_1^+ a_1^- = 1, \quad b_2^+ b_2^- = 1. \tag{3.35}$$

It is easy to show using (3.34) that these conditions imply the following relations on the coefficients α, β, γ, δ:

$$\alpha\delta - \beta\gamma = \frac{\alpha}{\delta} = \frac{\delta}{\alpha} = \frac{(T_2 a_1^+)(T_1 b_2^-)}{a_1^+ b_2^-} = \pm 1. \tag{3.36}$$

Condition (3.35) defines the constants a_1^+ and b_2^-, and hence the functions ψ_1 and ψ_2, only up to a factor of ± 1 that depends on n and m. This allows us to impose the following additional condition on the functions ψ_1 and ψ_2:

$$(T_2 a_1^+)(T_1 b_2^-) = a_1^+ b_2^-. \tag{3.37}$$

In other words, we can choose the sign for the function ψ_2 arbitrarily, and then choose the sign for the function ψ_1 using the above relation. With this condition, the sign in equation (3.36) is positive. Therefore, reductions (3.35) and (3.37) impose the following relations on the coefficients of the Dirac operator (3.31):

$$\alpha\delta - \beta\gamma = 1, \quad \alpha = \delta. \tag{3.38}$$

In other words, the coefficients of a general Dirac operator of the form (3.31) depend, up to gauge equivalence, on two arbitrary functions of the discrete variables.

We now introduce a reduction under which the coefficients of the Dirac operator (3.31) depend on only one function of the variables n, m. Suppose that, in addition to data A described above, the spectral curve X has the following:

Data B.

- A holomorphic involution $\sigma : X \to X$ that interchanges the marked points and the local parameters at the marked points as follows:

$$\sigma(P_i^{\pm}) = P_i^{\mp}, \quad \sigma(k_i^{\pm}) = k_i^{\mp}. \tag{3.39}$$

- A meromorphic 1-form ω on X which has simple poles at the marked points $P_i^{\pm}$ with residues ± 1 and no other singularities, whose zero divisor is $\mathcal{D} + \sigma(\mathcal{D})$, and which is odd with respect to the involution.

Consider the meromorphic 1-form $\psi_1(n,m,P)\psi_2(n,m,\sigma(P))\omega(P)$. Comparing the singularities of the three terms, we see that this 1-form has simple poles at P_1^+ and P_2^- with residues $a_1^+ b_1^-$ and $-a_2^- b_2^+$, respectively, and no other singularities. Hence, the existence of the additional data above implies that the coefficients of the functions ψ_1 and ψ_2 satisfy the following additional condition:

$$a_1^+ b_1^- = a_2^- b_2^+. \tag{3.40}$$

Using (3.34) and (3.35), it is easy to show that this condition implies the following additional relation on the coefficients of the Dirac operator:

$$\beta = \gamma. \tag{3.41}$$

Using the involution σ we can rewrite the normalization conditions (3.35) and (3.37) in the following equivalent form:

$$\psi_1(P)\psi_1(\sigma(P))|_{P=P_1^+} = 1, \tag{3.42}$$

$$\psi_2(P)\psi_2(\sigma(P))|_{P=P_2^+} = 1, \tag{3.43}$$

$$\left.\frac{T_2\psi_1(P)}{\psi_1(P)}\right|_{P=P_1^+} = \left.\frac{T_1\psi_2(P)}{\psi_2(P)}\right|_{P=P_2^-}. \tag{3.44}$$

Therefore, we can summarize the result of this reduction as follows.

Proposition 3.3.3 *Suppose that X is a Riemann surface with data A and data B as defined above, and suppose the functions $\psi_1(P)$ and $\psi_2(P)$ defined by (3.29) and (3.30) satisfy the normalization conditions (3.42)-(3.44). Then there exist functions of the discrete variables α and β that satisfy the relation*

$$\alpha^2 - \beta^2 = 1 \tag{3.45}$$

and such that the functions $\psi_1(P)$ and $\psi_2(P)$ satisfy the discrete Dirac equation:

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{3.46}$$

We now construct a further reduction of the discrete Dirac equation (3.46) which is the discrete analogue of the real-valued reduction in the differential case. Suppose that, in addition to data A and data B above, the spectral curve X has the following:

Data C.

- An anti-holomorphic involution $\tau : X \to X$ that interchanges the marked points and acts on the local parameters at the marked points as follows:

$$\tau(P_1^\pm) = P_2^\pm, \quad \tau(P_2^\pm) = P_1^\pm, \quad \tau(k_1^\pm) = \bar{k}_2^\pm, \quad \tau(k_2^\pm) = \bar{k}_1^\pm. \tag{3.47}$$

- A meromorphic function $f(P)$ on X with divisor $(f) = \mathcal{D} - \tau(\mathcal{D})$ satisfying the conditions

$$f(P)\bar{f}(\tau(P)) = -1 \text{ for all } P \in X, \quad f(P_1^+)f(P_1^-) = 1. \tag{3.48}$$

For a function $f(n,m)$ of the discrete variables, we introduce the notation $f^*(n,m) = \bar{f}(m,n)$. Consider the two functions $\psi_2^*(n,m,\tau(P))$ and $\psi_1(n,m,P)f(P)$. Both these functions are meromorphic and lie in the one-dimensional space $H^0(\tau(\mathcal{D}) + (n-1)P_2^+ - nP_2^- + mP_1^+ - mP_1^-)$, hence there exists a function $C(n,m)$ of n and m such that

$$\bar{\psi}_2(m,n,\tau(P)) = \psi_1(n,m,P)f(P)C(n,m). \tag{3.49}$$

Considering this equation at $P = P_1^+$ and $P = P_1^-$ and using conditions (3.35) and (3.48), we see that

$$C(n,m)^2 = 1 \text{ for all } n, m \in \mathbb{Z}. \tag{3.50}$$

We recall that the function ψ_2 was normalized by condition (3.35), which specifies it up to multiplication by a factor ± 1 dependent on n and m. Therefore, we can choose this factor in such a way that $C(n,m) = 1$ for all n and m, in other words we may impose the additional following condition:

$$\bar{\psi}_2(m,n,\tau(P)) = \psi_1(n,m,P)f(P). \tag{3.51}$$

Equation (3.48) then implies that the functions ψ_1 and ψ_2 chosen in this way satisfy the following relations:

$$\bar{\psi}_2(m,n,\tau(P)) = \psi_1(n,m,P)f(P), \quad \bar{\psi}_1(m,n,\tau(P)) = -\psi_2(n,m,P)f(P). \tag{3.52}$$

Plugging these relations into the reduced Dirac equation (3.46) gives us the following relations on the coefficients of the operator:

$$\alpha^* = \alpha, \quad \beta^* = -\beta. \tag{3.53}$$

We summarize the results of this reduction in the following proposition:

Proposition 3.3.4 *Suppose that X is a Riemann surface with data A, B and C as defined above, and suppose the functions $\psi_1(P)$ and $\psi_2(P)$ defined by (3.29) and (3.30) satisfy the normalization conditions (3.42)-(3.44) and (3.51). Then there exist functions of the discrete variables α and β that satisfy the relations*

$$\alpha^2 - \beta^2 = 1, \quad \alpha^* = \alpha, \quad \beta^* = -\beta \tag{3.54}$$

that the functions $\psi_1(P)$ and $\psi_2(P)$ satisfy the discrete Dirac equation:

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{3.55}$$

3.4 The discrete modified Novikov–Veselov hierarchy

In the previous section, we constructed algebro-geometric solutions of the discrete Dirac operator (3.31) and its reductions (3.46) and (3.55) by considering spaces of meromorphic functions $\Psi_{n,m}$ on a Riemann surface X with poles and zeroes determined by the numbers n and m. In this section, we embed these meromorphic solutions into a family of transcendental functions, called *Baker–Akhiezer functions*, and construct a hierarchy of commuting flows on the space of these functions. The set of compatibility conditions of these flows is the discrete analogue of the modified Novikov–Veselov hierarchy.

Let $\mathbf{t} = \left\{t_s^1, t_s^2, s = 1, 2, \ldots\right\} \in \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ denote two sequences of complex numbers, only finitely many of which are non-zero, which we think of as continuous time variables. We construct deformations $\Psi_{n,m,\mathbf{t}}$ of the function spaces $\Psi_{n,m}$ constructed in the previous section by considering functions which in addition have essential singularities at the marked points controlled by the times $\mathbf{t}$.

Proposition 3.4.1 *Suppose that X is a Riemann surface with data A and data B given as in the previous section. Denote by $\tilde{X} = X - P_1^+ - P_1^- - P_2^+ - P_2^-$ the surface X with the marked points removed. Consider the space $\Psi_{n,m,\mathbf{t}} \in Mer(\tilde{X})$ of functions on $\tilde{X}$ defined by the following conditions*

1. *For all $\psi(n, m, \mathbf{t}, P) \in \Psi_{n,m,\mathbf{t}}$ we have $(\psi) + \mathcal{D} \geq 0$, where (ψ) is the divisor of ψ.*

2. *At the marked points $P_i^\pm$ the elements $\psi(n, m, \mathbf{t}, P)$ of $\Psi_{n,m,\mathbf{t}}$ have essential singularities of the following form, where by k we denote the appropriate local coordinate $k_i^\pm$:*

$$\begin{aligned} \psi(n, m, \mathbf{t}, P) &= \exp\left(\pm \sum_{s=1}^{\infty} t_\mu^1 k^\mu\right) O(k^{\pm n}) \text{ as } P \to P_1^\pm, \\ \psi(n, m, \mathbf{t}, P) &= \exp\left(\pm \sum_{s=1}^{\infty} t_\mu^2 k^\mu\right) O(k^{\pm m}) \text{ as } P \to P_2^\pm. \end{aligned} \tag{3.56}$$

Then for sufficiently small $\mathbf{t}$ each of the spaces $\Psi_{n,m,\mathbf{t}}$ is two-dimensional:

$$\dim \Psi_{n,m,\mathbf{t}} = 2 \text{ for all } n, m \in \mathbb{Z}, \tag{3.57}$$

the intersection of two of these spaces at adjacent lattice points is one-dimensional:

$$\dim \Psi_{n,m,\mathbf{t}} \cap \Psi_{n,m-1,\mathbf{t}} = 1 \textit{ for all } n, m \in \mathbb{Z}, \tag{3.58}$$

$$\dim \Psi_{n,m,\mathbf{t}} \cap \Psi_{n-1,m,\mathbf{t}} = 1 \textit{ for all } n, m \in \mathbb{Z}, \tag{3.59}$$

and these two one-dimensional subspaces of $\Psi_{n,m,\mathbf{t}}$ *span the entire space, i.e. their intersection is trivial:*

$$\dim \Psi_{n,m,\mathbf{t}} \cap \Psi_{n,m-1,\mathbf{t}} \cap \Psi_{n-1,m,\mathbf{t}} = 0 \textit{ for all } n, m \in \mathbb{Z}. \tag{3.60}$$

Proof. The proof of this proposition is a standard application of the Riemann–Roch theorem.

This proposition allows us to define functions $\psi_1(n, m, \mathbf{t}, P)$ and $\psi_2(n, m, \mathbf{t}, P)$ using the same relations as in the previous section. We observe the normalization conditions (3.42)-(3.44) can be applied to elements of $\Psi_{n,m,\mathbf{t}}$, since the exponential singularities cancel out.

Proposition 3.4.2 *There exist unique functions* $\psi_1(n, m, \mathbf{t}, P)$ *and* $\psi_2(n, m, \mathbf{t}, P)$ *that form a basis for the vector space* $\Psi_{n,m,\mathbf{t}}$ *such that*

$$\psi_1(n, m, \mathbf{t}, P) \in \Psi_{n,m,\mathbf{t}} \cap \Psi_{n,m-1,\mathbf{t}} - \{0\}, \tag{3.61}$$

$$\psi_2(n, m, \mathbf{t}, P) \in \Psi_{n,m,\mathbf{t}} \cap \Psi_{n-1,m,\mathbf{t}} - \{0\}. \tag{3.62}$$

and which satisfy the normalization conditions (3.42)-(3.44). These functions satisfy the discrete Dirac equation

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{3.63}$$

where α *and* β *are functions of the variables* n*,* m*, and* $\mathbf{t}$ *satisfying the condition*

$$\alpha^2 - \beta^2 = 1. \tag{3.64}$$

We later give explicit formulas for the functions ψ_i in terms of the theta functions of the curve X

We now show that these functions satisfy a system of commuting linear equations. Let $\mathfrak{R}$ denote the ring of functions in the variables n, m and $\mathbf{t}$. We consider the ring $\mathfrak{O} = \mathfrak{R}[T_1, T_1^{-1}, T_2, T_2^{-1}]$ of finite difference operators with coefficients in $\mathfrak{R}$, and the ring $\mathfrak{M}$ of (2×2) matrix operators with coefficients in $\mathfrak{O}$. By ψ we denote the column vector $(\psi_1(n, m, \mathbf{t}, P), \psi_2(n, m, \mathbf{t}, P))^T$.

Proposition 3.4.3 *There exist unique matrix difference operators A^i_μ in $\mathfrak{M}$*

$$A^i_\mu = \begin{pmatrix} A^i_{\mu,1} & 0 \\ 0 & A^i_{\mu,2} \end{pmatrix}, \quad i = 1, 2, \quad \mu = 1, 2, \ldots \tag{3.65}$$

$$A^i_{\mu,j} = \sum_{s=-\mu}^{\mu} f^i_{\mu,j,s}(n, m, \mathbf{t}) T_i^s, \tag{3.66}$$

such that the functions $\psi_1(n, m, \mathbf{t}, P)$ and $\psi_2(n, m, \mathbf{t}, P)$ satisfy the following system of differential equations:

$$\frac{\partial}{\partial t^i_\mu} \psi = A^i_\mu \psi. \tag{3.67}$$

Proof. The proof is standard. For a given μ we show how to construct the operator $A^1_{\mu,1}$, the other cases being similar.

The derivative of the function $\psi_1(n, m, \mathbf{t}, P)$ with respect to t^1_μ has the following expansions at the marked points $P_i^\pm$, where by k we denote the appropriate local coordinate $k_i^\pm$:

$$\frac{\partial}{\partial t^1_\mu} \psi_1(n, m, \mathbf{t}, P) = \exp\left(\pm \sum_{\nu=1}^{\infty} t^1_\nu k^\nu\right) \cdot O(k^{\pm n + \mu}) \text{ as } P \to P_1^\pm, \tag{3.68}$$

$$\frac{\partial}{\partial t^1_\mu} \psi_1(n, m, \mathbf{t}, P) = \exp\left(\sum_{i=1}^{\infty} t^2_\nu k^\nu\right) O(k^{m-1}) \text{ as } P \to P_2^+, \tag{3.69}$$

$$\frac{\partial}{\partial t^1_\mu} \psi_1(n, m, \mathbf{t}, P) = \exp\left(-\sum_{\nu=1}^{\infty} t^2_\nu k^\nu\right) \cdot O(k^{-m}) \text{ as } P \to P_2^-. \tag{3.70}$$

Therefore, for an appropriate choice of functions $f^1_{\mu,i,s}(n, m, \mathbf{t})$, the function

$$\widetilde{\psi}(n, m, \mathbf{t}, P) = \frac{\partial}{\partial t^1_\mu} \psi_1(n, m, \mathbf{t}, P) - \sum_{s=-\mu}^{\mu} f^1_{\mu,1,s}(n, m, \mathbf{t}) \psi_1(n + s, m, \mathbf{t}, P) \tag{3.71}$$

has the following expansions at $P_1^{\pm}$:

$$\widetilde{\psi}(n, m, \mathbf{t}, P) = \exp\left(\sum_{\nu=1}^{\infty} t_\nu^1 k^\nu\right) \cdot O(k^{n-1}) \text{ as } P \to P_1^+, \tag{3.72}$$

$$\widetilde{\psi}(n, m, \mathbf{t}, P) = \exp\left(-\sum_{\nu=1}^{\infty} t_\nu^1 k^\nu\right) \cdot O(k^{-n}) \text{ as } P \to P_1^-, \tag{3.73}$$

and the same expansions (3.69)-(3.70) at $P_2^{\pm}$ as $\frac{\partial}{\partial t_\mu^1}\psi_1(n, m, \mathbf{t}, P)$. Therefore, by (3.60) this function is identically zero on X. Therefore, the function $\psi_1(n, m, \mathbf{t}, P)$ satisfies the system of equations (3.67).

Proposition 3.4.4 *The left ideal of matrix difference operators in $\mathfrak{M}$ that annihilate ψ is the principal left ideal generated by the operator D.*

Proof. Suppose that A and B are two operators in $\mathfrak{O}$ that satisfy the following equation:

$$A\psi_1 + B\psi_2 = 0. \tag{3.74}$$

We need to show that there exist operators $C, D \in \mathfrak{O}$ such that $A = C(T_2 - \alpha) - D\beta$ and $B = -C\beta + D(T_1 - \alpha)$.

First, we multiply equation (3.74) on the left by sufficiently high powers of T_1 and T_2 so that the operators A and B become polynomial in T_1 and T_2. Next, we show that we can eliminate all terms containing mixed powers of T_1 and T_2. Indeed, suppose

$$A = \sum_{i=1}^{n-1} a_i T_1^i T_2^{n-i} + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

$$B = \sum_{i=1}^{n-1} b_i T_1^i T_2^{n-i} + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

then we can write

$$A = \sum_{i=1}^{n-1} \left[a_i T_1^i T_2^{n-i-1}(T_2 - \alpha) - b_i T_1^i T_2^{n-i-1}\beta\right] + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

$$B = \sum_{i=1}^{n-1} \left[b_i T_1^i T_2^{n-i-1}(T_1 - \alpha) - a_i T_1^i T_2^{n-i-1}\alpha\right] + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

and proceeding in this way, we can eliminate all terms which are not powers of only T_1 or T_2. Therefore, we can assume that $A = A_1(T_1) + A_2(T_2)$, $B = B_1(T_1) + B_2(T_2)$, where the A_i, B_i are polynomials in only T_i.

Suppose that $A_1 = \sum_{i=0}^{n} a_i T_1^i$ and $B_1 = \sum_{j=0}^{m} b_j T_1^j$. Comparing the singularities in (3.74) at the point P_1^+, we see that $m = n+1$. Subtracting $b_{n+1}T_1^n\left[(T_1-\alpha)\psi_2 - \beta\psi_1\right]$ from (3.74), we reduce the degree of B_1, and hence of A_1. In this way we can eliminate A_1, and similarly B_2. Therefore, we are left with showing that if $A = A_2(T_2)$ and $B = B_1(T_1)$ are linear polynomials satisfying (3.74), then they can be expressed as $A = f(T_2-\alpha) - g\beta$ and $B = -f\beta + g(T_1-\alpha)$ for some functions f and g, which can be easily shown.

Proposition 3.4.5 *There exist matrix difference operators B_μ^i in $\mathfrak{M}$ such that the following equations are satisfied:*

$$-\frac{\partial}{\partial t_\mu^i} D = DA_\mu^i + B_\mu^i D \tag{3.75}$$

Proof. Equations (3.63) and (3.67) imply that

$$\left[\frac{\partial}{\partial t_\mu^i} - A_\mu^i, D\right]\psi = 0. \tag{3.76}$$

Since the operator in the left hand side does not contain derivation in time, it is inside $\mathfrak{M}$, hence by the above proposition it is a left multiple of D, which proves the statement.

Theorem 3.4.1 *The equations*

$$\frac{\partial}{\partial t_\mu^i} D + DA_\mu^i \equiv 0 \ mod\ D \tag{3.77}$$

define a commuting hierarchy of differential-difference equations.

We call this system the *discrete modified Novikov–Veselov (dmNV) hierarchy.* In the next section, we give the explicit form of the first two pairs of equations of the dmNV hierarchy.

3.5 First and second equations: explicit forms

In this section, we write down the explicit form of the dmNV hierarchy corresponding to times t_1^1, t_1^2, t_2^1 and t_2^2. We give the explicit calculations for t_1^1, the derivations for the other times being similar.

It is difficult to write down the dmNV as they are defined in (3.77), since this involves performing division with remainder in a matrix algebra over a non-commutative operator ring. To circumvent this difficulty, we notice that the discrete Dirac equation (3.63), which is a difference equation of degree one on the two functions ψ_1 and ψ_2, is equivalent to a degree two difference equation on one of the ψ_1 or ψ_2.

Proposition 3.5.1 *Suppose the functions ψ_1 and ψ_2 satisfy the discrete Dirac equation (3.63). Then these functions individually satisfy the following discrete Schrödinger equations*

$$H_1\psi_1 = \left[T_1T_2 - (T_1\alpha)T_1 - \frac{\alpha(T_1\beta)}{\beta}T_2 + \frac{T_1\beta}{\beta}\right]\psi_1 = 0 \tag{3.78}$$

$$H_2\psi_2 = \left[T_1T_2 - (T_2\alpha)T_2 - \frac{\alpha(T_2\beta)}{\beta}T_1 + \frac{T_2\beta}{\beta}\right]\psi_2 = 0. \tag{3.79}$$

Proof. This follows from excluding ψ_1 or ψ_2 from the system (3.63).

Conversely, we have an analogue of Prop.3.4.4 for the operators H_i:

Proposition 3.5.2 *The left ideal of difference operators in $\mathfrak{O}$ that annihilate ψ_i is the principal left ideal generated by the operator H_i.*

Proof. Suppose that $A \in \mathfrak{O}$ is an operator such that $A\psi_1 = 0$. Then Proposition 3.4 implies that there exist operators $C, D \in \mathfrak{O}$ such that

$$A = C(T_2 - \alpha) - D\beta, \quad -C\beta + D(T_1 - \alpha) = 0.$$

Expressing $C = D(T_1 - \alpha)(\beta)^{-1}$ from the second equation and plugging it in to the first, we get that $A = D(T_1\beta)^{-1}H_1$. The case of ψ_2 is similar.

These two propositions allow us to write our hierarchy as a system of rank one difference equations of degree two.

Proposition 3.5.3 *The discrete modified Novikov–Veselov hierarchy (3.77) is equivalent to either of the following two systems of equations*

$$\frac{\partial}{\partial t^i_\mu} H_1 + H_1 A^i_{\mu,1} \equiv 0 \ mod \ H_1, \tag{3.80}$$

$$\frac{\partial}{\partial t^i_\mu} H_2 + H_2 A^i_{\mu,2} \equiv 0 \ mod \ H_2. \tag{3.81}$$

We now use this approach to construct the equations corresponding to times t^1_1, t^2_1, t^1_2 and t^2_2.

The functions ψ_1 and ψ_2 have the following power series expansions at the marked points $P^\pm_i$, where by k we denote the appropriate local coordinate $k^\pm_i$:

$$\begin{aligned}
\psi_1(n,m,\mathbf{t},P) &= k^{\pm n} \exp\left(\pm\sum_{\mu=1}^\infty t^1_\mu k^\mu\right)\cdot\left(\sum_{j=0}^\infty \xi^\pm_{1,j}(n,m,\mathbf{t})k^{-j}\right) \text{ as } P\to P^\pm_1,\\
\psi_1(n,m,\mathbf{t},P) &= k^{\pm m} \exp\left(\pm\sum_{\mu=1}^\infty t^2_\mu k^\mu\right)\cdot\left(\sum_{j=0}^\infty \xi^\pm_{2,j}(n,m,\mathbf{t})k^{-j}\right) \text{ as } P\to P^\pm_2,\\
\psi_2(n,m,\mathbf{t},P) &= k^{\pm n} \exp\left(\pm\sum_{\mu=1}^\infty t^1_\mu k^\mu\right)\cdot\left(\sum_{j=0}^\infty \chi^\pm_{1,j}(n,m,\mathbf{t})k^{-j}\right) \text{ as } P\to P^\pm_1,\\
\psi_2(n,m,\mathbf{t},P) &= k^{\pm m} \exp\left(\pm\sum_{\mu=1}^\infty t^2_\mu k^\mu\right)\cdot\left(\sum_{j=0}^\infty \chi^\pm_{2,j}(n,m,\mathbf{t})k^{-j}\right) \text{ as } P\to P^\pm_2,
\end{aligned} \tag{3.82}$$

where the $\xi^\pm_{i,\mu}(n,m,\mathbf{t})$ and $\chi^\pm_{i,\mu}(n,m,\mathbf{t})$ are analytic functions in the variables $\mathbf{t}$, and $\xi^+_{2,0}=0$, $\chi^+_{1,0}=0$. To make our notation consistent with (3.32)-(3.33), we denote

$$a^\pm_i = \xi^\pm_{i,0}, \quad b^\pm_i = \chi^\pm_{i,0} \tag{3.83}$$

$$c^\pm_i = \xi^\pm_{i,1}, \quad d^\pm_i = \chi^\pm_{i,1} \tag{3.84}$$

Plugging these expressions into (3.63), we see that these coefficients satisfy the following system of equations:

$$T_2\xi^\pm_{1,j} = \alpha\xi^\pm_{1,j} + \beta\chi^\pm_{1,j} \tag{3.85}$$

$$T_2\xi^\pm_{2,j\pm1} = \alpha\xi^\pm_{2,j} + \beta\chi^\pm_{2,j} \tag{3.86}$$

$$T_1\chi^\pm_{1,j\pm1} = \beta\xi^\pm_{1,j} + \alpha\chi^\pm_{1,j} \tag{3.87}$$

$$T_1\chi^{\pm}_{2,j} = \beta\xi^{\pm}_{2,j} + \alpha\chi^{\pm}_{2,j} \tag{3.88}$$

Because the functions ψ_1 and ψ_2 satisfy the normalization conditions (3.42)-(3.44), we also have

$$a_1^+a_1^- = 1, \quad b_2^+b_2^- = 1. \tag{3.89}$$

We now derive the dmNV equation corresponding to time t_1^1 using its equivalent form (3.80). Let $\dot{f}$ denote differentiation by t_1^1. We denote $A_{1,1}^1 = AT_1 + BT_1^{-1} + C$ and $H_1 = T_1T_2 + xT_1 + yT_2 + z$. The equation in time T_1^1 has the form

$$-\dot{x}T_1 - \dot{y}T_2 - \dot{z} \equiv (T_1T_2 + xT_1 + yT_2 + z)(AT_1 + BT_1^{-1} + C) \bmod H_1. \tag{3.90}$$

First, we express all of the coefficients of the above equation in terms of the variables a_1^+, b_2^+, α and β. The coefficients x, y, z of H_1 were found above in Prop.3.5.1:

$$x = -T_1\alpha, \quad y = -\frac{\alpha(T_1\beta)}{\beta}, \quad z = \frac{T_1\beta}{\beta}. \tag{3.91}$$

To calculate the coefficients of the operator $A_{1,1}^1$, we use the method of Prop.3.4.3. Comparing singularities, we see that if

$$A = f_{1,1,1}^1 = \frac{a_1^+}{T_1a_1^+}, \quad B = f_{1,1,-1}^1 = -\frac{a_1^-}{T_1^{-1}a_1^-} = -\frac{T_1^{-1}a_1^+}{a_1^+}, \tag{3.92}$$

then the functions ψ_1 and $\dot{\psi}_1 - AT_1\psi_1 - BT_1^{-1}\psi$ are proportional. Hence we can determine the third coefficient $C = f_{1,1,0}^1$ by comparing these two functions at either P_2^+ or P_2^-, which gives us two alternative expressions:

$$\begin{aligned} C = f_{1,1,0}^1 &= \frac{1}{c_2^+}\left(\frac{\partial c_2^+}{\partial t_1^1} - \frac{a_1^+}{T_1a_1^+}T_1c_2^+ + \frac{a_1^-}{T_1^{-1}a_1^-}T_1^{-1}c_2^+\right) = \\ &= \frac{1}{a_2^-}\left(\frac{\partial a_2^-}{\partial t_1^1} - \frac{a_1^+}{T_1a_1^+}T_1a_2^- + \frac{a_1^-}{T_1^{-1}a_1^-}T_1^{-1}a_2^-\right). \end{aligned} \tag{3.93}$$

We first these expressions by removing the coefficients a_2^- and c_2^+. From the system (3.85)-(3.88) we get that $c_2^+ = (T_2^{-1}\beta)(T_2^{-1}b_2^+)$ and $a_2^- = -\beta/(\alpha b_2^+)$. Using $T_1b_2^+ = \alpha b_2^+$, the first expression becomes

$$C = \frac{T_2^{-1}\dot{\beta}}{T_2^{-1}\beta} + \frac{T_2^{-1}\dot{b}_2^+}{T_2^{-1}b_2^+} - \frac{a_1^+}{T_1a_1^+}\frac{(T_2^{-1}\alpha)(T_1T_2^{-1}\beta)}{T_2^{-1}\beta} + \frac{T_1^{-1}a_1^+}{a_1^+}\frac{T_1^{-1}T_2^{-1}\beta}{(T_2^{-1}\beta)(T_1^{-1}T_2^{-1}\alpha)}$$

and the second expression becomes

$$C = f^1_{1,1,0} = \frac{\dot\beta}{\beta} - \frac{\dot\alpha}{\alpha} - \frac{\dot b_2^+}{b_2^+} - \frac{a_1^+}{T_1 a_1^+}\frac{T_1\beta}{\beta(T_1\alpha)} + \frac{T_1^{-1}a_1^+}{a_1^+}\frac{\alpha(T_1^{-1}\beta)}{\beta}.$$

Expanding the right hand side of (3.90), we get

$$H_1A^1_{1,1} = (T_1T_2A)T_1^2T_2 + x(T_1A)T_1^2 + [T_1T_2C + y(T_2A)]\,T_1T_2 + [x(T_1C) + zA]\,T_1 +$$

$$+ [T_1T_2B + y(T_2C)]\,T_2 + x(T_1B) + zC + y(T_2B)T_1^{-1}T_2 + zBT_1^{-1}.$$

This expression is a Laurent polynomial in T_1 and T_2 whose terms have degrees i and j in T_1 and T_2, respectively, where $i = -1, 0, 1, 2$ and $j = 0, 1$. We need to express it as a left multiple of H_1 plus an operator containing terms of degrees $(0,0)$, $(0,1)$ and $(1,0)$. First, to cancel the term containing $T_1^2T_2$, we subtract the following left multiple of H_1:

$$(T_1T_2A)T_1H_1 = (T_1T_2A)T_1^2T_2 + (T_1T_2A)(T_1x)T_1^2 + (T_1T_2A)(T_1y)T_1T_2 + (T_1T_2A)(T_1z)T_1.$$

Using (3.78), (3.92) and (3.85), we see that the coefficient in front of T_1^2 in this difference vanishes:

$$x(T_1A) - (T_1x)(T_1T_2A) = -(T_1\alpha)\frac{T_1a_1^+}{T_1^2a_1^+} + (T_1^2\alpha)\frac{T_1T_2a_1^+}{T_1^2T_2a_1^+} = 0.$$

Similarly, to cancel the term containing $T_1^{-1}T_2$, we subtract

$$y(T_2B)T_1^{-1}y^{-1}H_1 = \frac{y(T_2B)}{T_1^{-1}y}T_2 + \frac{y(T_2B)}{T_1^{-1}y}(T_1^{-1}x) + y(T_2B)T_1^{-1}T_2 + \frac{y(T_2B)}{T_1^{-1}y}(T_1^{-1}z)T_1^{-1},$$

and using (3.78), (3.92), (3.85) and the relation (3.89), we show that the coefficient in front of T_1^{-1} vanishes:

$$zB - \frac{y(T_2B)}{T_1^{-1}y}(T_1^{-1}z) = 0.$$

Hence, we see that

$$H_1A^1_{1,1} \equiv [T_1T_2C + y(T_2A) - (T_1T_2A)(T_1y)]\,T_1T_2 + [x(T_1C) + zA - (T_1T_2A)(T_1z)]\,T_1 +$$

$$+ \left[T_1T_2B + y(T_2C) - \frac{y(T_2B)}{T_1^{-1}y}\right] T_2 + x(T_1B) + zC - \frac{y(T_2B)}{T_1^{-1}y}(T_1^{-1}x) \mod H_1.$$

Finally, to obtain the evolution equation, we subtract $[T_1T_2C + y(T_2A) - (T_1T_2A)(T_1y)]\,H_1$ from the right hand side of the equation, and obtain the following equations:

$$-\dot x = x(T_1C) + zA - (T_1T_2A)(T_1z) - x\,[T_1T_2C + y(T_2A) - (T_1T_2A)(T_1y)]\,, \qquad (3.94)$$

$$-\dot{y} = T_1T_2B + y(T_2C) - \frac{y(T_2B)}{T_1^{-1}y} - y\left[T_1T_2C + y(T_2A) - (T_1T_2A)(T_1y)\right], \tag{3.95}$$

$$-\dot{z} = x(T_1B) + zC - \frac{y(T_2B)}{T_1^{-1}y}(T_1^{-1}x) - \left[T_1T_2C + y(T_2A) - (T_1T_2A)(T_1y)\right]. \tag{3.96}$$

Since the coefficients x, y, z of H are expressed in terms of α and β, which are in turn related by the equation $\alpha^2 - \beta^2 = 1$, it is sufficient to find one of the derivatives, for example $\dot{x}$. Expanding the expression for $\dot{x}$ and using the expressions for the coefficients x, y, z and A, B, C obtained above (using the first expression for C in T_1T_2C and using the second one in T_1C), we obtain the following equation

$$\frac{T_1\dot{b}_2^+}{T_1b_2^+} = \frac{a_1^+}{T_1a_1^+}\frac{\beta(T_1\beta)}{T_1\alpha}, \tag{3.97}$$

which is the first equation of the dmNV hierarchy.

It seems natural to replace the variables a_1^+ and b_2^+ with their logarithms, i.e. to introduce new variables $a_1^+ = e^{\varphi}$ and $b_2^+ = e^{\psi}$. Since $\alpha = T_2a_1^+/a_1^+ = T_1b_2^+/b_2^+$, these variables are related by the equation

$$T_2\varphi - \varphi = T_1\psi - \psi. \tag{3.98}$$

Writing the evolution equation (3.97) in terms of these new variables, we get

$$\frac{\partial\psi}{\partial t_1^1} = \sqrt{\left(e^{2T_1^{-1}T_2\varphi} - e^{2T_1^{-1}\varphi}\right)\left(e^{-2\varphi} - e^{-2T_2\varphi}\right)} \tag{3.99}$$

To derive the evolution equation for time t_1^2, we use its equivalent form (3.81). The calculations in this case are identical to those performed above. In fact, since our problem is symmetric with respect to exchanging the marked points $P_1^{\pm}$ and $P_2^{\pm}$, we can obtain the desired equation simply by exchanging the functions a_1^+ and b_2^+ and simultaneously exchanging the shift operators T_1 and T_2 in the evolution equation in time t_1^1 (3.99). This gives us the following equation:

$$\frac{T_2\dot{a}_1^+}{T_2a_1^+} = \frac{\beta(T_2\beta)}{T_2\alpha}\frac{b_2^+}{T_2b_2^+}. \tag{3.100}$$

In terms of the logarithmic variables, this equation reads

$$\frac{\partial\varphi}{\partial t_1^2} = \sqrt{\left(e^{2T_1T_2^{-1}\psi} - e^{2T_2^{-1}\psi}\right)\left(e^{-2\psi} - e^{-2T_1\psi}\right)} \tag{3.101}$$

The derivation of the equations for times t_2^1 and t_2^2 involves similar calculations. For time t_2^1, we use the equivalent form (3.80):

$$-\frac{\partial H_1}{\partial t_2^1} = H_1 A_{2,1}^1 \bmod H_1. \tag{3.102}$$

Here $A_{2,1}$ is a Laurent polynomial in T_1 with terms of degree -2 to 3. As above, we successively subtract appropriate left multiples of H_1 to cancel the terms containing $T_1^i T_2$ for $i = 3, -2, 2, -1$. At every step, the corresponding T_1^i term vanishes. Finally, canceling the $T_1 T_2$ term gives us the following equation:

$$\begin{aligned}\frac{T_1 \dot{b}_2^+}{T_1 b_2^+} &= \frac{\beta(T_1\beta)}{T_1\alpha}\frac{1}{T_1 a_1^+}c_1^+ - \frac{\beta(T_1\beta)}{T_1\alpha}\frac{a_1^+}{(T_1 a_1^+)(T_1^2 a_1^+)}T_1^2 c_1^+ + \\ &+ \frac{\beta(T_1^2\beta)}{(T_1\alpha)(T_1^2\alpha)}\frac{a_1^+}{T_1^2 a_1^+} + \frac{\alpha(T_1^{-1}\beta)(T_1\beta)}{T_1\alpha}\frac{T_1^{-1}a_1^+}{T_1 a_1^+},\end{aligned} \tag{3.103}$$

where the functions a_1^+, b_2^+, c_1^+, α and β in the equation satisfy the following relations:

$$\alpha = \frac{T_2 a_1^+}{a_1^+} = \frac{T_1 b_2^+}{b_2^+}, \quad \alpha^2 - \beta^2 = 1, \quad T_2 c_1^+ = \alpha c_1^+ + \beta(T_1^{-1}\beta)(T_1^{-1}a_1^+). \tag{3.104}$$

3.6 Theta function formulas

In this section we give explicit formulas for the functions $\psi_i(n, m, \mathbf{t}, P)$ in terms of the theta functions of the surface X. Choose a basis a_j, b_j, $j = 1, \ldots, g$ of $H_1(X, \mathbb{Z})$ with canonical intersection form, i.e. such that $a_j \circ a_k = 0$, $b_j \circ b_k = 0$, $a_j \circ b_k = \delta_{jk}$. Let B be the period matrix of the surface X with respect to this basis. Let Ω_1 and Ω_2 denote Abelian differentials of the third kind with poles at $P_1^\pm$ and $P_2^\pm$:

$$\Omega_i = d(k_i^\pm)^{-1}\left(\mp k_i^\pm + O(1)\right) \text{ as } P \to P_i^\pm$$

which are normalized to have zero periods over the a-cycles. Let Ω_i^μ denote Abelian differentials of the second kind with poles at $P_i^\pm$ and principal parts

$$\Omega_i^\mu = d(k_i^\pm)^{-1}\left(\mp\mu(k_i^\pm)^{\mu+1} + O(1)\right) \text{ as } P \to P_i^\pm,$$

and with zero a-periods, and which are odd with respect to the involution σ. It is a standard fact that these differentials exist and are unique. Let U_i and U_i^μ denote the vectors of the

b-periods of these differentials:

$$(U_i)_j = \frac{1}{2\pi\sqrt{-1}} \oint_{b_j} \Omega_i, \quad (U_i^\mu)_j = \frac{1}{2\pi\sqrt{-1}} \oint_{b_j} \Omega_i^\mu.$$

Choose a base point $P_0 \in X$ away from the marked points $P_i^\pm$ and the divisor $\mathcal{D}$, and let $A : X \to J(X)$ denote the Abel map with base point P_0, where $J(X)$ is the Jacobian variety of X. Let $\theta(z|B)$ denote the theta function of $J(X)$ for $z \in \mathbb{C}^g$. Introduce the functions

$$r_1(P) = \frac{\theta(A(P) - A(P_2^+) - \sum_{i=2}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=1}^{g+1} A(P_i) + A(P_2^+) - K|B)}{\theta(A(P) - \sum_{i=1}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=2}^{g+1} A(P_i) - K|B)},$$

$$r_2(P) = \frac{\theta(A(P) - A(P_1^+) - \sum_{i=2}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=1}^{g+1} A(P_i) + A(P_1^+) - K|B)}{\theta(A(P) - \sum_{i=1}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=2}^{g+1} A(P_i) - K|B)}.$$

By construction, these are meromorphic functions on X whose pole divisor is $\mathcal{D} = \sum_{i=1}^{g+1} P_i$ and whose zero divisors are $P_2^+ + \mathcal{D}_1$ and $P_1^+ + \mathcal{D}_2$, respectively, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are some divisors of degree g.

We define the functions ψ_1 and ψ_2 by the following formulas:

$$\psi_i(n, m, \mathbf{t}, P) = r_i(P) C_i(n, m, \mathbf{t}) F_i(n, m, \mathbf{t}, P) \times$$
$$\times \exp\left[n \int_{P_0}^P \Omega_1 + m \int_{P_0}^P \Omega_2 + \sum_{\mu=1}^{\infty} \sum_{l=1}^{2} t_\mu^l \int_{P_0}^P \Omega_l^\mu \right], \quad (3.105)$$

where the function $F(n, m, \mathbf{t}, P)$ is defined as

$$F_i(n, m, \mathbf{t}; P) = \frac{\theta\left(A(P) - A(D_i) + nU_1 + mU_2 + \sum_{\mu=1}^{\infty} \sum_{l=1}^{2} t_\mu^l U_l^\mu \right)}{\theta\left(A(P) - A(D_i) - K \right)}$$

and the path of integration in the exponent is the same as in the Abel map in F_i. By construction, these are single-valued functions on the surface X, having the required meromorphic and exponential singularities at the marked points, and having pole divisor $\mathcal{D}$ away from the marked points.

The constants $C_i(n, m, \mathbf{t})$ are determined by the normalization conditions (3.42)-(3.44). Choose paths of integration $\gamma_i : [0, 1] \to X$ from P_0 to P_i^+ and a path γ from P_0 to $\sigma(P_0)$. We assume that the integration path in $\psi_i(P)$ is γ_i and that the path in $\psi_i(\sigma(P))$ is γ

followed by the image of γ_i under σ. Writing out the expression for $\psi_i(P)\psi_i(\sigma(P))$ using (3.105), we see that we need to choose the constants $C_i(n,m,\mathbf{t})$ as follows:

$$\frac{1}{C_i(n,m,\mathbf{t})^2} = r_i(P_i^+)r_i(P_i^-)F_i(n,m,\mathbf{t},P_i^+)F_i(n,m,\mathbf{t},P_i^-)\times$$
$$\times \exp\left[nI_i^1 + mI_i^2 + \sum_{\mu=1}^{\infty}\sum_{i=1}^{2} t_\mu^l \int_\gamma \Omega_l^\mu\right] \tag{3.106}$$

where the path of integration in the $F_i(n,m,\mathbf{t},P_i^-)$ factor is γ followed by $\sigma(\gamma_i)$, and the constants I_i^1 and I_i^2 are the principal values of the integrals of Ω_1 and Ω_2 along the path $-\gamma_i + \gamma + \sigma(\gamma_i)$:

$$I_i^k = \lim_{t\to 1}\left(\int_{\gamma(0)}^{\gamma(t)} \Omega_k + \int_\gamma \Omega_k + \int_{\sigma(\gamma(0))}^{\sigma(\gamma(t))} \Omega_k\right), \quad k = 1, 2. \tag{3.107}$$

Finally, we choose the signs of $C_i(n,m,\mathbf{t})$ in such a way that the functions ψ_i satisfy the equation (3.44).

Chapter 4

Discretization of the generalized Weierstrass representation

In this section, we use various reductions of the discrete Dirac operator (3.12) constructed in the previous section to construct discrete analogues of the generalized Weierstrass representations (2.12)-(2.17) of isotropic surfaces in pseudo-Euclidean spaces. These discrete Weierstrass representations are maps of the regular $\mathbb{Z}^2$ lattice into the pseudo-Euclidean spaces $\mathbb{R}^{2,1}$, $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ with the property that every edge of the lattice is an isotropic vector.

4.1 Discrete surfaces

Let V be a vector space. A *discrete surface* $\vec{X}$ in V is a map $\vec{X} : \mathbb{Z}^2 \to V$. The *edges* of a discrete surface $\vec{X}$ are the vectors

$$\vec{F}(n,m) = \vec{X}(n+1,m) - \vec{X}(n,m), \quad \vec{G}(n,m) = \vec{X}(n,m+1) - \vec{X}(n,m). \tag{4.1}$$

Conversely, a pair of functions $\vec{F} : \mathbb{Z}^2 \to V$ and $\vec{G} : \mathbb{Z}^2 \to V$ defines a discrete surface (up to translation) if and only if they satisfy the consistency condition

$$\vec{F}(n,m+1) - \vec{F}(n,m) = \vec{G}(n+1,m) - \vec{G}(n,m) \tag{4.2}$$

guaranteeing that the edges link up. We use the following notation for a discrete surface $\vec{X}$ defined in terms of its edges:

$$\vec{X} = \sum \left(\vec{F}\Delta_1 + \vec{G}\Delta_2 \right). \tag{4.3}$$

A discrete surface is called *non-degenerate* if its edges are linearly independent at every lattice point. A discrete surface in $\mathbb{R}^{n,m}$ is called *isotropic* if all of its edges are light-like vectors:

$$\langle \vec{F}(n,m), \vec{F}(n,m) \rangle = 0, \quad \langle \vec{G}(n,m), \vec{G}(n,m) \rangle = 0, \tag{4.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{n,m}$.

The main result of this chapter is that isotropic discrete surfaces in $\mathbb{R}^{2,1}$, $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ satisfying a certain monotonicity condition are described by solutions of a discrete Dirac equation, using essentially the same formulas as in the continuous case.

4.2 The $\mathbb{R}^{2,1}$ case

Proposition 4.2.1 *Suppose that the functions ψ_1, ψ_2 satisfy the following discrete Dirac equation:*

$$\left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{4.5}$$

where $\bar{\alpha} = \alpha$, $\bar{\beta} = \beta$, $\alpha^2 - \beta^2 = 1$. Then the formulas

$$X_1 = \frac{1}{2} \sum \left[(\psi_1^2 + \bar{\psi}_1^2)\Delta_1 + (\psi_2^2 + \bar{\psi}_2^2)\Delta_2 \right], \tag{4.6}$$

$$X_2 = \frac{i}{2} \sum \left[(\psi_1^2 - \bar{\psi}_1^2)\Delta_1 + (\psi_2^2 - \bar{\psi}_2^2)\Delta_2 \right], \tag{4.7}$$

$$X_3 = \sum \left[\psi_1\bar{\psi}_1\Delta_1 + \psi_2\bar{\psi}_2\Delta_2 \right], \tag{4.8}$$

define an isotropic discrete surface $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{2,1}$. Conversely, if $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{2,1}$ is a non-degenerate isotropic discrete surface that satisfies the following condition

$$X_3(n+1,m) - X_3(n,m) > 0, \quad X_3(n,m+1) - X_3(n,m) > 0 \text{ for all } n,m \in \mathbb{Z}^2, \tag{4.9}$$

then there exist functions ψ_1 and ψ_2 satisfying equation (4.5) such that equations (4.6)-(4.8) hold.

Proof. Given functions ψ_1, ψ_2 satisfying (4.5), a direct calculation shows that the edges given by equations (4.6)-(4.8) are isotropic (4.4) and satisfy the consistency condition (4.2), and therefore define an isotropic discrete surface in $\mathbb{R}^{2,1}$.

Conversely, suppose that $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{2,1}$ is an isotropic discrete surface satisfying the monotonicity condition (4.9). The edges of the lattice satisfy the equations

$$F_1^2 + F_2^2 = F_3^2, \quad G_1^2 + G_2^2 = G_3^2, \quad F_3 > 0, \quad G_3 > 0, \tag{4.10}$$

therefore there exist functions ψ_1 and ψ_2, defined up to multiplication by ± 1, such that the edges are given by the formulas (4.6)-(4.8). The consistency condition (4.2) implies that these functions satisfy the following equations

$$(T_2\psi_1)^2 - \psi_1^2 = (T_1\psi_2)^2 - \psi_2^2, \tag{4.11}$$

$$(T_2\psi_1)(T_2\bar{\psi}_1) - \psi_1\bar{\psi}_1 = (T_1\psi_2)(T_1\bar{\psi}_2) - \psi_2\bar{\psi}_2, \tag{4.12}$$

and the non-degeneracy condition implies that

$$\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2 \neq 0. \tag{4.13}$$

The above equation implies that there exist unique real-valued functions α, β, γ and δ such that the following system of equations is satisfied:

$$\left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{4.14}$$

Solving this system, we get

$$\alpha = \frac{\bar{\psi}_2(T_2\psi_1) - \psi_2(T_2\bar{\psi}_1)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \quad \beta = \frac{\psi_1(T_2\bar{\psi}_1) - \bar{\psi}_1(T_2\psi_1)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \tag{4.15}$$

$$\gamma = \frac{\bar{\psi}_2(T_1\psi_2) - \psi_2(T_1\bar{\psi}_2)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \quad \delta = \frac{\psi_1(T_1\bar{\psi}_2) - \bar{\psi}_1(T_1\psi_2)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \tag{4.16}$$

and a direct calculation using (4.11)-(4.12) shows that

$$\alpha^2 - \gamma^2 = 1, \quad \delta^2 - \beta^2 = 1, \quad \alpha\beta = \gamma\delta. \tag{4.17}$$

Solving this system we get that $\delta = \lambda\alpha$ and $\gamma = \lambda\beta$, where $\lambda = \pm 1$. Changing the signs of ψ_2 at every point if necessary, we can set $\lambda = 1$, so that the functions ψ_1 and ψ_2 satisfy the system (4.5). This proves the proposition.

4.3 The $\mathbb{R}^{3,1}$ case

Proposition 4.3.1 *Suppose that the functions φ_i, ψ_i, $i = 1, 2$ satisfy the following discrete Dirac equations:*

$$\left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{4.18}$$

where $|\alpha|^2 - |\beta|^2 = 1$. Then the formulas

$$X_1 = \frac{1}{2}\sum\left[(\varphi_1\bar{\psi}_1 + \bar{\varphi}_1\psi_1)\Delta_1 + (\varphi_2\bar{\psi}_2 + \bar{\varphi}_2\psi_2)\Delta_2\right], \tag{4.19}$$

$$X_2 = \frac{i}{2}\sum\left[(\varphi_1\bar{\psi}_1 - \bar{\varphi}_1\psi_1)\Delta_1 + (\varphi_2\bar{\psi}_2 - \bar{\varphi}_2\psi_2)\Delta_2\right], \tag{4.20}$$

$$X_3 = \frac{1}{2}\sum\left[(\varphi_1\bar{\varphi}_1 - \psi_1\bar{\psi}_1)\Delta_1 + (\varphi_2\bar{\varphi}_2 - \psi_2\bar{\psi}_2)\Delta_2\right], \tag{4.21}$$

$$X_4 = \frac{1}{2}\sum\left[(\varphi_1\bar{\varphi}_1 + \psi_1\bar{\psi}_1)\Delta_1 + (\varphi_2\bar{\varphi}_2 + \psi_2\bar{\psi}_2)\Delta_2\right], \tag{4.22}$$

define an isotropic discrete surface $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{3,1}$. Conversely, if $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{3,1}$ is a non-degenerate isotropic discrete surface that satisfies the following condition

$$X_4(n+1, m) - X_4(n, m) > 0, \quad X_4(n, m+1) - X_4(n, m) > 0 \text{ for all } n, m \in \mathbb{Z}^2, \tag{4.23}$$

then there exist functions φ_i, ψ_i, $i = 1, 2$ satisfying equation (4.18) such that equations (4.19)-(4.22) hold.

Proof. Given functions φ_i, ψ_i, $i = 1, 2$ satisfying (4.18), a direct calculation shows that the edges given by equations (4.19)-(4.22) are isotropic (4.4) and satisfy the consistency condition (4.2), hence define an isotropic discrete surface in $\mathbb{R}^{3,1}$.

Conversely, suppose that $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{3,1}$ is an isotropic discrete surface satisfying the monotonicity condition (4.23). The edges of the lattice satisfy the equations

$$F_1^2 + F_2^2 + F_3^2 = F_4^2, \quad G_1^2 + G_2^2 + G_3^2 = G_4^2, \quad F_4 > 0, \quad G_4 > 0, \tag{4.24}$$

therefore, there exist functions φ_i and ψ_i, where $i = 1, 2$, such that the edges are given by the formulas (4.19)-(4.22). These functions are defined up to the following local gauge equivalence:

$$\varphi_1 \to e^{i\zeta}\varphi_1, \quad \psi_1 \to e^{i\zeta}\psi_1, \quad \varphi_2 \to e^{i\xi}\psi_1, \quad \psi_2 \to e^{i\xi}\varphi_2, \tag{4.25}$$

where ζ and ξ are real-valued functions. The consistency condition (4.2) implies that these functions satisfy the following equations

$$(T_2\varphi_1)(T_2\bar{\varphi}_1) - \varphi_1\bar{\varphi}_1 = (T_1\psi_1)(T_1\bar{\psi}_1) - \psi_1\bar{\psi}_1, \tag{4.26}$$

$$(T_2\varphi_1)(T_2\bar{\psi}_1) - \varphi_1\bar{\psi}_1 = (T_1\varphi_2)(T_1\bar{\psi}_2) - \varphi_2\bar{\psi}_2, \tag{4.27}$$

$$(T_2\psi_1)(T_2\bar{\psi}_1) - \psi_1\bar{\psi}_1 = (T_1\psi_2)(T_1\bar{\psi}_2) - \psi_2\bar{\psi}_2, \tag{4.28}$$

and the non-degeneracy condition implies that

$$\varphi_1\psi_2 - \psi_1\varphi_2 \neq 0. \tag{4.29}$$

The above equation implies that there exist unique complex-valued functions α, β, γ and δ such that the following system of equations is satisfied:

$$\left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{4.30}$$

We can explicitly solve these equations to obtain

$$\alpha = \frac{\psi_2(T_2\varphi_1) - \varphi_2(T_2\psi_1)}{\varphi_1\psi_2 - \psi_1\varphi_2}, \quad \beta = \frac{\varphi_1(T_2\psi_1) - \psi_1(T_2\varphi_1)}{\varphi_1\psi_2 - \psi_1\varphi_2}, \tag{4.31}$$

$$\gamma = \frac{\psi_2(T_1\varphi_2) - \varphi_2(T_1\psi_2)}{\varphi_1\psi_2 - \psi_1\varphi_2}, \quad \delta = \frac{\varphi_1(T_1\psi_2) - \psi_1(T_1\varphi_2)}{\varphi_1\psi_2 - \psi_1\varphi_2}. \tag{4.32}$$

and a direct calculation using (4.26)-(4.28) shows that

$$\alpha\bar{\alpha} - \gamma\bar{\gamma} = 1, \quad \delta\bar{\delta} - \beta\bar{\beta} = 1, \quad \alpha\bar{\beta} - \gamma\bar{\delta} = 0. \tag{4.33}$$

Solving this system we get that $\delta = \lambda\bar{\alpha}$ and $\gamma = \lambda\bar{\beta}$, where $\lambda\bar{\lambda} = 1$. A gauge transformation (4.25) acts on λ as follows:

$$\lambda \to e^{i(\zeta+\xi-T_2\zeta-T_1\xi)}\lambda. \tag{4.34}$$

hence we can set $\lambda = 1$. Therefore, the functions φ_i and ψ_i satisfy the system (4.18). This proves the proposition.

We note that the $\mathbb{R}^{2,1}$ case can be obtained as a reduction by setting $\psi_1 = \bar{\varphi}_1$, $\psi_2 = \bar{\varphi}_2$.

4.4 The $\mathbb{R}^{2,2}$ case

Proposition 4.4.1 *Suppose that the functions φ_i, ψ_i, $i = 1,2$ satisfy the following discrete Dirac equation:*

$$\left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{4.35}$$

where α, β, γ and δ are real and $\alpha\delta - \beta\gamma = 1$. Then the formulas

$$X_1 = \frac{1}{2}\sum\left[(\varphi_1\psi_1 + \bar{\varphi}_1\bar{\psi}_1)\Delta_1 + (\varphi_2\psi_2 + \bar{\varphi}_2\bar{\psi}_2)\Delta_2\right], \tag{4.36}$$

$$X_2 = \frac{i}{2}\sum\left[(\varphi_1\psi_1 - \bar{\varphi}_1\bar{\psi}_1)\Delta_1 + (\varphi_2\psi_2 - \bar{\varphi}_2\bar{\psi}_2)\Delta_2\right], \tag{4.37}$$

$$X_3 = \frac{1}{2}\sum\left[(\varphi_1\bar{\psi}_1 + \bar{\varphi}_1\psi_1)\Delta_1 + (\varphi_2\bar{\psi}_2 + \bar{\varphi}_2\psi_2)\Delta_2\right], \tag{4.38}$$

$$X_4 = \frac{i}{2}\sum\left[(\varphi_1\bar{\psi}_1 - \bar{\varphi}_1\psi_1)\Delta_1 + (\varphi_2\bar{\psi}_2 - \bar{\varphi}_2\psi_2)\Delta_2\right], \tag{4.39}$$

define an isotropic discrete surface $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{2,2}$. Conversely, if $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{2,2}$ is a non-degenerate isotropic discrete surface, then there exist functions φ_i, ψ_i, $i = 1,2$ satisfying equation (4.35) such that equations (4.36)-(4.39) hold.

Proof. Given functions φ_i, ψ_i, $i = 1,2$ satisfying (4.35), a direct calculation shows that the edges given by equations (4.36)-(4.39) are isotropic (4.4) and satisfy the consistency condition (4.2), hence define an isotropic discrete surface in $\mathbb{R}^{2,2}$.

Conversely, suppose that $\vec{X} : \mathbb{Z}^2 \to \mathbb{R}^{2,2}$ is a non-degenerate isotropic discrete surface. The edges of the lattice satisfy the equations

$$F_1^2 + F_2^2 = F_3^2 + F_4^2, \quad G_1^2 + G_2^2 = G_3^2 + G_4^2, \tag{4.40}$$

therefore, there exist functions φ_i and ψ_i, where $i = 1,2$, such that the edges are given by the formulas (4.36)-(4.39). These functions are defined up to the following local gauge equivalence:

$$\varphi_1 \to \mu\varphi_1, \quad \psi_1 \to \mu^{-1}\psi_1, \; \varphi_2 \to \nu\varphi_2, \quad \psi_2 \to \nu^{-1}\psi_2, \tag{4.41}$$

where μ and ν are real-valued functions. The consistency condition (4.2) implies that these functions satisfy the following equations

$$(T_2\varphi_1)(T_2\psi_1) - \varphi_1\psi_1 = (T_1\varphi_2)(T_1\psi_2) - \varphi_2\psi_2, \tag{4.42}$$

$$(T_2\varphi_1)(T_2\bar{\psi}_1) - \varphi_1\bar{\psi}_1 = (T_1\varphi_2)(T_1\bar{\psi}_2) - \varphi_2\bar{\psi}_2, \tag{4.43}$$

and the non-degeneracy condition implies that

$$\varphi_1\bar{\varphi}_2 - \bar{\varphi}_1\varphi_2 \neq 0, \quad \psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2 \neq 0. \tag{4.44}$$

The above equations imply that there exist unique real-valued functions α_i, β_i, γ_i and δ_i, where $i = 1, 2$, such that the following system of equations is satisfied:

$$\left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}\right]\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{4.45}$$

We can explicitly solve these to obtain

$$\alpha_1 = \frac{\bar{\varphi}_2(T_2\varphi_1) - \varphi_2(T_2\bar{\varphi}_1)}{\varphi_1\bar{\varphi}_2 - \bar{\varphi}_1\varphi_2}, \quad \beta_1 = \frac{\varphi_1(T_2\bar{\varphi}_1) - \bar{\varphi}_1(T_2\varphi_1)}{\varphi_1\bar{\varphi}_2 - \bar{\varphi}_1\varphi_2}, \tag{4.46}$$

$$\gamma_1 = \frac{\bar{\varphi}_2(T_1\varphi_2) - \varphi_2(T_1\bar{\varphi}_2)}{\varphi_1\bar{\varphi}_2 - \bar{\varphi}_1\varphi_2}, \quad \delta_1 = \frac{\varphi_1(T_1\bar{\varphi}_2) - \bar{\varphi}_1(T_1\varphi_2)}{\varphi_1\bar{\varphi}_2 - \bar{\varphi}_1\varphi_2}, \tag{4.47}$$

$$\alpha_2 = \frac{\bar{\psi}_2(T_2\psi_1) - \psi_2(T_2\bar{\psi}_1)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \quad \beta_2 = \frac{\psi_1(T_2\bar{\psi}_1) - \bar{\psi}_1(T_2\psi_1)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \tag{4.48}$$

$$\gamma_2 = \frac{\bar{\psi}_2(T_1\psi_2) - \psi_2(T_1\bar{\psi}_2)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \quad \delta_2 = \frac{\psi_1(T_1\bar{\psi}_2) - \bar{\psi}_1(T_1\psi_2)}{\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2}, \tag{4.49}$$

and a direct calculation using (4.42)-(4.43) shows that

$$\alpha_1\alpha_2 - \gamma_1\gamma_2 = 1, \quad \delta_1\delta_2 - \beta_1\beta_2 = 1, \quad \alpha_1\beta_2 - \gamma_1\delta_2 = 0, \quad \alpha_2\beta_1 - \gamma_2\delta_1 = 0. \tag{4.50}$$

Solving this system we get that $\alpha_2 = \lambda\delta_1$, $\beta_2 = \lambda\gamma_1$, $\gamma_2 = \lambda\beta_1$ and $\delta_2 = \lambda\alpha_1$. A gauge transformation (4.41) acts on λ as follows:

$$\lambda \to (T_2\mu)(T_1\nu)\mu^{-1}\nu^{-1}\lambda. \tag{4.51}$$

hence we can set $\lambda = 1$. Therefore, the functions φ_i and ψ_i satisfy the system (4.35). This proves the proposition.

4.5 The continuous limit

In this section we show that in the continuous limit, the reductions (4.5), (4.18), (4.35) of the Dirac operator (3.12) converge to their continuous counterparts (2.13), (2.15) and (2.17).

First, consider the operator (4.5). Let h denote the size of the mesh, so that

$$\psi_1(x,y+h) = \alpha(x,y)\psi_1(x,y) + \beta(x,y)\psi_2(x,y), \tag{4.52}$$

$$\psi_2(x+h,y) = \beta(x,y)\psi_1(x,y) + \alpha(x,y)\psi_2(x,y), \tag{4.53}$$

where α and β are real and $\alpha^2 - \beta^2 = 1$. Setting $\beta = hp$, we get that $\alpha = 1 + O(h^2)$, and expanding the above equation up to $O(h^2)$ gives us

$$\psi_1 + h\partial_y\psi_1 = \psi_1 + hp\psi_2 + O(h^2), \quad \psi_2 + h\partial_x\psi_2 = hp\psi_1 + \psi_2 + O(h^2), \tag{4.54}$$

so in the limit $h \to 0$ we get equation (2.12).

Similarly, for the operator (4.18) introducing mesh size h we see

$$\varphi_1(x,y+h) = \alpha(x,y)\varphi_1(x,y) + \beta(x,y)\varphi_2(x,y), \tag{4.55}$$

$$\varphi_2(x+h,y) = \bar{\beta}(x,y)\varphi_1(x,y) + \bar{\alpha}(x,y)\varphi_2(x,y), \tag{4.56}$$

$$\psi_1(x,y+h) = \alpha(x,y)\psi_1(x,y) + \beta(x,y)\psi_2(x,y), \tag{4.57}$$

$$\psi_2(x+h,y) = \bar{\beta}(x,y)\psi_1(x,y) + \bar{\alpha}(x,y)\psi_2(x,y), \tag{4.58}$$

where $|\alpha|^2 - |\beta|^2 = 1$. Again, setting $\beta = hp$ gives is $\alpha = 1 + O(h^2)$, and expanding the above equation up to $O(h^2)$ gives us

$$\varphi_1 + h\partial_y\varphi_1 = \varphi_1 + hp\varphi_2 + O(h^2), \quad \varphi_2 + h\partial_x\varphi_2 = h\bar{p}\varphi_1 + \varphi_2 + O(h^2), \tag{4.59}$$

$$\psi_1 + h\partial_y\psi_1 = \psi_1 + hp\psi_2 + O(h^2), \quad \psi_2 + h\partial_x\psi_2 = h\bar{p}\psi_1 + \psi_2 + O(h^2), \tag{4.60}$$

so in the limit $h \to 0$ we get equation (2.14).

For the operator (4.35), we introduce a mesh size h to get

$$\varphi_1(x,y+h) = \alpha(x,y)\varphi_1(x,y) + \beta(x,y)\varphi_2(x,y), \tag{4.61}$$

$$\varphi_2(x+h,y) = \gamma(x,y)\varphi_1(x,y) + \delta(x,y)\varphi_2(x,y), \tag{4.62}$$

$$\psi_1(x,y+h) = \delta(x,y)\psi_1(x,y) + \gamma(x,y)\psi_2(x,y), \tag{4.63}$$

$$\psi_2(x+h,y) = \beta(x,y)\psi_2(x,y) + \alpha(x,y)\psi_2(x,y). \tag{4.64}$$

We now use the remaining gauge symmetry (4.41) to set $\alpha = \delta$. Therefore, if we have a mesh size of h, then setting $\beta = hp$, $\gamma = hq$, we see that $\alpha = 1 + O(h^2)$ and $\delta = 1 + O(h^2)$, so expanding the above equation up to $O(h^2)$ gives us

$$\varphi_1 + h\partial_y\varphi_1 = \varphi_1 + hp\varphi_2 + O(h^2), \quad \varphi_2 + h\partial_x\varphi_2 = hq\varphi_1 + \varphi_2 + O(h^2), \tag{4.65}$$

$$\psi_1 + h\partial_y\psi_1 = \psi_1 + hq\psi_2 + O(h^2), \quad \psi_2 + h\partial_x\psi_2 = hp\psi_1 + \psi_2 + O(h^2), \tag{4.66}$$

so in the limit $h \to 0$ we get equation (2.16).

Chapter 5

Conclusions

In this thesis we considered the problem of constructing an integrable discretization of the Dirac operator (2.7) and a discrete analogue of the Weierstrass representation. We saw that the generalized finite-gap Dirac operator (2.11) admits a natural discretization (3.12), which can be constructed by deforming the spectral properties of the eigenfunctions of (2.11). By introducing additional symmetries on the spectral curve and appropriate normalization conditions, we obtained the reduction (3.15) and constructed a discrete analogue of the modified Novikov–Veselov hierarchy (3.77).

The continuous limit of the reduced discrete Dirac operator (3.15) that we obtained is a hyperbolic differential operator (2.12), while the original Dirac operator (2.7) is elliptic. Therefore, the operator (3.15) cannot be used to discretize the classical Weierstrass representation. However, we saw that it can be used to construct a discrete analogue of the generalized Weierstrass representation in $\mathbb{R}^{2,1}$. Similar operators (4.18), (4.35) can be used to give discrete Weierstrass representations in $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$.

It remains an open problem to construct an integrable discretization of the original Dirac operator (2.7), which is an elliptic reduction of the generalized Dirac operator (2.11). In general, elliptic operators seem to have more complicated discretizations than hyperbolic ones, for example, the discretization of the Laplace operator involves a five-point scheme [6] defined on a sublattice of a four-point discretization of the Moutard equation. It also remains an open problem to construct the discrete analogue of the notion of a conformal coordinate system and the corresponding discrete analogue of the classical Weierstrass representation.

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