On the twisted Floer homology of mapping tori of periodic diffeomorphisms

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ABSTRACT

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Let $K \subset Y$ be a knot in a three manifold which admits a longitude-framed surgery such that the surgered manifold has first Betti number greater than that of Y. We find a formula which computes the twisted Floer homology of the surgered manifold, in terms of twisted knot Floer homology. Using this, we compute the twisted Heegaard Floer homology $\underline{HF}^+$ of the mapping torus of a diffeomorphism of a closed Riemann surface whose mapping class is periodic, giving an almost complete description of the structure of these groups. When the surface is of genus at least three and the mapping class is nontrivial, we find in particular that in the "second-to-highest level" of Spin^c structures, this is isomorphic to a free module (over a certain ring) whose rank is equal to the Lefschetz number of the diffeomorphism. After taking a tensor product with $\mathbb{Z}/2\mathbb{Z}$, this agrees precisely with the symplectic Floer homology of the diffeomorphism, as calculated by Gautschi.

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Chapter 1

Introduction

Floer homology is a generic name for a vast array of invariants associated to manifolds equipped with various types of structures. In its original incarnation, it was introduced in the 1980's by Andreas Floer [1, 2, 3, 4] as a means of attacking the Arnol'd conjecture. This conjecture proposed that the number of fixed points of a Hamiltonian symplectomorphism of a symplectic manifold would satisfy an inequality analogous to the classical Morse inequalities for the manifold. Floer realized that what was needed was a type of Morse theory for certain infinite-dimensional loop spaces, and furthermore that this theory could be realized by replacing the idea of gradient flow lines in the loop space with pseudoholomorphic curves in finite-dimensional manifolds associated to the original symplectic manifold itself.

Since then, this idea has grown tremendously, leading to applications useful to problems all over geometry and topology. One of the most interesting offshoots of this idea is Heegaard Floer homology. This is itself a generic name for a suite of invariants for three-manifolds [19], four-manifolds [24], and knots [17]. We briefly describe the idea of the original Heegaard Floer homologies, of three-manifolds. Any closed oriented three-manifold admits a Heegaard decomposition, into two three-dimensional handlebodies glued along their common boundary. These handlebodies can be specified by their attaching circles on the Heegaard surface; if the surface is of genus g, then specifically each handlebody can be specified by g circles in the surface which are homologically linearly independent. If we take the g-fold symmetric product of the surface, then these two g-tuples of circles form two g-dimensional tori in the symmetric product. Equipping everything with suitable structures, it makes sense to talk about the Floer homology of these tori in the manifold. This homology turns out to yield invariants of the original three-manifold.

Of course, if it could only be studied in the above terms, it would be very difficult to use. However, computations in the theory can be understood by examining the curves in the Heegaard surface itself. This leads to the theory having a strong combinatorial flavor, making it much simpler than many of the other Floer homologies, and also simpler than the related gauge-theoretic techniques of Donaldson theory and Seiberg-Witten theory that were previously used to study three- and four-manifolds. As such, progress in Heegaard Floer has been quite rapid. It has, for example, been applied to questions of unknotting numbers [22], lens space surgeries [21], and fiberedness of knots and manifolds [6, 12, 13]; and it has strong connections with Seiberg-Witten theory (which agrees with Heegaard Floer in all cases where both have been computed) and Khovanov homology (see for example [23]).

There are also connections with the symplectic Floer homology that it initially grew out of, but these are perhaps slightly less developed. One of these connections is conjectural. Symplectic Floer homology may be defined for a symplectomorphism of a symplectic surface. The generators of this homology are the fixed points of the symplectomorphism, and the differential counts certain pseudoholomorphic curves connecting the non-degenerate fixed points of the diffeomorphism. This homology is known to depend only on the mapping class of the diffeomorphism. The conjecture, then, is that the Heegaard Floer homology of the mapping torus of the diffeomorphism determines the symplectic Floer homology; more precisely, a certain "level" of the former is conjectured to agree with the latter. This has previously been shown to hold for a large number of mapping classes, and no counter examples have been found.

The main motivating purpose of this thesis is to show a version of this conjecture for periodic diffeomorphisms. For such a diffeomorphism ϕ of a closed Riemann surface Σ of genus g_Σ, we compute the twisted coefficient Heegaard Floer homology of its mapping torus, and compare this with the symplectic Floer homology of ϕ. We find that the "second-to-highest level" of the twisted Floer homology is $\mathbb{Z}^{\Lambda(\phi)}$ tensored with a certain module depending only on the genera of Σ and of the underlying surface of the quotient orbifold of Σ by the action of ϕ, where Λ denotes Lefschetz number. (Here, "nth-to-highest level" means the direct sum of the twisted Floer homologies for those Spin^c structures $\mathfrak{t}$ for which $\langle c_1(\mathfrak{t}), \Sigma\rangle = 2n - 2g_\Sigma$.) This matches with the computation of the symplectic Floer homology of ϕ over $\mathbb{Z}_2$, which is computed in [5] to be $\mathbb{Z}_2^{\Lambda(\phi)}$. In fact, this computation is part of a wider pattern encompassing all the levels of the homology, which is described in Theorem 1.1.

Along the way, we develop a surgery exact triangle for twisted Floer homology, applicable in situations when one of the surgeries raises the Betti number. This is given in Chapter 6, and described briefly below.

1.1 Results

We now would like to state precisely the main results of this thesis. Before doing so, it will be helpful to review some facts about Seifert fibered spaces and the mapping tori we study.

The following is described in [27]. The mapping torus of any periodic, orientation-preserving diffeomorphism of a closed Riemann surface is an orientable Seifert fibered space with orientable base orbifold, of degree 0. The demand that the degree be 0 is equivalent to saying that the space itself has odd first Betti number. In fact, any Seifert fibered space of this type can be realized as a mapping torus for such a diffeomorphism.

Any oriented Seifert fibered space over an orientable base can be realized also as a surgery on a connect sum of knots of the following two types. First, on the Borromean rings, perform 0-surgery on two of the components; then the third component is the Borromean knot $B_1 \subset \#^2 S^1 \times S^2$, and we write B_g for the g-fold connect sum of copies of B_1. Second, on the Hopf link, perform surgery with coefficient $-p/q$ on one component; then the other component is the O-knot $O_{p,q} \subset L(p,q)$. We always assume that $0 < q < p$ and that p and q are relatively prime. Let $K = B_g \#_{\ell=1}^n O_{p_\ell,q_\ell}$ be a knot in $Y = \#^{2g} S^1 \times S^2 \#_{\ell=1}^n L(p_\ell, q_\ell)$. Then if

$$\sum_{\ell=1}^{n} \frac{q_\ell}{p_\ell} \in \mathbb{Z},$$

K admits a longitude λ (unique up to isotopy) such that λ-framed surgery on K yields a manifold $Y_\lambda(K)$ with odd Betti number, which is therefore a mapping torus of the type we are interested in. The base orbifold will have genus g, and the genus g_Σ of the Riemann surface being mapped will be given by

$$g_\Sigma = 1 + d\left(g - 1 + \frac{1}{2}\sum_{\ell=1}^{n}\left(1 - \frac{1}{p_\ell}\right)\right),$$

where d is the order of K in $H_1(Y)$. It is not hard to see that d is the least common multiple of the p_ℓ values.

We can take a Seifert surface for dK in Y, and then cap this off in the obvious manner in $Y_\lambda(K)$ (hereafter denoted as Y_0) to get an element $[\widehat{dS}] \in H_2(Y_0)$. The element $[\widehat{dS}]$ depends on the specific choice of Seifert surface, but all results that make reference to this class are true for all such choices. Thinking of Y_0 as a mapping torus, the fiber Σ is one such choice (but we continue to refer to $[\widehat{dS}]$).

We now explain our main results more precisely. For Y_0 as above, we compute $\underline{HF}^+(Y_0)$, where the underscore denotes totally twisted coefficients (as described in

Section 8 of [18]). First, let us describe $\underline{HF}^+(Y_0)$ crudely, neglecting some of the finer structure (e.g. U-actions, relative gradings).

Let μ be a meridian of K, thought of as an element of $H_1(Y_0)$. If $\mathfrak{t}_0$ is a Spin^c structure on Y_0 for which $c_1(\mathfrak{t}_0)$ goes to an element of $\mathbb{Q}\cdot \mathrm{PD}[\mu]$ in $H^2(Y_0;\mathbb{Q})$, we say that $\mathfrak{t}_0$ is *μ-torsion.* Let $\mu\mathcal{T}_K$ denote the set of μ-torsion elements of Spin$^c(Y_0)$.

For $D, E \in \mathbb{Z}$, let $\mathcal{N}(D,E)$ denote the number of solutions $(i_1,\ldots,i_n)$ to the equation

$$\sum_{\ell=1}^{n} \frac{i_\ell}{p_\ell} = \frac{E}{d} - D - g + 1$$

for which $0 \le i_\ell < p_\ell$ for all ℓ.

The wider pattern alluded to above is given by the following. (To make the statement clearer, we ignore some the structure of the modules; but see Theorem 1.4 below.)

Theorem 1.1. *There are groups $\Omega^g(k)$, which depend only on k and g (and not on Y_0), and which are trivial when $|k| > g-1$, such that the following holds. For $0 \le i \le g_\Sigma - 2$, let*

$$\underline{HF}^+(Y_0,[-i]) = \bigoplus_{\left\{\mathfrak{t}\in\mu\mathcal{T}_K \,|\, \langle c_1(\mathfrak{t}),[\widehat{dS}]\rangle = 2g_\Sigma - 2i - 2\right\}} \underline{HF}^+(Y_0,\mathfrak{t}),$$

where the summands are thought of as ungraded $\mathbb{Z}[H^1(Y_0)]$-modules (and we forget about the U-action). Then we have a short exact sequence

$$\begin{aligned} 0 \;\rightarrow\; \left(\textstyle\bigoplus_k (\Omega^g(k))^{\mathcal{N}(k,i)}\right)\otimes\mathbb{Z}[T,T^{-1}] \;&\rightarrow\; \underline{HF}^+(Y_0,[-i]) \\ &\rightarrow\; \mathbb{Z}^{a_i}\otimes\mathbb{Z}[T,T^{-1}] \;\rightarrow\; 0 \end{aligned}$$

where

$$a_i = \sum_{D\in\mathbb{Z}} \max\{0, D, \lfloor \frac{g+D+1}{2} \rfloor\}\cdot\mathcal{N}(D,i),$$

and where $T \in \mathbb{Z}[H^1(Y_0)]$ represents the Poincaré dual of a fiber of Y_0 thought of as a mapping torus.

All the above holds for the reduced twisted homology $\underline{HF}_{red}$ when $i = g_\Sigma - 1$.

For torsion Spin^c structures, $\underline{HF}^+$ is isomorphic to the direct sum of $\underline{HF}_{red}$ with a summand $\mathcal{T}^+$, described below. Together with conjugation invariance and this observation, Theorem 1.1 describes $\underline{HF}^+$ for all those Spin^c structures where it is non-trivial.

The following Corollary gives the mentioned connection with symplectic Floer homology.

Corollary 1.2. *Let $\phi : \Sigma \to \Sigma$ be a periodic diffeomorphism of a closed Riemann surface, whose mapping class is not trivial, with g_Σ at least 3. Let its mapping torus be Y_0, and set $R = \mathbb{Z}[H^1(Y_0)]$. Then, as R-modules,*

$$\underline{HF}^+(Y_0, [-i]) \cong \begin{cases} R, & i = 0 \\ R^{\Lambda(\phi)}, & i = 1 \end{cases}$$

where Λ denotes Lefschetz number. Furthermore, we have: the U-action is trivial in each; for $i = 1$, each copy of R lies in a different Spin^c structure; and if T represents the Poincaré dual of a fiber in R, then T lowers this relative grading by $2d(g_\Sigma - 1 - i)$.

Informally, when $g_\Sigma < 3$, $\underline{HF}^+(Y_0)$ doesn't have enough levels for the statement of Corollary 1.2 to make sense. When ϕ is isotopic to the identity, both $\underline{HF}^+(Y_0, [-1])$ and the symplectic Floer homology $HF_*(\phi; \mathbb{Z}_2)$ diverge from the Lefschetz number description, and we currently cannot compute the former precisely. However, it is known that in this case, $HF^+(M_\phi, [-1]) \otimes \mathbb{Z}_2 \cong {\mathbb{Z}_2}^{2g+2} \cong HF_*(\phi; \mathbb{Z}_2)$, where the first group is untwisted Heegaard Floer homology (calculated in [17]) and the last is symplectic Floer homology (calculated in [5]).

The arguments used to prove Corollary 1.2 can be extended to lower levels, although the statements get progressively more cumbersome. We give the extension to the third-to-highest level in Theorem 9.3. The result we find agrees with that suggested by the periodic Floer homology developed by Hutchings [7]; it appears that if we were to formulate extensions to lower levels, we would also find the answers that periodic Floer homology would have us expect. There are connections to be made with other theories as well: we may compare the wider structure of Theorem 1.1 with the computations of [10], which tackles the analogous computation for Seiberg-Witten-Floer homology and finds results that have at least some similarity to ours. In yet another direction, in [28] and [29], computations of perturbed Floer homology – that is, Heegaard Floer homology with a special type of twisted coefficients – are carried out for some mapping tori, including $S^1 \times \Sigma_g$.

The groups $\Omega^g(k)$ are given, in Definition 8.4, in terms of the twisted knot Floer homology of B_g. We do not describe the structure of these groups explicitly, leaving them as mystery subgroups. We would like a better description of them, but their presence ends up as only a minor distraction here. In many instances, they don't show up at all: when we look at a mapping torus with first Betti number 1; in Corollary 1.2; and generally, in many relative grading levels of any space of the type we examine.

Now, we describe the full structure of $\underline{HF}^+(Y_0)$ (to the extent that we can), from which we extract Theorem 1.1 and Corollary 1.2. To do this, we have to introduce some machinery.

First, we introduce some notation to keep track of μ-torsion Spin^c structures. Define

$$\widetilde{\mathcal{MT}}_K = \mathbb{Z} \times \bigoplus_{\ell=1}^{n} \mathbb{Z}/p_\ell\mathbb{Z}.$$

We write elements of $\widetilde{\mathcal{MT}}_K$ as pairs $(Q; r_1, \ldots, r_n)$, where r_ℓ is an integer satisfying $0 \leq r_\ell \leq p_\ell - 1$; we usually shorten this and just write elements as $(Q; r)$ (with r understood as denoting an n-tuple), or simply as A. Let

$$S\ell(Q; r) = 2\left(Q - \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}\right) - \sum_{\ell=1}^{n}\left(1 - \frac{1}{p_\ell}\right). \tag{1.1}$$

Define an equivalence relation $\sim$ on $\widetilde{\mathcal{MT}_K}$, by setting $(Q;r) \sim (Q';r')$ if and only if $S\ell(Q;r) = S\ell(Q';r')$ and r and r' descend to the same element of the quotient group $\left(\bigoplus_{\ell=1}^{n} \mathbb{Z}/p_\ell\mathbb{Z}\right)/\mathbb{Z}(q_1 \oplus \ldots \oplus q_n)$. Then, let

$$\mathcal{MT}_K = \widetilde{\mathcal{MT}_K}/\sim;$$

we write equivalence classes as $[A]$. The function $S\ell$ obviously extends to $\mathcal{MT}_K$, as does the function $\epsilon : \widetilde{\mathcal{MT}_K} \to \mathbb{Q}$ given by

$$\epsilon(A) = g_\Sigma - 1 - \frac{d}{2} \cdot S\ell(A).$$

Lemma 1.3. *There is a bijective map*

$$\theta_K : \mathcal{MT}_K \to \mu\mathcal{T}_K,$$

which satisfies

$$S\ell([A]) = \frac{\langle c_1\left(\theta_K([A])\right), [\widehat{dS}]\rangle}{d} \tag{1.2}$$

for $[A] \in \mathcal{MT}_K$.

Next, to describe our answers neatly, we use (a slightly altered version of) the concept of wells introduced in [21], based on ideas in [11] and [16]. In our version, for a function $H : \frac{1}{2}\mathbb{Z} \to \mathbb{Z}$ and an **odd** integer n, we define a *well at height n for H* to be a pair of integers (i,j), $i \leq j$, which satisfy $n \geq H(s)$ for $i \leq s \leq j$, $\max\{H(i-1), H(i-\frac{1}{2})\} > n$, and $\max\{H(j+1), H(j+\frac{1}{2})\} > n$; essentially, wells correspond to maximally distant pairs of integers between which the graph of H runs at or below hieight n, hence the name wells. We also write such a pair as $(i,j)_n$ to denote the height of the well. Let $W_n(H)$ be the set of wells at height n for H; and let $\mathbb{W}_n(H)$ be the free abelian group generated by $W_n(H)$. Then we define

$$\mathbb{W}_*(H) = \bigoplus_{l \in \mathbb{Z}} \mathbb{W}_{2l+1}(H).$$

We write $(i', j')_{n-2} < (i, j)_n$ if $(i', j')_{n-2} \in W_{n-2}(H)$ and $i \leq i' < j' \leq j$. Then we can define an action of U on $\mathbb{W}_*(H)$ by

$$U \cdot (i, j)_n = \sum_{\{w \in W_{n-2}(H) | w < (i,j)_n\}} w,$$

and extending linearly. We endow $\mathbb{W}_*(H)$ with the $\mathbb{Z}$-grading given by height.

The well functions we need are given as follows. For $A \in \widetilde{\mathcal{MT}}_K$, we define a function

$$\eta_A(x) = \sum_{\ell=1}^{n} \left\{ \frac{q_\ell x - r_\ell}{p_\ell} \right\} + \frac{1}{d}\epsilon(A) - \left(g - 1 + \sum_{\ell=1}^{n} (1 - \frac{1}{p_\ell}) \right),$$

where the curly braces denote fractional part, $\{x\} = x - \lfloor x \rfloor$. We then define a function $G_A : \frac{1}{2}\mathbb{Z} \to \mathbb{Z}$ by

$$G_A(0) = 1,$$

$$G_A(x + 1) = G_A(x) - 2\eta_A(x) \text{ for } x \in \mathbb{Z},$$

$$G_A \left(x + \frac{1}{2} \right) = g + \frac{1}{2} \left(G_A(x) + G_A(x + 1) \right) \text{ for } x \in \mathbb{Z}.$$

(Unwrapping definitions, G_A is given by

$$G_A(x) = 1 + 2Qx - 2 \sum_{i=1}^{x} \sum_{\ell=1}^{n} \left(\left\{ \frac{q_\ell i - r_\ell}{p_\ell} \right\} + \frac{r_\ell}{p_\ell} \right)$$

for positive integers x.) The function G_A, very roughly, describes relative gradings of elements in a certain set of subcomplexes of $\underline{CF}^+(Y_0, \mathfrak{t}_0)$, this set being parametrized by x.

Recall that $\mathcal{T}^+_{(\ell)}$ denotes the $\mathbb{Z}[U]$-module $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$, equipped with a $\mathbb{Z}$-

grading so that U^{-i} lies in level $\ell + 2i$ for $i \geq 0$.

Theorem 1.4. *For $A \in \widetilde{\mathcal{MT}}_K$, let*

$$B_\ell(G_A) \cong \bigoplus_{\{i \in \mathbb{Z} | G_A(i+\frac{1}{2}) = \ell+1\}} \Omega^g\left(G_A(i) - G_A(i+\frac{1}{2}) + g\right),$$

and let $B_*(G_A) = \bigoplus_{\ell \in \mathbb{Z}} B_\ell(G_A)$ (where $\Omega^g(k)$ is the group from the statement of Theorem 1.1). Equip this group with trivial U-action. Let $b_A \in 2\mathbb{Z} \cup \{\infty\}$ be the least even upper bound of the function $G_A - 1$.

Then, if $\mathfrak{t}_0 = \theta_K([A])$ is μ-torsion, the relative $\mathbb{Z}$-grading on $\underline{HF}^+(Y_0, \mathfrak{t}_0)$ lifts to an absolute $\mathbb{Z}$-grading, such that there are short exact seqeunces of graded $\mathbb{Z}[H^1(Y_0)] \otimes \mathbb{Z}[U]$-modules

$$0 \to B_*(G_A) \oplus \mathcal{T}^+_{(b_A)} \to \underline{HF}^+(Y_0, \mathfrak{t}_0) \to \mathbb{W}_*(G_A) \to 0$$

if $b_A \neq \infty$ and

$$0 \to B_*(G_A) \to \underline{HF}^+(Y_0, \mathfrak{t}_0) \to \mathbb{W}_*(G_A) \to 0$$

if $b_A = \infty$; and otherwise $\underline{HF}^+(Y_0, \mathfrak{t}_0)$ is trivial. Furthermore, $b_A \neq \infty$ precisely when $\theta_K([A])$ is torsion.

The function G_A is the sum of a periodic function with a linear one. If we think of Y_0 as a mapping torus, and $T \in \mathbb{Z}[H^1(Y_0)]$ represents the Poincaré dual of a fiber, then T acts on $\mathbb{W}_*(G_A)$ by moving a well to the corresponding one a period to the right.

The short exact sequences above are not necessarily split over $\mathbb{Z}[H^1(Y_0)] \otimes \mathbb{Z}[U]$, but we have the following.

Corollary 1.5. *If Y_0 is the mapping torus of a periodic diffeomorphism, then $\underline{HF}^+(Y_0)$ contains no non-trivial elements of finite order (that is, it contains no torsion as an abelian group). Hence, the short exact sequences of Theorem 1.4 are split over $\mathbb{Z}[U]$; in particular, $\underline{HF}^+(S^1 \times \Sigma_g)$ contains no non-trivial elements of finite order.*

This corollary is also to be compared with [8], where it is found that $HF^+(S^1 \times \Sigma_g; \mathbb{Z})$ contains torsion for large enough g.

Together with the results of [21], Theorem 1.4 in a sense completes the calculation of the Heegaard Floer homology of Seifert-fibered spaces. The qualification comes from two sources: the fact that we don't describe the subgroups $\Omega^g(k)$ explicitly, and the fact that by "the calculation of Heegaard Floer homology", we mean the calculation of one of HF^+ or $\underline{HF}^+$.

Theorem 1.4 is shown using a twisted surgery formula, akin to the formulas of [25] and [21]. Before outlining the formula, we say a brief word about our use of twisted coefficients. Both our formula and those of [25] and [21] come about by relating knot Floer homology of a knot with the cobordism maps induced by attaching a two-handle along the knot, eventually arriving at the Floer homology of a three-manifold obtained by surgery along the knot. The twisted coefficient setting is useful for sorting out how the cobordism-induced maps break down into summands for each Spin^c structure on the cobordism, especially when we have surgeries that raise the first Betti number, as we encounter here. Indeed, an untwisted version of the formula we use, if it exists, would likely be much less user-friendly.

We now give a quick description of the formula, which computes $\underline{HF}^+(Y_\lambda(K), \mathfrak{t}_0)$ when λ is a longitude of K such that $b_1(Y_\lambda(K)) = b_1(Y) + 1$, and $\mathfrak{t}_0$ is μ-torsion with respect to the meridian μ of K (i.e., $c_1(\mathfrak{t}_0)$ goes to an element of $\mathbb{Q}\cdot\mathrm{PD}[\mu]$ in $H^2(Y_\lambda(K); \mathbb{Q})$). Consider the two handle cobordism W_0 obtained by attaching a λ-framed 2-handle to K. Let $\mathfrak{t}^0_\infty \in \mathrm{Spin}^c(Y)$ be some Spin^c structure cobordant to $\mathfrak{t}_0$, and let $\mathfrak{t}^i_\infty$ be $\mathfrak{t}^0_\infty - i\mathrm{PD}[K]$ for $i \in \mathbb{Z}$. If d is the order of K in $H_1(Y)$, then $\mathfrak{t}^{i+d}_\infty$ will be the same as $\mathfrak{t}^i_\infty$, but we nonetheless treat them as distinct in what follows.

We recall that there is a notion of relative Spin^c structures on (Y, K), which we denote by $\underline{\mathrm{Spin}}^c(Y,K)$; there is a natural map

$$G_K : \underline{\mathrm{Spin}}^c(Y,K) \to \mathrm{Spin}^c(Y),$$

whose fibers are the orbits of a $\mathbb{Z} \cdot \mathrm{PD}[\mu]$ action on $\underline{\mathrm{Spin}}^c(Y,K)$. To each relative Spin^c structure ξ we may associate the group $CFK^\infty(Y,K,\xi)$, which is generated over $\mathbb{Z}$ by elements of the form $[\mathbf{x},i,j]$, where $\mathbf{x}$ is a generator of $\widehat{CF}(Y,G_K(\xi))$ and i,j are integers, required to satisfy a certain condition which depends on $\mathbf{x}$. Then, as a group, $\underline{CFK}\{i \geq 0 \text{ or } j \geq 0\}(Y,K,\xi)$ is generated by elements of the form $[\mathbf{x},i,j] \otimes r$, where $[\mathbf{x},i,j]$ is one of the generators of $CFK^\infty(Y,K,\xi)$ for which $i \geq 0$ or $j \geq 0$, and r is an element of the group ring $\mathbb{Z}[H^1(Y)]$. The differential is analogous to that used in ordinary twisted Floer homology. We give more detail in Chapter 5.

There is a chain map

$$v_\xi : \underline{CFK}\{i \geq 0 \text{ or } j \geq 0\}(Y,K,\xi) \to \underline{CF}^+\big(Y, G_K(\xi)\big),$$

which simply takes the generator $[\mathbf{x},i,j] \otimes r$ to $[\mathbf{x},i] \otimes r$. There is also a map

$$h_\xi : \underline{CFK}\{i \geq 0 \text{ or } j \geq 0\}(Y,K,\xi) \to \underline{CF}^+\big(Y, G_K(\xi) - \mathrm{PD}[K]\big),$$

which takes the generator $[\mathbf{x},i,j] \otimes r$ to $[\mathbf{x},j] \otimes r$, the latter now belonging to the same Heegaard diagram but with different basepoint (and hence representing a different Spin^c structure).

Theorem 1.6. *Let K, λ, and $\mathfrak{t}_0$ be as above. There are elements $\xi_i \in \underline{Spin}^c(Y,K)$ for each $i \in \mathbb{Z}$ such that $G_K(\xi_i) = \mathfrak{t}^i_\infty$, $\xi_{i+d} = \xi_i$, and so that the following holds. Let*

$$\underline{f}^+_{K,\mathfrak{t}_0} : \bigoplus_{i\in\mathbb{Z}} \underline{CFK}\{i \geq 0 \text{ or } j \geq 0\}(Y,K,\xi_i) \to \bigoplus_{i\in\mathbb{Z}} \underline{CF}^+\big(Y,\mathfrak{t}_i\big)$$

be the map given by the direct sum of the maps v_{ξ_i} *and* h_{ξ_i} *over all* i*, where each* ξ_i *and* $\mathfrak{t}^i_\infty$ *is treated as distinct, and* v_{ξ_i} *and* h_{ξ_i} *are considered to take summand* i *on the left to summands* i *and* $i+1$ *on the right, respectively. Then, there is a quasi-isomorphism from* $M(\underline{f}^+_{K,\mathfrak{t}_0})$ *to* $\underline{CF}^+(Y_0, \mathfrak{t}_0)$*, where* M *denotes the mapping cone. Furthermore,* $M(\underline{f}^+_{K,\mathfrak{t}_0})$ *admits a relative* $\mathbb{Z}$*-grading and a* U*-action which the quasi-isomorphism respects.*

We give a more precise version of the formula in Theorem 6.1. In particular, we will describe the structures ξ_i precisely. We believe that the details of [21] can be mimicked along the lines of this paper, to get a twisted surgery formula that applies when λ is an arbitrary longitude, but we don't undertake this herein.

At least at the level of concept, twisted knot Floer homology is a more or less straightforward combination of twisted Floer homology and untwisted knot Floer homology. For this reason, twisted knot Floer homology has already been used in, for example, [9] and [14], despite having had only the barest of definitions written down (as far as the author can tell). As we rely on this more extensively, and for fairly delicate computations leading to Theorem 1.4, we give a slightly fuller treatment here.

We conclude the introduction by giving an outline of the organization of this thesis. In Chapter 2, we introduce special Heegaard diagrams that we use throughout, and treat relative Spin^c structures. We put some of the more tedious results of this section in the Appendix. In Chapter 3, we make some observations about certain triangles in our diagrams, and the Spin^c structures they represent. In Chapter 4, we prove a twisted coefficient long exact sequence. In Chapter 5, we introduce twisted knot Floer homology, and give analogues of the large N surgeries formula and the Künneth formula. In Chapter 6, we state and prove the twisted surgery formula. In Chapter 7, we make basic computations for the Borromean knots and O-knots, reducing the work for Theorem 1.4 to algebra, which is then carried out in Chapter 8. We prove Theorem 1.1 and Corollary 1.2 in Chapter 9. Examples are presented in Chapter 10.

Chapter 2

Standard Heegaard Diagrams and Relative Spin^c Structures

We introduce a special type of Heegaard diagram associated to a knot in a three manifold. We make reference to these diagrams when we develop twisted knot Floer homology, and when we prove the long exact sequence of Chapter 4. In the present section, we also use these to solidify a connection between relative Spin^c structures on a knot complement and Spin^c structures on a cobordism gotten by attaching a two-handle to the knot.

Throughout this paper, all knots we consider are oriented rationally nullhomologous knots $K \subset Y$, always implicitly equipped with a distinguished longitude λ. When K and λ are understood, we henceforth write Y_0 for $Y_\lambda(K)$, and more generally Y_N for $Y_{N\mu+\lambda}(K)$.

Much of what we say will be applicable to all such knots, but the main results concern knots of the following type.

Definition 2.1. *If an oriented rationally nullhomologous knot $K \subset Y$ admits a longitude λ satisfying $b_1\big(Y_\lambda(K)\big) = b_1(Y)+1$, then we call the knot (and the distinguished longitude)* special.

The reason that we tend to not explicitly mention longitudes lies in the fact that any knot admits at most one special longitude, so that for special knots the choice of longitude is canonical.

2.1 Standard Heegaard diagrams and cobordisms

For an (oriented rationally nullhomologous) knot $K \subset Y$ and a positive integer N, we can form a weakly admissible, doubly pointed Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w, z)$ on a genus g Riemann surface Σ such that $Y_{\alpha\beta} = Y_0$, $Y_{\alpha\gamma} = Y_N$, and $Y_{\alpha\delta} = Y$, and such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z)$ is a Heegaard diagram for (Y, K). We restrict attention to certain diagrams.

Definition 2.2. *Suppose that we have a weakly admissible, doubly pointed Heegaard quadruple as above, for which the following holds. A portion of the diagram can be drawn as in Figure 2.1, which takes place in a punctured torus. Also, the vertically-drawn curve δ_g is a meridian for K; the curve β_g, oriented as shown, is a special longitude for K; the points w and z flank δ_g as shown, with w on the left. Finally, the β, γ and δ curves not shown are all small isotopic translates of each other. Then, we call such a diagram* standard. *We will often forget parts of the data (e.g., we consider a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w)$, ignoring $\boldsymbol{\beta}$ and z); we will also refer to such diagrams as standard.*

It is not hard to see that for any oriented knot in a three manifold and any N, there exist standard diagrams.

We have usual "highest" intersection points $\Theta_{\beta\delta} \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ and $\Theta_{\gamma\delta} \in \mathbb{T}_\gamma \cap \mathbb{T}_\delta$, which have components as shown in Figure 2.1. We also choose a point $w_{\beta\gamma}$ on $\beta_g \cap \gamma_g$, which determines a "highest" intersection point $\Theta_{\beta\gamma}$ in $\widehat{CF}(Y_{\beta\gamma}) = \widehat{CF}(L(N,1)\#^{g-1}S^1 \times S^2)$, as follows. $L(N,1)$ is realized as the boundary of a disk bundle V over a sphere with Euler number N, and there is a unique Spin^c structure on $L(N,1)$ that extends to some $\mathfrak{s}$ on V such that $\langle c_1(\mathfrak{s}), V \rangle = N$. We choose the point $w_{\beta\gamma}$ so that $\mathfrak{s}_w(w_{\beta\gamma})$ is the unique

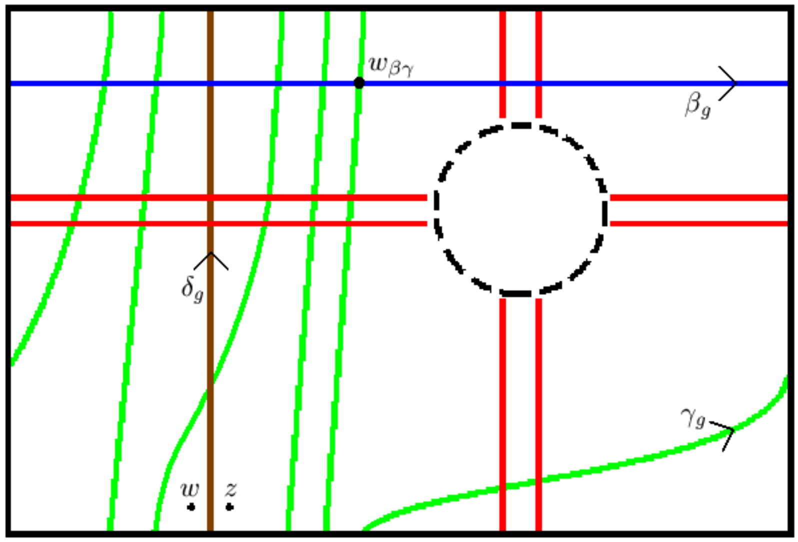

Figure 2.1: Local picture of a standard diagram inside a punctured torus region, where the sides of the rectangle are identified in the usual way, and the dashed circle represents the puncture. The red curves are strands of α circles. We supply β_g, γ_g and δ_g with orientations for later reference. The winding region is the portion of the above picture that lies to the left of the puncture.

Spin^c structure that is torsion and restricts to this Spin^c structure on $L(N,1)$.

For a standard diagram, we will also fix iterated small isotopies $\boldsymbol{\beta}_{(i)}$, $\boldsymbol{\gamma}_{(i)}$, and $\boldsymbol{\delta}_{(i)}$ for positive integers i, which will play a role in the proof of the long exact sequence. Thinking of $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ as $\boldsymbol{\beta}_{(0)}$, $\boldsymbol{\gamma}_{(0)}$, and $\boldsymbol{\delta}_{(0)}$, we will sometimes write $\boldsymbol{\eta}^0 = \boldsymbol{\alpha}$, $\boldsymbol{\eta}^{3i+1} = \boldsymbol{\beta}_{(i)}$, $\boldsymbol{\eta}^{3i+2} = \boldsymbol{\gamma}_{(i)}$, and $\boldsymbol{\eta}^{3i+3} = \boldsymbol{\delta}_{(i)}$. Likewise, we will have points $\Theta_{\eta^i\eta^j}$ for all $j > i > 0$, defined in the obvious manner.

The Figure also shows a natural partition of $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ into points that are *supported in the winding region* – those for which the point on γ_g is on one of the horizontal α strands – and those that are not. The *winding region* on γ_g itself is the portion of that circle that is visible in the Figure, running between the leftmost and rightmost intersections with the horizontal α strands. For $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$, we say a triangle in $\pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{y})$ is *small* if its boundary has component on γ_g contained entirely in the winding region.

Each point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ that is supported in the winding region has a unique *nearest* point $\mathbf{y}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$, for which there exists a small triangle in $\pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{y})$. We also say that $\mathbf{x}$ is a nearest point of $\mathbf{y}$.

Let W_N be the cobordism gotten by attaching a 2-handle to Y along the longitude $N\mu + \lambda$, with boundary $-Y \coprod Y_N$. Note that $H_2(Y_N)$ and $H_2(Y)$ inject into $H_2(W_N)$; furthermore, if $N\mu + \lambda$ is a longitude that is *not* special, they in fact have the same image in $H_2(W_N)$, since the inclusion of $Y \setminus K$ into Y and Y_N induces isomorphisms on second homology.

We write W'_N for $-W_N$, thought of as a cobordism from Y_N to Y. Then a standard diagram for Y, K yields a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w)$ for W'_N. The above observation then has consequences for the periodic domains appearing in a standard diagram, which are identified with elements of second homology (with respect to the basepoint w). Indeed, each $\boldsymbol{\alpha\gamma}-$periodic domains (that is, domains whose boundary is composed of an integral linear combination of entire α_i and γ_i circles) will correspond to a $\boldsymbol{\alpha\delta}-$periodic domain; the latter will have the same boundary as the former, except with γ circles replaced by the corresponding δ translates. In particular, none of the $\boldsymbol{\alpha\gamma}-$ or $\boldsymbol{\alpha\delta}-$periodic domains will have boundary including a nonzero multiple of γ_g or δ_g.

Likewise, there will also be $\boldsymbol{\alpha\beta}-$periodic domains corresponding to the $\boldsymbol{\alpha\gamma}-$periodic domains (again gotten by comparing boundaries). However, for a special longitude, we will need to add one extra $\boldsymbol{\alpha\beta}-$periodic domain to generate the set of all $\boldsymbol{\alpha\beta}-$periodic domains.

We can of course say corresponding things about the periodic domains between the α and η^i circles.

2.2 Relative Spin^c structures

There are a couple of slightly different (but equivalent) descriptions of relative Spin^c structures on three-manifolds with torus boundary. We review the description we use.

There is a unique homotopy class of nonwhere vanishing vector fields on the torus such that if we take a covering map of $\mathbb{R}^2$ to the torus, any representative vector field lifts to one that is homotopic through nowhere vanishing vector fields to a constant vector field (identifying the tangent space of a point on $\mathbb{R}^2$ with $\mathbb{R}^2$ itself). We refer to this as the canonical vector field on the torus. Given any Dehn filling of the torus, the canonical vector field extends to a nonwhere vanishing vector field on the filling.

Given an oriented rationally nullhomologous knot K in a three-manifold Y, let $\underline{\mathrm{Spin}}^c(Y, K)$ be the set of homology classes of nowhere-vanishing vector fields on $Y \setminus N(K)$ that agree with the canonical vector field on the boundary (and in particular point along the boundary), where $N(K)$ is a regular neighborhood of K; we refer to these as *relative* Spin^c *structures*. Here, two vector fields are homologous if they are homotopic through non-vanishing vector fields in the complement of a ball in the interior of $Y \setminus N(K)$. The set $\underline{\mathrm{Spin}}^c(Y, K)$ is an affine space for $H^2(Y, K)$, in the same way that absolute Spin^c structures form an affine space for $H^2(Y)$.

If we take the two-plane field $\vec{v}^\perp$ of vectors orthogonal to a field $\vec{v}$ which represents $\xi \in \underline{\mathrm{Spin}}^c(Y, K)$, we have a well-defined global section of $\vec{v}^\perp$ along the boundary given by unit normal vectors pointing outwards, and hence a trivialization τ of $\vec{v}^\perp$ along the boundary; this then gives a well-defined relative first Chern class $c_1(\xi) = c_1(\vec{v}^\perp, \tau) \in H^2(Y, K)$.

There is a natural projection map $G_K : \underline{\mathrm{Spin}}^c(Y, K) \to \mathrm{Spin}^c(Y)$, given as follows. Think of D^2 as the unit circle in the complex plane, and view $N(K)$ as $S^1 \times D^2$, where K is $S^1 \times \{0\}$, with direction of increasing angle agreeing with the orientation of K. Also, for $S \subset [0, 1]$, let D_S be the set $\{z \in D^2 | |z| \in S\}$. Extend the vector field over

$S^1 \times D_{[1/2,1]}$ so that the vector field points inward nowhere and so that it points in the positive direction on the S^1 factor on $S^1 \times D_{\{1/2\}}$. Then, extend over the rest of $N(K)$ so that the vector field is transverse to the D^2 factor on $S^1 \times D_{[0,1/2]}$, and so that on $K \approx S^1 \times \{0\}$ the vector field traces out a closed orbit whose orientation agrees with that of K.

The projection is $H^2(Y,K)$-equivariant with respect to the natural map j^* from $H^2(Y,K)$ to $H^2(Y)$. In [20], it is shown that for $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(Y,K)$, we have

$$c_1\big(G_K(\underline{\mathfrak{s}})\big) = j^*\big(c_1(\underline{\mathfrak{s}})\big) + \mathrm{PD}[K]. \tag{2.1}$$

The fibers of G_K are the orbits of the $\mathbb{Z}\cdot\mathrm{PD}[\mu]$ action on $\underline{\mathrm{Spin}}^c(Y,K)$, where $\mathrm{PD}[\mu]$ is the class in $H^2(Y,K)$ corresponding to the oriented meridian. Also, just like in the absolute case, we have that for $x \in H^2(Y,K)$

$$c_1(\xi + x) = c_1(\xi) + 2x.$$

2.3 Intersection points and relative Spin^c structures

Consider a doubly-pointed Heegard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z)$ associated to an oriented knot K, which is part of a standard diagram. Recall the conventions for this. In Σ, we draw two arcs connecting w and z: a small one η_α crossing δ_g once and none of the other α or δ circles, and another one η_δ which does not cross any of the δ circles. The former can be pushed into the α-handlebody, and the latter can be pushed into the δ-handlebody. The union of these two arcs (which intersect at the common boundary of the handlebodies) should then give K, and the orientation on K should be such that η_α goes from w to z. In particular, if we push a small segment of δ_g near the basepoints into the α-handlebody, we should have a meridian for K.

Remark We can draw an honest oriented meridian in Σ by taking a small counterclockwise circle around z. Confusingly, this appears to be at odds with the fact that, say, $\lambda + 2\mu$ surgery is given by drawing γ_g with slope $+2$ in a standard diagram, rather than slope -2. This is because when we draw our Heegaard diagram for the surgered manifold, we pretend that the knot passes through the handle attached by δ_g, even though according to the above discussion it does not. If one is bothered by this discrepancy, then he or she can think of the results contained herein as being the result of comparing a doubly-pointed Heegaard 2-tuple with a singly-pointed Heegaard triple with an extra "reference point" that just happens to look very similar.

Analogously to the constructions used for the three-manifold invariants, we can assign a relative Spin^c structure $\underline{\mathfrak{s}}_{w,z}(\mathbf{x})$ to a point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$. Specifically, we take a Morse function compatible with the pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w)$, and take its gradient vector field; we then remove the portions over neighborhoods of the flowlines corresponding to w and to the components of $\mathbf{x}$, replacing them with non-vanishing ones. We replace the field over the neighborhood of the flowline for w as depicted in Figure 2.3; together with the unaltered portion of the flowline for z, this will give a periodic orbit of the vector field which should coincide with the oriented knot K. After a homotopy, we can assume that there is a tubular neighborhood of K over which the vector field is of the form described on $S^1 \times D^2$ in the construction of the projection map G_K. Then $\underline{\mathfrak{s}}_{w,z}(\mathbf{x})$ is the homology class of this vector field restricted to the complement of $S^1 \times D_{[0,1/2]}$ as in that construction.

As in the absolute case, we have a straightforward way to calculate $\underline{\mathfrak{s}}_{w,z}(\mathbf{x}_2) - \underline{\mathfrak{s}}_{w,z}(\mathbf{x}_1)$ for two intersection points $\mathbf{x}_1, \mathbf{x}_2$. We take arcs from $\mathbf{x}_1$ to $\mathbf{x}_2$ along the $\boldsymbol{\alpha}$ circles, and from $\mathbf{x}_2$ to $\mathbf{x}_1$ along the $\boldsymbol{\delta}$ circles, yielding a closed path in Σ and hence a homology class in $H_1(\Sigma \setminus \{w, z\})$. The difference will then be the image of this homology class in $H_1(Y \setminus K)$, whose Poincaré dual lives in $H^2\big(Y \setminus K, \partial(Y \setminus K)\big)$. Then $\underline{\mathfrak{s}}_{w,z}(\mathbf{x}_2) - \underline{\mathfrak{s}}_{w,z}(\mathbf{x}_1)$

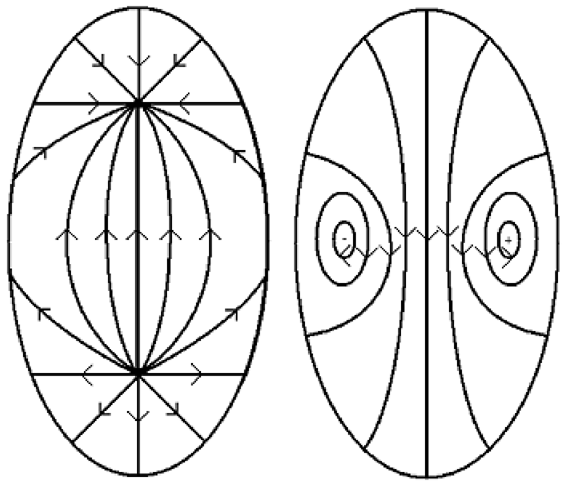

Figure 2.2: The left side (when rotated about a vertical axis) shows the original vector field in a neighborhood of the flowline from the index 0 critical point to the index 3 critical point; the right shows what we replace it with. At the points marked + and − on the right, the vector field points out of and into the page, respectively.

is this Poincaré dual, thought of as an element of $H^2(Y, K)$. From this, it is shown in [21] that if there is a disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, then

$$\underline{\mathfrak{s}}_{w,z}(\mathbf{y}) - \underline{\mathfrak{s}}_{w,z}(\mathbf{x}) = \big(n_z(\phi) - n_w(\phi)\big) \cdot \mathrm{PD}[\mu]. \tag{2.2}$$

We also want to be able to evaluate the Chern class of the relative Spin^c structure of an intersection point on an arbitrary class in $H_2\big(Y \setminus N(K), \partial(Y \setminus N(K))\big)$, and so we will construct geometric representatives of elements of this group.

To start, consider a standard Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w, z)$ for W'_N for any value of N. To a $\boldsymbol{\alpha\gamma\delta}$-periodic domain $\mathcal{P}$ such that $\partial\mathcal{P}$ has no components in $\boldsymbol{\gamma} \setminus \{\gamma_g\}$, the usual construction of representives of $H_2(W'_N)$ yields an orientation-preserving map $\Phi : S \to W'_N$ (for some oriented surface S with boundary) of the following form. There is a disjoint union of disks $D \subset S$, such that Φ maps each component of D diffeomorphically to the core of the 2-handle attached to Y to form W'_N, each component of ∂D into γ_g, and $S \setminus D$ into Y.

Choose open balls B_w and B_z around w and z in Σ, such that γ_g is tangent to ∂B_w and ∂B_z at precisely one point each. Then, we may choose our regular neighborhood $N(K)$ carefully so that B_w and B_z are the intersections of $N(K)$ with Σ, and so that $\gamma_g \subset \Sigma$ lies in $\partial N(K)$. (Roughly, the portion of γ_g lying between the two tangencies is thought of as being on the top of a tunnel in the α-handlebody, and the rest is thought of as being on the bottom of a tunnel in the β-handlebody.) So, let $D' \subset S$ be D union with $\Phi^{-1}\big(\{B_w \cup B_z\}\big)$; then, we think of $\mathcal{P}$ as corresponding to the map $\Phi_0 : (S \setminus D', \partial(S \setminus D')) \to \big(Y \setminus N(K), \partial(Y \setminus N(K))\big)$ defined by restricting Φ to $S \setminus D'$, which gives the desired homology class.

It is easy to see that all classes in $H_2\big(Y \setminus N(K), \partial(Y \setminus N(K))\big)$ can be represented by triply-periodic domains via this construction; and triply periodic domains in turn are in correspondence with classes of $H_2(W'_N)$. We can make this identification more explicit as follows. Write the handle attached to form W_N as $D^2 \times D^2$, where the first factor is the core of the handle. Let U be the $B \times D^2$, where B is a small neighborhood of the center of the disk, and think of U as sitting in W'_N. Then the pair $(W'_N \setminus U, \partial U)$ is homotopy equivalent to the pair (Y, K); and by excision, the (co)homology of $(W'_N \setminus U, \partial U)$ is the same as that of (W'_N, U). Thus, for $i \neq 0$, there are canonical isomorphisms $\phi_* : H_i(Y, K) \to H_i(W'_N)$ and $\phi^* : H^i(W'_N) \to H^i(Y, K)$. Then, it is clear that in fact $\phi_*\big(\Phi_{0*}([S \setminus D'])\big) = \Phi_*([S])$.

The following largely follows as in Proposition 7.5 of [18]. Recall that in that paper quantities $\widehat{\chi}(\mathcal{D})$ and $\mu_{\mathbf{y}}(\mathcal{D})$ are defined for two-chains $\mathcal{D}$.

Proposition 2.3. *Fix some value of N, and a standard (doubly-pointed) Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w, z)$ for W'_N. Let $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ and $h \in H_2(Y, K)$. Set $\mathcal{P}$ to be the unique $\boldsymbol{\alpha\gamma\delta}$-periodic domain with no boundary components in $\boldsymbol{\gamma} \setminus \{\gamma_g\}$ (and no local* *multiplicity at w) that represents $\phi_*(h)$. Then*

$$\langle c_1\big(\underline{\mathfrak{s}}_{w,z}(\mathbf{y})\big), h\rangle = \widehat{\chi}(\mathcal{P}) + 2\mu_{\mathbf{y}}(\mathcal{P}) - n_w(\mathcal{P}) - n_z(\mathcal{P}).$$

Proof. Let v be a vector field representing $\underline{\mathfrak{s}}_{w,z}(\mathbf{y})$. By construction, this vector field should point along γ_g, and $v^\perp$ is trivialized along γ_g by vectors pointing out of Σ (after a homotopy). Calling this trivialization τ, we want to calculate $\langle e(v^\perp|_{S\setminus D'}, \tau), [S\setminus D', \partial(S\setminus D')]\rangle$. This proceeds almost identically to the proof of Proposition 7.5 of [18]. There it is found that

$$\langle c_1\big(\mathfrak{s}_w(\mathbf{y})\big), h\rangle = \widehat{\chi}(\mathcal{P}) + 2\mu_{\mathbf{y}}(\mathcal{P}) - 2n_w(\mathcal{P})$$

for a $\boldsymbol{\alpha\delta}$-periodic domain $\mathcal{P}$ representing $h \in H_2(Y)$. For us, the boundary components of $\mathcal{P}$ along γ_g don't affect the calculation. However, we must subtract 1 for each point in $\Phi^{-1}(\{z\})$ and subtract -1 for each point in $\Phi^{-1}(\{w\})$ due to the fact that v is trivialized in a neighborhood of K. Hence, we get $\widehat{\chi}(\mathcal{P}) + 2\mu_{\mathbf{y}}(\mathcal{P}) - n_w(\mathcal{P}) - n_z(\mathcal{P})$. □

2.4 Spin^c structures on cobordisms

Given K, $N > 0$, and $\mathfrak{t}_0 \in \mathrm{Spin}^c(Y_0)$, let $\mathfrak{S}_0^N(\mathfrak{t}_0)$ denote the set $\mathfrak{t}_0 + \mathbb{Z}\cdot \mathrm{PD}[N\mu]$, thinking of μ as an element of $H_1(Y_0)$. In a standard diagram for K, consider the set of Spin^c structures on $X_{\alpha\beta\gamma}$ that restrict to an element of $\mathfrak{S}_0^N(\mathfrak{t}_0)$ on $Y_{\alpha\beta} = Y_0$ and to the canonical Spin^c structure on $Y_{\beta\gamma} = L(N,1)$. Define $\mathfrak{S}_N(\mathfrak{t}_0)$ to be the set of restrictions of these structures to $Y_{\alpha\gamma} = Y_N$.

Also, let $\mathfrak{S}_{N\infty}(\mathfrak{t}_0) \subset \mathrm{Spin}^c(W_N)$ be the set of structures that restrict to an element of $\mathfrak{S}_N(\mathfrak{t}_0)$, and let $\mathfrak{S}_\infty(\mathfrak{t}_0) \subset \mathrm{Spin}^c(Y)$ be the restrictions of $\mathfrak{S}_{N\infty}(\mathfrak{t}_0)$ to Y.

We banish the proof of the following to the Appendix.

Proposition 2.4. *Assume that $N\mu+\lambda$ is not special. Then the sets $\mathfrak{S}_\infty(\mathfrak{t}_0)$ and $\mathfrak{S}_N(\mathfrak{t}_0)$ are finite, and independent of the family of diagrams we use. The former is even inde-*

pendent of N: precisely, it is the set of restrictions to Y of structures on W_0' that also restrict to $\mathfrak{t}_0$. Furthermore, if there exist $i, j \in \mathbb{Z}$ such that $ic_1(\mathfrak{t}_0) + j\mathrm{PD}[\mu]$ is torsion, then both $\mathfrak{S}_\infty(\mathfrak{t}_0)$ and $\mathfrak{S}_N(\mathfrak{t}_0)$ consist entirely of torsion Spin^c structures; otherwise, they both consist entirely of non-torsion ones.

If $ic_1(\mathfrak{t}_0) + j\mathrm{PD}[\mu]$ is torsion for some $i, j \in \mathbb{Z}$, let us call $\mathfrak{t}_0$ *μ-torsion*. Obviously, a torsion structure is μ-torsion.

We wish to speak of the sets $\mathfrak{S}_N(\mathfrak{t}_0)$ on a common ground for all N – specifically, we want to identify them all with subsets of $\underline{\mathrm{Spin}}^c(Y, K)$. As a first step toward this end, we make the following definition. For $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$, and $\psi \in \pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{y})$, define

$$E_{K,N}(\psi) = \underline{\mathfrak{s}}_{w,z}(\mathbf{y}) + \big(n_w(\psi) - n_z(\psi)\big)\mathrm{PD}[\mu]. \tag{2.3}$$

We will show that this in fact gives a well-defined, diagram-independent map

$$E_{K,N} : \mathrm{Spin}^c(W_N') \to \underline{\mathrm{Spin}}^c(Y, K).$$

Recall the definition of the spider number $\sigma(\psi, \mathcal{P})$. Orient the $\boldsymbol{\alpha}, \boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ circles so that every circle appears in $\partial\mathcal{P}$ with nonnegative multiplicity. Let $\partial_\alpha'\mathcal{P}$ be gotten by taking an inward translate of each $\boldsymbol{\alpha}$ circle with respected to the endowed orientation, and then taking a linear combination of these circles with multiplicities given by the corresponding multiplicities in $\partial\mathcal{P}$.

Think of $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ as a map from Δ to Σ, where Δ is a triangle depicted as in Figure 2.4. A spider is a point u in Δ together with three segments a, b and c from u to the α, γ and δ portions of the boundary of Δ, respectively, each oriented outward from u. Then set

$$\sigma(\psi, \mathcal{P}) = n_{\psi(u)}(\mathcal{P}) + \#\big(\partial_\alpha'\mathcal{P} \cap \psi(a)\big) + \#\big(\partial_\gamma'\mathcal{P} \cap \psi(b)\big) + \#\big(\partial_\delta'\mathcal{P} \cap \psi(c)\big),$$

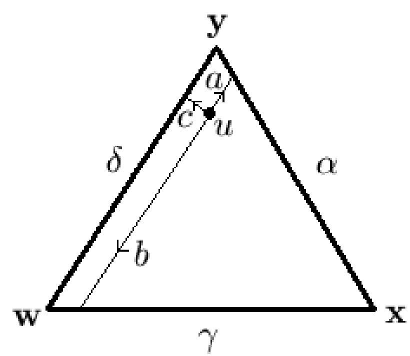

Figure 2.3: A triangle Δ, with sides and vertices marked by where their images lie under a map ψ, and a spider. We will think of our spiders as having the point near $\mathbf{x}$, with two short legs a and c and one long one b running parallel to δ.

where all the intersection numbers are oriented (with x-axis $\cap$ y-axis $= 1$).

Proposition 2.5. *Given a knot K, choose N such that $N\mu+\lambda$ is not a special longitude, and form a standard diagram. Let $h \in H_2(Y, K)$, and let $\mathcal{P}$ be the unique $\boldsymbol{\alpha\gamma\delta}$-periodic domain with no boundary components in $\boldsymbol{\gamma} \setminus \{\gamma_g\}$ (and no local multiplicity at w) that represents $\phi_*(h)$. Then, writing λ_N for $N\mu + \lambda \in H_1(Y \setminus K)$, we have*

$$\langle c_1\big(\mathfrak{s}_w(\psi)\big), \phi_*(h)\rangle = \langle c_1\big(E_{K,N}(\psi)\big) + \mathrm{PD}[\lambda_N - \mu], h\rangle.$$

Proof. The Chern class formula from Section 6 of [24] gives

$$\langle c_1\big(\mathfrak{s}_w(\psi)\big), \phi_*(h)\rangle = \widehat{\chi}(\mathcal{P}) + \#\partial\mathcal{P} - 2n_w(\mathcal{P}) + 2\sigma(\psi, \mathcal{P}). \tag{2.4}$$

Let us calculate the quantity $\langle c_1\big(\mathfrak{s}_w(\psi)\big), \phi_*(h)\rangle - \langle c_1\big(\underline{\mathfrak{s}}_{w,z}(\mathbf{y})\big), h\rangle$, which is equal to

$$\#\partial\mathcal{P} - n_w(\mathcal{P}) + n_z(\mathcal{P}) + 2\sigma(\psi, \mathcal{P}) - 2\mu_{\mathbf{y}}(\mathcal{P}),$$

with $\mathbf{y}$ a corner of ψ in $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$ as above.

For the calculation of σ, choose the spider in our triangle ψ as depicted in Figure 2.4, with the spider point near the intersection of the α and δ edges, and the leg b running parallel and close to the δ edge. Then it is easy to check that

$$2\mu_{\mathbf{y}}(\mathcal{P}) = \#\partial\mathcal{P} - \#\partial_\gamma\mathcal{P} + 2\sigma(\psi,\mathcal{P}) - 2(\#\partial'_\gamma\mathcal{P}\cap b)$$

where $\#\partial_\gamma\mathcal{P}$ is the number of γ circles in $\partial\mathcal{P}$. So, we need only calculate

$$\#\partial_\gamma\mathcal{P} + 2(\#\partial'_\gamma\mathcal{P}\cap b) - n_w(\mathcal{P}) + n_z(\mathcal{P}).$$

Suppose that $\partial\mathcal{P} = k\gamma_g + L$, where L is a linear combination of circles in $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, and we break our convention and orient γ_g to agree with the orientation of K. Assume for now that $k \geq 0$. Then examination of a standard diagram shows that $\#\partial_\gamma\mathcal{P} = k$ and $2(\#\partial'_\gamma\mathcal{P}\cap b) = -2k(n_z(\psi) - n_w(\psi) + 1)$. To see the latter, note that we assume that the only $\boldsymbol{\gamma}$ boundary component of $\mathcal{P}$ is γ_g, and it is easy to see that if a component of the image of b intersects γ_g, it must run parallel to δ_g, and we just count the number of times it circles the meridian.

It is not hard to see from our construction of a surface representing the homology class h that $k = \langle \mathrm{PD}[\mu], h\rangle$ and that $n_z(\mathcal{P}) - n_w(\mathcal{P}) = \langle \mathrm{PD}[\lambda_N], h\rangle$. So, we conclude that

$$\langle c_1\big(\mathfrak{s}_w(\psi)\big), \phi_*(h)\rangle = \langle c_1\Big(\underline{\mathfrak{s}}_{w,z}(\mathbf{y}) + \big(n_w(\psi) - n_z(\psi)\big)\mathrm{PD}[\mu]\Big) + \mathrm{PD}\big[\lambda_N - \mu], h\rangle.$$

The case where $k < 0$ is similar, and we get the same class. □

As a map from $\mathrm{Spin}^c(W'_N)$ to $\underline{\mathrm{Spin}}^c(Y,K)$, let $E_{K,N}$ be defined as follows. We claim that given $\mathfrak{s} \in \mathrm{Spin}^c(W'_N)$, there is a unique element $\xi \in \underline{\mathrm{Spin}}^c(Y,K)$ such that $G_K(\xi) = \mathfrak{s}|_Y$ and such that $c_1(\xi) = \phi^*\big(c_1(\mathfrak{s})\big) - \mathrm{PD}[\lambda_N - \mu]$; then let $E_{K,N}(\mathfrak{s}) = \xi$.

Proposition 2.6. *The map described above is indeed well-defined, and* $E_{K,N}\big(\mathfrak{s}_w(\psi)\big) = E_{K,N}(\psi)$ *for any triangle* ψ *in a standard diagram, where the right hand side uses the definition of* $E_{K,N}$ *as given in Equation 2.3. This map is* $H^2(W'_N)$*-equivariant, where the action on the right is via the map* ϕ^**.*

Proof. First note that the condition that $G_K(\xi) = \mathfrak{s}|_Y$ determines ξ up to adding some multiple of $\mathrm{PD}[\mu]$. Examining the long exact sequence

$$H^1(Y) \to H^1(K) \to H^2(Y,K) \xrightarrow{j^*} H^2(Y),$$

we see that the first arrow is the zero map, since K is rationally nullhomologous; hence, the second map is an injection of $H^1(K) \cong \mathbb{Z}$ onto $\mathrm{Ker}(j^*) \subset H^2(Y,K)$. Thus, there is at most one ξ that satisfies the first conditions and the demand that $c_1(\xi) - \phi^*\big(c_1(\mathfrak{s})\big) + \mathrm{PD}[\lambda_N - \mu]$ is torsion. Assuming existance of such a ξ, then Equation 2.1 and the fact that $j^*(\mathrm{PD}[\lambda_N - \mu]) = \mathrm{PD}[K]$ then imply that this class is actually 0.

Proposition 2.5 ensures that such a ξ does exist if there is a triangle ψ representing $\mathfrak{s}$ in some standard diagram. It is easy to see that if ξ works for $\mathfrak{s}$, then $\xi + \phi^*(x)$ works for $\mathfrak{s} + x$, which establishes the map as well-defined on all of $\mathrm{Spin}^c(W'_N)$ as well as the $H^2(W'_N)$-equivariance. □

2.5 Squares of Chern classes

The intersection form on the cobordism W_N is defined via the cup product

$$H^2(W_N, \partial W_N) \otimes H^2(W_N, \partial W_N) \to H^4(W_N, \partial W_N) \cong \mathbb{Z},$$

where the latter isomorphism is evaluation on a fundamental class. If j denotes the composition of maps $H^2(W_N;\mathbb{Z}) \to H^2(\partial W_N;\mathbb{Z}) \to H^2(\partial W_N;\mathbb{Q})$, then the intersection form can be extended to a pairing $\mathrm{Ker}\ j \otimes \mathrm{Ker}\ j \to \mathbb{Q}$ as follows. If $x \in \mathrm{Ker}\ j$, then

$x \otimes 1 \in H^2(W_N;\mathbb{Z}) \otimes \mathbb{Q}$ goes to an element in $H^2(W_N;\mathbb{Q})$ which clearly lifts to some $\widetilde{x} \in H^2(W_N, \partial W_N;\mathbb{Q})$. The square of $\widetilde{x}$ is easily seen to be independent of the choice of lift $\widetilde{x}$, and so we define $x^2 = \widetilde{x}^2$.

We can also square elements of $H_2(W_N)$ in the usual manner. The group $H_2(W_N, Y)$ is isomorphic to $\mathbb{Z}$, generated by the oriented core F of the attached handle, oriented to agree with $-K$ on the boundary; $H_2(W_N)$ splits (non-canonically) as the direct sum of $H_2(Y)$ and $\mathbb{Z} \cdot [\widetilde{dF}]$, where $[\widetilde{dF}]$ is gotten by taking d copies of F (recalling that d is the order of K in $H_1(Y)$) and then capping off the boundaries with a Seifert surface for dK.

We write F' or $[\widetilde{dF'}]$ when thinking of the above as elements of $H_2(W'_N, Y)$ or $H_2(W'_N)$; this distinction only matters when we consider intersection forms on cobordisms. Note that for any rationally nullhomologous knot K equipped with longitude λ, there are unique relatively prime integers p and q with $p > 0$ such that surgery on the framing $p\mu + q\lambda$ increase the first Betti number; let $\kappa = -\frac{p}{q}$. In particular, if K is special, then $\kappa = 0$. The proof of the following is also given in the Appendix.

Proposition 2.7. *The order of μ in $H_1(Y_0)$ is $|d\kappa|$, and for any lift $[\widetilde{dF'}]$ of F', we have that*

$$\frac{[\widetilde{dF'}]^2}{d^2} = -\kappa - N.$$

In particular, for special K, μ is not torsion in $H_1(Y_0)$ and the square of $[\widetilde{dF'}]$ is $-d^2N$.

We restrict attention now to Spin^c structures on W'_N that restrict to torsion or (if $N\mu + \lambda$ is special) μ-torsion structures on the boundary; we refer to these Spin^c structures, as well as the relative structures on (Y, K) over them, as *boundary-torsion.* Our interest in these stems from the fact that the generators of $\underline{HF}^+(Y)$ and $\underline{HF}^+(Y_N)$ (for non-special $N\mu + \lambda$) that represent them will inherit absolute $\mathbb{Q}$-gradings from their untwisted counterparts.

For torsion $\mathfrak{t}_N \in \mathrm{Spin}^c(Y_N)$, choose some fixed $\mathfrak{s}_0 \in \mathrm{Spin}^c(W'_N)$ that restricts to $\mathfrak{t}_N$. Then the subset of elements in $\mathrm{Spin}^c(W'_N)$ that restrict to $\mathfrak{t}_N$ will be $\mathfrak{s}_0 + \mathbb{Z} \cdot \big(\mathrm{PD}[F']|_{W'_N}\big)$.

(Here, $\mathrm{PD}[F'] \in H^2(W'_N, Y_N)$ is the Poincaré dual of F' thought of as an element of $H_2(W'_N, Y)$.) For boundary-torsion $\mathfrak{s}$, the evaluation of $c_1(\mathfrak{s})$ on any element of the kernel of the map from $H_2(W'_N)$ to $H_2(W'_N, Y)$ vanishes; thus, the quantity $\frac{\langle c_1(\mathfrak{s}), [\widetilde{dF'}]\rangle}{d}$ is the same for any choice of $[\widetilde{dF'}]$ in the preimage of $d[F']$. Define

$$Q_K(x; \mathfrak{s}_0) = x^2 \frac{[\widetilde{dF'}]^2}{d^2} + x\left(\frac{\langle c_1(\mathfrak{s}_0), [\widetilde{dF'}]\rangle}{d}\right) = \frac{c_1^2(\mathfrak{s}_0 + x\mathrm{PD}[F']|_{W'_N}) - c_1^2(\mathfrak{s}_0)}{4}.$$

For a fixed $\mathfrak{s}_0$ which restricts to $\mathfrak{t}_N$, we can interpret $Q_K(x; \mathfrak{s}_0)$ as a quadratic function of $x \in \mathbb{Q}$; let $x_* \in \mathbb{Q}$ be the value at which this function achieves its maximum, and set $x_0 = \lfloor x_* \rfloor$. We write $\mathfrak{s}_{K+}(\mathfrak{t}_N) = \mathfrak{s}_0 + x_0\mathrm{PD}[F']|_{W'_N}$ and $\mathfrak{s}_{K-}(\mathfrak{t}_N) = \mathfrak{s}_0 + (x_0+1)\mathrm{PD}[F']|_{W'_N}$. These structures depend on K, N and $\mathfrak{t}_N$, but clearly not on $\mathfrak{s}_0$, since they are simply the maximizers of the square of the Chern class among those Spin^c structures which restrict to $\mathfrak{t}_N$.

We are interested also in the quantity

$$q_K(\xi) = Q_K\big(1; E_{K,N}^{-1}(\xi)\big) = \frac{[\widetilde{dF'}]^2}{d^2} + \left(\frac{\langle c_1\left(E_{K,N}^{-1}(\xi)\right), [\widetilde{dF'}]\rangle}{d}\right) \tag{2.5}$$

for $\xi \in \underline{\mathrm{Spin}}^c(Y, K)$. We also abuse notation and sometimes write q_K for $q_K \circ E_{K,N} = Q_K(1; \cdot)$, which makes sense for any boundary-torsion element of $\mathrm{Spin}^c(W'_N)$ for any N. In light of the following, we sometimes write a relative Spin^c structure ξ in the form $[G_K(\xi), q_K(\xi)]$.

Proposition 2.8. *The quantity $q_K(\xi)$ is independent of N, for $\xi \in \underline{Spin}^c(Y, K)$, and the map $\xi \mapsto \big(G_K(\xi), q_K(\xi)\big)$ is injective.*

We defer the proof to Chapter 3.

We would hope that the above makes the set of relative Spin^c structures corresponding to $\mathfrak{S}_{N\infty}(\mathfrak{t}_0)$ independent of N. This is not the case, but when K is special we can say something that will be of use to us later.

Proposition 2.9. *Fix some oriented knot K and some μ-torsion $\mathfrak{t}_0 \in Spin^c(Y_0)$. If K is special, we have*

$$q_K\big(\mathfrak{s}_{K+}(\mathfrak{t}_N)\big) = -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}$$

for any $\mathfrak{t}_N$ in $\mathfrak{S}_N(\mathfrak{t}_0)$, where $[\widehat{dS}]$ is any Seifert surface for dK capped off in Y_0; and

$$E_{K,N} \circ \mathfrak{s}_{K+}\big(\{\mathfrak{S}_N(\mathfrak{t}_0)\}\big) = \big\{[\mathfrak{s}_{K+}(\mathfrak{t}_N)|_Y - i\mathrm{PD}[K], -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}]\big|0 \leq i < d\big\}.$$

We give the proof in the Appendix.

Chapter 3

Families of Standard Heegaard Diagrams and Small Triangles

In this section, we make a convenient generalization of the standard Heegaard diagrams, and note some facts that will be useful in proving Theorem 5.2, a twisted version of the large-N surgery formula that is the first step towards the more specific formula of Theorem 6.1.

Note that the winding region of a standard diagram, as depicted in Figure 2.1, has a fairly rigid form: essentially, the picture is determined by the number of horizontal $\boldsymbol{\alpha}$ strands, the number of turns in γ_g, and the location of δ_g (along with the flanking basepoints). Given a particular standard diagram, we can then cut out the winding region, and replace it with any other possible winding region that has the same number of $\boldsymbol{\alpha}$ strands. Such a replacement has little effect on the manifolds Y_0 and Y that the diagram represents, or the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ or $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$. The new diagram will represent Y_N and W'_N for new values of N, of course.

Likewise, the small triangles appearing in a standard diagram have a rigid form; specifically, they are determined by their corner $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ and the value of $n_w(\psi) - n_z(\psi)$.

Definition 3.1. *A* family *of special Heegaard diagrams for (Y, K) is the set of all diagrams gotten from a given one by making this type of replacement.*

If two small triangles appearing in two Heegaard diagrams in the same family have the same corner $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ and the same value of $n_w(\psi) - n_z(\psi)$, we call them *similar*.

Proof of Proposition 2.8. We show that raising N by 1 doesn't change $q_K(\xi)$. When N is increased by 1, the quantity $\frac{[\widetilde{dF'}]^2}{d^2}$ decreases by 1 by Proposition 2.7.

As for $\frac{\langle c_1(E^{-1}_{K,N}(\xi)),[\widetilde{dF'}]\rangle}{d}$, we appeal to the Chern class evaluation formula, Equation 2.4. Any lift $[\widetilde{dF'}]$ in $H_2(W'_N)$ has a representative periodic domain in a standard diagram whose boundary is of the form $d(\gamma_g - N\delta_g) + L$, where L is a linear combination of α circles and $\gamma_1, \ldots, \gamma_{g-1}$. If we take a diagram for W'_{N+1} from the same family, we will have a periodic domain with boundary $d(\gamma_g - (N+1)\delta_g) + L$ for the same value of L, which will also represent some class $[\widetilde{dF'}] \in H_2(W'_{N+1})$. Likewise, for some ξ, we have similar triangles in the two diagrams that respectively represent $E^{-1}_{K,N}(\xi)$ and $E^{-1}_{K,N+1}(\xi)$. So, we can calculate the difference using the Chern class formula; we find that $\langle c_1(E^{-1}_{K,N+1}(\xi)), [\widetilde{dF'}]\rangle - \langle c_1(E^{-1}_{K,N}(\xi)), [\widetilde{dF'}]\rangle = -d$ for such values of ξ. By the characterization of $E_{K,N}(\xi)$ given in Proposition 2.6, it is then clear that this equation must hold for all ξ.

To see that ξ is determined by $G_K(\xi)$ and $q_K(\xi)$, simply note that $q_K(\xi + i\mathrm{PD}[\mu]) = q_K(\xi) + 2i$. $\square$

Fix a family $\mathcal{F}$ of standard diagrams $\{(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w, z)\}$ for (Y, K). If we forget about the basepoints, we can still talk about equivalence classes of intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$: two intersection points are equivalent if and only if, upon adding basepoints, they represent the same Spin^c structure. Of course, the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$ for any two of these diagrams are the same. Equivalence classes of points still form an affine set for the action of $H_1(Y_N)$.

Over all the diagrams in $\mathcal{F}$, the total number of intersection points not supported in the winding region is constant, since the components of these points lie in the portions of the diagram that don't change throughout the family. Hence, the number of equivalence classes that contain points not supported in the winding region is bounded independent of N.

Lemma 3.2. *There is an integer $\epsilon > 0$ such that if N is sufficiently large, then for all $\mathfrak{t} \in Spin^c(Y_N)$, there is a diagram in $\mathcal{F}$ such that $\mathfrak{t}$ is represented by an equivalence class of points which are all supported in the winding region, and such that we have*

$$2|n_w(\psi) - n_z(\psi)| \leq N + \epsilon$$

for all small ψ with a corner representing $\mathfrak{t}$.

Proof. We assume as always that K is rationally nullhomologous in Y, say of order d. Then, it is not hard to show that there is a constant c such that μ will be of order $|dN - c|$ in $H_1(Y_N)$.

There are $N+1$ members of $\mathcal{F}$ that represent W'_N, corresponding to the $N+1$ placements of δ_g and accompanying placements of the basepoint w. Consider the ϵ innermost placements of δ_g – that is, disregard the $\frac{N-\epsilon}{2}$ leftmost and rightmost placements. We claim that if N is large enough, then amongst these ϵ different placements, a given Spin^c structure $\mathfrak{t}$ will be represented by ϵ distinct equivalence classes of intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. To see this, suppose we have two adjacent placements of δ_g, with corresponding basepoints w_1 and w_2; then if e is the equivalence class of $\mathfrak{t}$ with respect to w_1, then $e + \text{PD}[\mu]$ is the equivalence class with respect to w_2. Then for large enough N, $|dN - c| \geq \epsilon$, so that the ϵ equivalence classes will therefore all be distinct.

Hence, if ϵ is larger than the number of equivalence classes that contain points not supported in the winding region, then with respect to one of these placements of δ_g, $\mathfrak{t}$ must be represented by an equivalence class of points which all are supported in the winding region. Furthermore, for these placements, it is easy to see that

$$|n_w(\psi) - n_z(\psi)| \leq \epsilon + \frac{N - \epsilon}{2}$$

(since for small triangles, at most one of $n_w(\psi)$ and $n_z(\psi)$ is nonzero), from which the claim follows. □

Call a value of ϵ valid if it makes Lemma 3.2 hold. Given $\mathfrak{t} \in \mathrm{Spin}^c(Y_N)$, we call a standard diagram $\mathfrak{t}$-*proper* if it is of the type described in the statement of Lemma 3.2 with respect to some valid ϵ.

Lemma 3.3. *For any small triangle ψ in* any *diagram in $\mathcal{F}$, the quantity*

$$|q_K(\mathfrak{s}_w(\psi)) - 2(n_w(\psi) - n_z(\psi))|$$

is bounded independent of the particular diagram. In particular, there exists a constant C_q depending only on $\mathcal{F}$ such that

$$|q_K(\mathfrak{s}_w(\psi))| \leq C_q + N$$

for any small triangle ψ in a $\mathfrak{t}$-proper standard diagram in $\mathcal{F}$ representing W'_N, where $\mathfrak{t} = \mathfrak{s}_w(\psi)|_{Y_N}$.

Proof. Fix a point $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$, and consider some small triangle ψ with $\mathbf{y}$ as a corner in a diagram in $\mathcal{F}$. If we choose a different small triangle ψ' in the same diagram, with $\mathbf{y}$ still a corner but such that $n_w(\psi') - n_z(\psi')$ increased by 1, then we can use the Chern class formula to show that $\langle c_1(\mathfrak{s}_w(\psi)), [\widetilde{dF'}]\rangle$ increases by $2d$, so that $q(\mathfrak{s}_w(\psi))$ is increased by 2. On the other hand, if we add a turn to γ_g and consider the similar triangle ψ'' to ψ in this diagram, Proposition 2.8 shows that $q_K(\mathfrak{s}_w(\psi))$ doesn't change. Thus, $|q_K(\mathfrak{s}_w(\psi)) - 2(n_w(\psi) - n_z(\psi))|$ depends only on $\mathbf{y}$; since the number of points in

$\mathbb{T}_\alpha \cap \mathbb{T}_\delta$ is fixed, the first claim follows.

The second claim follows from the first together with Lemma 3.2. □

Proposition 3.4. *For a family $\mathcal{F}$ of standard diagrams for K, there is a constant $N(\mathcal{F})$ such that the following holds. Take any $\mathfrak{t} \in Spin^c(Y_N)$, and let ψ be some small triangle with a corner representing $\mathfrak{t}$, in a standard $\mathfrak{t}$-proper diagram for W'_N. Then if $N \geq N(\mathcal{F})$, we have*

$$\mathfrak{s}_w(\psi) = \mathfrak{s}_{K+}(\mathfrak{t})$$

and

$$\mathfrak{s}_z(\psi) = \mathfrak{s}_{K-}(\mathfrak{t}).$$

Proof. First, note that $\mathfrak{s}_z(\psi) - \mathfrak{s}_w(\psi) = \mathrm{PD}[F']|_{W'_N}$. To see this, note that the Chern class formula gives that for $\mathcal{P}$ representing $h \in H_2(W'_N)$, $\langle c_1(\mathfrak{s}_z(\psi)) - c_1(\mathfrak{s}_w(\psi)), h\rangle = 2n_w(\mathcal{P}) - 2n_z(\mathcal{P})$. Thus, we can easily show that $c_1(\mathfrak{s}_z(\psi)) - c_1(\mathfrak{s}_w(\psi)) - 2\mathrm{PD}[F']|_{W'_N}$ is trivial as an element of $\mathrm{Hom}(H_2(W'_N), \mathbb{Z})$, hence torsion in $H^2(W'_N)$, hence 0 since $\mathfrak{s}_z(\psi)|_{Y_N} = \mathfrak{s}_w(\psi)|_{Y_N}$ and the torsion subgroup of $H^2(W'_N)$ injects into $H^2(Y_N)$.

We claim that if N is large enough, we must have

$$c_1^2(\mathfrak{s}_w(\psi)) \geq c_1^2(\mathfrak{s}_w(\psi) - \mathrm{PD}[F']|_{W'_N})$$

and

$$c_1^2(\mathfrak{s}_w(\psi) + \mathrm{PD}[F']|_{W'_N}) \geq c_1^2(\mathfrak{s}_w(\psi) + 2\mathrm{PD}[F']|_{W'_N})$$

for ψ having a corner representing any $\mathfrak{t} \in \mathrm{Spin}^c(Y_N)$. Assuming this, the fact that $c_1^2(\mathfrak{s}_w(\psi) + x\mathrm{PD}[F']|_{W'_N})$ depends quadratically on x means that this function is greater at $x = 0$ and $x = 1$ than at any other values of x, from which it follows that $\mathfrak{s}_w(\psi) = \mathfrak{s}_{K+}(\mathfrak{t})$ and $\mathfrak{s}_z(\psi) = \mathfrak{s}_w(\psi) + \mathrm{PD}[F']|_{W'_N} = \mathfrak{s}_{K-}(\mathfrak{t})$.

To show the claim, note that the two inequalities above are respectively equivalent to

$$Q_K\big(0;\mathfrak{s}_w(\psi_{\mathfrak{t}})\big) \geq Q_K\big(-1;\mathfrak{s}_w(\psi_{\mathfrak{t}})\big)$$

and

$$Q_K\big(1;\mathfrak{s}_w(\psi_{\mathfrak{t}})\big) \geq Q_K\big(2;\mathfrak{s}_w(\psi_{\mathfrak{t}})\big).$$

In turn, these can be reduced further, to

$$q_K\big(\mathfrak{s}_w(\psi)\big) - 2\frac{[\widetilde{dF'}]^2}{d^2} \geq 0 \tag{3.1}$$

and

$$q_K\big(\mathfrak{s}_w(\psi)\big) + 2\frac{[\widetilde{dF'}]^2}{d^2} \leq 0. \tag{3.2}$$

Lemma 3.3 says that there is a number C_q depending only on $\mathcal{F}$ such that

$$-C_q - N \leq q_K\big(\mathfrak{s}_w(\psi)\big) \leq C_q + N$$

always holds for all small triangles ψ. By Proposition 2.8, $2\frac{[\widetilde{dF'}]^2}{d^2}$ will be equal to $-2\kappa - 2N$. So we have

$$q_K\big(\mathfrak{s}_w(\psi)\big) - 2\frac{[\widetilde{dF'}]^2}{d^2} \geq 2\kappa - C_q + N,$$

$$q_K\big(\mathfrak{s}_w(\psi)\big) + 2\frac{[\widetilde{dF'}]^2}{d^2} \leq -2\kappa + C_q - N;$$

for large enough N, these imply that inequalities (3.1) and (3.2) hold, which proves the claim. $\square$

Chapter 4

Twisted Coefficients and a Long Exact Sequence

We wish to prove a twisted analogue of the surgery long exact sequence relating Y, Y_0, and Y_N when K is a special knot. To accomplish this, we introduce a system of coefficients adapted to a given standard diagram.

4.1 Additive functions on polygons

Suppose that we have a standard Heegaard diagram with translates $\mathcal{H} = (\Sigma, \{\boldsymbol{\eta}^i\}, w)$ for W'_N with $N > 0$, recalling that we denote the tuples $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ associated to a standard diagram, as well as their translates, by $\boldsymbol{\eta}^i$ for $i \geq 0$. Again, we assume that K is special.

To properly relate the twisted Floer homologies of the various manifolds $Y_{\eta^i\eta^j}$, we find it useful to have the following. Let $C(\mathcal{H})$ be the free abelian group generated by the set $\bigcup_{i\geq 0}\{\eta^i_1, \ldots, \eta^i_g\}$. Define $L(\mathcal{H})$ to be the quotient of $C(\mathcal{H})$ by the equivalence relation $\sim$, where $\sim$ is generated by $\eta^i_k \sim \eta^j_k$ when $i, j \neq 0$ and $k \neq g$; $\eta^i_g \sim \eta^j_g$ when $i, j \neq 0$ and $i \equiv j \bmod 3$; and $\gamma_g \sim \beta_g + N\delta_g$. Informally, we are identifying elements that are *a priori* equivalent in $H_1(\Sigma)$.

Then, let $K(\mathcal{H})$ be the kernel of the obvious homomorphism from $L(\mathcal{H})$ to $H_1(\Sigma)$.

Denote by $\Delta(\mathcal{H})$ the set of all homotopy classes of polygons in this diagram, i.e., the disjoint union of $\pi_2(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ for $k \geq 2$, over all tuples of points such that $\pi_2(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ makes sense. We wish to define a function $A_K : \Delta(\mathcal{H}) \to K(\mathcal{H})$ that is additive under splicing, and has appropriate equivariance properties (which we make precise later).

To begin the construction, we first make the following choices:

- points $\mathbf{p}_i \in \mathbb{T}_{\eta^0} \bigcap \mathbb{T}_{\eta^i}$ for each i;
- for each $\mathbf{x} \in \mathbb{T}_{\eta^i} \bigcap \mathbb{T}_{\eta^j}$, an oriented multiarc $q_i(\mathbf{x})$ from $\mathbf{x}$ to $\mathbf{p}_i$ along $\boldsymbol{\eta}^i$ (and similarly a multiarc $q_j(\mathbf{x})$); and
- multiarcs m_i from $\mathbf{p}_i$ to $\mathbf{p}_0$ along $\boldsymbol{\eta}^0$, letting m_0 be the trivial multiarc.

We choose the multiarcs $q_i(\mathbf{x})$ so that if $\mathbf{x} \in \mathbb{T}_{\eta^i} \bigcap \mathbb{T}_{\eta^j}$ and $\mathbf{x}' \in \mathbb{T}_{\eta^{i'}} \bigcap \mathbb{T}_{\eta^{j'}}$ with $i \equiv i'$ and $j \equiv j'$ mod 3 are corresponding points, then $q_i(\mathbf{x})$ and $q_{i'}(\mathbf{x}')$ are corresponding multiarcs.

For any point $\mathbf{x} \in \mathbb{T}_{\eta^i} \bigcap \mathbb{T}_{\eta^j}$ with $j > i$, define $\ell_0(\mathbf{x}) = q_j(\mathbf{x}) - q_i(\mathbf{x}) + m_j - m_i$. Note that this realizes a closed oriented multiarc in Σ, and if $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are such that $\mathbf{x}_k \in \mathbb{T}_{\eta^i_k} \bigcap \mathbb{T}_{\eta^i_{k+1}}$ with $i_{k+1} = i_1 < i_2 < \ldots < i_k$, then the sum $\ell_0(\mathbf{x}_1) + \ldots + \ell_0(\mathbf{x}_k) - \ell_0(\mathbf{x}_{k+1})$ is homotopic (within the circles of $\mathcal{H}$) to a multiarc supported along circles in $\boldsymbol{\eta}^{i_n}$ for $n = 1, \ldots, k$.

Let $L(i,j)$ denote $\ell_0(\Theta_{i,j})$. It is not hard to see for any point $\mathbf{x} \in \mathbb{T}_{\eta^i} \cap \mathbb{T}_{\eta^j}$ with $j > i > 0$, that $L(i, i+1) + L(i+1, i+2) + \ldots + L(j-1, j) - \ell_0(\mathbf{x})$ is homologous in $H_1(\Sigma)$ to some element $\ell_1(\mathbf{x}) \in C(\mathcal{H})$. Then, define $\ell(\mathbf{x})$ to be the closed oriented multiarc $\ell_0(\mathbf{x}) - \ell_1(\mathbf{x})$.

Say that two points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^i}$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^{i+k}}$ are *homologous* if $\pi_2(\mathbf{x}, \Theta_{i,i+1}, \ldots, \Theta_{i+k-1,i+k}, \mathbf{y})$ is nonempty, and extend this to an equivalence relation on $\bigcup_i \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^i}$. Of course, generators that are Spin^c-equivalent will be homologous.

For each such homology class c, choose a representative point $\mathbf{x}_c \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^1}$. Then, it is also not hard to see for any other point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^i}$ for any i representing c, that $\ell_0(\mathbf{x}_c) + L(1,2) + \ldots + L(i-1,i) - \ell_0(\mathbf{x})$ is homologous in $H_1(\Sigma)$ to some element $\ell_1(\mathbf{x}) \in C(\mathcal{H})$; so let $\ell(\mathbf{x})$ be $\ell_0(\mathbf{x}) - \ell_1(\mathbf{x})$.

The upshot of the above is that for any polygon $\psi \in \pi_2(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{y})$, we have $\partial\psi = \ell(\mathbf{x}_1) + \ldots + \ell(\mathbf{x}_k) - \ell(\mathbf{y}) + Z$ if ψ is not a bigon or $\partial\psi = \ell(\mathbf{y}) - \ell(\mathbf{x}_1) + Z$ if ψ is a bigon, where $Z \in C(\mathcal{H})$ vanishes in $H_1(\Sigma)$. So, choose points $p_{i,s}$ on η^i_s, in a suitably generic position (away from intersections between isotopes), and orient each curve. Let $A_0(c) = \sum_{i,s} m_{p_{i,s}}(c) \cdot \eta^i_s$ for any closed multiarc c, where $m_{p_{i,s}}$ is the oriented intersection number; and let

$$A_0(\psi) = A_0\left(\partial\psi - \ell(\mathbf{x}_1) - \ldots - \ell(\mathbf{x}_k) + \ell(\mathbf{y})\right)$$

if ψ is not a bigon, and

$$A_0(\psi) = A_0\left(-\partial\psi - \ell(\mathbf{x}_1) + \ell(\mathbf{y})\right)$$

if ψ is a bigon. Letting A_K denote the composition of A_0 with the map taking $C(\mathcal{H})$ to $L(\mathcal{H})$, the image of A_K actually lies in $K(\mathcal{H})$. This map will clearly be additive under splicing. Furthermore, note that $H^1(Y_{0,j})$ naturally embeds into $K(\mathcal{H})$ (via a choice of basepoint in our Heegaard diagram). With respect to this, the map A_K is also $H^1(Y_{0,j})$-equivariant.

We also define one more function: let $M(\mathcal{H})$ be the $\mathbb{Z}/N\mathbb{Z}$-module freely generated by $\{\eta^{3i}_g | i \geq 1\}$ (i.e., δ_g and all its isotopic translates). For $\psi \in \Delta(\mathcal{H})$, define $A_M(\psi)$ to be the summands of $A_0(\partial\psi)$ corresponding to these circles. Again, this is clearly additive under splicing.

Going forward, we write M_i for η^{3i}_g; and we write $A(\psi)$ for $A_K(\psi) \oplus A_M(\psi) \in K(\mathcal{H}) \oplus M(\mathcal{H})$.

4.2 Standard diagram coefficients

Let $R_K = \mathbb{Z}[K(\mathcal{H})]$ and $R_M = \mathbb{Z}[M(\mathcal{H})]$; set $R = R_K \otimes R_M$, which will be equal to the group ring of $K(\mathcal{H}) \oplus M(\mathcal{H})$. Of course, R_K and hence R will also be an algebra over $\mathbb{Z}[H^1(Y_{i,j})]$.

We now define the *chain complex with standard diagram coefficients,* $\underline{CF}^+(Y_{i,j}; R)$, to be the group $CF^+(Y_{i,j}) \otimes R$ equipped with the differential given by

$$\underline{\partial}^+\big([\mathbf{x}, i] \otimes r\big) = \sum_{\mathbf{y} \in \mathbb{T}_{\eta^i} \cap \mathbb{T}_{\eta^j}} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y}) | \mu(\phi)=1\}} \#\widehat{\mathcal{M}}(\phi) \cdot [\mathbf{y}, i - n_w(\phi)] \otimes (e^{A(\phi)} \cdot r),$$

where as usual we use exponential notation for elements of the group ring. The fact that A is additive under splicing ensures that this indeed defines a chain complex. This chain complex is not *a priori* an invariant, but rather depends on the diagram and the function A. However, the relationship with the twisted coefficient chain complex should be not difficult to see.

When $i, j \neq 0$, we will be interested in

$$\underline{CF}^{\leq 0}(Y_{i,j}; \mathbb{Z}[[U]] \otimes R) = CF^{\leq 0}(Y_{i,j}) \otimes \mathbb{Z}[[U]] \otimes R,$$

where $\mathbb{Z}[[U]]$ denotes a ring of formal power series, and where $H^1(Y_{i,j})$ acts trivially on $\mathbb{Z}[[U]] \otimes R$. The differential is defined in the same way as above. It is not hard to find an isomorphism from the trivially-twisted complex $CF^{\leq 0}(Y_{i,j}; \mathbb{Z}[[U]]) \otimes R$ to $\underline{CF}^{\leq 0}(Y_{i,j}; \mathbb{Z}[[U]] \otimes R)$.

For a set of g-tuples of circles $\boldsymbol{\eta}^{i_1}, \ldots, \boldsymbol{\eta}^{i_k}$ with k equal to 3 or 4, we define maps

$$\underline{f}^+_{i_1,\ldots,i_k} : \underline{CF}^+(Y_{i_1,i_2}; R) \bigotimes_{n=2}^{k-1} \underline{CF}^{\leq 0}(Y_{i_n,i_{n+1}}; \mathbb{Z}[[U]] \otimes R) \to \underline{CF}^+(Y_{i_1,i_k}; R)$$

if $i_1 = 0$. If $i_1 \neq 0$, we also define a similar map, except replacing $\underline{CF}^+(Y_{i_1,i_j}; R)$ with

$\underline{CF}^{\leq 0}(Y_{i_1,i_j};\mathbb{Z}[[U]]\otimes R)$ for $j=2$ and $j=k$. In both cases, the map is given by

$$\underline{f}^+_{i_1,\ldots,i_k}\Big(\bigotimes_{n=1}^{k-1}([\mathbf{x}_n,j_n]\otimes r_n)\Big)=$$

$$\sum_{\mathbf{w}\in\mathbb{T}_{\eta^{i_1}}\cap\mathbb{T}_{\eta^{i_k}}}\ \sum_{\substack{\psi\in\pi_2(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{w})\\ \mu(\psi)=k-3}}\#\mathcal{M}(\psi)\cdot\big[\mathbf{w},\sum_{n=1}^{k-1}j_n-n_w(\psi)\big]\otimes\big(e^{A(\psi)}\cdot\prod_{n=1}^{k-1}r_n\big).$$

In general, when we refer to a moduli space of rectangles or pentagons, we will mean the moduli space of those rectangles or pentagons that are pseudoholomorphic with respect to a one-parameter family of almost-complex structures on Σ; in particular, for $k=4$, we take $\#\mathcal{M}(\psi)$ in the above to be a count of such rectangles.

Given $\mathfrak{s}\in\mathrm{Spin}^c(X_{i_1,\ldots,i_k})$, we also have maps $\underline{f}^+_{i_1,\ldots,i_k,\mathfrak{s}}$ defined similarly, except counting only those polygons ψ representing $\mathfrak{s}$.

Lemma 4.1. *We have for each $i\geq 1$*

$$\underline{f}^+_{i,i+1,i+2,i+3}(\Theta_{i,i+1}\otimes\Theta_{i+1,i+2}\otimes\Theta_{i+2,i+3})=\Theta_{i,i+3}\otimes r_i,$$

with $r_i\in\mathbb{Z}[[U]]\otimes R$. There are constants $c\in\mathbb{Z}/N\mathbb{Z}$ and $k_i\in K(\mathcal{H})$, such that the U^0 coefficient of r_i is $e^{k_i+cM_j}$ if i equals $3j-2$ or $3j-1$, and $e^{k_i}\cdot\sum_{n=0}^{N-1}e^{nM_j+(c-n)M_{j+1}}$ if $i=3j$.

Proof. We have three cases to look at, according to the value of i mod 3. In each case, every holomorphic quadrilateral passing through $\Theta_{i,i+1}$, $\Theta_{i+1,i+2}$, and $\Theta_{i+2,i+3}$ that the map counts has last corner $\Theta_{i,i+3}$, for Maslov index reasons.

First, consider the case where i equals $3j-2$ or $3j-1$. Examining periodic domains, we see that precisely one of these quadrilaterals, say ψ_0, will have zero multiplicity at the base point w. Thinking of $\underline{f}^+_{i,i+1,i+2,i+3}(\Theta_{i,i+1}\otimes\Theta_{i+1,i+2}\otimes\Theta_{i+2,i+3})$ as a sum of terms corresponding to each homotopy class of quadrilateral, the term corresponding to ψ_0 will be $\Theta_{i,i+3}\otimes e^{A_K(\psi_0)}\otimes e^{A_M(\psi_0)}$, and all the other nonzero summands will be $U^n\cdot\Theta_{i,i+3}\otimes r$

with $n \geq 1$ and $r \in R$. We immediately see that $A_M(\psi_0)$ is a multiple of M_j. It is also not hard to see that this multiple really only depends on the position of the component of $\Theta_{\beta\gamma}$ in the torus portion of a standard diagram; in particular, this multiple should be the same for all such i.

When $i = 3j$, things work slightly differently. If we arrange our diagram appropriately, there will now be N holomorphic rectangles with Maslov index -1 and zero multiplicity at w which don't cancel; these can be labelled as $\psi_0, \ldots, \psi_{N-1}$ so that $A_M(\psi_n) = nM_j + (c-n)M_{j+1}$, as these triangles only differ by $\boldsymbol{\eta}^i\boldsymbol{\eta}^{i+3}$-periodic domains. The rest of the calculation proceeds as before; in particular, $A_K(\psi_n)$ is independent of n. The result follows. □

We define chain maps $\underline{f}^+_{(i)} : \underline{CF}^+(Y_{0,i}; R) \to \underline{CF}^+(Y_{0,i+1}; R)$ by

$$\underline{f}^+_{(i)}([\mathbf{x}, j] \otimes r) = \underline{f}^+_{0,i,i+1}\big(([\mathbf{x}, j] \otimes r) \otimes \Theta_{\eta^i\eta^{i+1}}\big).$$

That these are indeed chain maps follows from the usual untwisted arguments, together with the fact that the quantity A used in the definition of $\underline{f}^+_{0,i,i+1}$ is additive under splicing. When $i \equiv 0, 2 \bmod 3$, there are also maps $\underline{f}^+_{(i),\mathfrak{s}}$ for each $\mathfrak{s}$ in $\mathrm{Spin}^c(W_0)$ if $i \equiv 0$ or $\mathrm{Spin}^c(W'_N)$ if $i \equiv 2$; these only count triangles which represent $\mathfrak{s}$ (identifying the appropriate fillings of $X_{0,i,i+1}$ with W_0 or W'_N).

For each $i > 0$, we have a map $\underline{H}_i : \underline{CF}^+(Y_{0,i}; R) \to \underline{CF}^+(Y_{0,i+2}; R)$ by

$$\underline{H}_i([\mathbf{x}, j] \otimes r) = \underline{f}^+_{0,i,i+1,i+2}\big(([\mathbf{x}, j] \otimes r) \otimes \Theta_{i,i+1} \otimes \Theta_{i+1,i+2}\big),$$

which is also a chain map.

Furthermore, there are chain maps

$$\underline{g}_i : \underline{CF}^+(Y_{0,i}; R) \to \underline{CF}^+(Y_{0,i+3}; R)$$

given by

$$\underline{g}_i([\mathbf{x}, j] \otimes r) = \underline{f}^+_{0,i,i+3}\big(([\mathbf{x}, j] \otimes r) \otimes (\Theta_{i,i+3} \otimes r_i)\big),$$

with r_i as furnished by Lemma 4.1.

We need a number of minor results to establish the long exact sequence, as well as for later; we break them up into the next two Propositions. Henceforth, we write $i \equiv j$ to mean that i and j are equivalent mod 3. We say that two generators $x = [\mathbf{x}, j] \otimes e^a$ and $x' = [\mathbf{x}', j'] \otimes e^{a'}$ are *connected by a disk* if there exists $\phi \in \pi_2(\mathbf{x}, \mathbf{x}')$ such that $n_w(\phi) = j - j'$ and $A(\phi) = a' - a$. There is a similar notion of $\mathbf{x}'$ being connected to $\mathbf{x}$ and some of the $\Theta_{i,j}$ by a polygon. If it is clear from context which $\Theta_{i,j}$ we mean, we simply say that $\mathbf{x}$ and $\mathbf{x}'$ are *connected by a polygon.*

Proposition 4.2. *For $i \geq 1$, there are subcomplexes C_i of $\underline{CF}^+(Y_{0,i}; R)$ and projection maps $\pi_i : \underline{CF}^+(Y_{0,i}; R) \to C_i$ such that the following hold.*

a) The maps π_i are chain maps.

b) If $i \equiv 0$, then $C_i \cong \underline{CF}^+(Y_{0,i}) \otimes \mathbb{Z}[\mathbb{Z}] \otimes \mathbb{Z}[\mathbb{Z}/N\mathbb{Z}] = \bigoplus_{l,n} C_i(l, n)$, where the sum is over $l \in \mathbb{Z}$ and $n \in \mathbb{Z}/N\mathbb{Z}$, and $C_i(l, n) \cong \underline{CF}^+(Y_{0,i})$.

c) If $i \equiv 1$, then $C_i \cong \underline{CF}^+(Y_{0,i})$.

d) If $i \equiv 2$, then $C_i \cong \underline{CF}^+(Y_{0,i}) \otimes \mathbb{Z}[\mathbb{Z}] = \bigoplus_l C_i(l)$, where the sum is over $l \in \mathbb{Z}$, and $C_i(l) \cong \underline{CF}^+(Y_{0,i})$.

e) If $i \equiv 0$, then $\pi_{i+1} \circ \underline{f}^+_{(i)} \circ \pi_i = \pi_{i+1} \circ \underline{f}^+_{(i)}$, and $\pi_{i+2} \circ \underline{H}^+_i \circ \pi_i = \pi_{i+2} \circ \underline{H}^+_i$.

f) If $i \equiv 1$, then $\pi_{i+1} \circ \underline{f}^+_{(i)} = \underline{f}^+_{(i)} \circ \pi_i$.

g) If $i \equiv 1, 2$, then $\pi_{i+3} \circ \underline{g}_i$ takes C_i isomorphically to C_{i+3}.

Proof. Recall the discussion from the end of Section 2.1, from which we cull the following. The $\boldsymbol{\alpha\gamma}$- and $\boldsymbol{\alpha\delta}$-periodic domains will each correspond to a subgroup of $K(\mathcal{H})$ (the same for each); call this $K_0(\mathcal{H})$. For periodic domains $\mathcal{P}$, $A_K(\mathcal{P})$ is the image of $\partial\mathcal{P}$ in $K(\mathcal{H})$; thus for these periodic domains, $A_K(\mathcal{P}) \in K_0(\mathcal{H})$. There is also some periodic domain

P, whose boundary contains β_g with multiplicity d, such that $K(\mathcal{H}) = K_0(\mathcal{H}) \oplus \mathbb{Z}P$, and any element of $K(\mathcal{H})$ corresponds to a $\boldsymbol{\alpha\beta}$-periodic domain. For any of these periodic domains, $A_M(\mathcal{P}) = 0$.

Therefore, the value of A_M on a bigon is determined by its endpoints; the same is true of A_K up to K_0 if $i \equiv 0, 1$. So, there is a function z_0 from intersection points in our diagram to $K(\mathcal{H}) \oplus M(\mathcal{H})$, such that $[\mathbf{x}, j] \otimes e^a$ and $[\mathbf{x}', j'] \otimes e^{a'}$ are connected by a disk if and only if $z_0(\mathbf{x}') - z_0(\mathbf{x}) - (a' - a)$ is in the image of the $\boldsymbol{\alpha\eta}^i$-periodic domains in $K(\mathcal{H}) \oplus 0$ for appropriate i. So, for any fixed element s of $K(\mathcal{H}) \oplus M(\mathcal{H})$, let $C_i^0(s)$ be the subgroup of $\underline{CF}^+(Y_{0,i}; R)$ generated over $\mathbb{Z}$ by elements of the form $[\mathbf{x}, j] \otimes e^{z_0(\mathbf{x})+s+k}$, for $k \in K(\mathcal{H})$ if $i \equiv 0$, and for $k \in K_0(\mathcal{H})$ otherwise. Then this group is a subcomplex, isomorphic to $\underline{CF}^+(Y_{0,i})$, and in fact $\underline{CF}^+(Y_{0,i}; R)$ splits as a direct sum of subcomplexes of this form (for varying s).

Let $s_j = c \cdot \sum_{l=1}^{j-1} M_j$, where c is as given by Lemma 4.1. If $i = 3j-2$, let $C_i = C_i^0(s_j)$. If $i = 3j - 1$, let $C_i = \bigoplus_{l \in \mathbb{Z}} C_i^0(s_j + lP)$; denote the summands by $C_i(l)$. If $i = 3j$, let $C_i = \bigoplus_{\substack{l \in \mathbb{Z} \\ n \in \mathbb{Z}/N\mathbb{Z}}} C_i^0(s_j + lP + nM_j)$; denote the summands by $C_i(l, n)$. Let π_i be the projection from the full complexes down to these subcomplexes. It is clear that these maps are chain maps, and that these groups satisfy the isomorphisms of claims b), c), and d).

Let us examine the maps $\underline{f}^+_{(i)}$ and $\underline{H}_i$. If $i = 3j$, for any $m \in M(\mathcal{H})$, the image of $C_i \otimes e^m$ under the former map will lie in $\bigoplus_{n=0}^{N-1} C_{i+1} \otimes e^{m+nM_j}$, since the counted triangles may have boundaries traversing M_j, but not any other translates of δ_g. In particular, $\pi_{i+1} \circ \underline{f}^+_{(i)}$ will only be nontrivial on elements of C_i. The image of $C_i \otimes e^m$ under the latter map will likewise lie in $\bigoplus_{n=0}^{N-1} C_{i+2} \otimes e^{m+nM_j}$, so we can say the same for $\pi_{i+1} \circ \underline{H}_i$, showing claim e).

If $i \equiv 1$, then the image of $C_i \otimes e^m$ under $\underline{f}^+_{(i)}$ will lie in $C_{i+1} \otimes e^m$, since none of the triangles this map counts will traverse any of the δ_g translates. This gives f).

For each point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^i}$, there is a canonical nearest point $\mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta^{i+3}}$ and small triangle $\psi(\mathbf{x}) \in \pi_2(\mathbf{x}, \Theta_{i,i+3}, \mathbf{x}')$ that admits a single holomorphic representative. Let $\underline{\widetilde{g}}_i$ be the summand of $\underline{g}_i$ which counts only this triangle. If the U^0 coefficient of r_i is r_i^0, then $\underline{\widetilde{g}}_i$ will take $[\mathbf{x}, j] \otimes r_{\mathbf{x}}$ to $[\mathbf{x}', j] \otimes r_i^0 \cdot r_{\mathbf{x}}$, since the points which measure $A_M(\psi(\mathbf{x}))$ are arranged to lie away from the small triangles. Therefore, when i equals $3j-2$ or $3j-1$, for each $m \in M(\mathcal{H})$ the map $\underline{\widetilde{g}}_i$ gives an isomorphism from $C_i \otimes e^m$ to $C_{i+3} \otimes e^m$, in light of Lemma 4.1 and the definition of s_j. In particular $\underline{\widetilde{g}}_i$ and hence $\pi_{i+3} \circ \underline{\widetilde{g}}_i$ take C_i to C_{i+3} isomorphically. By a standard area-filtration argument, it follows that the same is true for $\pi_{i+3} \circ \underline{g}_i$, giving the last assertion. □

Proposition 4.3. *If $i \equiv 0$, then there is a function $n_0^i : Spin^c(W_0) \to \mathbb{Z}/N\mathbb{Z}$ such that $\underline{f}^+_{(i),\mathfrak{s}} = \underline{f}^+_{(i),\mathfrak{s}}|_{C_i(l, n_0^i(\mathfrak{s}))}$. If $\mathfrak{s}$ and $\mathfrak{s}'$ both have the same restriction to Y, this function has the property that $n_0^i(\mathfrak{s}) = n_0^i(\mathfrak{s}')$ if and only $\mathfrak{s}|_{Y_0}$ and $\mathfrak{s}'|_{Y_0}$ belong to the same* $\mathrm{PD}[N\mu]$-*orbit in $Spin^c(Y_0)$.*

If $i \equiv 2$, the image of $C_i(l)$ under $\underline{f}^+_{(i),\mathfrak{s}}$ lies in $C_{i+1}(l_N^i(l, \mathfrak{s}), n_N^i(\mathfrak{s}))$ for some functions $l_N^i : \mathbb{Z} \times Spin^c(W_N') \to \mathbb{Z}$ and $n_N^i : Spin^c(W_N') \to \mathbb{Z}/N\mathbb{Z}$. If $\mathfrak{s}$ and $\mathfrak{s}'$ both have the same restriction to Y, n_N^i has the property that $n_N^i(\mathfrak{s}) = n_N^i(\mathfrak{s}')$ if and only $\mathfrak{s}|_{Y_0}$ and $\mathfrak{s}'|_{Y_0}$ belong to the same $PD[N\mu]$-orbit in $Spin^c(Y_N)$. The function l_N^i satisfies $l_N^i(l + k, \mathfrak{s} + m\mathrm{PD}[\widetilde{dF}]) = l_N^i(l, \mathfrak{s}) + k + m$.

Proof. If $i = 3j$, suppose that generators x and x' of C_i admit triangles ψ and ψ' representing $\mathfrak{s} \in \mathrm{Spin^c}(W_0)$ that connect them to respective generators y and y' of C_{i+1}. Then y and y' are certainly connected by a disk, so x will also be connected to y' by ψ spliced with this disk, which represents $\mathfrak{s}$. Clearly x must be connected to $x' \otimes e^{a_K + a_M}$ by a disk for some $a_K \in K(\mathcal{H})$ and $a_M \in M(\mathcal{H})$, and we can splice this disk with ψ' to get another triangle representing $\mathfrak{s}$ connecting x and y'. Since any two triangles connecting x and y' that represent the same $\mathrm{Spin^c}$ structure will have the same value of A_M, it follows quickly that $a_M = 0$. Thus, if $x \in C_i(\ell, n)$ and $x' \in C_i(\ell', n')$, then $n = n'$, and we set this value to be $n_0^i(\mathfrak{s})$.

Suppose that $\mathfrak{s}$ and $\mathfrak{s}'$ both have the same restriction to Y and satisfy $\mathfrak{s}'|_{Y_0} - \mathfrak{s}|_{Y_0} = k\mathrm{PD}[\mu]$. It is not hard to see that if we choose the function z_0 of the last proposition carefully, then we may ensure that for $\psi \in \pi_2(\mathbf{x}, \Theta_{i,i+1}, \mathbf{y})$ and $\psi' \in \pi_2(\mathbf{x}, \Theta_{i,i+1}, \mathbf{y}')$ representing these two Spin^c structures, $A_M(\psi') - A_M(\psi) - (z_0(\mathbf{y}') - z_0(\mathbf{y}))$ equals kM_j plus an element of $K(\mathcal{H})$. It follows that $n_0^i(\mathfrak{s}) = n_0^i(\mathfrak{s}')$ if and only if k is a multiple of N.

The corresponding claims when $i \equiv 2$ follows along similar lines; the only real difference is that we now note that for two triangles ψ, ψ' connecting the same two points representing the same Spin^c structure, in addition to $A_M(\psi) = A_M(\psi')$, we also have $A_K(\psi') - A_K(\psi) \in K_0(\mathcal{H})$. Also, note that $\mathrm{PD}[\widetilde{dF}]$ is the Poincaré dual of a class in $H_2(W_N')$ whose associated periodic domain represents P; from this, the claim about l_N^i is clear. $\square$

4.3 The long exact sequence

We first prove a long exact sequence in terms of the above, and then we translate it into a more invariant result. In the following, for any map $\underline{f}$ whose source and target are respectively $\underline{CF}^+(Y_{0,i}; R)$ and $\underline{CF}^+(Y_{0,j}; R)$, we write $\underline{\pi f}$ for $\pi_j \circ \underline{f} \circ \pi_i$, and similarly for induced maps on homology. The precomposition by π_i can be thought of as a restriction of the domain.

Theorem 4.4. *There is a long exact sequence*

$$H_*(C_1) \xrightarrow{\underline{\pi F}^+_{(1)}} H_*(C_2) \xrightarrow{\underline{\pi F}^+_{(2)}} H_*(C_3) \xrightarrow{\underline{\pi F}^+_{(3)}} H_*(C_4) \to \ldots;$$

furthermore, there exists a quasi-isomorphism

$$\psi^+ : M(\underline{\pi F}^+_{(2)}|_{C_2}) \to C_4,$$

where M denotes the mapping cone.

Proof. We follow the strategy of [23]. To do this, we will show that there are chain nullhomotopies $\underline{\pi H}_i : C_i \to C_{i+2}$ of $\underline{\pi f}^+_{(i+1)} \circ \underline{\pi f}^+_{(i)}$, and that the maps $\underline{\pi f}^+_{(i+2)} \circ \underline{\pi H}_i - \underline{\pi H}_{i+1} \circ \underline{\pi f}^+_{(i)}$ are quasi-isomorphisms.

We consider the moduli space of quadrilaterals $\psi \in \pi_2(\mathbf{x}, \Theta_{i,i+1}, \Theta_{i+1,i+2}, \mathbf{w})$ with $\mu(\psi) = 0$. This space is compact and oriented of dimension 1 (recall our use of the phrase "moduli space"), so the signed count of its boundaries vanishes. Noting that the Θ points are cycles, we see that with appropriate orientation conventions, a twisted count of the ends yields

$$\begin{aligned} \underline{\partial}^+_{\eta^0\eta^{i+2}} \circ \underline{H}_i + \underline{H}_i \circ \underline{\partial}^+_{\eta^0\eta^i} &= \underline{f}^+_{(i+1)} \circ \underline{f}^+_{(i)} \\ &+ \textstyle\sum_{\mathfrak{s}} \underline{f}^+_{0,i,i+2,\mathfrak{s}_1}\big(\ \cdot \ \otimes \underline{f}^+_{i,i+1,i+2,\mathfrak{s}_2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2})\big), \end{aligned}$$

where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are the appropriate restrictions of $\mathfrak{s}$. It is not hard to see that

$$\pi_{i+2} \circ \left(\underline{\partial}^+_{\eta^0\eta^{i+2}} \circ \underline{H}_i + \underline{H}_i \circ \underline{\partial}^+_{\eta^0\eta^i}\right) \circ \pi_i = \underline{\pi\partial}^+_{\eta^0\eta^{i+2}} \circ \underline{\pi H}_i + \underline{\pi H}_i \circ \underline{\pi\partial}^+_{\eta^0\eta^i}$$

and that

$$\pi_{i+2} \circ \left(\underline{f}^+_{(i+1)} \circ \underline{f}^+_{(i)}\right) \circ \pi_i = \underline{\pi f}^+_{(i+1)} \circ \underline{\pi f}^+_{(i)}.$$

So as usual, to show that $\underline{\pi H}_i$ is a chain nullhomotopy, it suffices to show that for each $\mathfrak{s}_1 \in \mathrm{Spin}^c(X_{0,i,i+2})$, we have

$$\sum_{\{\mathfrak{s} \left| \mathfrak{s}|_{X_{0,i,i+2}} = \mathfrak{s}_1\right\}} \underline{f}^+_{i,i+1,i+2,\mathfrak{s}|_{X_{i,i+1,i+2}}}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2}) = 0. \tag{4.1}$$

To verify this equation, in fact, the untwisted arguments work with few changes. Suppose first that we have two triangles ψ and ψ' with $n_w(\psi) = n_w(\psi')$ connecting the same three points $\mathbf{x}_{i+j} \in \mathbb{T}_{\eta^{i+j}} \cap \mathbb{T}_{\eta^{i+j+1}}$ for $i \geq 1$ and $j = 0, 1, 2$. Then $\partial\psi'$ will equal $\partial\psi$

plus a number of doubly-periodic domains plus some triply-periodic domain $\mathcal{P}$, where $\partial\mathcal{P}$ is a multiple of $\gamma_g - \beta_g - N\delta_g$ (up to replacing circles with corresponding translates). All of these domains go to 0 in both $L(\mathcal{H})$ and $M(\mathcal{H})$, the latter because δ_g is of order N in $M(\mathcal{H})$; and so $A_K(\psi') - A_K(\psi) = A_M(\psi') - A_M(\psi) = 0$. Thus, for triangles through any three such points, $n_w(\psi)$ determines $A_K(\psi)$ and $A_M(\psi)$.

Having shown this, the usual arguments show that for all $i \neq 0$ and each $k > 0$, there are two homotopy classes of triangles

$$\psi_k^{\pm} \in \pi_2(\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i,i+2})$$

with $\mu(\psi_k^{\pm}) = 0$ and $n_w(\psi_k^+) = n_w(\psi_k^-)$ (both of which equal a quadratic function of k, depending on N and the precise position of $\Theta_{\beta\gamma}$), and each of these admit a single holomorphic representative. Since $n_w(\psi_k^+) = n_w(\psi_k^-)$, we know that $A_K(\psi_k^+) = A_K(\psi_k^-)$ and $A_M(\psi_k^+) = A_M(\psi_k^-)$; hence, the two corresponding terms in $\underline{f}^+_{i,i+1,i+2,\mathfrak{s}}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2})$ will appear with the same twisting coefficient. Furthermore, orientations can be arranged so that the terms appear with opposite signs. Likewise, when $i \equiv 2$, any of the other points Θ' in $\mathbb{T}_\gamma \cap \mathbb{T}_\beta$ will have two homotopy classes of triangles $\psi_k^{\pm}(\Theta') \in \pi_2(\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta')$ with $\mu(\psi_k^{\pm}(\Theta')) = 0$ and $n_w(\psi_k^+(\Theta')) = n_w(\psi_k^-(\Theta'))$, and we can ensure that these yield cancelling terms as well. Thus, $\underline{f}^+_{i,i+1,i+2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2}) = 0$; the left hand side of Equation 4.1 is equal to sum of terms equal to $\underline{f}^+_{i,i+1,i+2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2})$, and so this equation holds, proving that the $\underline{\pi H}_i$ are nullhomotopies.

Next, we examine the desired quasi-isomorphisms. Consider the moduli space of pentagons $\psi \in \pi_2(\mathbf{x}, \Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3}, \mathbf{w})$ with $\mu(\psi) = 0$. We count signed boundary components, noting again that the Θ points are cycles and that $\underline{f}^+_{i,i+1,i+2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2}) = 0$. Ignoring the terms that vanish due to these observations, and orienting appropriately, we have

$$\underline{f}^+_{(i+2)} \circ \underline{H}_i - \underline{H}_{i+1} \circ \underline{f}^+_{(i)} = \underline{\partial}^+_{\eta^{i+3}} \circ \underline{J} + \underline{J} \circ \underline{\partial}^+_{\eta^i} +$$

$$\sum_{\mathfrak{s}} \underline{f}^+_{0,i,i+3,\mathfrak{s}_1}\big(\ \cdot \ \otimes \underline{f}^+_{i,i+1,i+2,i+3,\mathfrak{s}_2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2} \otimes \Theta_{i+2,i+3})\big),$$

where $\underline{J}$ counts pentagons ψ with $\mu(\psi) = -1$. In fact, we can dispense with the Spin^c structures in the second line.

First, consider the case where i is not 0. In these cases, it is easy to see that the above equation together with 4.2 e) and f) imply that

$$\underline{\pi f}^+_{(i+2)} \circ \underline{\pi H}_i - \underline{\pi H}_{i+1} \circ \underline{\pi f}^+_{(i)} = \underline{\pi \partial}^+_{\eta^{i+3}} \circ \underline{\pi J} + \underline{\pi J} \circ \underline{\pi \partial}^+_{\eta^i} + \\ \underline{\pi f}^+_{0,i,i+3}\big(\pi_i(\cdot) \otimes \underline{f}^+_{i,i+1,i+2,i+3}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2} \otimes \Theta_{i+2,i+3})\big).$$

Hence, in these cases, it suffices to show that the term in the second line of this expression is a quasi-isomorphism when considered as a map from C_i to C_{i+3}; but this is immediate from Proposition 4.2 g).

We have only to show the analogous result for $i = 3j$. This requires a little bit of finesse.

Any element of C_i is of the form $x = [\mathbf{x}, i] \otimes e^{k+s_j+nM_j}$, for some $k \in K(\mathcal{H})$ and $n \in \mathbb{Z}/N\mathbb{Z}$. Then $\pi_{i+1} \circ \underline{f}^+_{(i)}(x)$ gives a count of triangles ψ originating at $\mathbf{x}$ with $m_{p^i}(\partial\psi) = c - n$ (where c is as in Lemma 4.1), and $\pi_{i+2} \circ \underline{H}_i(x)$ does the same for rectangles.

Given $\ell \in \mathbb{Z}/N\mathbb{Z}$, define $\underline{J'}_\ell$ to be a map which counts holomorphic pentagons $\psi \in \pi_2(\mathbf{x}, \Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3}, \mathbf{w})$ with $\mu(\psi) = -1$, but *only* those pentagons for which $m_{p^i}(\partial\psi) = \ell$. We bundle these into a single map $\underline{J'}$, by having $\underline{J'}(x) = \underline{J'}_{c-n}(x)$ when x is of the form given above. Then, examining boundary components of moduli spaces of pentagons ψ with $\mu(\psi) = 0$ and $m_{p^i}(\partial\psi) = c_j - n$, we see that

$$\pi_{i+3} \circ \underline{f}^+_{(i+2)} \circ \pi_{i+2} \circ \underline{H}_i(x) \quad - \quad \pi_{i+3} \circ \underline{H}_{i+1} \circ \pi_{i+1} \circ \underline{f}^+_{(i)}(x) = \\ \underline{\pi\partial}^+_{\eta^{i+3}} \circ \underline{J'}(x) + \underline{J'} \circ \underline{\pi\partial}^+_{\eta^i}(x) + G(x),$$

where

$$G(x) = \sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\eta^{i+3}}} \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{i,i+3},\mathbf{w})\\ \mu(\psi)=0}} C(\psi)\cdot\#\mathcal{M}(\psi)\cdot[\mathbf{w}, i-n_w(\psi)]\cdot e^{k+s_j+nM_j+A(\psi)},$$

for appropriate constants $C(\psi)\in\mathbb{Z}[[U]]\otimes R$. Precisely, write

$$\underline{f}^+_{i,i+1,i+2,i+3,\mathfrak{s}_2}(\Theta_{i,i+1}\otimes\Theta_{i+1,i+2}\otimes\Theta_{i+2,i+3})) = \sum_{q\in\mathbb{Z}/N\mathbb{Z}} e^{qM_j}\cdot P(q)\cdot\Theta_{i,i+3},$$

where $P(q)\in\mathbb{Z}[[U]]\otimes R$ contains no powers of e^{M_j}; then if $m_{p^i}(\partial\psi)=a$, $C(\psi)$ should equal $P(c-n-a)$. Indeed, the coefficients of $P(q)$ all arise from triangles $\psi_1\in\pi_2(\Theta_{i,i+1},\Theta_{i+1,i+2},\Theta_{i+2,i+3},\Theta_{i,i+3})$ for which $m_{p^i}(\partial\psi_1)=q$, and we essentially want to count pairs (ψ,ψ_1) such that $m_{p^i}(\partial(\psi*\psi_1))=c-n$.

Let $\widetilde{G}$ be the count of triangles that G performs, with the second sum restricted to canonical small triangles. If ψ_0 is such a small triangle, recall that we have arranged so that $m_{p^i}(\partial\psi_0)=0$. Hence, Lemma 4.1 implies that $C(\psi_0)$ will have U^0 coefficient equal to $e^{k+(c-n)M_j+nM_{j+1}}$ for some $k\in K(\mathcal{H})$ and $n\in\mathbb{Z}/N\mathbb{Z}$. Therefore, $\widetilde{G}$ is an isomorphism; as usual, an area filtration argument can then be used to show that G is an isomorphism as well, proving the quasi-isomorphism statement for $i\equiv 0$.

Thus, the mapping cone Lemma of [23] finishes the proof. □

We now refine Theorem 4.4. Recall that the usual cobordism-induced map $\underline{F}^+_{W'_N,\mathfrak{s}}$: $\underline{HF}^+(Y_N,\mathfrak{s}|_{Y_N})\to\underline{HF}^+(Y,\mathfrak{s}|_Y)$, and indeed the groups themselves, are well-defined only up to a sign and the actions of $H^1(Y_N)$ and $H^1(Y)$. We may fix these so that the following holds.

Theorem 4.5. *For special K, there is a long exact sequence*

$$\ldots\xrightarrow{\underline{F}^+_0}\underline{HF}^+(Y_0,\mathfrak{S}^N_0(\mathfrak{t}_0))\xrightarrow{\underline{F}^+_N}\underline{HF}^+(Y_N,\mathfrak{S}_N(\mathfrak{t}_0))\otimes\mathbb{Z}[\mathbb{Z}]$$
$$\xrightarrow{\underline{F}^+}\underline{HF}^+(Y,\mathfrak{S}_\infty(\mathfrak{t}_0))\otimes\mathbb{Z}[\mathbb{Z}]\to\ldots,$$

where each group is taken with totally twisted coefficients. We have

$$\underline{F}^+ = \sum_{\mathfrak{s}\in\mathfrak{S}_{N\infty}(\mathfrak{t}_0)} \underline{F}^+_{W'_N,\mathfrak{s}} \otimes \mathbb{I}$$

where $\underline{F}^+_{W'_N}$ is the usual twisted coefficient map induced by W'_N, appropriately fixed; and so there exists a quasi-isomorphism

$$\psi^+ : M\left(\sum_{\mathfrak{s}\in\mathfrak{S}_{N\infty}(\mathfrak{t}_0)} \underline{f}^+_{\mathfrak{s}}\right) \to \underline{CF}^+\left(Y_0, \mathfrak{S}^N_0(\mathfrak{t}_0)\right),$$

where $\underline{f}^+_{\mathfrak{s}}$ is the chain map inducing $\underline{F}^+_{\mathfrak{s}}$.

Furthermore, we can choose $\mathfrak{t}_N \in \mathfrak{S}_N(\mathfrak{t}_0)$ and $\mathfrak{t}_\infty \in \mathfrak{S}(\mathfrak{t}_0)$, and identifications

$$\underline{CF}^+(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)) \otimes \mathbb{Z}[\mathbb{Z}] \cong \bigoplus_{i\in\mathbb{Z}} \underline{CF}^+(Y_N, \mathfrak{t}_N + i\mathrm{PD}[F']|_{Y_N})$$

and

$$\underline{CF}^+(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)) \otimes \mathbb{Z}[\mathbb{Z}] \cong \bigoplus_{i\in\mathbb{Z}} \underline{CF}^+(Y, \mathfrak{t}_\infty + i\mathrm{PD}[F']|_Y),$$

where we treat each summand as distinct, so that if $\underline{f}^+_{\mathfrak{s}}$ takes summand i to summand j, then $\underline{f}^+_{\mathfrak{s}+k\mathrm{PD}[F']|_{W'_N}}$ takes summand i to summand $j+k$.

Proof. For $\mathfrak{t}_0 \in \mathrm{Spin}^c(Y_0)$, let $\mathfrak{S}^*_0(\mathfrak{t}_0) = \mathfrak{t}_0 + \mathbb{Z}\cdot\mathrm{PD}[\mu]$ and $\mathfrak{S}^*_N(\mathfrak{t}_0) = \mathfrak{S}_N(\mathfrak{t}_0) + \mathbb{Z}\cdot\mathrm{PD}[\mu]$. It follows quickly from Theorem 4.4 that there is a long exact sequence of the form

$$\begin{aligned} \ldots \xrightarrow{\underline{F}^+_0} \underline{HF}^+(Y_0, \mathfrak{S}^*_0(\mathfrak{t}_0)) &\xrightarrow{\underline{F}^+_N} \underline{HF}^+(Y_N, \mathfrak{S}^*_N(\mathfrak{t}_0)) \otimes \mathbb{Z}[\mathbb{Z}] \\ &\xrightarrow{\underline{F}^+} \bigoplus^N \underline{HF}^+(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)) \otimes \mathbb{Z}[\mathbb{Z}] \to \ldots, \end{aligned}$$

by identifying each C_i with one of the above groups and noting that each map of the long exact sequence naturally splits along Spin^c structures as given.

All three groups above further decompose into N subgroups: the first two by breaking into sums indexed by the N different $\mathbb{Z}\cdot N\mathrm{PD}[\mu]$-suborbits of Spin^c structures, and the last in the obvious manner. So to show that the exact sequence of the statement exists, it suffices to show that the maps in the long exact sequence respect these decompositions (i.e. each map takes each summand of its source to a distinct summand of its target). It is not difficult to see that $\underline{F}^+_N$ does, by (for example) examining Lemma A.1. That the other two do follows from the statement about n_0^i and n_N^i in Proposition 4.3.

The identification of $\underline{F}^+_{W'_N}$ and the mapping cone statement are both clear. Finally, the last statement follows from the second paragraph of Proposition 4.3. □

For an integer $\delta > 0$, let $\underline{CF}^\delta$ denote the subcomplex of $\underline{CF}^+$ consisting of elements in the kernel of U^δ. In the statements and proofs of all the above, we can go through line by line and systematically replace $\underline{CF}^+$ and $\underline{HF}^+$ with $\underline{CF}^\delta$ and $\underline{HF}^\delta$ (interpreting every map with one of these groups as source as having appropriately restricted domain). It is straightforward to then go through and check that everything still makes sense and holds true. We will specifically need the quasi-isomorphism statement, so we state it precisely, and enhance it a bit.

Corollary 4.6. *If K is special, then there is a quasi-isomorphism*

$$\psi^\delta : M\left(\sum_{\mathfrak{s}\in\mathfrak{S}_{N\infty}(\mathfrak{t}_0)} \underline{f}^+_{W'_N}|_{\underline{CF}^\delta(Y_N,\mathfrak{s}|_{Y_N})} \otimes \mathbb{I}\right) \to \underline{CF}^\delta(Y_0,\mathfrak{S}_0^N(\mathfrak{t}_0)).$$

The mapping cone inherits a U action from the summands of its chain group; with respect to this action, the quasi-isomorphisms are U-equivariant.

Proof. We explain the last statement quickly. The chain group of the mapping cone is $\underline{CF}^\delta(Y_N,\mathfrak{S}_N(\mathfrak{t}_0)) \oplus \underline{CF}^\delta(Y,\mathfrak{S}_\infty(\mathfrak{t}_0))$. Examining the proof of the mapping cone Lemma as given in [23], we can express ψ^δ in terms of this decomposition by

$$\psi^\delta(x,y) = \underline{H}_2(x) + \underline{f}_0^+(y);$$

these maps are U-equivariant, and thus so is ψ^δ. □

Chapter 5

Twisted Knot Floer Homology

Knot Floer homology was originally defined in [17] and [26] for nullhomologous knots; in [21] the definition is extended to knots that are only rationally nullhomologous. We recall the construction here, extending it to the case of twisted coefficients. We then prove analogues of some of the results in [21] in the twisted setting, which relate the knot filtration with the homologies of large N surgeries on the knot. We write out a large portion of the details in this, even though most of the results here follow in a similar manner to previous results; we do this mainly so that we may be unambiguous when referring to these results afterwards.

5.1 The knot filtration

Take a doubly-pointed Heegard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z)$ for an oriented, rationally nullhomologous knot $K \subset Y$, which need not be standard. We can form the usual chain complex $CF^\circ(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w)$ (where $\circ$ denotes any of $\widehat{\ }, +, -$, or ∞), but the extra point endows this with an additional $\mathbb{Z}$ filtration, via the ordering on the fibers of $\underline{\mathrm{Spin}}^c(Y, K)$. In [21], it is asserted that the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy type of $CF^\circ(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z, \xi) = CF^\circ(Y, K, \xi)$ is an invariant of Y, K and $\xi \in \underline{\mathrm{Spin}}^c(Y, K)$.

There is an obvious alteration of this construction to get a filtration on the twisted complex. One still needs an invariance result, but it is easy to see that the filtration "cares" only about the generators of the complex, and not about the coefficients appearing next to them, so that invariance comes from the respective invariance results for the twisted three-manifold invariant and the untwisted knot filtration.

Let us be more precise about this. Recall the conventional set up for twisted coefficients: one chooses complete sets of paths in the sense of Section 3 of [19], which (together with the choice of basepoint w) yield a surjective additive assignment h from $\pi_2(\mathbf{x}, \mathbf{y})$ (when it is nonempty) to $H^1(Y_{\alpha\delta})$. Then we take our universal twisted coefficient ring to be $\mathbb{Z}[H^1(Y_{\alpha\delta})]$, with twisting specified by h.

So, if $\xi \in \underline{\mathrm{Spin}}^c(Y, K)$, we let $\mathfrak{T}(\xi)$ be the set of $[\mathbf{x}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\delta) \times \mathbb{Z} \times \mathbb{Z}$ such that

$$\underline{\mathfrak{s}}_{w,z}(\mathbf{x}) - (i-j)\mathrm{PD}[\mu] = \xi. \tag{5.1}$$

Then, for any $\mathbb{Z}[H^1(Y_{\alpha\delta})]$-module M, define $\underline{CFK}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z, \xi; M)$ to be the $\mathbb{Z}[H^1(Y_{\alpha\delta})]$-module $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z, \xi) \otimes_{\mathbb{Z}} M$, where $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z, \xi)$ is the free abelian group generated by $[\mathbf{x}, i, j] \in \mathfrak{T}(\xi)$, with differential

$$\begin{aligned}&\underline{\partial}^\infty([\mathbf{x}, i, j] \otimes m) = \\ &\quad \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) | \mu(\phi) = 1\}} \# \widehat{\mathcal{M}}(\phi) \cdot [\mathbf{y}, i - n_w(\phi), j - n_z(\phi)] \otimes e^{h(\phi)} \cdot m,\end{aligned}$$

which is a finite sum if the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w)$ is strongly-$G_K(\xi)$ admissible. The differential takes $\underline{CFK}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z, \xi; M)$ to itself in light of Equation 2.2, and the usual arguments show that $(\underline{\partial}^\infty)^2 = 0$. If we don't specify M, we take M to be $\mathbb{Z}[H^1(Y_{\alpha\delta})]$.

Give this group the obvious $\mathbb{Z} \oplus \mathbb{Z}$-grading, by declaring $[\mathbf{x}, i, j] \otimes m$ to be in grading (i, j), for $m \in M$. While this grading is not an invariant, it naturally induces a $\mathbb{Z} \oplus \mathbb{Z}$-filtration that is, with filtration level (i, j) consisting of the direct sum of summands with grading (i', j') with $i' \leq i$ and $j' \leq j$.

Theorem 5.1. *The $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy type of the $\mathbb{Z}[U] \otimes \mathbb{Z}[H^1(Y_{\alpha\delta})]$-module $\underline{CFK}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, w, z, \xi; M) = \underline{CFK}^\infty(Y, K, \xi; M)$ is an invariant of Y, K, and ξ.*

Proof. Considering the above remarks, this is a routine adaptation of the invariance arguments from [17]. □

We abbreviate $\underline{CFK}^\infty(Y, K, \xi; M)$ to $\underline{C}(Y, K, \xi; M)$. If S is a subset of $\mathbb{Z} \oplus \mathbb{Z}$ such that $(i', j') \in S$ whenever $(i, j) \in S$ with $i' \leq i, j' \leq j$ – or if S is the complement of one such region in another – then we have a submodule $\underline{C}\{S\}(Y, K, \xi; M)$ generated by elements of the form $[\mathbf{x}, i, j] \otimes m$ with $(i, j) \in S$, which is naturally a subquotient chain complex of $\underline{C}(Y, K, \xi; M)$. Such a chain complex will also be an invariant of Y, K, and ξ. Of particular interest is the set $S = \{(i, j) | i \geq 0 \text{ or } j \geq 0\}$; for this S, we write $\underline{C}^+(Y, K, \xi; M)$. Later, we will also use $S = \{(i, j) | \delta \geq \max\{i, j\} \geq 0\}$, for some integer δ; for this S, we write $\underline{C}^\delta(Y, K, \xi; M)$. If we only wish to calculate $\underline{C}^+(Y, K, \xi; M)$ or $\underline{C}^\delta(Y, K, \xi; M)$, we need only have a weakly admissible Heegaard diagram to have a well-defined differential.

5.2 Relationship with large N surgeries

Note that for $\xi \in \underline{\text{Spin}}^c(Y, K)$, the map from $\underline{C}\{i \geq 0\}(Y, K, \xi)$ to $\underline{CF}^+(Y, G_K(\xi))$ that takes $[\mathbf{x}, i, j] \otimes e^\ell$ to $[\mathbf{x}, i] \otimes e^\ell$ is an isomorphism. This map essentially forgets the second component of the filtration. Let $v_{\xi,K}$ be the composition of the quotient map from $\underline{C}^+(Y, K, \xi)$ to $\underline{C}\{i \geq 0\}(Y, K, \xi)$ with this isomorphism.

On the other hand, the complex $\underline{C}\{j \geq 0\}(Y, K, \xi)$ is isomorphic to $\underline{CF}^+(Y, G_{-K}(\xi))$, by the map taking $[\mathbf{x}, i, j] \otimes e^\ell$ to $[\mathbf{x}, j] \otimes e^\ell$. The reason we have $G_{-K}(\xi)$ here is that we are keeping track of intersections with z in this complex, which is only meaningful if we use the Spin^c structure for our points relative to z. Let $h_{\xi,K}$ be the composition of the quotient map from $\underline{C}^+(Y, K, \xi)$ to $\underline{C}\{j \geq 0\}(Y, K, \xi)$ with this isomorphism.

Of course, all of this works with an arbitrary coefficient module as well. We henceforth drop K from the notation and write v_ξ and h_ξ.

Now, fix a knot $K \subset Y$, and a family $\mathcal{F}$ of diagrams for K. Suppose N is large enough so that there is a $\mathfrak{t}$-proper diagram in $\mathcal{F}$ for each torsion $\mathfrak{t} \in \mathrm{Spin}^c(Y_N)$; in such a $\mathfrak{t}$-proper diagram, choose a small triangle $\psi_\mathfrak{t}$ with a corner representing $\mathfrak{t}$, for each $\mathfrak{t}$. Recall that for all N such that $N\mu + \lambda$ is not special, there are canonical identifications of the $\boldsymbol{\alpha\gamma}-$periodic domains in the Heegaard triple for W'_N with the $\boldsymbol{\alpha\delta}-$periodic domains; and that we have a canonical identification of $H^1(Y)$ and $H^1(Y_N)$, via the images of $H_2(Y)$ and $H_2(Y_N)$ in $H_2(W'_N)$. Any triangle representing $E_{K,N}\big(\mathfrak{s}_w(\psi_\mathfrak{t})\big)$ can be written as $\psi = \psi_\mathfrak{t} + \phi_{\alpha\gamma} + \phi_{\alpha\delta} + \phi_{\gamma\delta}$. We define $h'(\psi)$ to be $h(\phi_{\alpha\gamma}) + h(\phi_{\alpha\delta})$, where h denotes the additive assignments used to define $\underline{CF}^+(Y_N, \mathfrak{t})$ and $\underline{CF}^+(Y, \mathfrak{s}_w(\psi_\mathfrak{t})|_Y)$, and both $h(\phi_{\alpha\gamma})$ and $h(\phi_{\alpha\delta})$ are considered as elements of $H^1(Y)$.

We now can define a map $\Psi^+_{\mathfrak{t},N} : \underline{CF}^+(Y_N, \mathfrak{t}; M) \to \underline{C}^+(Y, K, \Xi(\mathfrak{t}); M)$ by

$$\Psi^+_{\mathfrak{t},N}([\mathbf{x}, i] \otimes m) =$$

$$\sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta} \sum_{\left\{\psi \in \pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{w}) \,\middle|\, \substack{\mu(\psi) = 0 \\ \mathfrak{s}_w(\psi) = \mathfrak{s}_+(\mathfrak{t})}\right\}} \#\widehat{\mathcal{M}}(\psi) \cdot [\mathbf{y}, i - n_w(\psi), i - n_z(\psi)] \otimes e^{h'(\psi)} \cdot m,$$

where M can be considered to be a module over both $\mathbb{Z}[H^1(Y)]$ and $\mathbb{Z}[H^1(Y_N)]$.

Theorem 5.2. *Fix a family of doubly-pointed standard Heegaard triples for K, and a $\mathbb{Z}[H^1(Y)]$-module M. For torsion $\mathfrak{t} \in Spin^c(Y_N)$, write $\xi(\mathfrak{t})$ for $E_{K,N}\big(\mathfrak{s}_{K+}(\mathfrak{t})\big)$. If N is sufficiently large, then there are $\mathfrak{t}$-proper diagrams inducing commutative squares*

$$\begin{array}{ccc}
\underline{CF}^+(Y_N, \mathfrak{t}; M) & \xrightarrow{f^+_{W'_N, \mathfrak{s}_{K+}(\mathfrak{t})}} & \underline{CF}^+(Y, \mathfrak{s}_{K+}(\mathfrak{t})|_Y; M) \\
\Psi^+_{\mathfrak{t},N} \big\downarrow & & \big\downarrow \\
\underline{C}^+(Y, K, \xi(\mathfrak{t}); M) & \xrightarrow{v_{\xi(\mathfrak{t})}} & \underline{CF}^+(Y, \mathfrak{s}_{K+}(\mathfrak{t})|_Y; M)
\end{array}$$

and

$$\begin{array}{ccc} \underline{CF}^+(Y_N, \mathfrak{t}; M) & \xrightarrow{\underline{f}^+_{W'_N, \mathfrak{s}_{K-}(\mathfrak{t})}} & \underline{CF}^+(Y, \mathfrak{s}_{K-}(\mathfrak{t})|_Y; M) \\ \Psi^+_{\mathfrak{t},N}\big\downarrow & & \big\downarrow \\ \underline{C}^+(Y, K, \xi(\mathfrak{t}); M) & \xrightarrow{h_{\xi(\mathfrak{t})}} & \underline{CF}^+(Y, \mathfrak{s}_{K-}(\mathfrak{t})|_Y; M). \end{array}$$

In each square, the vertical maps are isomorphisms of relatively $\mathbb{Z}$-graded complexes over $\mathbb{Z}[U] \otimes \mathbb{Z}[H^1(Y)]$; the righthand maps are each multiplication by elements of $H^1(Y)$, which depend on precisely how the maps $\underline{f}^+_{W'_N}$ are fixed.

Proof. Let us start with the claim that $\Psi^+_{\mathfrak{t},N}$ is an isomorphism. It is clear from the definition that it is a chain map. We also need to check that, as defined, $\Psi^+_{\mathfrak{t},N}$ indeed does take $\underline{CF}^+(Y_N, \mathfrak{t}; M)$ to $\underline{C}^+(Y, K, \xi(\mathfrak{t}); M)$. For any homotopy class ψ of triangles in $\pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{y})$ representing $\mathfrak{s}_{K+}(\mathfrak{t})$, we have that

$$E_{K,N}(\psi) = \underline{\mathfrak{s}}_{w,z}(\mathbf{y}) + \big(n_w(\psi) - n_z(\psi)\big) \cdot \mathrm{PD}[\mu] = E_{K,N}\big(\mathfrak{s}_{K+}(\mathfrak{t})\big) = \xi(\mathfrak{t}).$$

Then, according to Equation 5.1, $[\mathbf{y}, i - n_w(\psi), i - n_z(\psi)] \in \mathfrak{T}(\xi(\mathfrak{t}))$. Of course, twisting plays no role here.

For each point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ representing $\mathfrak{t}$, we have a canonical smallest triangle $\psi_{\mathbf{x}} \in \pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{x}')$ supported in the winding region. According to Proposition 3.4, assuming N is sufficiently large, this triangle satisfies $\mathfrak{s}_w(\psi_{\mathbf{x}}) = \mathfrak{s}_{K+}(\mathfrak{t})$ and $\mathfrak{s}_z(\psi_{\mathbf{x}}) = \mathfrak{s}_{K-}(\mathfrak{t})$.

Let Ψ^+_0 be the map which takes $[\mathbf{x}, i] \otimes m$ to $[\mathbf{x}', i - n_w(\psi_{\mathbf{x}}), i - n_z(\psi_{\mathbf{x}})] \otimes e^{h'(\psi_{\mathbf{x}})} \otimes m$. This map is an isomorphism, owing to the fact that only one of $n_w(\psi_{\mathbf{x}})$ and $n_z(\psi_{\mathbf{x}})$ is nonzero (indeed, this is why we need to work with $\underline{CFK}\{i \geq 0 \text{ or } j \geq 0\}$); twisting also has to be considered here, but ends up having no effect due to the canonical isomorphism $H^1(Y) \cong H^1(Y_N)$. Since $\#\mathcal{M}(\psi_{\mathbf{x}}) = 1$, the map Ψ^+_0 is also a summand of $\Psi^+_{\mathfrak{t},N}$. In fact, making the area of the winding region sufficiently small, every other summand of $\Psi^+_{\mathfrak{t},N}$ will be of lower order with respect to the energy filtration, as defined in [19]. The argument from that paper then applies, showing that $\Psi^+_{\mathfrak{t},N}$ must also be an isomorphism.

Hence, it is clear that the top square commutes, for both the top and bottom horizontal maps count precisely the same triangles, with coefficients differing by a constant factor in $H^1(Y)$.

Switching the basepoints w and z, we can say the same for the bottom square: in the context of this diagram, the map $\Psi^+_{\mathfrak{t},N}$ counts the same triangles, but with respect to the basepoint z these triangles represent $\mathfrak{s}_{K-}(\mathfrak{t})$ instead, again by Proposition 3.4. (Of course, w and z lie in the same component of $\Sigma \setminus \boldsymbol{\alpha} \setminus \boldsymbol{\gamma}$, so that $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}_z(\mathbf{x})$ for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$.) □

5.3 The Künneth Formula

Finally, we will need a version of the Künneth formula for behavior under connect sums. Recall that given two relative Spin^c structures $\xi_i \in \underline{\mathrm{Spin}}^c(Y_i, K_i)$ for $i = 1, 2$, there is a notion of the connect sum $\xi_1 \# \xi_2 \in \underline{\mathrm{Spin}}^c(Y_1 \# Y_2, K_1 \# K_2)$. To define this, fill in ξ_i to get a particular vector field on Y_i, which is tangent to K_i. Consider points $p_i \in K_i$, and take sufficiently small balls B_i around p_i so that the filled in vector fields are normal to ∂B_i at two points, one "going in", the other "going out". Now, remove these balls and glue the complements together along the boundaries so that the vector fields match up; remove a small neighborhood of $K_1 \# K_2$ to get $\xi_1 \# \xi_2$.

The gluing is equivariant with respect to the inclusion maps on relative second cohomology; e.g., if $a \in H^2(Y_1, K_1)$, and $i^* : H^2(Y_1, K_1) \to H^2(Y_1 \# Y_2, K_1 \# K_2)$ is induced by inclusion, then $(\xi_1 + a) \# \xi_2 = (\xi_1 \# \xi_2) + i^*(a)$.

Of course, the connect sum of two knots equipped with reference longitudes is canonically equipped with a longitude. With respect to this, we have the following.

Lemma 5.3. $q_{K_1 \# K_2}(\xi_1 \# \xi_2) = q_{K_1}(\xi_1) + q_{K_2}(\xi_2)$.

Proof. In the definition of q_K, Equation 2.5, both terms add under connect sums. That the first term adds can be seen by explicit drawing a diagram for the connect sum; that the second term adds is straightforward. □

Note that $H^1(Y_1\#Y_2) \cong H^1(Y_1) \oplus H^1(Y_2)$ canonically. Thus, if M_i is a module over $\mathbb{Z}[H^1(Y_i)]$ for $i = 1, 2$, then $M_1 \otimes_{\mathbb{Z}} M_2$ is a module over $\mathbb{Z}[H^1(Y_1)] \otimes \mathbb{Z}[H^1(Y_2)] \cong \mathbb{Z}[H^1(Y_1\#Y_2)]$.

Theorem 5.4. *For rationally null-homologous knots $K_1 \subset Y_1$ and $K_2 \subset Y_2$, and relative $Spin^c$ structures $\xi_1 \in \underline{Spin}^c(Y_1, K_1)$ and $\xi_2 \in \underline{Spin}^c(Y_2, K_2)$, there is a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain homotopy equivalence of complexes over $\mathbb{Z}[U] \otimes \mathbb{Z}[H^1(Y_1\#Y_2)]$*

$$\begin{aligned} \underline{CFK}^\infty(Y_1, K_1, \xi_1; M_1) \otimes_{\mathbb{Z}[U]} \quad & \underline{CFK}^\infty(Y_2, K_2, \xi_2; M_2) \to \\ & \underline{CFK}^\infty(Y_1\#Y_2, K_1\#K_2, \xi_1\#\xi_2; M_1 \otimes_{\mathbb{Z}} M_2). \end{aligned}$$

Proof. This is a routine adaptation of the argument from [17]. □

Chapter 6

Twisted Surgery Formula

We now combine the results of the previous two sections, to achieve results akin to Theorem 6.1 of [21].

Let K be a special knot, and $\mathfrak{t}_0 \in \mathrm{Spin}^c(Y_0)$ an μ-torsion Spin^c structure. Choose some $\mathfrak{t}_\infty \in \mathfrak{S}_\infty(\mathfrak{t}_0)$, and write ξ_i for $[\mathfrak{t}_\infty - i\mathrm{PD}[K], -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}] \in \underline{\mathrm{Spin}}^c(Y, K)$.

Define the map

$$\underline{f}^+_{K,\mathfrak{t}_0} : \bigoplus_{i\in\mathbb{Z}} \underline{C}^+(Y, K, \xi_i) \to \bigoplus_{i\in\mathbb{Z}} \underline{CF}^+\big(Y, G_K(\xi_i)\big)$$

so that $x \in \underline{C}^+(Y, K, \xi_i)$ goes to

$$v_{\xi_i}(x) \oplus h_{\xi_i}(x) \in \underline{CF}^+\big(Y, G_K(\xi_i)\big) \oplus \underline{CF}^+\big(Y, G_K(\xi_{i+1})\big).$$

Even though $G_K(\xi_i) = G_K(\xi_{i+d})$, we treat the corresponding summands as distinct.

Theorem 6.1. *There is a quasi-isomorphism from $M(\underline{f}^+_{K,\mathfrak{t}_0})$ to $\underline{CF}^+(Y_0, \mathfrak{t}_0)$. Furthermore, $M(\underline{f}^+_{K,\mathfrak{t}_0})$ admits a relative $\mathbb{Z}$-grading and a U-action which the quasi-isomorphism respects.*

We prove this in a number of steps. For the first, we set some notation.

Fix a family $\mathcal{F}$ of standard diagrams for (Y, K). For the rest of this section, every chain complex we speak of hereafter will be isomorphic to one calculated from an element of $\mathcal{F}$.

For $\mathfrak{t} \in \mathrm{Spin}^c(Y_N)$, we write

$$\mathfrak{s}_+^k(\mathfrak{t}) \equiv \mathfrak{s}_{K+}(\mathfrak{t}) - k \cdot \mathrm{PD}[F']|_{W_N'},\ \mathfrak{s}_-^k(\mathfrak{t}) \equiv \mathfrak{s}_{K-}(\mathfrak{t}) + k \cdot \mathrm{PD}[F']|_{W_N'}$$

for $k \geq 0$, where $\mathfrak{s}_{K\pm}(\mathfrak{t}) \in \mathrm{Spin}^c(W_N')$ are as described in Section 2.5; when $k = 0$, we drop the superscript. We can write

$$\underline{f}^+ = \sum_{\mathfrak{t} \in \mathfrak{S}_N} \sum_{k \geq 0} \big(\underline{f}^+_{\mathfrak{s}_+^k(\mathfrak{t})} + \underline{f}^+_{\mathfrak{s}_-^k(\mathfrak{t})}\big),$$

with notation as in the statement of Theorem 4.5. Then define

$$\underline{f}^+_* = \sum_{\mathfrak{t} \in \mathfrak{S}_N} \big(\underline{f}^+_{\mathfrak{s}_+(\mathfrak{t})} + \underline{f}^+_{\mathfrak{s}_-(\mathfrak{t})}\big),$$

i.e. the summands of $\underline{f}^+$ for $k = 0$.

Proposition 2.4 says that under the assumption that $\mathfrak{t}_0$ is μ-torsion, $\mathfrak{S}_\infty(\mathfrak{t}_0)$ and $\mathfrak{S}_N(\mathfrak{t}_0)$ consist of torsion structures, so that $\underline{CF}^+\big(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)\big)$ and $\underline{CF}^+\big(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)\big)$ come equipped with absolute $\mathbb{Q}$-gradings, which extend to $\underline{CF}^+\big(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)\big) \otimes \mathbb{Z}[\mathbb{Z}]$ and $\underline{CF}^+\big(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)\big) \otimes \mathbb{Z}[\mathbb{Z}]$. Furthermore, the usual untwisted grading shift formula still holds, so that if x is a homogenous element of $\underline{CF}^+\big(Y_N, \mathfrak{t}\big) \otimes \mathbb{Z}[\mathbb{Z}]$, then

$$\widetilde{\mathrm{gr}}\big(\underline{f}^+_{\mathfrak{s}}(x)\big) - \widetilde{\mathrm{gr}}(x) = \frac{c_1^2(\mathfrak{s}) - 2\chi(W_N') - 3\sigma(W_N')}{4} = \frac{c_1^2(\mathfrak{s}) + 1}{4} \tag{6.1}$$

if N is large enough so that W_N' is negative definite.

Proposition 6.2. *Fix an integer δ. Then for all sufficiently large N, if $U^\delta x = 0$ for x in $\underline{CF}^+\big(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)\big) \otimes \mathbb{Z}[\mathbb{Z}]$, then $\underline{f}^+_*(x) = \underline{f}^+(x)$.*

Proof. Choose a particular value of N. Let us find sufficient conditions so that $\underline{f}^+_*(x) = \underline{f}^+(x)$ when $U^\delta x = 0$, in terms of N.

Take x to be a homogenous generator of $\underline{CF}^+(Y_N, \mathfrak{t}) \otimes \mathbb{Z}[\mathbb{Z}]$ which satisfies $U^\delta x = 0$, where $\mathfrak{t} \in \mathfrak{S}_N(\mathfrak{t}_0)$ and the chain complex is constructed from a $\mathfrak{t}$-proper diagram. If x itself is nonzero, then Theorem 5.2 guarantees in particular that $\underline{f}^+_{\mathfrak{s}_+(\mathfrak{t})}(x) \neq 0$. Of course, we also have $U^\delta \underline{f}^+_{\mathfrak{s}_+(\mathfrak{t})}(x) = 0$. Since there are a finite number of intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$, it follows that there are two constants L^-_∞ and L^+_∞, depending only on δ and on the family $\mathcal{F}$, such that

$$L^-_\infty \leq \widetilde{\mathrm{gr}}\big(\underline{f}^+_{\mathfrak{s}_+(\mathfrak{t})}(x)\big) \leq L^+_\infty.$$

The same can be said of $\underline{f}^+_{\mathfrak{s}_-(\mathfrak{t})}(x)$.

Using the grading shift formula (6.1) and the definition of Q_K, we then have that

$$\widetilde{\mathrm{gr}}\big(\underline{f}^+_{\mathfrak{s}^k_+(\mathfrak{t})}(x)\big) = \widetilde{\mathrm{gr}}\big(\underline{f}^+_{\mathfrak{s}_+(\mathfrak{t})}(x)\big) + Q_K\big(-k; \mathfrak{s}_+(\mathfrak{t})\big)$$

and

$$\widetilde{\mathrm{gr}}\big(\underline{f}^+_{\mathfrak{s}^k_-(\mathfrak{t})}(x)\big) = \widetilde{\mathrm{gr}}\big(\underline{f}^+_{\mathfrak{s}_-(\mathfrak{t})}(x)\big) + Q_K\big(k; \mathfrak{s}_-(\mathfrak{t})\big)$$

for $k \geq 0$. Then, a sufficient condition for $\underline{f}^+_*(x) = \underline{f}^+(x)$ is that $Q_K\big(-k; \mathfrak{s}_+(\mathfrak{t})\big)$ and $Q_K\big(k; \mathfrak{s}_-(\mathfrak{t})\big)$ are both less than $L^-_\infty - L^+_\infty$ for $k \neq 0$.

Let us examine $Q_K\big(-k; \mathfrak{s}_+(\mathfrak{t})\big)$, which we can think of as a quadratic function of $k \in \mathbb{Q}$. It is not hard to see that

$$Q_K\big(-k; \mathfrak{s}_+(\mathfrak{t})\big) = k^2 \frac{[\widetilde{dF'}]^2}{d^2} + k\Big(\frac{[\widetilde{dF'}]^2}{d^2} - q_K\big(\mathfrak{s}_+(\mathfrak{t})\big)\Big).$$

Note first that if $N > 0$, then we have

$$\frac{-\left(\frac{[\widetilde{dF'}]^2}{d^2} - q_K\left(\mathfrak{s}_+(\mathfrak{t})\right)\right)}{2\left(\frac{[\widetilde{dF'}]^2}{d^2}\right)} \leq \frac{-2N - C_q}{-2N} = 1 + \frac{-C_q}{-2N},$$

recalling Lemma 3.3, Proposition 2.7, and the fact that $\mathfrak{s}_+(\mathfrak{t})$ is represented by a small triangle. Hence, for any value of $\epsilon > 0$, the value of $k \in \mathbb{Q}$ that maximizes $Q\big(-k; \mathfrak{s}_+(\mathfrak{t})\big)$ will be bounded above by $1+\epsilon$ if our value of N is large enough. If our value of N is large enough so that this holds with some $\epsilon < \frac{1}{2}$, we will then have

$$Q_K\big(-k; \mathfrak{s}_+(\mathfrak{t})\big) \leq Q_K\big(-1; \mathfrak{s}_+(\mathfrak{t})\big)$$

for all integers k greater than 1.

We also have (for all N) that

$$Q_K\big(-1; \mathfrak{s}_+(\mathfrak{t})\big) = 2\frac{[\widetilde{dF'}]^2}{d^2} - q_K\left(\mathfrak{s}_+(\mathfrak{t})\right) \leq C_q - N,$$

where the inequality again comes from Lemma 3.3 and Proposition 2.7. Hence, putting the two previous inequalities together yields

$$Q_K\big(-k; \mathfrak{s}_+(\mathfrak{t})\big) \leq C_q - N$$

for k an integer greater than 0 if N is large enough.

Let us turn to the other function $Q_K\big(k; \mathfrak{s}_-(\mathfrak{t})\big)$. In a completely analogous manner, we find that if our N is large enough, then we will have

$$Q_K\big(k; \mathfrak{s}_-(\mathfrak{t})\big) \leq Q_K\big(1; \mathfrak{s}_-(\mathfrak{t})\big)$$

for all integers k greater than 1. To find an upper bound for $Q_K\big(1; \mathfrak{s}_-(\mathfrak{t})\big)$, recall that this is equal to

$$\frac{c_1^2(\mathfrak{s}_-^1(\mathfrak{t}))-c_1^2(\mathfrak{s}_-(\mathfrak{t}))}{4}=\frac{c_1^2(\mathfrak{s}_-^1(\mathfrak{t}))-c_1^2(\mathfrak{s}_+(\mathfrak{t}))}{4}-\frac{c_1^2(\mathfrak{s}_-(\mathfrak{t}))-c_1^2(\mathfrak{s}_+(\mathfrak{t}))}{4}$$

$$=Q_K\big(2;\mathfrak{s}_+(\mathfrak{t})\big)-Q_K\big(1;\mathfrak{s}_+(\mathfrak{t})\big).$$

The latter is equal to $q_K\big(\mathfrak{s}_w(\psi_\mathfrak{t})\big)+2\frac{\widetilde{[dF']}^2}{d^2}$, and we observed in the proof of Proposition 3.4 that

$$q_K\big(\mathfrak{s}_w(\psi_\mathfrak{t})\big)+2\frac{\widetilde{[dF']}^2}{d^2}\leq C_q-N.$$

Hence,

$$Q_K\big(k;\mathfrak{s}_-(\mathfrak{t})\big)\leq Q_K\big(1;\mathfrak{s}_-(\mathfrak{t})\big)\leq C_q-N$$

for $k>0$ if N is large enough.

Therefore, if our N is sufficiently large, we will have

$$Q_K\big(-k;\mathfrak{s}_+(\mathfrak{t})\big)<L_\infty^- - L_\infty^+$$

and

$$Q_K\big(k;\mathfrak{s}_-(\mathfrak{t})\big)<L_\infty^- - L_\infty^+$$

for $k\neq 0$. There are a finite number of $\mathfrak{t}$ in $\mathfrak{S}_N(\mathfrak{t}_0)$; hence, for large enough N, it follows that $\underline{f}_*^+(x)=\underline{f}^+(x)$ for all $x\in\underline{CF}^+\big(Y_N,\mathfrak{S}_N(\mathfrak{t}_0)\big)\otimes\mathbb{Z}[\mathbb{Z}]$ that satisfy $U^\delta x=0$. $\square$

Corollary 6.3. *Write $\xi(\mathfrak{t})$ for $E_{K,N}\big(\mathfrak{s}_{K+}(\mathfrak{t})\big)$. Then for all $\delta\geq 0$, there is a quasi-isomorphism from $M(\underline{f}_{K,N,\mathfrak{t}_0}^{(\delta)})$ to $\underline{CF}^\delta\big(Y_0,\mathfrak{S}_0^N(\mathfrak{t}_0)\big)$ for large enough N, where*

$$\underline{f}_{K,N,\mathfrak{t}_0}^{(\delta)}:\bigoplus_{i\in\mathbb{Z}}\underline{C}^\delta(Y,K,\xi(\mathfrak{t}+i\mathrm{PD}[F']|_{Y_N}))\to\bigoplus_{i\in\mathbb{Z}}\underline{CF}^\delta\Big(Y,G_K\left(\xi\left(\mathfrak{t}+i\mathrm{PD}[F']|_{Y_N}\right)\right)\Big)$$

is given by taking x in summand i to

$$v_{\xi(\mathfrak{t}+i\mathrm{PD}[F']|_{Y_N})}(x) + h_{\xi(\mathfrak{t}+i\mathrm{PD}[F']|_{Y_N})}(x)$$

in summands i and $i+1$.

Proof. This follows from Proposition 6.2 and Corollary 4.6, via Theorem 5.2. More precisely, given δ, choose N large enough so that both Proposition 6.2 and Theorem 5.2 hold. In the commutative squares of Theorem 5.2, we can clearly replace $+$ with δ in all the groups, and we take M to be $\mathbb{Z}[H^1(Y)] \otimes \mathbb{Z}[\mathbb{Z}]$. Summing over Spin^c structures then gives a commutative square

$$\begin{array}{ccc}
\underline{CF}^\delta\big(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)\big) \otimes \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\sum\left(\underline{f}^+_{W'_N,\mathfrak{s}_{K+}(\mathfrak{t})} + \underline{f}^+_{W'_N,\mathfrak{s}_{K-}(\mathfrak{t})}\right)\otimes \mathbb{I}} & \underline{CF}^\delta\big(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)\big) \otimes \mathbb{Z}[\mathbb{Z}] \\
\Big\downarrow {\scriptstyle \sum \Psi^+_{\mathfrak{t},N}\otimes\mathbb{I}} & & \Big\downarrow \\
\bigoplus \underline{C}^\delta(Y, K, \xi(\mathfrak{t})) \otimes \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\sum\left(v_{\xi(\mathfrak{t})} + h_{\xi(\mathfrak{t})}\right)\otimes\mathbb{I}} & \underline{CF}^\delta\big(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)\big) \otimes \mathbb{Z}[\mathbb{Z}]
\end{array}$$

where the vertical maps are isomorphisms of chain complexes, and each sum is taken over $\mathfrak{t} \in \mathfrak{S}_N(\mathfrak{t}_0)$. Proposition 6.2 then says that replacing the upper horizontal map by $\underline{f}^+$ doesn't change the commutativity. Hence, the mapping cone of the bottom is isomorphic to the mapping cone of $\underline{f}^+|_{\underline{CF}^\delta\left(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)\right)\otimes\mathbb{Z}[\mathbb{Z}]}$, which in turn is quasi-isomorphic to $\underline{CF}^\delta\big(Y_0, \mathfrak{S}_0^N(\mathfrak{t}_0)\big)$ by Corollary 4.6. Finally, comparing the commutative square above with the last claim of Theorem 4.5 establishes the Corollary. □

Proof of Theorem 6.1. Proposition 2.9 shows that $M(\underline{f}^{(\delta)}_{K,N,\mathfrak{t}_0})$ and $M(\underline{f}^{(\delta)}_{K,\mathfrak{t}_0})$ are the same. So combining this with Corollary 6.3, we have for each value of δ a quasi-isomorphism

$$\psi^{(\delta)} : M(\underline{f}^{(\delta)}_{K,\mathfrak{t}_0}) \to \underline{CF}^\delta\big(Y_0, \mathfrak{S}_0^N(\mathfrak{t}_0)\big)$$

for all large N. The latter is just $\underline{CF}^\delta(Y_0, \mathfrak{t}_0)$. Specifically, since there are only finitely many Spin^c structures for which the homology is non-trivial, we can choose large N so that $\mathfrak{S}_0^N(\mathfrak{t}_0) \setminus \{\mathfrak{t}_0\}$ doesn't contain any of them.

We need only replace each δ with a $+$. But, in fact, this can be done directly: it is straightforward to work on the chain level to show that the corresponding map

$$\psi^+ : M(\underline{f}^+_{K,\mathfrak{t}_0}) \to \underline{CF}^+\big(Y_0, \mathfrak{S}_0^N(\mathfrak{t}_0)\big)$$

is in fact injective and surjective on homology.

Each summand of $\bigoplus_{i\in\mathbb{Z}} \underline{C}^+(Y, K, \xi_i)$ and $\bigoplus_{i\in\mathbb{Z}} \underline{CF}^+\big(Y, G_K(\xi_i)\big)$ admits a relative $\mathbb{Z}$-grading, as is always the case for twisted coefficient Floer homology: if two generators are connected by a disk, that disk is unique, and the grading difference is the Maslov index of this disk. We can extend this to a relative $\mathbb{Z}$-grading on the entire mapping cone, by demanding that $\underline{f}^+_{K,\mathfrak{t}_0}$ lowers grading by one. It is now easy to see that ψ^+ respects relative $\mathbb{Z}$-gradings, by simply inspecting Maslov indices of polygons in the diagram. There is also the U-action on the mapping cone induced by that of the summands; ψ^+ clearly respects this also, and the action lowers the relative $\mathbb{Z}$-grading by 2. $\square$

Chapter 7

Computations

For the computations we are interested in, we will need to compute the twisted filtered chain complexes for two different families of knots: the Borromean knots B_g and the O-knots $O_{p,q}$. The Borromean knot B_1 is gotten by performing 0-surgery on two of the components of the Borromean link in S^3, thinking of the last component as a knot in $S^1 \times S^2 \# S^1 \times S^2$; the Borromean knot B_g is the g-fold connect sum of B_1. The knot $O_{p,q}$ is obtained by performing $-p/q$-surgery along one component of the Hopf link in S^3, and thinking of the other component as a knot in $L(p,q)$.

We make computations for these knots, and then we proceed to our final results.

7.1 The Borromean knots B_g

We want to calculate $\underline{C}^+(\#^{2g} S^1 \times S^2, B_g, \xi)$ for $\xi \in \underline{\mathrm{Spin}}^c(\#^{2g} S^1 \times S^2, B_g)$. Our calculations are essentially carried out in [9], but we want to detail the result.

Let Z_g hereafter denote $\#^{2g} S^1 \times S^2$.

We begin with the case $g = 1$, by drawing the weakly admissible Heegaard diagram for (Z_1, B_1) of Figure 7.1. It is not hard to verify that there are four intersection points in our diagram that represent the torsion Spin^c structure $\mathfrak{t}_{B_1} \in \mathrm{Spin}^c(Z_1)$, which we

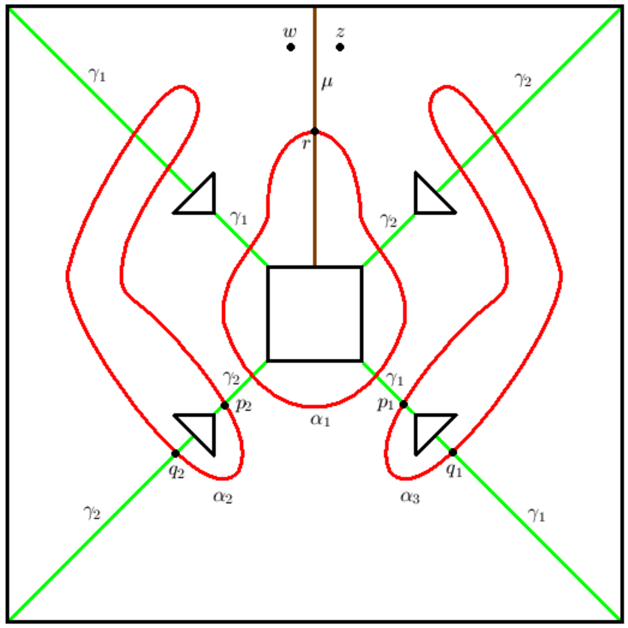

Figure 7.1: An admissible doubly-pointed Heegaard diagram for (Z_1, B_1). The outer square and inner square are feet of a 1-handle, the upper left and lower right triangles are feet of another and the upper right and lower left triangles are feet of a third. The points p_1, q_1, p_2, q_2 and r are components of the generators $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, and $\mathbf{y}_4$.

will denote $\mathbf{y}_1 = \{p_1, p_2, r\}, \mathbf{y}_2 = \{p_1, q_2, r\}, \mathbf{y}_3 = \{q_1, p_2, r\}$, and $\mathbf{y}_4 = \{q_1, q_2, r\}$. There are four other generators representing different Spin^c structures, but it is not difficult to adjust this diagram to get a different weakly admissible diagram with these structures not represented. Hence, $\underline{C}^+(Z_1, B_1, \xi)$ is trivial unless $G_{B_1}(\xi) = \mathfrak{t}_{B_1}$.

Of course, since $\mathfrak{t}_{B_1}$ is torsion, this diagram is also $\mathfrak{t}_{B_1}$-strongly admissible, and so going forward we will concern ourselves with $\underline{C}(Z_1, B_1, \xi) = \underline{CFK}^\infty(Z_1, B_1, \xi)$, instead of the quotient version.

Thinking of the $\mathbf{y}_i$ as generators of the complex $\widehat{CF}(Z_1, \mathfrak{t}_{B_1})$ obtained with respect to the basepoint w, it is easy to compute their relative gradings. Comparing this with the computation of $\widehat{HF}(Z_1, \mathfrak{t}_{B_1})$, we find that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, and $\mathbf{y}_4$ have respective absolute $\mathbb{Q}$-gradings of $1, 0, 0$ and -1.

Let $\xi^0_{B_1} \in \underline{\mathrm{Spin}}^c(Z_1, B_1)$ be the unique relative Spin^c structure which extends to torsion ones on both Z_1 and T^3 (which is 0-surgery on B_1); let $\xi^k_{B_1} = \xi^0_{B_1} + k\mathrm{PD}[\mu]$. It is easy to see that $q_{B_1}(\xi^0_{B_1}) = 0$, and hence that $q_{B_1}(\xi^k_{B_1}) = 2k$. We can also see that $\underline{\mathfrak{s}}_{w,z}(\mathbf{y}_1) = \xi^1_{B_1}, \underline{\mathfrak{s}}_{w,z}(\mathbf{y}_2) = \underline{\mathfrak{s}}_{w,z}(\mathbf{y}_3) = \xi^0_{B_1}$, and $\underline{\mathfrak{s}}_{w,z}(\mathbf{y}_4) = \xi^{-1}_{B_1}$.

Let us focus on $\underline{C}(Z_1, B_1, \xi^0_{B_1})$. According to the above, this will be generated over $R = \mathbb{Z}[H^1(Z_1)]$ by elements of the form $[\mathbf{y}_1, i, i+1], [\mathbf{y}_2, i, i], [\mathbf{y}_3, i, i]$, and $[\mathbf{y}_4, i, i-1]$. So the group $\underline{C}(Z_1, B_1, \xi^0_{B_1})$ can be realized as

$$\underline{C}(Z_1, B_1, \xi^0_{B_1}) \cong \bigoplus_{i,j} A_{i,j},$$

where $A_{i,j}$ is the free R-module generated by $[\mathbf{y}_1, i, i+1]$ if $j = i+1$, by $[\mathbf{y}_2, i, i]$ and $[\mathbf{y}_3, i, i]$ if $j = i$, and by $[\mathbf{y}_4, i, i-1]$ if $j = i-1$ (and otherwise $A_{i,j}$ is trivial). The filtration on $\underline{C}(Z_1, B_1, \xi^0_{B_1})$ is the naturally-induced one: $\underline{C}\{S\})(Z_1, B_1, \xi^0_{B_1}) \cong \bigoplus_{(i,j)\in S} A_{i,j}$. Furthermore, the action of U clearly takes $A_{i,j}$ isomorphically to $A_{i-1,j-1}$.

Thinking of $\underline{C}(Z_1, B_1, \xi^0_{B_1})$ as a filtration on the complex $\underline{CF}^+(Z_1, \mathfrak{t}_0)$ gotten from the diagram with respect to w, the elements of the former inherit an absolute $\mathbb{Q}$-grading from those of the latter: $[\mathbf{y}_n, i, j]$ is assigned the grading of $\mathbf{y}_n$ plus $2i$. This lifts the natural relative $\mathbb{Z}$-grading on the complex. Notice, then, that, the elements of $A_{i,j}$ are all supported in absolute grading $i+j$. Since $A_{i,j}$ is non-empty only when $|i-j| \leq 1$, this immediately implies that the differential on our realization of $\underline{C}(Z_1, B_1, \xi^0_{B_1})$ takes $A_{i,j}$ to $A_{i-1,j} \oplus A_{i,j-1}$ for all i, j. Let $V_{i,j} : A_{i,j} \to A_{i,j-1}$ and $H_{i,j} : A_{i,j} \to A_{i-1,j}$ denote the appropriate restrictions/projections of the differential.

Let M_0 (respectively M_{-1}, M_1) denote the free R-module generated by $\mathbf{y}_2$ and $\mathbf{y}_3$ (resp. $\mathbf{y}_1$, $\mathbf{y}_4$); let M_i be trivial for other values of i. Define $\phi_{i,j} : A_{i,j} \to M_{i-j}$ to be the obvious isomorphisms. It is easy to see that $V_{i,j}$ and $H_{i,j}$ can be understood uniformly via the maps $\phi_{i,j}$. To put this more precisely, there are maps $V : M_i \to M_{i+1}$ and

$H : M_i \to M_{i-1}$ such that we have commutative diagrams

$$\begin{array}{ccc} A_{i,j} & \xrightarrow{V_{i,j}} & A_{i,j-1} \\ \phi_{i,j}\downarrow & & \downarrow\phi_{i,j-1} \\ M_{i-j} & \xrightarrow{V} & M_{i-j+1} \end{array}$$

and

$$\begin{array}{ccc} A_{i,j} & \xrightarrow{H_{i,j}} & A_{i-1,j} \\ \phi_{i,j}\downarrow & & \downarrow\phi_{i-1,j} \\ M_{i-j} & \xrightarrow{H} & M_{i-j-1} \end{array}$$

for all i, j. This holds simply because the counts of disks leaving from $[\mathbf{y}_n, i, j]$ don't depend on i and j, as we are working with the infinity version of knot Floer homology. These maps clearly satisfy $\phi_{i,j} = \phi_{i-1,j-1} \circ U$.

We claim that there is an exact sequence of R-modules

$$0 \to M_{-1} \xrightarrow{V} M_0 \xrightarrow{V} M_1 \to \mathbb{Z} \to 0$$

where each element of $H^1(Z_1)$ acts on $\mathbb{Z}$ by ± 1. To see this, we simply compare the chain complex $\oplus_j A_{0,j}$ with the computation $\underline{\widehat{HF}}(Z_1, \mathfrak{t}_{B_1}) \cong \mathbb{Z}_{(-1)}$ in [9], where the parenthetical subscript denotes the absolute grading.

If we think with respect to the basepoint z instead, it is immediate that we likewise have an exact sequence of R-modules

$$0 \to M_1 \xrightarrow{H} M_0 \xrightarrow{H} M_{-1} \to \mathbb{Z} \to 0.$$

So, we have an understanding of $\underline{C}(Z_1, B_1, \xi^0_{B_1})$. We now turn to understanding $\underline{C}(Z_g, B_g, \xi)$. We take advantage of the Kunneth formula; the first thing that this tells us is that $\underline{C}(Z_g, B_g, \xi)$ is trivial unless $G_{B_g}(\xi) = \mathfrak{t}_{B_g}$, the latter being the torsion Spin^c structure on Z_g. So, again, define $\xi^0_{B_g} \in \underline{\text{Spin}}^c(Z_g, B_g)$ to be the unique structure which extends to torsion ones on Z_g and $S^1 \times \Sigma_g$; and let $\xi^k_{B_g} = \xi^0_{B_g} + k\mathrm{PD}[\mu]$. Then $q_{B_g}(\xi^k_{B_g}) = 2k$.

We focus on $\underline{C}(Z_g, B_g, \xi^0_{B_g})$. The Kunneth formula tells us that this is a g-fold tensor product of complexes isomorphic to $\underline{C}(Z_1, B_1, \xi^0_{B_1})$. Hence, taking $R_g = \mathbb{Z}[H^1(Z_1)]$, we find the following.

Proposition 7.1. *For each g, there exist free R_g-modules M^g_n for $n \in \mathbb{Z}$ and maps $V : M^g_i \to M^g_{i+1}$ and $H : M^g_i \to M^g_{i-1}$ such that the following hold.*

1. *M^g_n is trivial unless $|n| \leq g$, in which case M^g_n is of rank $\begin{pmatrix} 2g \\ g+n \end{pmatrix}$.*

2. *There are exact sequences of R_g-modules*

$$0 \to M^g_{-g} \xrightarrow{V} M^g_{-g+1} \xrightarrow{V} \dots \xrightarrow{V} M^g_g \to \mathbb{Z} \to 0$$

and

$$0 \to M^g_g \xrightarrow{H} M^g_{g-1} \xrightarrow{H} \dots \xrightarrow{H} M^g_{-g} \to \mathbb{Z} \to 0.$$

3. *We may write*

$$\underline{C}(Y, K, \xi) = \bigoplus_{i,j} A_{i,j}$$

so that the filtration on the left side is induced by the grading on the right side, with U taking $A_{i,j}$ to $A_{i-1,j-1}$; the $A_{i,j}$ may be chosen such that there are canonical isomorphisms

$$\phi_{i,j} : A_{i,j} \to M^g_{i-j},$$

and such that $\phi_{i,j} = \phi_{i-1,j-1} \circ U$.

4. *The restriction of the differential on $\underline{C}(Y, K, \xi)$ to $A_{i,j}$ is equal to $V_{i,j} + H_{i,j}$, where*

$V_{i,j}$ and $H_{i,j}$ are defined so as to make the diagrams

$$\begin{array}{ccc} A_{i,j} & \xrightarrow{V_{i,j}} & A_{i,j-1} \\ {\scriptstyle\phi_{i,j}}\downarrow & & \downarrow{\scriptstyle\phi_{i,j-1}} \\ M^g_{i-j} & \xrightarrow{V} & M^g_{i-j+1} \end{array}$$

and

$$\begin{array}{ccc} A_{i,j} & \xrightarrow{H_{i,j}} & A_{i-1,j} \\ {\scriptstyle\phi_{i,j}}\downarrow & & \downarrow{\scriptstyle\phi_{i-1,j}} \\ M^g_{i-j} & \xrightarrow{H} & M^g_{i-j-1} \end{array}$$

commute.

5. *The complex $\underline{C}(Y, K, \xi)$ admits a $\mathbb{Z}$-grading such that $A_{i,j}$ is supported in grading $i + j$.*

Proof. This all follows from the computation for $g = 1$ and the Kunneth formula, except for the exact sequences; these follow as before by comparing with the calculation $\underline{\widehat{HF}}(Z_g) \cong \mathbb{Z}_{(-g)}$ of [9]. □

7.2 O-knots

For the O-knot, of course, non-trivial twisting doesn't occur, as the ambient manifolds are rational homology spheres. Nonetheless, we take a close look at them, since we want to carefully write down what relative Spin^c structures the generators lie in.

Let $K = O_{p,q}$; we restrict to the case $p, q > 0$. In Figure 7.2, we depict a standard doubly-pointed Heegaard triple for K, equipped with the 0-framing as the longitude λ (i.e., surgery along this longitude is the same as surgering the Hopf link in S^3 with coefficents $-\frac{p}{q}$ and 0). Hence, $Y_{\alpha\delta}$ is $L(p, q)$; and, for ease of computation, we choose γ so that $Y_{\alpha\gamma}$ is surgery on K with framing $\mu+\lambda$. We write W'_K for the cobordism $X_{\alpha\gamma\delta}$ filled in by B^4 along $Y_{\gamma\delta}$. This cobordism can also be described as the orientation reversal of

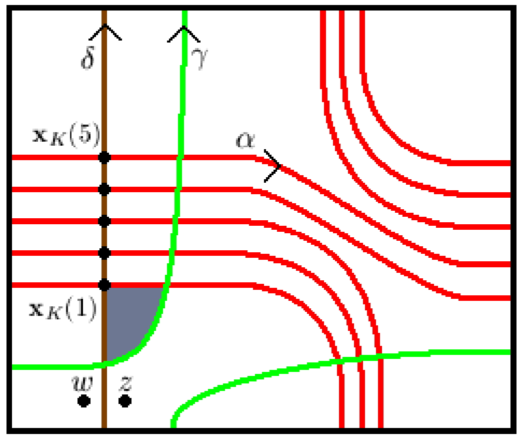

Figure 7.2: A standard Heegaard diagram for W'_K, where $K = O_{5,3}$. The marked points are $\mathbf{x}_K(n)$ for $n = 1, \ldots, 5$. The domain of the triangle ψ_n is shaded in.

the cobordism obtained by attaching a 1-framed 2-handle to K. We fix orientations for the circles, and label the points of $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$ by $\mathbf{x}_K(n), n = 1, \ldots, p$, as shown.

Let $\mathcal{P}$ be the periodic domain such that $\partial\mathcal{P} = -\alpha + p\gamma - (p+q)\delta$ and $n_w(\mathcal{P}) = 0$. The placement of basepoints specifies an orientation of K; with respect to this, the class $[\widetilde{dF'}]$ which generates $H_2(W'_K)$ corresponds to this domain.

For each n, we have a small triangle $\psi_n \in \pi_2(\mathbf{y}, \Theta_{\gamma\delta}, \mathbf{x}_K(n))$ through one of the points $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, which has multiplicity 0 at both basepoints. We calculate using Proposition 2.3 that

$$\langle c_1(\underline{\mathfrak{s}}_{w,z}(\mathbf{x}_K(n)), \phi_*^{-1}([\widetilde{dF'}])\rangle = 2p - 2n + 1,$$

recalling that we have an isomorphism $\phi_* : H_2(Y, K) \to H_2(W'_K)$. If $[\mu] \in H_1(L(p,q)\backslash K)$ is the homology class of an oriented meridian of K, then $\langle \mathrm{PD}[\mu], \phi_*^{-1}([\widetilde{dF'}])\rangle = p$, thinking of $\mathrm{PD}[\mu]$ as an element of $H^2(L(p,q), K)$. Thus,

$$\langle c_1\big(\underline{\mathfrak{s}}_{w,z}(\mathbf{x}_K(n)) + m\mathrm{PD}[\mu]\big), \phi_*^{-1}([\widetilde{dF'}])\rangle = 2(m+1)p - 2n + 1.$$

We can also identify $\mathfrak{s}_w(\psi_n)$ using the Chern class evaluation formula (6.1); we get that

$$\langle c_1(\mathfrak{s}_w(\psi_n)), [\widetilde{dF'}]\rangle = 2p + q - 2n + 1.$$

Furthermore, the initial defining Equation 2.3 of $E_{K,N}$ shows that $E_{K,1}\big(\mathfrak{s}_w(\psi_n)\big) = \underline{\mathfrak{s}}_{w,z}\big(\mathbf{x}_K(n)\big)$; then Proposition 2.6 shows that

$$E_{K,1}\big(\mathfrak{s}_w(\psi_n) + m\phi^{*-1}(\mathrm{PD}[\mu])\big) = \underline{\mathfrak{s}}_{w,z}\big(\mathbf{x}_K(n)\big) + m\mathrm{PD}[\mu].$$

With respect to the orientation on $O_{p,q}$, it is not difficult to see that $\kappa = \frac{q}{p}$, so that $\frac{[\widetilde{dF'}]^2}{d^2} = -\frac{q}{p} - 1$. Hence,

$$\begin{aligned} q_K\big(\underline{\mathfrak{s}}_{w,z}\big(\mathbf{x}_K(n)\big) + m\mathrm{PD}[\mu]\big) &= -\tfrac{q}{p} - 1 + \tfrac{2p+q-2n+1+2pm}{p} \\ &= \tfrac{p-2n+1}{p} + 2m. \end{aligned}$$

In addition, observe that $\underline{\mathfrak{s}}_{w,z}\big(\mathbf{x}_K(1)\big) - \underline{\mathfrak{s}}_{w,z}\big(\mathbf{x}_K(q+1)\big) = \mathrm{PD}[\lambda]$. From this, it is easy to extract that

$$q_K(\xi + \mathrm{PD}[\lambda]) = q_K(\xi) + \frac{2q}{p}.$$

Each of the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ lies in a different absolute Spin^c structure; hence we infer that all differentials in the complex vanish. So, it is straightforward to manipulate the above to get the following summation.

Proposition 7.2. *Take $p, q > 0$, and let $K = O_{p,q}$, equipped with the 0-framed longitude λ (as described above), with oriented meridian μ. Then for each $r \in \mathbb{Z}$, there is precisely one relative Spin^c structure $\xi^r_{p,q}$ with $q_K(\xi^r_{p,q}) = \frac{2r-p-1}{p}$, and $\underline{\mathrm{Spin}}^c(L(p,q),K)$ is composed of precisely these structures. (So, denoting $G_K(\xi^r_{p,q})$ by $\mathfrak{t}^r_{p,q}$, we have that $\mathfrak{t}^{r+p}_{p,q} = \mathfrak{t}^r_{p,q}$, and of course $\xi^r_{p,q} = \big[\mathfrak{t}^r_{p,q}, \frac{2r-p-1}{p}\big]$.) Furthermore, we have*

$$\underline{C}(L(p,q), K, \xi^r_{p,q}) \cong \mathbb{Z}[U, U^{-1}],$$

generated over $\mathbb{Z}$ *by a single generator in grading* $\left(i, i - \lfloor \frac{p-r}{p} \rfloor\right)$ *for each integer* i*; and*

$$\xi^r_{p,q} + \mathrm{PD}[\lambda] = \xi^{r+q}_{p,q}.$$

7.3 The full filtered complex for connected sums

Now, we describe the chain complex we are ultimately interested in, that of $B_g \#_{\ell=1}^n O_{p_\ell,q_\ell}$. Denote this knot by K, and let

$$Y = Z_g \#_{\ell=1}^n L(p_\ell, q_\ell)$$

be the ambient manifold for K.

Let $\xi(k; r_1, \dots, r_n) = \xi^k_{B_g} \#_{\ell=1}^n \xi^{r_\ell}_{p_\ell,q_\ell}$. Also, let

$$\eta\big(\xi(k; r_1, \dots, r_n)\big) = k - \sum_{\ell=1}^n \lfloor \frac{p_\ell - r_\ell}{p_\ell} \rfloor.$$

Let $R_g = \mathbb{Z}[H^1(Z_g)]$. Note that $H^1(Y)$ is canonically isomorphic to $H^1(Z_g)$, and so we can identify their group rings.

Putting the results of the previous two sections together yields the following.

Theorem 7.3. *The statement of Proposition 7.1 still holds when* $Y = Z_g \#_{\ell=1}^n L(p_\ell, q_\ell)$, $K = B_g \#_{\ell=1}^n O_{p_\ell,q_\ell}$, *and* $\xi = \xi(k; r_1, \dots, r_n)$, *with the following changes: in (3) and (4),* M^g_{i-j+s} *is replaced with* $M^g_{i-j+\eta(\xi)+s}$ *for* $s = -1, 0, 1$, *and in (5), we take* $A_{i,j}$ *to be supported in grading level* $i + j - \eta(\xi)$. *In particular, the same modules* M^g_n *and maps* V *and* H *may be used for all such pairs* (Y, K) *with* $b_1(Y) = 2g$.

Proof. First, let $n = 0$, i.e. $K = B_g$. When $k \neq 0$, the only thing that changes from Proposition 7.1 is that a generator of the form $[\mathbf{y}, i, j]$ gets replaced by one of the form $[\mathbf{y}, i, j + k]$. So we are done in this case.

If $n \neq 0$, the Kunneth formula together with Proposition 7.2 show that the only difference between this case and the above is an additional summand shift – specifically, one can easily check that a generator of the form $[\mathbf{y}, i, j]$ now gets replaced by one of the form $[\mathbf{y}, i, j + \eta(\xi)]$. □

Note that the $\mathbb{Z}$-grading of the theorem (item (5) of Proposition 7.1) is *one* lift of the natural relative $\mathbb{Z}$-grading on the complex; this will differ by a constant from the natural absolute $\mathbb{Q}$-grading (given by thinking of the complex as a filtration on $\underline{CF}^+(Y, G_K(\xi))$).

Chapter 8

Proof of Main Theorem

Let Y and K be as in Section 7.3. Take $\mathfrak{t}_0 \in \mathrm{Spin}^c(Y_0)$, and let us assume that $\mathfrak{t}_0$ is μ-torsion. We have only to compute the homology of the mapping cone of the map

$$\underline{f}^+_{K,\mathfrak{t}_0} : \bigoplus_{i\in\mathbb{Z}} \underline{C}^+(Y, K, \xi_i) \to \bigoplus_{i\in\mathbb{Z}} \underline{CF}^+\big(Y, G_K(\xi_i)\big)$$

of Theorem 6.1. This entails identifying the relative Spin^c structures ξ_i of that Theorem, and understanding the maps v_{ξ_i} and h_{ξ_i} (or at least the induced maps on homology).

8.1 The relative Spin^c structures ξ_i

Let $K = B_g \#_{\ell=1}^n O_{p_\ell,q_\ell}$, $Y = Z_g \#_{\ell=1}^n L(p_\ell, q_\ell)$. Suppose that a μ-torsion structure $\mathfrak{t}_0 \in \mathrm{Spin}^c(Y_0)$ extends over W_0 to a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W_0)$ for which $\mathfrak{s}|_Y = G_K\left(\xi(j; r_1, \ldots, r_n)\right)$, for some j and $r_1, \ldots, r_n$. Let the structure $\mathfrak{t}_\infty$ of Theorem 6.1 be $\mathfrak{s}|_Y$. Then, the relative Spin^c structure ξ_i used in Theorem 6.1 will be given by $\left[\mathfrak{s}|_Y - i\mathrm{PD}[K], -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}\right]$. Note that $G_{-K}(\xi_i) = G_K(\xi_{i+1})$.

Given these choices, it is clear that we can also write ξ_i as $\xi(j_i; r_1 - iq_1, \ldots, r_n - iq_n)$ for some j_i. Furthermore, since $q_K(\xi_i) = -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}$, the value of j_i must satisfy the equation

$$2j_i + \sum_{\ell=1}^{n} \frac{2(r_\ell - iq_\ell) - p_\ell - 1}{p_\ell} = -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}.$$

Hence,

$$j_i = -\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{2d} + \sum_{\ell=1}^{n} \frac{1-p_\ell}{2p_\ell} + \frac{p_\ell - (r_\ell - iq_\ell)}{p_\ell}.$$

Let $\eta(i) = \eta(\xi_i)$. We compute

$$\begin{aligned} \eta(i) &= -\tfrac{\langle c_1(\mathfrak{t}_0),[\widehat{dS}]\rangle}{2d} + \textstyle\sum_{\ell=1}^{n} \tfrac{1-p_\ell}{2p_\ell} + \tfrac{p_\ell-(r_\ell-iq_\ell)}{p_\ell} - \left\lfloor \tfrac{p_\ell-(r_\ell-iq_\ell)}{p_\ell} \right\rfloor \\ &= -\tfrac{\langle c_1(\mathfrak{t}_0),[\widehat{dS}]\rangle}{2d} + \textstyle\sum_{\ell=1}^{n} \tfrac{1-p_\ell}{2p_\ell} + \left\{ \tfrac{iq_\ell-r_\ell}{p_\ell} \right\}, \end{aligned}$$

where the curly braces denote the fractional part, $\{x\} = x - \lfloor x \rfloor$.

Proof of Lemma 1.3. Let $\mathfrak{t}_0$ be an μ-torsion Spin^c structure on Y_0, which extends over W_0 to some $\mathfrak{s}$ for which $\mathfrak{s}|_Y = G_K(\xi(j; r_1, \ldots, r_n))$. We can assume that $0 \le r_\ell < p_\ell$ for each ℓ. Given such a choice, let Q be such that

$$\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{2d} = Q - \frac{1}{2}\sum_{\ell=1}^{n}\left(1 - \frac{1}{p_\ell}\right) - \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}.$$

In light of the above, it is clear that Q will be integral. Then for $A = (Q; r_1, \ldots, r_n) \in \widetilde{\mathcal{MT}}_K$, set $\theta_K([A]) = \mathfrak{t}_0$. To show that this is well-defined, first note that A determines $\mathfrak{t}_0$, since all the Spin^c structures on Y_0 cobordant to a given torsion one on Y are distinguished by their Chern class evaluations. This shows that $\theta_K(A)$ is well-defined; to show that this descends to equivalence classes, note that if $A \sim A'$, then both A and A' will be realized by the above construction applied to $\mathfrak{t}_0$, except starting with different choices for r_ℓ.

We check the properties of this map. First, it is easy to see that any two A and A' realized by the same $\mathfrak{t}_0$ will have $A \sim A'$, showing that θ_K is in fact injective. If $S\ell$ is defined as in Equation 1.1, then Equation 1.2 clearly holds. Finally, to see that the image of θ_K contains all the μ-torsion structures, note that by Proposition 2.4, any μ-torsion structure will be cobordant to a torsion structure of Y, and all the torsion structures on Y are of the form $G_K\left(\xi(j; r_1, \ldots, r_n)\right)$. □

8.2 The homology of $\underline{C}^+(Z_g, B_g, \xi^k_{B_g})$

We now address the maps v_{ξ_*} and h_{ξ_*}, starting with the special case $Y = Z_g$, $K = B_g$, and $\xi = \xi^k_{B_g}$. Write

$$\underline{C}(Z_g, B_g, \xi) \cong \bigoplus_{i,j} A_{i,j},$$

as in the statement of Proposition 7.3. Then

$$\underline{C}^+(Z_g, B_g, \xi) \cong \bigoplus_{\max\{i,j\} \geq 0} A_{i,j};$$

the differential is the same as for $\underline{C}(Z_g, B_g, \xi)$, except that components mapping to summands no longer present are dropped. The summand $A_{i,j}$ is isomorphic to M^g_{i-j+k} (hereafter we drop the superscript); this will be non-trivial if and only if $-g \leq i-j+k \leq g$, or equivalently, if $-g-k \leq i-j \leq g-k$.

Denote $\underline{C}(Z_g, B_g, \xi)$ and $\underline{C}^+(Z_g, B_g, \xi)$ by C and C^+, respectively. Forgetting the second component of the $\mathbb{Z}\oplus\mathbb{Z}$ filtration, the complex C can be identified with $\underline{CF}^\infty(Z_g, \mathfrak{t}_{B_g})$ (recall that $\mathfrak{t}_{B_g} \in \mathrm{Spin}^c(Z_g)$ is the unique torsion structure). The latter complex has an absolute $\mathbb{Q}$-grading, with summand $A_{i,j}$ supported in grading level $i+j-k$, and so C inherits this grading.

Denote by ${}^b\underline{HFK}_\ell(Z_g, B_g, \xi)$ the homology of $\underline{C}^+(Z_g, B_g, \xi)$ in grading level ℓ. We would like to understand the maps $v_{\xi_*} : {}^b\underline{HFK}(Z_g, B_g, \xi) \to \underline{HF}^+(Z_g, \mathfrak{t}_{B_g})$ and $h_{\xi_*} : {}^b\underline{HFK}(Z_g, B_g, \xi) \to \underline{HF}^+(Z_g, \mathfrak{t}_{B_g})$.

The chain complex $C\{i \geq 0\}$, which is a quotient of C^+, is identified with $\bigoplus_{i\geq 0} A_{i,j}$, and represents $\underline{CF}^+(Z_g, \mathfrak{t}_{B_g})$. Note that

$$\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g}) \cong \begin{cases} \mathbb{Z}, & \ell \equiv g (\text{mod } 2) \text{ and } \ell \geq -g \\ 0, & \text{otherwise,} \end{cases}$$

where the subscript ℓ denotes $\mathbb{Q}$-grading (see [9]). So there is an isomorphism of graded $R_g[U]$-modules $H_*(C\{i \geq 0\}) \cong \underline{HF}^+(Z_g, \mathfrak{t}_{B_g})$. The map v_{ξ_*} is the projection ${}^b\underline{HFK}_*(Z_g, B_g, \xi) \to H_*(C\{i \geq 0\})$ followed by this isomorphism.

The chain complex $C\{j \geq 0\}$ is also a quotient of C^+, and is identified with $\bigoplus_{j \geq 0} A_{i,j}$. This complex also represents $\underline{CF}^+(Z_g, \mathfrak{t}_{B_g})$, but the identification is now gotten by forgetting the *first* component of the $\mathbb{Z} \oplus \mathbb{Z}$-filtration rather than the second. In particular, this means that the identification only preserves relative $\mathbb{Z}$-grading, as the absolute grading is defined by forgetting the second component. Still, it is easy to see that $H_\ell(C\{j \geq 0\}) \cong \underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$; and the map h_{ξ_*} is the projection ${}^b\underline{HFK}(Z_g, B_g, \xi) \to H_*(C\{j \geq 0\})$ followed by this isomorphism.

Remark. We will need to use the identification, via $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$, of $H_{\ell-2k}(C\{j \geq 0\})$ with $H_\ell(C\{i \geq 0\})$. The particular identification is canonical, up to sign. Of course, up to sign, there is only one automorphism of $\mathbb{Z}$, so we need not think more about this. However, it should be noted that in other similar situations, there are many isomorphisms between two realizations of the same Floer homology group, while there is only one particular isomorphism induced by Heegaard diagram operations. When using surgery formulas, it is essential to be careful about this: identifying groups by the wrong isomorphism can lead to an incorrect calculation. See [8] for an example illustrating the importance of this point.

For i, j with $i \leq j$ and $i \equiv j \pmod 2$, let $M_{i,j}$ denote $M_i \oplus M_{i+2} \oplus \ldots \oplus M_{j-2} \oplus M_j$.

Lemma 8.1. *Let $\xi = \xi^k_{B_g}$. If $\ell \geq -k$, then at least one of*

$$v_{\xi_*} : {}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \to \underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$$

and

$$h_{\xi_*} : {}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \to \underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$$

is an isomorphism. Precisely, if both $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ and $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$ are trivial, then so is ${}^b\underline{HFK}_\ell(Z_g, B_g, \xi)$; if either $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ or $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$ is non-trivial, then ${}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \cong \mathbb{Z}$; if both $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ and $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$ are non-trivial, then both maps v_{ξ_*} and h_{ξ_*} are isomorphisms.

Proof. If ℓ is sufficiently large, then all the nontrivial summands $A_{i,j}$ of C supported in level ℓ will have both $i > 0$ and $j > 0$. Hence, for such values of ℓ, the complexes C^+, $C\{i \geq 0\}$ and $C\{j \geq 0\}$ will all be exactly the same in level ℓ. More precisely, the quotient map from C^+ to the latter two complexes, restricted to the portions lying in grading levels $\ell - 1, \ell$ and $\ell + 1$, will just be the identity map. So in this case, we have the result.

Now, assume that $\ell \geq -k$, and that $\ell \equiv g \pmod 2$. This means that any summand $A_{i,j}$ of C supported in level ℓ will all have $i + j - k \geq -k$, which implies that $i + j \geq 0$, and hence that $i \geq 0$ or $j \geq 0$. Then we have an identification of chain complexes

$$\begin{array}{ccccc}
C^+_{\ell+1} & \xrightarrow{\partial} & C^+_\ell & \xrightarrow{\partial} & C^+_{\ell-1} \\
\downarrow & & \downarrow & & \downarrow \\
M_{-g+1,g-1} & \xrightarrow{V+H} & M_{-g,g} & \xrightarrow{V+H} & M_{-g+1,g-1}
\end{array}$$

where the vertical maps are isomorphisms induced by the various maps $\phi_{i,j}$, and the maps $V + H$ act in the obvious manner. In particular, since the homology of the bottom is fixed, this means that the homology of the top, $H_*(C^+_\ell)$, is independent of ℓ, as long as ℓ satisfies our assumptions. For large ℓ satisfying our assumptions, the previous paragraph shows that $H_*(C^+_\ell) \cong \mathbb{Z}$, and so the same is true for any ℓ satisfying these assumptions.

If $\ell \geq -k$ and $\ell \not\equiv g \pmod 2$, then we have a similar argument showing that $H_*(C^+_\ell)$ is trivial.

Recall that the maps $\phi_{i,j}$ satisfy the relations $\phi_{i,j} = \phi_{i-1,j-1} \circ U$. From this, it easily follows that for all $\ell \geq -k$, the map $U : C^+_{\ell+2} \to C^+_\ell$ induces an isomorphism on homology. The same holds for $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ whenever ℓ is non-trivial. Since v_{ξ_*} is a U-equivariant isomorphism when ℓ is large, this implies that it is an isomorphism in all degrees ℓ where $\ell \geq -k$ and $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ is non-trivial; a similar statement holds for v_{ξ_*}, except this one holds when $\ell \geq -k$ and $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$ is non-trivial.

Finally, it is clear that for $\ell \geq -k$, ${}^b\underline{HFK}_\ell(Z_g, B_g, \xi)$ is non-trivial if and only if at least one of $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ and $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$ is. □

Lemma 8.2. *For all ℓ, the maps*

$$v_{\xi_*} : {}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \to \underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$$

and

$$h_{\xi_*} : {}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \to \underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$$

are surjective.

Proof. The action of U^n on $\underline{HF}^+(Z_g, \mathfrak{t}_{B_g})$ is surjective for all $n > 0$. Therefore, the claims follows from U-equivariance of the maps, together with the previous Lemma. □

Lemma 8.3. *If $\ell \leq -k-2$, then*

$${}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \cong \underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g}) \oplus \underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g});$$

the maps v_{ξ_*} and h_{ξ_*} are the projections to the first and second summands, respectively. The same holds for $\ell = -k-1$ if, in addition, $|\ell+1| > g-1$.

Proof. We assume that $\ell \equiv g \pmod 2$; the other case is similar. In both cases, the idea is that precisely the same calculations go into each side of the isomorphism.

The non-zero summands $A_{i,j}$ supported in level ℓ are the ones for which $i+j = k+\ell$, $|i-j+k| \leq g$ and at least one of i and j is nonnegative. If $\ell < -k$, then exactly one of i and j must be nonnegative for $A_{i,j}$ to be nonzero. We further separate our calculations into three subcases.

$\ell < -2k - g$: One can verify that in this case, the non-zero summands $A_{i,j}$ supported in grading level ℓ all satisfy $i \geq 0$, $j < 0$. Then for some q between $-g$ and g, we have an identification of chain complexes

$$\begin{array}{ccccc} C^+_{\ell+1} & \xrightarrow{\partial} & C^+_{\ell} & \xrightarrow{\partial} & C^+_{\ell-1} \\ \downarrow & & \downarrow & & \downarrow \\ M_{q-1,g-1} & \xrightarrow{V+H} & M_{q,g} & \xrightarrow{V+H} & M_{q+1,g-1} \end{array}$$

where the vertical maps are isomorphisms induced by the various maps $\phi_{i,j}$, and the maps $V+H$ act in the obvious manner. We also have a similar diagram, but with C^+_n in the top row replaced by $\underline{CF}^+_n(Z_g, \mathfrak{t}_{B_g})$. Hence, ${}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \cong \underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ in this case, with v_{ξ_*} inducing the isomorphism. Clearly, $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$ is trivial in this case, as is the map h_{ξ_*}.

$\ell < -g$: In this case, the non-zero summands $A_{i,j}$ supported in grading level ℓ all satisfy $j \geq 0$, $i < 0$. The argument here runs exactly like the previous case, and leaves us with an isomorphism $h_{\xi_*} : {}^b\underline{HFK}_\ell(Z_g, B_g, \xi) \to \underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g}) \cong \mathbb{Z}$; and $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g})$ and the map v_{ξ_*} are both trivial.

$\ell + g - k \geq |k|, \ell \leq -k-2$: In this case, there are non-zero summands $A_{k+\ell,0}$ and $A_{0,k+\ell}$ supported in grading level ℓ. Note that $\phi_{k+\ell,0}$ identifies $A_{k+\ell,0}$ with $M_{\ell+2k}$, and similarly $A_{0,k+\ell}$ is identified with $M_{-\ell}$. Since we take $k+\ell \leq -2$, then, it is easy to see that we have an identification of chain complexes

$$\begin{array}{ccccc} C^+_{\ell+1} & \xrightarrow{\partial} & C^+_{\ell} & \xrightarrow{\partial} & C^+_{\ell-1} \\ \downarrow & & \downarrow & & \downarrow \\ M_{\text{-}g+1,2k+\ell+1} & & M_{\text{-}g,2k+\ell} & & M_{\text{-}g+1,2k+\ell-1} \\ \oplus & \xrightarrow{V+H} & \oplus & \xrightarrow{V+H} & \oplus \\ M_{\text{-}\ell-1,g-1} & & M_{\text{-}\ell,g} & & M_{\text{-}\ell+1,g-1} \end{array}$$

where both maps $V + H$ take first summand to first summand and second summand to second. In particular, the chain complex splits as the direct sum of two subcomplexes. (Note that this does not hold when $k + \ell = -1$: in this case, $A_{0,0}$ has non-trivial maps to both $A_{0,-1}$ and $A_{-1,0}$.) The two complexes are of course the same as those which calculate $\underline{HF}^+_\ell(Z_g, \mathfrak{t}_{B_g}) \cong \mathbb{Z}$ and $\underline{HF}^+_{\ell+2k}(Z_g, \mathfrak{t}_{B_g})$. So the result clearly holds in this case. □

The three Lemmas, together with certain facts noted in their proofs, tell us almost all that we want to know about v_ξ and h_ξ, except sometimes when $\ell = -k-1$. In this case, we introduce the following.

Definition 8.4. *With $\xi = \xi^k_{B_g}$ and $\mathfrak{t} = \mathfrak{t}_{B_g}$, define*

$$\Omega^g_0(k) = {}^b\underline{HFK}_{-k-1}(Z_g, B_g, \xi),$$

and define $\Omega^g(k)$ by

$$\Omega^g(k) = Ker \left(\begin{array}{ll} v_{\xi_*} \oplus h_{\xi_*}: & {}^b\underline{HFK}_{\text{-}k-1}(Z_g, B_g, \xi) \to \\ & \qquad \underline{HF}^+_{\text{-}k-1}(Z_g, \mathfrak{t}) \oplus \underline{HF}^+_{k-1}(Z_g, \mathfrak{t}) \end{array} \right).$$

We wrap everything up in the following.

Proposition 8.5. *Let $\xi = \xi^k_{B_g}$ and $\mathfrak{t} = \mathfrak{t}_{B_g}$. Consider the diagram*

$$\begin{array}{ccccc} \underline{HF}^+_{\ell}(Z_g,\mathfrak{t}) & \xleftarrow{v_{\xi *}} & {}^b\underline{HFK}_{\ell}(Z_g,B_g,\xi) & \xrightarrow{h_{\xi *}} & \underline{HF}^+_{\ell+2k}(Z_g,\mathfrak{t}) \\ & & \Big\downarrow{\scriptstyle v_{\xi *}\oplus h_{\xi *}} & & \\ & & \underline{HF}^+_{\ell}(Z_g,\mathfrak{t}) & & \\ & & \oplus & & \\ & & \underline{HF}^+_{\ell+2k}(Z_g,\mathfrak{t}) & & \end{array}$$

If $\ell \equiv g \pmod 2$, then in this diagram:

1. *If $\ell \geq -g-2k$, $\ell \geq -g$, and $\ell > -k-1$, the three modules across the top are all isomorphic to $\mathbb{Z}$; the left and right maps are isomorphisms while the middle map is injective.*

2. *If $\ell \geq -g-2k$, $\ell \geq -g$, and $\ell = -k-1$, the three modules across the top, from left to right, are respectively isomorphic to $\mathbb{Z}$, $\Omega^g_0(k)$ and $\mathbb{Z}$; all three maps are surjective, the vertical one with kernel $\Omega^g(k)$.*

3. *If $\ell \geq -g-2k$, $\ell \geq -g$, and $\ell < -k-1$, the three modules across the top from left to right are respectively isomorphic to $\mathbb{Z}$, $\mathbb{Z}\oplus\mathbb{Z}$, and $\mathbb{Z}$, the horizontal maps are surjective, and the verical map is an isomorphism.*

4. *If $-g > \ell \geq -g-2k$, the modules across the top are respectively isomorphic to 0, $\mathbb{Z}$, and $\mathbb{Z}$, and the center and right maps are isomorphisms.*

5. *If $-g-2k > \ell \geq -g$, the modules across the top are respectively isomorphic to $\mathbb{Z}$, $\mathbb{Z}$, and 0, and the center and left maps are isomorphisms.*

6. *If $\ell < -g-2k$, and $\ell < -g$, all the modules are trivial.*

If $\ell \not\equiv g \pmod 2$, then all the modules are trivial unless $\ell = -k-1$, $\ell \geq -g-2k$, and $\ell \geq -g$, in which case only ${}^b\underline{HFK}_{\ell}(Z_g,B_g,\xi) \cong \Omega^g(k) = \Omega^g_0(k)$ is non-trivial.

8.3 The homology of $\underline{C}^+(Y, K, \xi)$

Now, consider the more general case of $K = B_g \#_{\ell=1}^n O_{p_\ell,q_\ell}$, $Y = Z_g \#_{\ell=1}^n L(p_\ell, q_\ell)$, and $\xi = \xi_{B_g}^k \#_{\ell=1}^n \xi_{p_\ell,q_\ell}^{r_\ell}$. Let $\eta(\xi)$ be as in Chapter 7. According to Theorem 7.3, it is clear that $\underline{C}^+(Y, K, \xi)$ is isomorphic to $\underline{C}^+(Z_g, B_g, \xi_{B_g}^{\eta(\xi)})$ as relatively $\mathbb{Z}$-graded complexes; and we assign an absolute grading to the former via this isomorphism.

It follows straightforwardly than an analogue of Proposition 8.5 holds in this case. Specifically, if k is replaced with $\eta(\xi_i)$, and the diagram of that Proposition is replaced with

$$\begin{array}{ccccc} \underline{HF}_\ell^+(Y, G_K(\xi)) & \xleftarrow{v_{\xi*}} & {}^b\underline{HFK}_\ell(Y, K, \xi) & \xrightarrow{h_{\xi*}} & \underline{HF}^+_{\ell+2\eta(\xi)}(Y, G_{\text{-}K}(\xi)) \\ & & \Big\downarrow{\scriptstyle v_{\xi*} \oplus h_{\xi*}} & & \\ & & \underline{HF}_\ell^+(Y, G_K(\xi)) & & \\ & & \oplus & & \\ & & \underline{HF}^+_{\ell+2\eta(\xi)}(Y, G_{\text{-}K}(\xi)) & & \end{array}$$

then Proposition 8.5 holds precisely. Observe that with our grading, $\underline{HF}_\ell^+(Y, G_K(\xi))$ is non-trivial if and only if $\ell \geq -g$ and $\ell \equiv g \pmod 2$.

8.4 An absolute $\mathbb{Z}$-grading for the mapping cone

Each of the constituent parts of the mapping cone of $\underline{f}^+_{K,\mathfrak{t}_0}$ carries a relative $\mathbb{Z}$-grading, which we have lifted to an absolute grading at our convenience. Theorem 6.1 states that the entire mapping cone supports an overall relative $\mathbb{Z}$-grading. We fix, once and for all, an absolute lift of this grading, which will also agree with the canonical $\mathbb{Z}_2$-grading.

Suppose that $\mathfrak{t}_0$ is realized as $\theta_K([A])$, where $A = (Q; r_1, \ldots, r_n)$. Let $G_A(i)$ be defined by

$$G_A(0) = 1, \quad G_A(i+1) = G_A(i) - 2\eta(i).$$

Then, define

$$\mathbb{A}_\ell(i) = \underline{C}^+_{\ell - G_A(i) - g}(Y, K, \xi_i)$$

and

$$\mathbb{B}_\ell(i) = \underline{CF}^+_{\ell - G_A(i) - g + 1}\left(Y, G_K(\xi_i)\right),$$

where the gradings on the right side are endowed as described previously, by identification with a complex $\underline{CF}^+(Z_g, B_g, \xi_{B_g}^{\eta(\xi)})$. When there is no risk of confusion, we just write G for G_A. Unravelling definitions, we have for $k \in \mathbb{Z}$,

$$G(k) = 1 + 2k\left(Q - \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}\right) - 2\sum_{i=0}^{k-1}\sum_{\ell=1}^{n}\left\{\frac{q_\ell i - r_\ell}{p_\ell}\right\} + 2\sum_{i=k}^{-1}\sum_{\ell=1}^{n}\left\{\frac{q_\ell i - r_\ell}{p_\ell}\right\}.$$

For later, we extend h by defining it on half integers, by

$$G(i + \frac{1}{2}) = g + \frac{1}{2}\left(G(i) + G(i+1)\right) = g + G(i) - \eta(i),\ \ i \in \mathbb{Z}.$$

Notice that $G(i)$ is always odd for $i \in \mathbb{Z}$, so $G(i + \frac{1}{2})$ is an integer. Also, if ℓ is even, then $\ell - G(i) - g + 1 \equiv g \pmod 2$. Hence, given i, for large enough even ℓ we have $\mathbb{B}_\ell(i) \cong \mathbb{Z}$ and $\mathbb{A}_\ell(i)$ vanishes. The reverse holds for odd ℓ.

Set $\mathbb{A} = \bigoplus_{i \in \mathbb{Z}} \mathbb{A}_i$ and $\mathbb{B} = \bigoplus_{i \in \mathbb{Z}} \mathbb{B}_i$, and let $f : \mathbb{A} \to \mathbb{B}$ be defined to make the square

$$\begin{array}{ccc} \bigoplus_{i \in \mathbb{Z}} {}^b\underline{CFK}(Y, K, \xi_i) & \xrightarrow{f^+_{K,\mathfrak{t}_0}} & \bigoplus_{i \in \mathbb{Z}} \underline{CF}^+\left(Y, G_K(\xi_i)\right) \\ \downarrow & & \downarrow \\ \mathbb{A} & \xrightarrow{f} & \mathbb{B} \end{array}$$

commute, where the vertical maps are the direct sums of the obvious relatively graded isomorphisms $\mathbb{A}(i) \cong {}^b\underline{CFK}(Y, K, \xi_i)$ and $\mathbb{B}(i) \cong \underline{CF}^+\left(Y, G_K(\xi_i)\right)$. Denote the portions corresponding to v_{ξ_i} and h_{ξ_i} respectively by

$$v_i : \mathbb{A}(i) \to \mathbb{B}(i)$$

and

$$h_i : \mathbb{A}(i) \to \mathbb{B}(i+1).$$

Lemma 8.6. *The map f is graded of degree -1.*

Proof. Simply note that

$$\underline{C}^+_\ell(Y, K, \xi_i) \cong \mathbb{A}_{\ell+G(i)+g}(i),$$

$$\underline{CF}^+_\ell(Y, G_K(\xi_i)) \cong \mathbb{B}_{\ell+G(i)+g-1}(i),$$

and

$$\begin{aligned}\underline{CF}^+_{\ell+2\eta(i)}(Y, G_{-K}(\xi_i)) &= \underline{CF}^+_{\ell+2\eta(i)}(Y, G_K(\xi_{i+1})) \\ &\cong \mathbb{B}_{\ell+2\eta(i)+G(i+1)+g-1}(i+1) = \mathbb{B}_{\ell+G(i)+g-1}(i+1).\end{aligned}$$

Hence v_i and h_i take $\mathbb{A}_{\ell+G(i)+g}(i)$ to $\mathbb{B}_{\ell+G(i)+g-1}(i)$ and $\mathbb{B}_{\ell+G(i)+g-1}(i+1)$, respectively. □

8.5 The homology of the mapping cone

Our goal is to compute the homology of $\underline{f}^+_{K,\mathfrak{t}_0}$, or equivalently, of f. Since $\mathbb{A}$ and $\mathbb{B}$ are graded and f is a graded map of degree -1, we have an induced grading on $M(f)$; and in particular, we have a short exact sequence

$$0 \to \mathrm{Coker}\left(f_* : H_{\ell+1}(\mathbb{A}) \to H_\ell(\mathbb{B})\right) \to H_\ell\left(M(f)\right) \to \\ \mathrm{Ker}\left(f_* : H_\ell(\mathbb{A}) \to H_{\ell-1}(\mathbb{B})\right) \to 0.$$

Let

$$\Omega_\ell(G) = \bigoplus_{\{i\in\mathbb{Z} | G(i+\frac{1}{2})=\ell+1\}} \Omega^g\left(G(i) - G(i+\frac{1}{2}) + g\right),$$

thought of as a graded module supported in degree ℓ. Note that $\Omega^g(k)$ vanishes if $|k| > g-1$.

Proposition 8.7. *There is a short exact sequence*

$$0 \to \Omega_\ell(G) \to Ker\left(f_* : H_\ell(\mathbb{A}) \to H_{\ell-1}(\mathbb{B})\right) \to \mathbb{W}_\ell(G) \to 0.$$

Proof. We factor the map f_*. To do this, let

$$\tilde{f} : \bigoplus_{i\in\mathbb{Z}} \mathbb{A}_\ell(i) \to \bigoplus_{i\in\mathbb{Z}} \left(\mathbb{B}_{\ell-1}(i) \oplus \mathbb{B}_{\ell-1}(i+1)\right)$$

be the map taking $x \in \mathbb{A}_\ell(i)$ to $v_i(x) \oplus h_i(x) \in \left(\mathbb{B}_{\ell-1}(i) \oplus \mathbb{B}_{\ell-1}(i+1)\right)$. Also, let

$$\pi : \bigoplus_{i\in\mathbb{Z}} \left(\mathbb{B}_{\ell-1}(i) \oplus \mathbb{B}_{\ell-1}(i+1)\right) \to \bigoplus_{i\in\mathbb{Z}} \mathbb{B}_{\ell-1}(i)$$

be such that the restriction of π to summand i is the identity map (i.e., summand i on the left, which is $\left(\mathbb{B}_{\ell-1}(i) \oplus \mathbb{B}_{\ell-1}(i+1)\right)$, goes to summands i and $i+1$ on the right). Then it is clear that $f = \pi \circ \tilde{f}$, and hence that there is a short exact sequence

$$0 \to \text{Ker } \tilde{f}_* \to \text{Ker } f_* \to \text{Ker } \pi_*|_{\text{Im } \tilde{f}_*} \to 0.$$

We first focus on the map $\tilde{f}_*$. By Proposition 8.5, the restriction of $\tilde{f}_*$ to $H_\ell\left(\mathbb{A}(i)\right)$ is injective unless

$$\ell - G(i) - g = -\eta(i) - 1,$$

$$\ell - G(i) - g \geq -g,$$

and

$$\ell - G(i) - g \geq -g - 2\eta(i),$$

in which case the restriction has kernel identified with $\Omega^g\left(\eta(i)\right)$. The three conditions above can be rephrased as

$$\ell = G(i + \frac{1}{2}) - 1,\ \ell \geq G(i),\ \ell \geq G(i + 1).$$

Note that

$$\eta(i) = G(i) - G(i + \frac{1}{2}) + g = G(i + \frac{1}{2}) - G(i + 1) - g,$$

and that $\Omega^g(\eta(i))$ is trivial unless $|\eta(i)| \leq g - 1$. It follows that if $\Omega^g(\eta(i))$ is non-trivial, then $G(i + \frac{1}{2}) - 1 \geq \max\{G(i), G(i + \frac{1}{2})\}$. Hence, we may rephrase the above again, to say that the kernel of the restriction of $\tilde{f}_*$ to $H_\ell(\mathbb{A}(i))$ is $\Omega^g(G(i) - G(i + \frac{1}{2}) + g)$ if $\ell = G(i + \frac{1}{2}) - 1$, and trivial otherwise.

The kernel of $\tilde{f}_*$ is the direct sum of the kernels of these restrictions; so the identification of Ker $\tilde{f}_*$ with $\Omega_\ell(h)$ follows.

Now, we turn to the map π_*. Since $H_\ell(\mathbb{B}(i))$ is either isomorphic to $\mathbb{Z}$ or trivial, write elements of $\bigoplus_{i \in \mathbb{Z}} (H_{\ell-1}(\mathbb{B}(i)) \oplus H_{\ell-1}(\mathbb{B}(i+1)))$ as

$$b = \ldots \oplus (b_{-1}(-1) \oplus b_{-1}(0)) \oplus (b_0(0) \oplus b_0(1)) \oplus (b_1(1) \oplus b_1(2)) \oplus \ldots$$

where $(b_i(i) \oplus b_i(i+1)) \in (H_{\ell-1}(\mathbb{B}(i)) \oplus H_{\ell-1}(\mathbb{B}(i+1)))$ is the component in summand i; we take the $b_i(j)$ to be integers, and the element is subject to the condition that only finitely many of the $b_i(j)$ are non-zero and that if $H_{\ell-1}(\mathbb{B}(i))$ is trivial, then $b_i(i)$ and $b_{i-1}(i)$ are both zero.

The kernel of π_* is the subset of elements of the above form for which $b_i(i) = -b_{i-1}(i)$. Hence, it is easy to see that a generating set (as a group) for this kernel is given by

$\{b^k | H_{\ell-1}(\mathbb{B}(k)) \cong \mathbb{Z}\}$, where b^k has

$$b_i^k(j) = \begin{cases} 1, & i = j = k \\ -1, & i+1 = j = k \\ 0, & \text{otherwise.} \end{cases} \tag{8.1}$$

Furthermore, the k for which $H_{\ell-1}(\mathbb{B}(k)) \cong \mathbb{Z}$ are those for which $\ell - G(k) - g \geq -g$ and $\ell - G(k) - g \equiv g \pmod 2$; or, equivalently, the values of k for which

$$\ell \geq G(k), \text{ and } \ell \equiv 1 \,(\text{mod } 2)$$

(using the fact that $G(k)$ is odd for $k \in \mathbb{Z}$).

By Proposition 8.5, the restriction

$$\tilde{f}_* : H_\ell(\mathbb{A}(i)) \to H_{\ell-1}(\mathbb{B}(i)) \oplus H_{\ell-1}(\mathbb{B}(i+1))$$

is surjective unless $\ell - G(i) - g > -\eta(i) - 1$, $\ell - G(i) - g \geq \max\{-g, -g - 2\eta(i)\}$, and $\ell - G(i) - g \equiv g \pmod 2$; or, equivalently, unless

$$\ell \geq G(i + \frac{1}{2}), \tag{8.2}$$

$$\ell \geq \max\{G(i), G(i+1)\}, \text{ and} \tag{8.3}$$

$$\ell \equiv 1 \,(\text{mod } 2). \tag{8.4}$$

In this case, $H_\ell(\mathbb{A}(i)) \cong \mathbb{Z}$, while $(H_{\ell-1}(\mathbb{B}(i)) \oplus H_{\ell-1}(\mathbb{B}(i+1))) \cong \mathbb{Z} \oplus \mathbb{Z}$; the composition of the restriction of $\tilde{f}_*$ with projection to each summand is an isomorphism. We can assume without loss of generality, then, that the intersection of Im $\tilde{f}_*$ with $(H_{\ell-1}(\mathbb{B}(i)) \oplus H_{\ell-1}(\mathbb{B}(i+1)))$ consists of elements $(b_i(i) \oplus b_i(i+1))$ such that $b_i(i) = -b_i(i+1)$, with notation as above.

Thus, Ker $\pi_*|_{\text{Im } \tilde{f}_*}$ (if ℓ is odd) is the group of finite sums

$$\sum_{\{k|\ell \geq G(k)\}} c_k b^k$$

with $c_k \in \mathbb{Z}$, such that if ℓ and k satisfy inequalities 8.2, 8.3 and 8.4, then $c_k = c_{k+1}$. Hence, a set generating Ker $\pi_*|_{\text{Im } \tilde{f}_*}$ freely is given by

$$\left\{ \sum_{i=k_1}^{k_2} b^i \,\middle|\, [k_1, k_2] \text{ is a maximal run at or below } \ell \right\},$$

where $[k_1, k_2]$ being a run at or below ℓ means that $\ell \geq G(i)$ for i between integers k_1 and k_2, and such a run being maximal means that $[k_1 - 1, k_1]$ and $[k_2, k_2 + 1]$ are not runs below ℓ. Now, note that there is a one-to-one correspondence between such maximal runs and wells of G at height ℓ. So we have the identification of Ker $\pi_*|_{\text{Im } \tilde{f}_*}$ with $\mathbb{W}_\ell(G)$.

□

Proposition 8.8. *We have*

$$\textit{Coker}\,(f_* : H_{\ell+1}(\mathbb{A}) \to H_\ell(\mathbb{B})) \cong \begin{cases} \mathbb{Z} & \ell \textit{ is even and } \ell \geq G(i) - 1 \textit{ for all } i \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Keeping notation from the previous proof, we are interested in the image of $\pi_*|_{\text{Im } \tilde{f}_*}$. Let us look at the image of π_* itself.

Write an element of $\bigoplus_{i\in\mathbb{Z}} \mathbb{B}_\ell(i)$ as $x = \oplus_{i\in\mathbb{Z}} x_i$, where $x_i \in \mathbb{B}_\ell(i)$ is an integer, equal to zero if $\mathbb{B}_\ell(i)$ is trivial. Then $x = \pi_* \left(\oplus_{i\in\mathbb{Z}} \left(b_i(i) \oplus b_i(i+1)\right)\right)$ if and only if

$$b_{i-1}(i) + b_i(i) = x_i$$

for all i.

Now, x being in the image of $\pi_*|_{\mathrm{Im}\ \tilde{f}_*}$ is equivalent to the existance of solutions $b_{i-1}(i), b_i(i)$ to the above which also satisfy the demand that if $\ell+1 \geq \max\{G(i), G(i+\frac{1}{2}), G(i+1)\}$ and ℓ is even, then $b_i(i-1) = -b_i(i)$. It is not difficult to see that if there is even one i for which the demand allows $b_i(i-1) \neq -b_i(i)$, then we will have solutions for any x, and hence the cokernel will vanish. Clearly, if ℓ is odd, then the cokernel vanishes simply because $\mathbb{B}_\ell$ does.

On the other hand, if we have that ℓ is even and that the demand requires that $b_i(i-1) = -b_i(i)$ for all i, it is not hard to show that this implies that $\sum_{i\in\mathbb{Z}} x_i$ must equal 0. Conversely, if $\sum_{i\in\mathbb{Z}} x_i = 0$, it is easy to find such solutions. In this case, therefore, the cokernel is $\mathbb{Z}$. This case occurs if and only if ℓ is even and $\ell+1 \geq G(i)$ for all i. □

Proof of Theorem 1.4. Let $\mathfrak{t}_0 = \theta_K([A]) \in \mathrm{Spin}^c(Y_0)$ be μ-torsion. Choose a representative $A = (Q; r_1, \ldots, r_n) \in \widetilde{\mathcal{MT}_K}$ of $[A]$, and construct the function G_A as above.

Think of Y_0 as the mapping torus of some periodic diffeomorphism $\phi : \Sigma \to \Sigma$ of order d. We have an action of $\mathrm{PD}[\Sigma]$ on $\underline{HF}^+(Y_0, \mathfrak{t}_0)$. It is not hard to see that $\mathrm{PD}[\Sigma]$ in fact acts on $\mathbb{A}$ and $\mathbb{B}$ by taking $\mathbb{A}(i)$ isomorphically to $\mathbb{A}(i+d)$ and $\mathbb{B}(i)$ isomorphically to $\mathbb{B}(i+d)$. It then follows from the arguments of the previous two propositions that $\mathrm{PD}[\Sigma]$ acts as the identity on $\mathrm{Coker}\,(f_* : H_*(\mathbb{A}) \to H_*(\mathbb{B}))$, and that it acts non-trivially on $\mathrm{Ker}\,(f_* : H_*(\mathbb{A}) \to H_*(\mathbb{B}))$. To see that the sequence

$$0 \to \mathrm{Coker}\,(f_* : H_{\ell+1}(\mathbb{A}) \to H_\ell(\mathbb{B})) \to \quad H_\ell\,(M(f)) \to \\ \mathrm{Ker}\,(f_* : H_\ell(\mathbb{A}) \to H_{\ell-1}(\mathbb{B})) \quad \to 0$$

splits, we note that the first and last terms are free as abelian groups, so the sequence splits as a sequence of abelian groups; then, by the properties we have noted, we can alter any such splitting so that the image of the last group in $H_\ell\,(M(f))$ is an R_g-submodule.

Observe that $\mathrm{Coker}\,(f_* : H_*(\mathbb{A}) \to H_*(\mathbb{B}))$ is isomorphic to $\mathcal{T}^+_{b_A}$ as a graded abelian group, with b_A given as in the statement. Hence, we have that the graded group $\underline{HF}^+(Y_0, \mathfrak{t}_0)$ sits in a short exact sequence as stated when $\mathfrak{t}_0$ is μ-torsion. It is straightforward to check that this isomorphism respects the U action as well, by examining all of the above.

For the claim about the boundedness of G_A, note that this function is the sum of a periodic function with period d, and a linear function with slope $S\ell(A) = \frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}]\rangle}{d}$. To see the latter, note that since p_ℓ divides d, and p_ℓ and q_ℓ are relatively prime,

$$\sum_{i=0}^{d-1} \left\{ \frac{q_\ell i - r_\ell}{p_\ell} \right\} = \frac{d(p_\ell - 1)}{2p_\ell}.$$

Hence, G_A will be bounded precisely when $\mathfrak{t}_0$ is torsion.

For Spin^c structures $\mathfrak{t}_0$ that are not μ-torsion, we note that $\mathfrak{S}_N(\mathfrak{t}_0)$ and $\mathfrak{S}_\infty(\mathfrak{t}_0)$ will both consist entirely of non-torsion Spin^c structures by Proposition 2.4. The adjunction inequality of [18] works the same in the twisted coefficient setting as it does in the untwisted setting. Since the Thurston semi-norms on Y_N and Y will both be trivial, the inequality implies that $\underline{HF}^+(Y_N, \mathfrak{S}_N(\mathfrak{t}_0)) \otimes \mathbb{Z}[T, T^{-1}]$ and $\underline{HF}^+(Y, \mathfrak{S}_\infty(\mathfrak{t}_0)) \otimes \mathbb{Z}[T, T^{-1}]$ will both be trivial. Hence, the long exact sequence shows that $\underline{HF}^+(Y_0, \mathfrak{S}_0^N(\mathfrak{t}_0))$ will be trivial as well.

Finally, the statement about the action of T follows from Theorem 4.5, via the results of this section. □

8.6 The modules $\Omega^g(k)$

While we cannot compute $\Omega^g(k)$ precisely in general, we can say some things about these modules.

Lemma 8.9. *The module $\Omega^g(k)$ contains no non-trivial elements of finite order.*

Proof. Let C^+ denote $\underline{C}^+(Z_g, B_g, \xi)$, with $\xi = \xi^k_{B_g}$ equipped with its natural grading, so that $\Omega^g(k) = \text{Ker } \left(v_{\xi_*} \oplus h_{\xi_*}\right) \cap {}^b\underline{HFK}_{-k-1}(Z_g, B_g, \xi)$. Suppose that $x \in C^+_{-k-1}$ is a cycle which descends to an element of finite order in homology, so that $nx = \partial y$ for some $y \in C^+_{-k}$.

Note that multiplication by U gives an isomorphism for $C^+_{\ell+2}$ to C^+_ℓ for all $\ell \geq -k-1$. Hence, there are unique elements $x' \in C^+_{-k+1}$, $y' \in C^+_{-k+2}$, such that $Ux' = x$ and $Uy' = y$. Since $U\partial y' = Unx' \in C^+_{-k-1}$, we have that $\partial y' = nx'$. But we know that ${}^b\underline{HFK}_{-k+1}(Z_g, B_g, \xi)$ contains no finite order non-trivial elements; so there exists $z \in C^+_{-k+2}$ such that $\partial z = x'$. Thus $\partial Uz = x$, and so x is trivial in homology. □

Proof of Corollary 1.5. By Lemma 8.9, $\Omega^g(k)$ contains no non-trivial elements of finite order. The well groups and $\mathcal{T}^+$ clearly both contain no finite order elements, and so the same can be said of $\underline{HF}^+(Y_0, \mathfrak{t}_0)$ for any $\mathfrak{t}_0$. The well groups, when thought of as abelian groups, are free, and so we have a splitting of the short exact sequences of Theorem 1.4 as $\mathbb{Z}$-modules, and it is easy to see that such a splitting will respect the U-action. □

Proposition 8.10. *Let $\xi = \xi^{1-g}_{B_g}$. The module $\Omega^g(1-g)$ fits into a short exact sequence*

$$0 \to \Omega^g(1-g) \to Ker\ v_{\xi_*} \cap {}^b\underline{HFK}_{g-2}(Z_g, B_g, \xi) \xrightarrow{h_{\xi_*}} \mathbb{Z} \to 0,$$

where the middle term is a rank one free $\mathbb{Z}[H^1(Z_g)]$-module.

Proof. Let C^+ denote $\underline{C}^+(Z_g, B_g, \xi)$, and write C^+_{g-2} as

$$C^+_{g-2} \cong A_{-1,0} \oplus A_{0,-1} \oplus \ldots \oplus A_{g-1,-g} \cong M_{-g} \oplus M_{-g+2} \oplus \ldots M_g,$$

using the notation and grading of Section 8.2 (and where $\phi_{i,-1-i}$ gives the isomorphism $A_{i,-1-i} \cong M_{2i+2-g}$). Note again that multiplication by U gives an isomorphism from C^+_g

to C^+_{g-2}. It is easy to see also that multiplication by U gives an isomorphism Im $\partial \cap C^+_g$ to Im $\partial \cap C^+_{g-2}$.

However, there is a splitting of $\mathbb{Z}[H^1(Z_g)]$-modules

$$\text{Ker } \partial \cap C^+_{g-2} \cong (U \cdot \text{Ker } \partial \cap C^+_g) \oplus R,$$

where R is rank one and free, gotten as follows. Write an element of C^+_{g-2} as $m = m_{-g} \oplus m_{-g+2} \oplus \ldots \oplus m_g$, where $m_i \in M_i$. It is not difficult to show that there is a unique element in M_{-g}, which we denote by $\tilde{m}(m_{-g+2}, \ldots, m_g)$, such that, letting $\pi_1(m) = \tilde{m}(m_{-g+2}, \ldots, m_g) \oplus m_{-g+2} \oplus \ldots \oplus m_g$, we have $\pi_1(m) = Ux$ for some $x \in \text{Ker } \partial \cap C^+_g$. Then, we can write any $m \in \text{Ker } \partial \cap C^+_{g-2}$ uniquely as $\pi_1(m) + (\pi_2(m) \oplus 0 \oplus \ldots \oplus 0)$, for some $\pi_2(m) \in M_{-g}$. Any element of the form $p_1 + (p_2 \oplus 0 \oplus \ldots \oplus 0)$ with $p_1 \in U \cdot \text{Ker } \partial \cap C^+_g$ and $p_2 \in M_{-g}$ will lie in Ker $\partial \cap C^+_{g-2}$. Recalling that M_{-g} is rank one and free over $\mathbb{Z}[H^1(Z_g)]$, we have our splitting, with R the module of elements $p_2 \oplus 0 \oplus \ldots \oplus 0$.

It is clear that the inclusion Im ∂C^+_{g-2} into Ker ∂C^+_{g-2} followed by projection to the summand R is trivial, since any element of Im ∂C^+_g lies in Ker ∂C^+_g. Hence, the portion of C^+ lying around C^+_{g-2} splits as the direct sum of two subcomplexes. It follows that ${}^b\underline{HFK}_{g-2}(Z_g, B_g, \xi) \cong \mathbb{Z} \oplus R$, and that v_{ξ_*} is the projection onto the first summand. So Ker $v_{\xi_*} \cong R$; and it is clear that $\Omega^g(k)$ injects into the kernel, with quotient isomorphic to the image of h_{ξ_*}, which is $\mathbb{Z}$. $\square$

Chapter 9

Proofs of Theorem 1.1 and Corollary 1.2

Choose a function $\rho : \mathcal{MT}_K \to \widetilde{\mathcal{MT}_K}$ such that $[\rho([A])] = [A]$ for $[A] \in \mathcal{MT}_K$. We start by calculating the number of pairs $([A], x) \in \mathcal{MT}_K \times \mathbb{Z}/d\mathbb{Z}$ that satisfy

$$\epsilon([A]) = E \tag{9.1}$$

and

$$\eta_{\rho([A])}(x) = D \tag{9.2}$$

for a given pair of integers D, E, with notation as in Chapter 1 (and x taken to be an integer between 0 and $d-1$). Let $\mathcal{N}_0(D, E)$ denote this number.

Lemma 9.1. *The number $\mathcal{N}_0(D, E)$ is independent of the choice of ρ.*

Proof. Let $\rho([A]) = A_1 = (Q; r_1, \ldots, r_n)$ and $\rho'([A]) = A_2 = (Q'; r'_1, \ldots, r'_n)$. Then there is a unique i with $0 \le i < d$ such that $r'_\ell = r_\ell + iq_\ell$ for all ℓ. Note that for this i,

$$\sum_{\ell=1}^{n} \left\{ \frac{q_\ell x - r_\ell}{p_\ell} \right\} = \sum_{\ell=1}^{n} \left\{ \frac{q_\ell (x+i) - r'_\ell}{p_\ell} \right\}.$$

So $\eta_{A_1}(x) = D$ if and only if $\eta_{A_2}(x + i) = D$. The functions η_A are all periodic with period d; hence, the solutions to $\eta_{A_1}(x) = D$ and $\eta_{A_2}(x) = D$ are in 1-1 correspondence. The total number of solutions therefore does not depend on ρ. □

The argument of Lemma 9.1 also shows that $\eta_{\rho([A])}(x) = D$ if and only $\eta_{A_x}(0) = D$ for $A_x = (Q_x; r_1 - q_1 x, \ldots, r_n - q_n x)$, where $r_\ell - q_\ell x$ is taken to mean the equivalent value mod p_ℓ between 0 and $p_\ell - 1$, and Q_x is chosen so that $[A_x] = [A]$. Therefore, the pairs $([A], x)$ that satisfy Equations 9.1 and 9.2 are in 1-1 correspondence with elements $A \in \widetilde{\mathcal{MT}_K}$ that satisfy

$$\epsilon(A) = E$$

and

$$\eta_A(0) = D.$$

For $A = (Q; r_1, \ldots, r_n)$ to satisfy these equations means that

$$E = g_\Sigma - 1 - \frac{d}{2} \cdot S\ell(A) = d\left(g - 1 + \sum_{\ell=1}^{n}(1 - \frac{1}{p_\ell}) - Q + \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}\right) \tag{9.3}$$

and

$$D = \sum_{\ell=1}^{n} \left\{-\frac{r_\ell}{p_\ell}\right\} + \frac{E}{d} - \left(g - 1 + \sum_{\ell=1}^{n}(1 - \frac{1}{p_\ell})\right). \tag{9.4}$$

Given values of r_ℓ, the value of Q is then determined by Equation 9.3; it is not hard to show that for any values of r_ℓ that make Equation 9.4 hold, the value of Q thus determined will be an integer. So, the number $\mathcal{N}_0(D, E)$ of solutions $(Q; r_1, \ldots, r_n)$ to Equations 9.3 and 9.4 is the same as the number of solutions $(r_1, \ldots, r_n)$ to Equation 9.4.

For each ℓ, the function $r_\ell \mapsto 1 - \frac{1}{p_\ell} - \left\{-\frac{r_\ell}{p_\ell}\right\}$ gives a bijection of the possible values of r_ℓ with the set $\left\{\frac{0}{p_\ell}, \ldots, \frac{p_\ell - 1}{p_\ell}\right\}$. So, in summation, we have the following.

Lemma 9.2. *The number $\mathcal{N}_0(D, E)$ of solutions to Equations 9.1 and 9.2 is the same* *as the number* $\mathcal{N}(D,E)$ *of solutions* $(i_1,\ldots,i_n)$ *to*

$$\sum_{\ell=1}^{n} \frac{i_\ell}{p_\ell} = \frac{E}{d} - D - g + 1$$

with $0 \le i_\ell < p_\ell$.

Proof of Theorem 1.1. We first look at the case where $g = 0$. Fix a value of i between 0 and $g_\Sigma - 2$. If $[A] \in \mathcal{MT}_K$ satisfies $\epsilon([A]) = i$, then $S\ell([A]) = \frac{2(g_\Sigma - i - 1)}{d}$, which will hence be positive. Then, it is clear that every decline that G_A takes between consecutive integers will "open" a well, and that all wells opened will eventually be closed. Precisely, for each integer x for which $G_A(x+1) < G_A(x)$, we have $\frac{1}{2}\left(G_A(x) - G_A(x+1)\right)$ wells, the ones whose left coordinates are at x. Note that this uses the fact that $G_A(x)$ is odd for $x \in \mathbb{Z}$.

So, the number of wells encountered per period of G_A will be the sum of $\frac{1}{2}\left(G_A(x) - G_A(x+1)\right)$ over those x between 0 and $d-1$ (or between any two values $d-1$ apart) for which it is positive. This is equal to

$$\sum_{x=0}^{d-1} \max\{0, \eta_A(x)\} = \sum_{D>0} D \cdot \#\{x | \eta_A(x) = D, 0 \le x \le d-1\}.$$

If we do this for all $[A] \in \mathcal{MT}_K$ for which $\epsilon([A]) = i$, the sum of the values we get will be

$$\sum_{D>0} D \cdot \mathcal{N}(D,i),$$

which is equivalent to the stated expression for the $g = 0$ case.

When $i = g_\Sigma - 1$, the above calculations still hold, but we also have the $\mathcal{T}^+$ subgroup to take into consideration for the calculation of $\underline{HF}^+$. However, since this subgroup is the image of $\underline{HF}^\infty$, what we have found is precisely $\underline{HF}_{red}$ in this case.

For the case of higher genus base orbifold, we note the effect of adding "spikes" that are g tall at half-integers on the number of $\mathbb{Z}$ summands. This is best done by inspection, but we note the general idea: if g is less than half the difference between $G_A(x)$ and $G_A(x+1)$, the spike doesn't change the number of wells initiated between x and $x+1$; if g is greater than half this difference, then the number of wells initiated depends on the difference between $g + \frac{1}{2}(G_A(x) + G_A(x+1))$ and $G_A(x+1)$ (and the parity of g). All told, if $\frac{1}{2}(G_A(x) - G_A(x+1)) = D$, then the spike will yield $\left\lfloor \frac{g+D+1}{2} \right\rfloor$ initiated wells if $|D| \leq g$, and D initiated wells if $D \geq g$. This count agrees with the formula given for a_i.

The remaining terms, for the $\Omega^g(k)$ subgroups, are easy to count; the terms contributed between x and $x+1$ depend on $G_A(x) - G_A(x+\frac{1}{2}) + g$, which equals $\eta_A(x)$. An $\Omega^g(k)$ subgroup will be contributed when $k = \eta_A(x)$, and the number of times this occurs is counted by $\mathcal{N}(k, i)$. □

Proof of Corollary 1.2. Let $\phi : \Sigma \to \Sigma$ be a periodic diffeomorphism, whose mapping class is of order d. Then there is a representative ϕ' of the mapping class of ϕ such that ϕ'^n either has only isolated fixed points or is the identity. Since the mapping torus $M(\phi)$ of ϕ depends only on its mapping class, we can assume that ϕ is such a diffeomorphism, and we assume that ϕ is not the identity.

In this case, the quotient of Σ by the action of ϕ is an orbifold B, and $M(\phi)$ is a degree 0 Seifert fibered space with this orbifold has its base. In other words, $M(\phi)$ is realized by special surgery on some knot $B_g \#_{\ell=1}^n O_{p_\ell, q_\ell}$; and the base orbifold B of the Seifert fibration will have underlying surface of genus g, with one cone point of angle $\frac{2\pi}{p_\ell}$ for each ℓ. These cone points will correspond to the n points fixed by ϕ^i for $0 < i < d$: more precisely, point ℓ will be fixed by $\phi^{i \cdot \frac{d}{p_\ell}}$ for integers i. In particular, the number of fixed points of ϕ is the number of ℓ for which $p_\ell = d$. This number is just $\mathcal{N}(1-g, 1)$; by the Lefschetz fixed point theorem, it also equals $\Lambda(\phi)$.

We start with the case $g = 0$, and assume that $g_\Sigma > 2$. The claim for $i = 0$ is clear from Theorem 1.1. For the claims when $i = 1$, it is easy to see that the T-orbits of wells furnished by the proof of Theorem 1.1 all belong to different Spin^c structures. Indeed, there will be one structure of the form $\theta_K(Q_\ell; 0, \ldots, 0, d-1, 0, \ldots, 0)$ for each ℓ with $p_\ell = d$, and a little algebra rules out the possibility of any of these being equal to one another. From this, the other claims follow for $i = 1$ (the last coming from detailed observation of Theorem 1.4).

For the case when $g > 0$, the count of $\mathbb{Z}$ summands is the same as before; but now, for each $\mathbb{Z}$ summand, there is also an $\Omega^g(1-g)$ subgroup, since the number of each is given by $\mathcal{N}(1-g, 1)$. Closely examining the proof of Theorem 1.4, however, it is not hard to see that the $\mathbb{Z}$ summands and $\Omega^g(1-g)$ subgroups come in pairs each lying in the same relative grading. To be more precise, there will be $\mathcal{N}(1-g,1)$ Spin^c structures $\theta_K([A_i])$, such that G_{A_i} will have

$$G_{A_i}(x) = s,\; G_{A_i}(x + \frac{1}{2}) = s + 2g - 1,\; G_{A_i}(x+1) = s + 2g - 2$$

for some integer s and exactly one integer x between 1 and d. Furthermore, it is not hard to see that under the assumption that $g_\Sigma > 2$ and $g > 0$ that $G_{A_i}(x + \frac{3}{2}) > s + 2g - 2$. This means that we have a well at height $s + 2g - 2$ and an $\Omega^g(1-g)$ subgroup at the same height.

Borrowing notation from the proof of Proposition 8.7, the $\Omega^g(1-g)$ summand will arise as

$$\text{Ker}\ \left(\tilde{f}_* : H_{s+2g-1}(\mathbb{A}(x)) \to H_{s+2g-2}(\mathbb{B}(x)) \oplus H_{s+2g-2}(\mathbb{B}(x+1))\right).$$

Meanwhile, the $\mathbb{Z}$ in the quotient will be the portion of Ker π_* consisting of elements of the form $n \cdot b^{x+1}$, where $n \in \mathbb{Z}$, and b^{x+1} is as descriped in Equation 8.1. Examining Proposition 8.10, it is easy to see that Ker $\tilde{f}_*$ is a rank one free $\mathbb{Z}[H^1(Z_g)]$-module, and that the $\Omega^g(1-g)$ and $\mathbb{Z}$ terms are just a submodule of Ker $\tilde{f}_*$ and the corresponding quotient, respectively.

This means that we have that the twisted Floer homology is $\mathbb{Z}[H^1(Z_g)] \otimes \mathbb{Z}[T, T^{-1}] \cong$

R for each Spin^c structure where the homology is non-trivial. Hence, the identification of the groups still holds up, and the other claims follow as before. □

We now extend Corollary 1.2 to the third-to-highest level.

Theorem 9.3. *Keep notation from Corollary 1.2, and assume that the mapping class of ϕ is not of order 1 or 2, and that $g_\Sigma > 3$. We have*

$$\underline{HF}^+(Y_0, [-2]) \cong R^{\frac{\Lambda(\phi^2)+\Lambda(\phi)^2}{2}}$$

where Λ denotes Lefschetz number. If the mapping class is not of order 4, we have in addition that the U-action is trivial, each summand lies in a different Spin^{c}structure, and T lowers this relative grading by $2d(g_\Sigma - 1 - i)$.

Proof. This follows along similar lines as Corollary 1.2. We again leave out the case where the mapping class of ϕ is trivial. We also assume that $g = 0$; the modifications for higher genus are exactly as above. We assume without loss of generality that ϕ and its iterates all have isolated fixed points (or are the identity map).

The number of orbits of the action of ϕ that are fixed pointwise by ϕ^n will be the number of ℓ for which $\frac{d}{p_\ell}$ divides n. Hence, the number of fixed points of ϕ^2, will be the number of fixed points of ϕ plus twice the number of ℓ for which $p_\ell = \frac{d}{2}$. So, the number of ℓ for which $p_\ell = \frac{d}{2}$ is equal to $\frac{\Lambda(\phi^2)-\Lambda(\phi)}{2}$.

The rank of $\underline{HF}^+(Y_0, [-2])$ will be equal to $\sum_{D>0} D \cdot \mathcal{N}(D, 2)$, where $\mathcal{N}(D, 2)$ is the number of solutions to

$$\sum_{\ell=1}^{n} \frac{i_\ell}{p_\ell} = \frac{2}{d} - D + 1$$

for which $0 \le i_\ell < p_\ell$ for all ℓ. If $D > 0$ (and $d \neq 1$), the right side will be positive only if $D = 1$ or if $d = 2$ and $D = 2$. Leaving the latter case aside, we see that for $d \neq 2$, the rank is simply $\mathcal{N}(1,2)$, and we calculate this to be

$$\frac{\Lambda(\phi^2)-\Lambda(\phi)}{2}+\Lambda(\phi)+\begin{pmatrix}\Lambda(\phi)\\ 2\end{pmatrix}=\frac{\Lambda(\phi^2)+\Lambda(\phi)^2}{2}.$$

In this case, a little algebra shows that the only time that different wells furnished by the proof of Theorem 1.1 land in the same Spin^c structure is when ϕ is isotopic to a diffeomorphism of order 4 with two fixed points, for which ϕ^2 has an odd number of fixed points. This case and the case of ϕ of order 1 or 2 aside, the other claims follow again. □

The rank here agrees with the rank suggested by comparison with the periodic Floer homology of Hutchings. Specifically, the periodic Floer homology of a periodic diffeomorphism ϕ (with ϕ^d being exactly the identity, and with non-identity iterates of ϕ having isolated fixed points) will have $\frac{\Lambda(\phi^2)+\Lambda(\phi)^2}{2}$ generators, and an implicit consequence of Proposition 1.6 of [7] is that this complex will have no non-trivial differentials. Furthermore, it appears that for a diffeomorphism ϕ of order d, the ranks of the first d levels of $\underline{HF}^+(M_\phi)$ should be equal to the rank of the corresponding portions of periodic Floer homology, as long as the genus of the underlying surface is large enough.

Chapter 10

Sample Calculations

We give two examples, which admit comparisons with known results.

10.1 0-surgery on the (2,7) torus knot

Let

$$K = O_{2,1}\#O_{7,3}\#O_{14,1} \subset Y = L(2,1)\#L(7,3)\#L(14,1),$$

as depicted in Figure 10.1. Surgery on K with coefficient -1 gives the same manifold as 0-surgery on the (2,7) torus knot.

So, $(p_1, q_1) = (2, 1)$, $(p_2, q_2) = (7, 3)$, $(p_3, q_3) = (14, 1)$. It is easy to see that the set $\mu\mathcal{T}_K$ of μ-torsion structures in $\mathrm{Spin}^c(Y_0)$ consists of $\theta_K([A])$ for where $A = (Q; 0, 0, i) \in \widetilde{AT_K}$ for $Q \in \mathbb{Z}$ and $i = 0, 1, \ldots, 13$.

Let us look at the case of $A_Q = (Q; 0, 0, 0)$ in depth. Using the process described in the introduction, we graph the function G_{A_Q} for $Q = 0, 1, 2$. The knot K is of order 14 in $H_1(Y)$, so these functions will all be the sum of linear functions plus functions that are periodic of period 14. We show the graphs of each function in Figure 10.2. It is clear that of these three, only the graph of G_{A_1} possesses any wells. In fact, since G_{A_Q} will be gotten from G_{A_1} by adding a linear function whose slope depends on Q, it is easy to see that G_{A_Q} will have no wells for any other value of $Q \in \mathbb{Z}$.

We can identify the wells in the function for $Q = 1$: there will be one at each height $3 - 4n$ for $n \in \mathbb{Z}$, each having trivial U-action. So, there is an absolute lift of the relative $\mathbb{Z}$-grading on $\underline{HF}^+(Y_0, \theta_K([A_1]))$, so that

$$\underline{HF}^+(Y_0, \theta_K([A_1])) \cong \mathbb{Z}_{(-1)} \otimes \mathbb{Z}[T_{(-4)}, T^{-1}_{(-4)}],$$

where $T_{(4)}$ takes a well to the corresponding well one period to the right and raises $\mathbb{Z}$-grading by 4. We see that $14 \cdot S\ell(A_1) = -4 = \langle c_1(\theta_K(A_1)), [\widehat{dS}]\rangle$.

The elements $A = (Q; 0, 0, i)$ for the other values of i admit a similar analysis. All told, we end up with five elements $\mathfrak{t} \in \mathrm{Spin}^c(Y_0)$ for which $\underline{HF}^+(Y_0, \mathfrak{t})$ is nontrivial. We label these as $\mathfrak{t}_4, \mathfrak{t}_2, \mathfrak{t}_0, \mathfrak{t}_{-2}$, and $\mathfrak{t}_{-4}$, where $\langle c_1(\mathfrak{t}_i), [\widehat{dS}]\rangle = i$. Let $\mathcal{T}^n_{(s)} \cong \mathbb{Z}[U^{-1}]/U^{-n} \cdot \mathbb{Z}[U^{-1}]$ as $\mathbb{Z}[U]$ modules, graded so that U^{-i} lies in level $s + 2i$ for $0 \leq i < n$ (so that the bottom degree non-trivial elements live in level s). Then, there is a lift of the

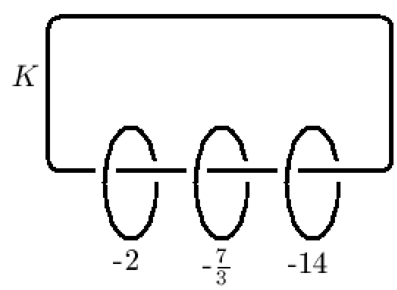

Figure 10.1: The knot $K = O_{2,1} \# O_{7,3} \# O_{14,1}$.

relative-grading on $\underline{HF}^+(Y_0, \mathfrak{t}_i)$ for which

$$\underline{HF}^+(Y_0, \mathfrak{t}_i) \cong \begin{cases} \mathcal{T}^1_{(-1)} \otimes \mathbb{Z}[T_{(4)}, T^{-1}_{(4)}] & i = 4 \\ \mathcal{T}^1_{(-1)} \otimes \mathbb{Z}[T_{(2)}, T^{-1}_{(2)}] & i = 2 \\ \left(\mathcal{T}^2_{(-3)} \otimes \mathbb{Z}[T_{(0)}, T^{-1}_{(0)}]\right) \oplus \mathcal{T}^+_{(2)} & i = 0 \\ \mathcal{T}^1_{(-1)} \otimes \mathbb{Z}[T_{(-2)}, T^{-1}_{(-2)}] & i = -2 \\ \mathcal{T}^1_{(-1)} \otimes \mathbb{Z}[T_{(-4)}, T^{-1}_{(-4)}] & i = -4, \end{cases}$$

where U lowers the grading by 2, and all the groups have trivial U action except for the one for $\mathfrak{t}_0$. Forgetting about relative $\mathbb{Z}$-gradings, this can be summarized as saying that

$$\underline{HF}^+(Y_0) \cong HF^+(Y_0) \otimes \mathbb{Z}[T, T^{-1}]$$

in light of Proposition 8.1 of [15].

10.2 $S^1 \times \Sigma_g$

Let us consider special surgery on the knot B_g itself, which yields $S^1 \times \Sigma_g$. Obviously, there are no O-knot connect summands, which is fine. So, we just write elements of $\widetilde{\mathcal{MT}}_{B_g}$ as (Q); if $\mathfrak{t} = \theta_{B_g}([(Q)])$, then $\langle c_1(\mathfrak{t}), [\widehat{dS}]\rangle = 2Q$.

Let us consider just $\mathfrak{t}_g = \theta_{B_g}([(0)])$, the unique torsion Spin^c structure on $S^1 \times \Sigma_g$. Writing G_g for the well function corresponding to $\mathfrak{t}_g$, we have

$$G_g(x) = \begin{cases} 1, & x \in \mathbb{Z} \\ g+1, & x \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

We show the well function in Figure 10.3 for the case $g = 1$.

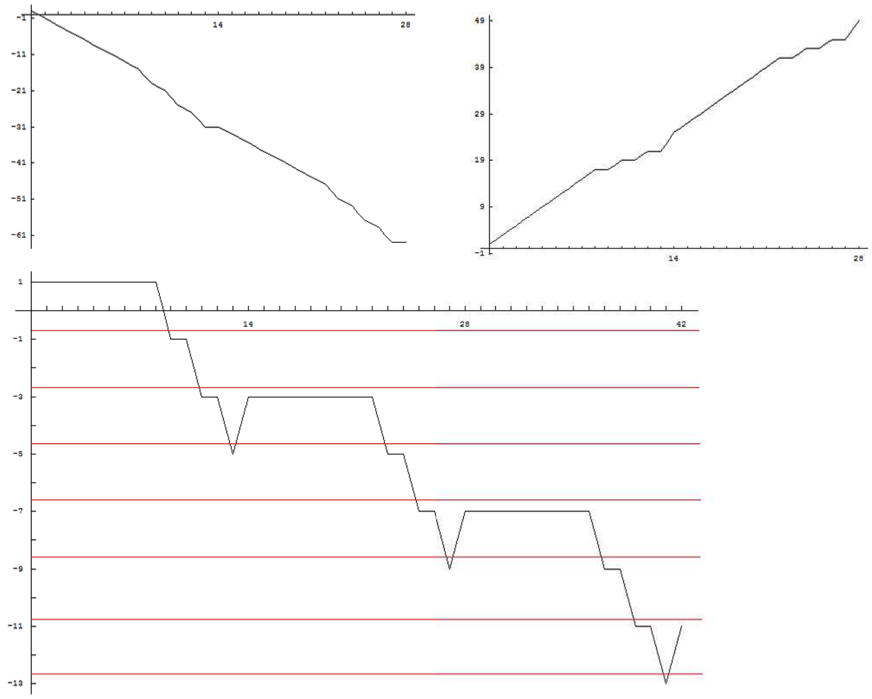

Figure 10.2: The graphs of the functions G_{A_Q} for $Q = 0$ (top left), $Q = 1$ (bottom) and $Q = 2$ (top right). We show two periods for $Q = 0, 2$, and three period for $Q = 1$. Only the last has any wells, which correspond to finite dips under any of the red lines, placed just above the odd integers; in the graph, we see three wells, at heights -5, -9, and -13.

It is easy to see that the short exact sequence of Theorem 1.4 becomes

$$0 \to \bigoplus_{i\in\mathbb{Z}} (\Omega^g(0))_{(g)} \oplus \mathcal{T}^+_{(2g')} \to \underline{HF}^+(S^1 \times \Sigma_g, \mathfrak{t}_g) \to \bigoplus_{i\in\mathbb{Z}} \mathcal{T}^{g'}_{(1)} \to 0$$

where $g' = \left\lfloor \frac{g+1}{2} \right\rfloor$.

In the particular case $g = 1$, we have

$$0 \to \bigoplus_{i\in\mathbb{Z}} \left(\Omega^1(0)\right)_{(1)} \oplus \mathcal{T}^+_{(2)} \to \underline{HF}^+(T^3, \mathfrak{t}_1) \to \bigoplus_{i\in\mathbb{Z}} \mathcal{T}^1_{(1)} \to 0.$$

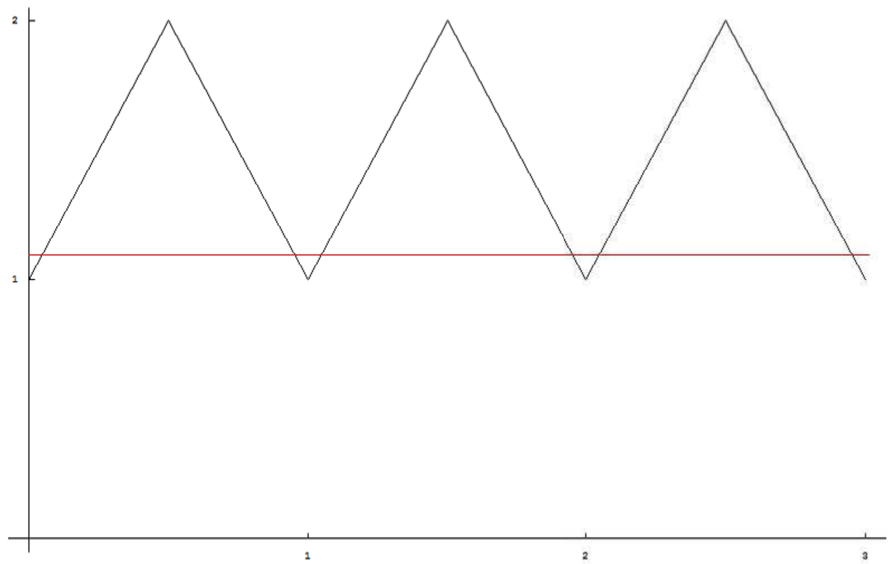

Figure 10.3: The graph of the function G_1 associated with the torsion Spin^c structure on T^3, over three periods. We see three wells at height 1. In addition, the fact that the graph goes up one from i to $i + \frac{1}{2}$, $i = 0, 1, 2$, contributes three $\Omega^g(-1+g) = \Omega^1(0)$ subgroups.

Of course, this is for some lift of the relative $\mathbb{Z}$-grading. In particular, we have that the only odd grading in which $\underline{HF}^+(T^3, \mathfrak{t}_1)$ is non-trivial is one below the lowest even grading where it is non-trivial; and in all the even grading levels above this, the homology is $\mathbb{Z}$. At least this much clearly agrees with the explicit calculation given in [15],

$$\underline{HF}^+_i(T^3, \mathfrak{t}_1) \cong \begin{cases} \mathbb{Z}, & i \equiv \frac{1}{2} (\mathrm{mod}\ 2) \text{ and } i \geq \frac{1}{2} \\ \mathrm{Ker}\ \epsilon, & i = -\frac{1}{2} \\ 0, & \text{otherwise,} \end{cases}$$

where i denotes absolute $\mathbb{Q}$-grading and $\epsilon : \mathbb{Z}[H^1(T^3)] \to \mathbb{Z}$ is the map which sends every element of $H^1(T^3)$ to 1.

In the sole odd non-trivial level, with absolute grading $-\frac{1}{2}$, we have

$$0 \to \bigoplus_{i \in \mathbb{Z}} \Omega^1(0) \to \underline{HF}^+_{-\frac{1}{2}}(T^3, \mathfrak{t}_1) \to \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \to 0.$$

From this it can be shown that $\underline{HF}^+_{-\frac{1}{2}}(T^3, \mathfrak{t}_1) \cong \mathrm{Ker}\ \epsilon$, although this requires some care, as the PD$[T^2]$-action is a little subtle in this case. As the agreement of our results with those of [15] is certainly plausible, we omit further verification.

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Appendix

We now return to some results from Chapter 2 whose proofs are fairly straightforward, but somewhat technical.

Take an oriented, rationally nullhomologous knot $K \subset Y$. We want to shift to a concrete setting to talk about algebraic topology issues more clearly. So, present Y as surgery on a framed link $L \subset S^3$; this also serves as a Kirby diagram for a 2-handlebody U_∞ with boundary Y. Order the components of the link, and assign each an orientation; call the linking matrix G_∞. This specifies a basis for $H_2(U_\infty)$, with respect to which the intersection form is just G_∞. To be precise, the basis is given by taking oreinted Seifert surfaces for the link components pushed into the interior of U_∞, and then capped off.

We can represent K as an extra component on this link, which comes equipped with orientation; the implicit framing λ will be some integer, which we call I_0. Let K_{+N} be K with framing $I_0 + N$. The framed link $L_N = L \cup K_{+N}$ is a Kirby diagram for a handlebody U_N with boundary Y_N (allowing $N = 0$, of course). Again, $H_2(U_N)$ has a canonical ordered basis, with respect to which the intersection form on U_N is equal to the linking matrix G_N of L_N. Written in block form, we have

$$G_N = \begin{pmatrix} G_\infty & \vec{k} \\ \vec{k}^{\,T} & I_0 + N \end{pmatrix},$$

where the last row and column correspond to the component K_{+N}. Note that $U_N = U_\infty \cup_Y W_N$.

We form another 2-handlebody as follows. To L_0, add a 0-framed meridian to K, and an N-framed meridian of the 0-framed meridian; call this link L_{0N}, the represented 2-handlebody U_{0N}, and the linking matrix G_{0N}. Ordering and orienting the components of L_{0N}, we have

$$G_{0N} = \begin{pmatrix} G_\infty & \vec{k} & \vec{0} & \vec{0} \\ \vec{k}^{\,T} & I_0 & 1 & 0 \\ \vec{0} & 1 & 0 & 1 \\ \vec{0} & 0 & 1 & N \end{pmatrix}.$$

It is not hard to see that, in fact, the cobordism $X_{\alpha\beta\gamma}$ of Section 2.5, when glued to a tubular neighborhood of a sphere with self intersection N along $L(N,1)$, gives a cobordism W_{0N} such that $U_{0N} = U_0 \cup_{Y_0} W_{0N}$. We define $\mathfrak{S}_{0N}(\mathfrak{t}_0) \subset \mathrm{Spin}^c(W_{0N})$ to be those structures whose first Chern class evaluates to N on this sphere. Then, of course, $\mathfrak{S}_N(\mathfrak{t}_0)$ can be described as the restrictions of $\mathfrak{S}_{0N}(\mathfrak{t}_0) \subset \mathrm{Spin}^c(W_{0N})$ to Y_N.

We denote elements of second homology and cohomology of U_* by vectors and covectors respectively (where U_* denotes any of U_∞, U_N, and U_{0N}), so that evaluation of a cohomology class on a homology class is given by the normal dot product. We also denote elements of $H^2(U_*, \partial U_*)$ by covectors, so that the Poincaré dual of $(\vec{a})_{U_*} \in H_2(U_*)$ is just $(\vec{a}^{\,T})_{U_*,\partial U_*}$. We will use the same notation for the corresponding cohomology groups with rational coefficients. In the long exact cohomology sequence for $(U_*, \partial U_*)$, we have

$$H^2(U_*, \partial U_*) \to H^2(U_*) \to H^2(\partial U_*) \to 0, \tag{A.1}$$

with the first map given in terms of our bases by right multiplication by G_*. We often write elements of $H^2(Y_N)$ as $h = (\vec{b}^{\,T}, c)_{U_N}|_{Y_N}$, so as to cooperate with the block form expression of G_N.

Since all the handlebodies are simply-connected, the Spin^c structures on each can be identified via the first Chern class with characteristic covectors of the corresponding matrices (i.e., covectors whose i^{th} component is congruent mod 2 to the i^{th} diagonal entry of the matrix). We denote the Spin^c structures by $\langle\vec{a}\,^T\rangle_{U_*}$ so that $c_1(\langle\vec{a}\,^T\rangle_{U_*}) = (\vec{a}\,^T)_{U_*}$. (In general, angle brackets will be used to signify that we are talking about a Spin^c structure rather than a cohomology class.)

The restriction maps $\text{Spin}^c(U_*) \to \text{Spin}^c(\partial U_*)$, $\text{Spin}^c(U_N) \to \text{Spin}^c(W_N)$, and $\text{Spin}^c(U_{0N}) \to \text{Spin}^c(W_{0N})$ are all surjections, and the restriction maps are all equivariant with respect to the action of $H^2(U_*)$ (acting on the targets via the restriction maps). It follows that every Spin^c structure on W_* or Y_* can be specified as the restriction of one on some U_*, and, identifying $\text{Spin}^c(U_*)$ with $\text{Char}(G_*)$, that we can view $\text{Spin}^c(\partial U_*)$ as $\text{Char}(G_*)/2\cdot\text{Im}G_*$. (The same can be said with Spin^c structures replaced by elements of second cohomology.)

It is not hard to see that U_{0N} is diffeomorphic to $U_N\#S$, with S denoting either $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ or $S^2\times S^2$; the handleslides necessary to realize this are shown in Figure A.1. We can also perform all the same moves if we bracket the framings for all but the two meridians shown in Figure A.1a, which would depict W_{0N}. We can see a spanning disk for the N-framed component in $-Y_0$ in Figure A.1d, since this component is unlinked from the link of bracketed components; gluing this to the core of the corresponding handle gives a distinguished sphere V embedded in W_{0N}.

Now, we can describe the sets mentioned at the beginning of Section 2.4.

Proposition A.1. *Suppose* $\mathfrak{t}_0 = \langle\vec{b}\,^T, c\rangle_{U_0}|_{Y_0} \in \mathit{Spin}^c(Y_0)$. *Then we have*

$$\mathfrak{S}_0^N(\mathfrak{t}_0) = \left\{\langle\vec{b}\,^T, c+2iN\rangle_{U_0}|_{Y_0}\middle| i\in\mathbb{Z}\right\},$$

$$\mathfrak{S}_{0N}(\mathfrak{t}_0) = \left\{\langle\vec{b}\,^T, c+2iN, 2j, N\rangle_{U_{0N}}|_{W_{0N}}\middle| i,j\in\mathbb{Z}\right\},$$

$$\mathfrak{S}_N(\mathfrak{t}_0) = \left\{\langle\vec{b}\,^T, c+(2i-1)N\rangle_{U_N}|_{Y_N}\middle| i\in\mathbb{Z}\right\},$$

$$\mathfrak{S}_{N\infty}(\mathfrak{t}_0) = \left\{\langle\vec{b}\,^T + 2i\vec{k}\,^T, c+2iI_0+(2j-1)N\rangle_{U_N}|_{W_N}\middle| i,j\in\mathbb{Z}\right\},$$

and

$$\mathfrak{S}_\infty(\mathfrak{t}_0) = \left\{ \langle \vec{b}^{\,T} + 2i\vec{k}^{\,T} \rangle_{U_\infty}|_Y \middle| i \in \mathbb{Z} \right\}.$$

In particular, $\mathfrak{S}_\infty(\mathfrak{t}_0)$ *is independent of* N*.*

Proof. The first claim is clear. The second claim follows by looking at the preimage of $\mathfrak{S}_0^N(\mathfrak{t}_0)$ under the restriction map induced by the inclusion $Y_0 \to U_0 \to U_{0N}$. We must have the last component equal to N since structures in $\mathfrak{S}_{0N}(\mathfrak{t}_0)$ have specified evaluation on V, and it is not hard to see that all rows of the matrix G_{0N} besides the last two vanish in $H^2(W_{0N})$.

The fourth claim follows from the third similarly, and the fifth follows straightforwardly from the fourth. To identify the restriction of $\mathfrak{S}_{0N}(\mathfrak{t}_0)$ to $\mathrm{Spin}^c(Y_N)$, we perform the change of basis corresponding to the moves shown in Figure A.1. In terms of the new basis, where the last three components correspond respectively to the $I_0 + N$-, I_0-, and 0-framed components, $c_1\left(\langle \vec{b}^{\,T}, c + 2iN, 2j, N\rangle_{U_{0N}}|_{W_{0N}}\right)$ will be written as $(\vec{b}^{\,T} - 2j\vec{k}^{\,T}, c - 2jI_0 + (2i-1)N, c + 2iN, 2j)$. The first two components give the restriction of this to $H^2(U_N)$, and the third claim follows straightforwardly from this after noting that $(\vec{k}^{\,T}, I_0)_{U_N}|_{Y_N} = (\vec{0}^{\,T}, -N)_{U_N}|_{Y_N}$. □

Let us explicitly identify some (co)homology classes. We may choose a surface P in U_∞ with $\partial P = K$ (i.e., a "pushed-in" Seifert surface), so that $(\vec{k})_{U_\infty, \partial U_\infty}$ represents P in $H_2(U_\infty, \partial U_\infty)$. Let us also choose a Seifert surface dS for $d \cdot K$ in Y, where d is the order of K in $H_1(Y)$. Gluing $-dS$ to $d \cdot P$ yields a class S_1 of $H_2(U_\infty; \mathbb{Z})$. Choose some $(\vec{p}_0)_{U_\infty} \in H_2(U_\infty; \mathbb{Q})$ so that $S_1 = (d\vec{p}_0)_{U_\infty}$. This choice must satisfy $G_\infty \vec{p}_0 = \vec{k}$.

We can also glue P to F, recalling that the latter is the core of the 2-handle of W_N. The resulting class will be the oriented generator of $H_2(U_N)$ corresponding to the link component K, and so it will be written as

$$\begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}_{U_N} \in H_2(U_N).$$

Now choose $\widetilde{dF}$ to be $d \cdot F$ glued to dS; it follows that if $i_* : H_2(W_N) \to H_2(U_N)$ is induced by inclusion,

$$i_*\left([\widetilde{dF}]\right) = \begin{pmatrix} \text{-}d\vec{p}_0 \\ d \end{pmatrix}_{U_N}.$$

We will also need to identify $\mathrm{PD}[F]|_{W_N}$ (we think of F as a generator of $H_2(W_N, Y)$, so its dual technically lives in $H^2(W_N, Y_N)$). To find this, we take advantage of the commutative diagram

$$\begin{array}{ccccc} H_2(W_N) & \cong & H^2(W_N, \partial W_N) & \xrightarrow{j^*_{W_N}} & H^2(W_N) \\ i_* \downarrow & & & & i^* \uparrow \\ H_2(U_N) & \cong & H^2(U_N, \partial U_N) & \xrightarrow{j^*_{U_N}} & H^2(U_N) \end{array}$$

where the isomorphisms are from Poincaré duality and the other maps are induced from inclusions. (The claimed commutativity is not quite as obvious as obvious as it seems, but is a reasonable exercise in diagram chasing.) Given this, it follows that $j^*_{W_N}\left(\mathrm{PD}[\widetilde{dF}]\right) = \left(\vec{0}^{\ T}, d(I_0 - \vec{p}_0^{\ T} G_\infty \vec{p}_0 + N)\right)_{U_N}|_{W_N}$, since $j^*_{U_N}$ is represented by multiplication by G_N. It is easy to see that $(d\vec{k}^{\ T}, d\vec{p}_0^{\ T} G_\infty \vec{p}_0)_{U_N}|_{W_N} = 0$, noting that $(d\vec{k}^{\ T}, d\vec{p}_0^{\ T} G_\infty \vec{p}_0)_{U_N} = j^*_{U_N}(\mathrm{PD}[S_1])$ and that $\mathrm{PD}[S_1]$ restricts to the trivial element of $H^2(W_N, Y_N) \cong \mathbb{Z}$. Hence $j^*_{W_N}\left(\mathrm{PD}[\widetilde{dF}]\right)$ can also be written as $\left(d\vec{k}^{\ T}, d(I_0 + N)\right)_{U_N}|_{W_N}$, and so at least up to torsion,

$$\mathrm{PD}[F]|_{W_N} = (\vec{k}^{\ T}, I_0 + N)_{U_N}|_{W_N}. \tag{A.2}$$

That this actually holds precisely follows from the fact that it restricts to $\mathrm{PD}[K]$ in $H^2(Y)$, and the kernel of $H^2(W_N) \to H^2(Y)$ contains no nontrivial torsion elements.

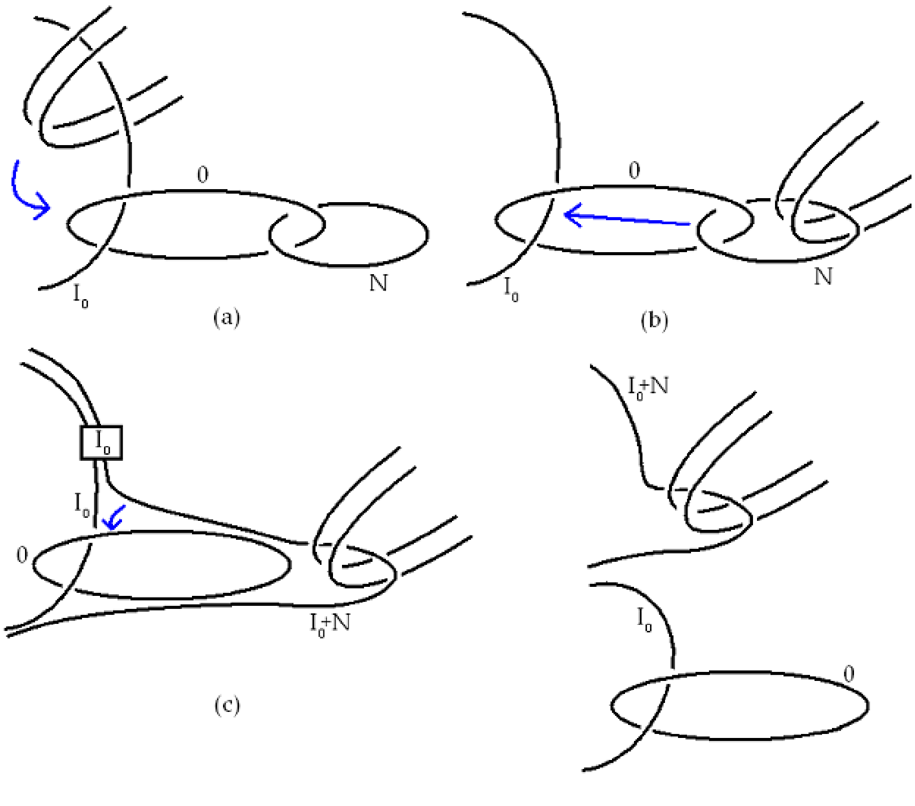

Figure A.1: We start with the diagram for X_N, and then we handleslide all of the components linking the I_0-framed knot over the 0-framed unknot, so that afterward this unknot is the only component that the I_0-framed knot links with. We then handleslide the N-framed unknot over the I_0-framed knot, turning the unknot into an $I_0 + N$ framed knot of the same type. Finally, we handle slide the $I_0 + N$-framed knot over the 0-framed component enough times to unlink it from the I_0-framed knot. We are left with the Kirby diagram for X_N together with a knot with a 0-framed meridian, which presents $X_N \# \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ or $X_N \# S^2 \times S^2$.

Now, let us note the following facts.

Lemma A.2. *The term κ is equal to $I_0 - \vec{p}_0{}^T G_\infty \vec{p}_0$, and is independent of the particular choice of $\vec{p}_0$. The longitude $N\mu + \lambda$ is special if and only if $\kappa = -N$. The order of μ*

in $H_1(Y_0)$ *is* $d|\kappa|$*. The element* $(\vec{b}^{\,T}, c)_{U_N}$ *of* $H^2(U_N)$ *restricts to a torsion element of* $H^2(Y_N)$ *if and only if* $(\vec{b}^{\,T})_{U_\infty}|_Y$ *is torsion and either* $\kappa \neq -N$ *or* $c = \vec{b}^{\,T} \cdot \vec{p}_0$ *for any choice of* $\vec{p}_0$*.*

Proof. All of these follow from straightforward matrix algebra, using the long exact sequence (A.1). Note that if $\kappa = -N$, then the quantity $c - \vec{b}^{\,T} \cdot \vec{p}_0 = \langle (\vec{b}^{\,T}, c)_{U_N}|_{W_N}, [\widetilde{dF}] \rangle$ depends only on $(\vec{b}^{\,T}, c)_{U_N}|_{Y_N}$. □

Proof of Proposition 2.5. Straightforward, utilizing Proposition A.1 and Lemma A.2. □

We now describe how to compute squares of elements of $H^2(W_N')$; we first compute $[\widetilde{dF'}]^2$.

Proof of Proposition 2.7. We can square elements of $H_2(W_N)$ using the intersection form on U_N, since W_N is a submanifold of U_N. Using our identification of $[\widetilde{dF}]$ from above, we compute that $[\widetilde{dF}]^2 = d\big(-d\vec{k}^{\,T}\vec{p}_0 + d(I_0 + N)\big) = d^2(\kappa + N)$. Hence $[\widetilde{dF'}]^2 = -d^2(\kappa + N)$ in $H_2(W_N')$, due to the reversal of orientation. □

Next, we compute squares of general boundary-torsion elements of $H^2(W_N)$ and $H^2(W_N')$. Denote by α the inclusion of $\mathrm{PD}[\widetilde{dF}] \in H^2(W_N, \partial W_N)$ into $H^2(W_N)$, $\alpha = j_{W_N}^*\big(\mathrm{PD}[\widetilde{dF}]\big)$. Then of course $\alpha^2 = d^2(\kappa + N)$.

We will also need to know the evaluation $\langle \alpha, [\widetilde{dF}] \rangle$. According to the diagram α equals $i^*\Big(j_{U_N}^*\big(\mathrm{PD}[i_*([\widetilde{dF}])]\big)\Big)$. Thus,

$$\langle \alpha, [\widetilde{dF}] \rangle = \big\langle j_{U_N}^*\big(\mathrm{PD}[i_*([\widetilde{dF}])]\big), i_*([\widetilde{dF}]) \big\rangle = \Big((-d\vec{p}_0^{\,T}, d) \cdot G_N\Big) \cdot \begin{pmatrix} -d\vec{p}_0 \\ d \end{pmatrix} = d^2(\kappa + N).$$

By the universal coefficients theorem, we have that

$$H^2(W_N)/\text{Torsion} \cong \text{Hom}\big(H_2(W_N),\mathbb{Z}\big) \cong \text{Hom}\big(H_2(Y),\mathbb{Z}\big) \oplus \text{Hom}\big(\mathbb{Z}\cdot[\widetilde{dF}],\mathbb{Z}\big).$$

Recalling the map $j : H^2(W_N;\mathbb{Z}) \to H^2(\partial W_N;\mathbb{Z}) \to H^2(\partial W_N;\mathbb{Q})$, if $\phi \in \text{Ker } j$ and $\langle \phi, [\widetilde{dF}]\rangle = i$, it then follows that $\phi = \frac{i}{d^2(\kappa+N)}\alpha$ in $H^2(W_N)/\text{Torsion}$ since elements of Ker j evaluate trivially on $H_2(Y)$. So $\phi^2 = \left(\frac{i}{d^2(\kappa+N)}\right)^2 d^2(\kappa+N) = \frac{i^2}{d^2(\kappa+N)}$.

By excision and the long exact sequence of (U_N, W_N), the restriction map $i^* : H^2(U_N) \to H^2(W_N)$ is surjective, and of course we have

$$\langle(\vec{b}\,^T, c)_{U_N}, i_*([\widetilde{dF}])\rangle = \langle i^*\big((\vec{b}\,^T, c)_{U_N}\big), [\widetilde{dF}]\rangle.$$

These facts, together with the above, allow us to compute the square of any element in $H^2(W_N)$.

Of course, squaring elements of $H^2(W'_N)$ is exactly the same, except that every Poincaré dual gets a minus sign. So we arrive at the value $-\frac{(c-\vec{b}\,^T\vec{p}_0)^2}{(\kappa+N)}$ for the square of the class $(\vec{b}\,^T, c)_{U_N}|_{W'_N}$. Since the evaluation of this class on any lift $[\widetilde{dF'}]$ is equal to $d(c-\vec{b}\,^T\vec{p}_0)$, we have the following. (Recall that j is the composition of obvious maps $H^2(W'_N;\mathbb{Z}) \to H^2(\partial W'_N;\mathbb{Z}) \to H^2(\partial W'_N;\mathbb{Q})$.)

Lemma A.3. *The square of $\alpha \in$ Ker $j \subset H^2(W'_N)$ is given by*

$$\alpha^2 = -\frac{\big(\langle\alpha, [\widetilde{dF}]\rangle\big)^2}{d^2(\kappa+N)}.$$

Proof of Proposition 2.9. Let $\mathfrak{t}_0 = \langle\vec{b}\,^T, c\rangle_{U_0}|_{Y_0}$, and set $\mathfrak{s}^i_N = \langle\vec{b}\,^T, c+(2i-1)N\rangle_{U_N}|_{W'_N}$ and $\mathfrak{t}^i_N = \mathfrak{s}^i_N|_{Y_N}$.

We want to compute $\mathfrak{s}_{K+}(\mathfrak{t}^i_N)$. Note that we have

$$\frac{[\widetilde{dF'}]^2}{d^2} = -N, \quad \frac{\langle c_1(\mathfrak{s}^i_N), [\widetilde{dF'}]\rangle}{d} = c - \vec{b}\,^T\vec{p}_0 + (2i-1)N.$$

Therefore, the function $Q_K(j, \mathfrak{s}_N^i) = 0$ when

$$j = \frac{c - \vec{b}^{\,T}\vec{p}_0 + (2i-1)N}{2N};$$

and so $\mathfrak{s}_{K+}(\mathfrak{t}_N^i) = \mathfrak{s}_N^i + \lfloor j \rfloor \mathrm{PD}[F']|_{W'_N}$.

Now,

$$\lfloor j \rfloor = i + \left\lfloor \frac{c - \vec{b}^{\,T}\vec{p}_0}{2N} - \frac{1}{2} \right\rfloor,$$

and for large N this will just equal $i - 1$. So,

$$\mathfrak{s}_{K+}(\mathfrak{t}_N^i) = \langle \vec{b}^{\,T} - 2(i-1)\vec{k}^{\,T}, c - 2(i-1)I_0 + N \rangle_{U_N}|_{W'_N}.$$

Also, $\mathrm{PD}[F']|_{W'_N}$ is represented by $\left(-\vec{k}^{\,T}, -(I_0+N)\right)_{U_N}|_{W'_N}$, where we pick up a negative sign from (A.2) since we are taking duals with respect to the orientation of W'_N; hence,

$$\mathfrak{s}_{K+}(\mathfrak{t}_N^i) + \mathrm{PD}[F']|_{W'_N} = \langle \vec{b}^{\,T} - 2i\vec{k}^{\,T}, c - 2iI_0 - N \rangle_{U_N}|_{W'_N}.$$

Therefore, we can just use Lemma A.3 to compute

$$q_K\left(\mathfrak{s}_{K+}(\mathfrak{t}_N^i)\right) = \frac{c_1^2\left(\mathfrak{s}_{K+}(\mathfrak{t}_N^i) + \mathrm{PD}[F']|_{W'_N}\right) - c_1^2\left(\mathfrak{s}_{K+}(\mathfrak{t}_N^i)\right)}{4} = c - \vec{b}^{\,T}\vec{p}_0.$$

Notice that this value is independent of $\mathfrak{t}_N^i$. In fact, it is not hard to see that there is a Seifert surface dS for dK, which can be capped off in Y_0 to give a surface $\widehat{dS}$, which will be represented by

$$\left.\begin{pmatrix} d\vec{p}_0 \\ -d \end{pmatrix}_{U_0}\right|_{Y_0},$$

and so

$$\frac{\langle c_1(\mathfrak{t}_0), [\widehat{dS}] \rangle}{d} = -(c - \vec{b}^{\,T}\vec{p}_0),$$

which completes the proof. □