Action-Maslov Homomorphism for Monotone Symplectic Manifolds

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Submitted in partial fulfillment of the

requirements for the degree

of Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2010

ABSTRACT

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The action-Maslov homomorphism $I : \pi_1(\mathrm{Ham}(X, \omega)) \to \mathbb{R}$ is an important tool for understanding the topology of the Hamiltonian group of monotone symplectic manifolds. We explore conditions for the vanishing of this homomorphism, and show that it is identically zero when the Seidel element has finite order and the homology satisfies property $\mathcal{D}$ (a generalization of having homology generated by divisor classes). These properties hold for products of projective spaces, the Grassmannian of 2 planes in $\mathbb{C}^4$, and toric 4-manifolds. We show that these properties do not hold for all Grassmannians. Finally, the relationship between these statements and the geometry of $\pi_1(\mathrm{Ham}(X, \omega))$ is explored.

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Acknowledgments

I would like to thank my thesis advisor, Dusa McDuff, for inspiring this work and guiding me through the process of writing this thesis. Also, I would like to thank Martin Pinsonnault, Egor Shelukhin, Yaron Ostrover, Leonid Polterovich, and Aleksey Zinger for conversations and correspondence regarding the material herein.

To my parents, who taught me that learning was the most valuable thing in the world; to my teachers and professors, who pushed me to do what I could instead of what was required; but most of all to Jorge, who never complained when I took my frustration with my research out on him and who always believed I could do this - even when I didn't.

Chapter 1

Introduction

Let (X, ω) be a monotone symplectic manifold. Polterovich introduced the action-Maslov homomorphism $I : \pi_1(Ham(X, \omega)) \to \mathbb{R}$ in (Polterovich (1997)). Manifolds where $I = 0$ have many interesting properties. For instance, the vanishing of I has implications on the Hofer norm (Polterovich (1997)) and displaceability of Lagrangian fibers of torus actions (Entov and Polterovich (2009)). It has been known for some time that $I = 0$ when $X = \mathbb{C}P^n$ and for $X = S^2 \times S^2$. In (McDuff (2010), McDuff gives several conditions which imply $I = 0$. Essentially, these criteria specify manifolds where most of the genus zero Gromov-Witten invariants vanish. We extend these results by exploring the form of the Seidel element more deeply. For monotone symplectic manifolds, the Seidel element always has integral coefficients and a finite number of terms. By studying these constraints on the Seidel element and properties of the quantum homology, we can show that I vanishes for products of projective spaces and the Grassmannian $G(2, 4)$.

Theorem 1.0.1 *$I = 0$ for $\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_k}$ with a monotone symplectic form.*

Theorem 1.0.2 *$I = 0$ for $G(2, 4)$.*

Theorem 1.0.1 is related to results of Pedroza (Pedroza (2008)) and Leclercq (Leclercq (2009)). They showed that, for X' and X'' monotone symplectic manifolds, $\gamma' \in \pi_1(Ham(X'))$, $\gamma'' \in \pi_1(Ham(X''))$, then $\mathcal{S}(\gamma' \times \gamma'') = \mathcal{S}(\gamma') \otimes \mathcal{S}(\gamma'')$, where $\mathcal{S}(\gamma)$ is the Seidel element. This is sufficient to show that $I = 0$ for any loop $\gamma' \in \pi_1(\mathrm{Ham}(\mathbb{C}P^m \times \mathbb{C}P^n))$ which is a product of loops in the hamiltonian groups of $\mathbb{C}P^m$ and $\mathbb{C}P^n$. Our result shows that $I = 0$ for all loops in $\pi_1(\mathrm{Ham}(\mathbb{C}P^m \times \mathbb{C}P^n))$ and for all products of projective spaces. Our method for proving that the action-Maslov homomorphism vanishes depends on showing two facts. First we show that, when X is one of the above manifolds and $\gamma \in \pi_1(\mathrm{Ham})$, there exists $k > 0$ such that the Seidel element $\mathcal{S}(k\gamma) = \mathbb{1} \otimes \lambda$, where λ is in the Novikov coefficient ring Λ. Then, we must prove that $\nu(\mathbb{1} \otimes \lambda) = 0$, where ν is the valuation map on quantum homology. These terms will be defined in Section 2. This second condition ends up following from the first, provided one of two technical assumptions is satisfied (we will explain in the text what these conditions mean, especially property $\mathcal{D}$):

Proposition 1.0.3 *Let (X, ω) be a symplectic manifold. Suppose that one of the following conditions holds:*

- *All genus 0 Gromov-Witten invariants of the form $\langle a, b \rangle_A^X$ vanish when $A \neq 0$.*
- *The quantum homology $\mathrm{QH}_*(X, \Lambda)$ has property $\mathcal{D}$.*

Then for all $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$ such that $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda$, $\nu(\mathbb{1} \otimes \lambda) = 0$.

The following theorem is an immediate consequence. Let $\mathbb{1} = [X] \in H_*(X)$.

Theorem 1.0.4 *Let (X, ω) be a monotone symplectic manifold. Assume that (X, ω) satisfies one of the two conditions given in Proposition 1.0.3 and that for all $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$, $\exists n$ such that the Seidel element $\mathcal{S}(n\gamma) = \mathbb{1} \otimes \lambda$ for some $\lambda \in \Lambda$. Then $I = 0$.*

These conditions are rather restrictive, but they are satisfied for almost all manifolds where $I = 0$ (with the possible exception of $\mathbb{C}P^2 \times 3\overline{\mathbb{C}P^2}$, which may not have $\mathcal{S}(n\gamma) = \mathbb{1} \otimes \lambda$).

Property $\mathcal{D}$ is trivially satisfied when the even homology classes are generated by divisors, so it includes many well-studied examples, such as toric varieties. In many of these cases, it is difficult to show that that $\mathcal{S}(n\gamma) = \mathbb{1} \otimes \lambda$. We will say that such Seidel elements have *finite order* (this is not strictly true, but the reason why this is a good term will be discussed in Chapter 2), and that Seidel elements without this property have *infinite order*.

We can state a few propositions regarding these conditions for other Grassmannians. Let $G(k, n)$ be the Grassmannian of complex k-planes in $\mathbb{C}^n$.

Proposition 1.0.5 *Let $\gamma \in \pi_1(\mathrm{Ham}(G(2,6),\omega))$. Then $\mathcal{S}(6\gamma) = \mathbb{1} \otimes \lambda$.*

However, we cannot show the more general result (that $I = 0$) for $G(2, 6)$, because the quantum homology - and that of most other Grassmannians - do not satisfy property $\mathcal{D}$:

Proposition 1.0.6 *Let $G(k, n)$ be a Grassmannian, $k \neq \{1, n-1\}$ and $G(k, n) \neq G(2, 4)$. Then the small quantum homology $\mathrm{QH}_*(G(k, n), \Lambda)$ does not satisfy property $\mathcal{D}$.*

Even the statement that all Seidel elements have finite order is not true for all Grassmannians, as shown in the following result:

Proposition 1.0.7 *Let $X = G(2, 2n+1)$. Then the element $x_1 = PD(c_1(X))$ has infinite order in $\mathrm{QH}_*(X, \Lambda)$.*

However, these results do tell us more about the structure of the quantum homology of $G(k, n)$, and are of interest on their own. They will be proven in section 5.2

Chapter 2

Definitions

Let X be a $2N$ dimensional symplectic manifold. All of these definitions can be found in (McDuff and Salamon (2004)) - see index for exact locations. Let K^{eff}, the effective cone of (X, ω), be the additive cone generated by the spherical homology classes $A \in H_2^S(X)$ with nonvanishing genus zero Gromov-Witten invariants $\langle a, b, c\rangle_A^X \neq 0$. Consider the Novikov ring Λ_{enr} given by formal sums

$$\lambda = \sum_{A \in K^{\text{eff}}} \lambda(A) e^{-A} \tag{2.0.1}$$

with the finiteness condition that, $\forall c \in \mathbb{R}$

$$\#\{A \in K^{\text{eff}} | \lambda(A) = 0, \omega(A) \leq c\} < \infty \tag{2.0.2}$$

where $\lambda(A) \in \mathbb{R}$. Λ_{enr} has a grading given by $|e^A| = 2c_1(A)$. We will call this the enriched Novikov ring. The universal ring Λ_{univ} is $\Lambda[q, q^{-1}]$, where Λ is generated by formal power series of the form:

$$\lambda = \sum_{\epsilon \in \mathbb{R}} \lambda_\epsilon t^{-\epsilon} \tag{2.0.3}$$

with the same finiteness condition and $\lambda_\epsilon \in \mathbb{R}$. The grading on Λ_{univ} is given by setting $\deg(q) = 2$. Note that Λ is a field. The map $\varphi : \Lambda_{\text{enr}} \to \Lambda_{\text{univ}}$ is given by taking

$$\varphi(e^{-A}) = q^{-c_1(A)} t^{-\omega(A)}$$

and extending by linearity. While these two rings are thus related, different properties of the quantum homology become apparent when different coefficient rings are used. In section 5.1 the enriched Novikov ring Λ_{enr} will be used to show that the units are of a specific form. In chapter 3, calculations will be carried out using the universal ring Λ. The quantum homology with respect to the Novikov ring Λ_{enr} is given by $\mathrm{QH}_*(X, \Lambda_{\text{enr}}) = H_*(X, \mathbb{R}) \otimes \Lambda_{\text{enr}}$. The grading on $\mathrm{QH}_*(X, \Lambda_{\text{enr}})$ will be given by the sum of the grading on $H_*(X, \mathbb{R})$ and the grading on Λ_{enr}. The quantum homology admits a product structure, called the quantum product. Given a basis ξ_i of $H_*(X)$ and a dual basis ξ_i^*, the quantum product of $a, b \in H_*(X, \mathbb{R})$ is defined by:

$$a * b = \sum_{i, A \in K^{\text{eff}}} \langle a, b, \xi_i \rangle_A^X \xi_i^* e^{-A} \tag{2.0.4}$$

We can then extend this to $\mathrm{QH}_*(X, \Lambda_{\text{enr}})$ by linearity. The quantum homology $\mathrm{QH}_*(X, \Lambda_{\text{univ}})$ is defined analogously and the map $\mathrm{id} \otimes \varphi$ extends to a ring homomorphism $\Phi : \mathrm{QH}_*(X, \Lambda_{\text{enr}}) \to \mathrm{QH}_*(X, \Lambda_{\text{univ}})$. We define the valuation map $\nu : \mathrm{QH}_*(X, \Lambda_{\text{univ}}) \to \mathbb{R}$ by

$$\nu(\lambda_i \otimes q^{a_i} t^{b_i}) = \max\{b_i | \lambda_i \neq 0\}.$$

Next, we will discuss $\mathcal{S}(\gamma)$, the Seidel element (defined in Seidel (1997)). Given a loop $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$ with $\gamma = \{\phi_t\}$, we define a bundle P_γ over S^2 with fiber X. This bundle is given by the clutching construction - take two copies of D^2, D_+ and D_-. Then take $X \times D_+$ and glue it to $X \times D_-$ (where D_- has the opposite orientation from D_+) via the map

$$(\phi_t(x), e^{2\pi i t})_+ \cong (x, e^{2\pi i t})_-.$$

When the loop γ is clear from context, we will refer to this bundle as P. P has two canonical classes - the vertical Chern class, denoted c_1^{vert} and the coupling class, denoted u_γ. c_1^{vert} is the first Chern class of the vertical tangent bundle. u_γ is the unique class such that $u_\gamma|_X = \omega$ and $u_\gamma^{n+1} = 0$. Given a section class $\sigma \in H_2(P_\gamma, \mathbb{Z})$, we can define the Seidel element in $\mathrm{QH}_*(X, \Lambda_{\text{enr}})$ by taking

$$\mathcal{S}(\gamma, \sigma) = \sum_{A \in H_2^S(X), i} \langle \xi_i \rangle_{\sigma+A}^{P_\gamma} \xi_i^* \otimes e^{-A} \tag{2.0.5}$$

where $\{\xi_i\}$ is a basis for $H_*(X, \mathbb{R})$, $\{\xi_i^*\}$ is the dual (in $H_*(X)$) of that basis, and H_2^S is the image of $\pi_2(X)$ in $H_2(X)$ (the spherical homology classes). Note that $\sigma + A$ is a slight abuse of notation; we should actually write $\sigma + \iota_*(A)$, where $\iota : X \to P_\gamma$ is the inclusion map. We will continue this abuse throughout the paper. The Seidel element can also be defined in $\mathrm{QH}_*(X, \Lambda)$ - in this case, the dependence on σ is eliminated by an averaging process.

$$\mathcal{S}(\gamma) = \sum_{\sigma, i} \langle \xi_i \rangle_\sigma^{P_\gamma} \xi_i^* \otimes q^{-c_1^{\text{vert}}(\sigma)} t^{-u_\gamma(\sigma)} \tag{2.0.6}$$

Although we have defined the Seidel element differently in these two rings, note that the first determines the second, via the following lemma.

Lemma 2.0.8 *For any section class $\sigma \in P_\gamma$, there exists an additive homomorphism $\Phi_\sigma : \mathrm{QH}_*(X, \Lambda_{\text{enr}}) \to \mathrm{QH}_*(X, \Lambda)$ which takes $\mathcal{S}(\gamma, \sigma)$ to $\mathcal{S}(\gamma)$. This homomorphism restricts to the identity on $H_*(X)$.*

Proof : Define

$$\Phi_\sigma(\xi_i \otimes e^{-A}) = \xi_i \otimes q^{-c_1^{\text{vert}}(\sigma+A)} t^{-u_\gamma(\sigma+A)}.$$

Extend this map over $\mathrm{QH}_*(X, \Lambda_{\text{enr}})$ by linearity. This is clearly an additive homomorphism, and $\Phi_\sigma(\xi_i) = \xi_i$. $\square$

We now explain what we mean when we say that the Seidel element is finite order.

Definition 2.0.9 *Let Λ be any Novikov ring, and let $\eta \in \mathrm{QH}_{2N}(X, \Lambda)$. We say that η has finite order if there exists k such that $\eta^k = \mathbb{1} \otimes \lambda$ for some $\lambda \in \Lambda$, $\lambda \neq 0$.*

This is not strictly the traditional sense of order, as some power is equal to $\mathbb{1} \otimes \lambda$ rather than $\mathbb{1}$. However, by a result of Fukaya-Oh-Ohta-Ono (Fukaya et al. (2010), Lemma A.1), we know that any Novikov ring with coefficients in an algebraically closed field of characteristic 0 is algebraically closed. Therefore, by enlarging Λ to have coefficients in $\mathbb{C}$ (we will call this $\Lambda^{\mathbb{C}}$, we can find $\eta \in \Lambda$ such that $(\mathcal{S}(\gamma) \otimes \eta)^n = \mathbb{1}$. Therefore, the statement that $\mathcal{S}(\gamma)$ has finite order is true in the classic sense, up to multiplication by some $\eta \in \Lambda^{\mathbb{C}}$. If $\mathcal{S}(\gamma)$ does not have finite order in this sense, we will say that it has *infinite order*.

Note that the Seidel element $\mathcal{S}(\gamma)$ is in degree $2N$ for dimensional reasons. But we can identify $\mathrm{QH}_{2N}(X, \Lambda_{\mathrm{univ}})$ with the ring $\mathrm{QH}_{\mathrm{ev}}(X, \Lambda)$ by taking

$$\psi(aq^{\epsilon_a}t^{\delta_a}) = at^{\delta_a}$$

This map is an isomorphism, since $\psi^{-1}(a) = aq^{N - \frac{1}{2}\deg(a)}$. Since Λ is a field, working in this ring is more convenient for us. Therefore, we will frequently use this isomorphism implicitly, especially in Chapter 3 and Section 5.2.

Now we can define the action-Maslov homomorphism I of Polterovich (Polterovich (1997)). Although the original definition is the difference of the action functional and the Maslov class, Polterovich shows in (Polterovich (1997), Proposition 3.a) that the homomorphism can also be defined as the difference between the vertical Chern class and the coupling class. Namely,

$$u_\gamma = \kappa c_1^{\mathrm{vert}} + I(\gamma) PD^{P_\gamma}(X). \tag{2.0.7}$$

Here, κ is the same constant of monotonicity from before: $\omega = \kappa c_1(X)$. We will use this alternate definition of the action-Maslov homomorphism, because it is more directly related to our results. Note in particular that if σ is a section class with $c_1^{\mathrm{vert}}(\sigma) = 0$, then $I(\gamma) PD^{P_\gamma}(X) = u_\gamma(\sigma)$.

Finally, we will define Property $\mathcal{D}$. This should be seen as a generalization of the statement that the even degree homology classes of X are generated by divisors. Here, we will use the conventions that $\cdot$ represents the intersection product

$$H_d(X) \otimes H_{2N-d}(X) \to H_0(X) \equiv \mathbb{R},$$

and that all Gromov-Witten invariants are genus-zero invariants. We will use these conventions throughout this paper.

Definition 2.0.10 $\mathrm{QH}_*(X, \Lambda)$ *satisfies Property* $\mathcal{D}$ *if there exists an additive complement* $\mathcal{V}$ *in* $H_{ev}(X, \mathbb{Q})$ *to the subring* $\mathcal{D} \subset H_{ev}(X, \mathbb{Q})$ *generated by the divisors such that:*

- $d \cdot v = 0$
- $\langle d, v \rangle_\beta = 0$ *for all* $\beta \in H_2^S(X)$

for all $d \in \mathcal{D}, v \in \mathcal{V}$.

Chapter 3

Seidel Elements with Vanishing Valuation

Let (X, ω) be a $2N$ dimensional symplectic manifold, $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$, and P_γ the bundle coming from the coupling construction. Quantum homology and Seidel elements in this chapter will always refer to those with respect to the universal Novikov ring Λ defined in (2.0.3). In this chapter we will prove Proposition 1.0.3, which states that if X has either property $\mathcal{D}$ or vanishing two-point invariants, and every γ has some power $\mathcal{S}(k\gamma) = \mathbb{1} \otimes \lambda$, then $\nu(\lambda) = 0$. We begin by defining a few specific terms which will we use throughout this chapter.

Definition 3.0.11 *Let* $Q_-(X, \Lambda) = \bigoplus_{i<2N} H_i(X) \otimes \Lambda$.

Definition 3.0.12 *Let X and γ be as above, and suppose that $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda + x$, where x is any element in $Q_-(X, \Lambda)$ and $\lambda = 0$. Consider the sections $\{\sigma\}$ with $c_1^{vert}(\sigma) = 0$ which contribute to the Seidel element $\mathcal{S}(\gamma)$. Then define σ_0 to be the section such that $u_\gamma(\sigma_0) = \min\{u_\gamma(\sigma)\}$.*

Note 3.0.13 *Since Λ is a field, the condition that $\lambda = 0$ is equivalent to λ being a unit in Λ.*

The main thrust of our argument will be that knowing the Seidel element of γ tells us a great deal about the Gromov-Witten invariants of P_γ. We use this knowledge to construct a homology representative of the Poincaré dual of u_γ, and to show that this homology representation has certain properties.

Lemma 3.0.14 *(Lemma 3.4, (McDuff (2010))) Suppose $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda + x$, where x is any element in $Q_-(X, \Lambda)$, and there is an element $H \in H_{2N}(P_\gamma)$ such that*

1. *$H \cap [X]$ is Poincaré dual in X to $[\omega]$.*
2. *$H \cdot \sigma_0 = 0$.*
3. *$H^{N+1} = 0$.*

Then $\nu(\mathbb{1} \otimes \lambda) = 0$.

Here conditions (1) and (3) imply that H is a representative of the Poincaré dual of the coupling class, so that (2) implies $\nu(\mathbb{1} \otimes \lambda) = 0$. We wish to construct such an H. We will do so by "fattening up" a representative of the dual of ω in the fibre. Define a map $s : H_*(X, \mathbb{R}) \to H_{*+2}(P, \mathbb{R})$ by the identity

$$s(a) \cdot_P v = \frac{1}{\langle pt \rangle^P_{\sigma_0}} \langle av \rangle^P_{\sigma_0}, \tag{3.0.1}$$

for all $v \in H_*(P)$. Let $H_P = s(PD_X(\omega))$. Now we need to show that this H_P satisfies the properties in Lemma 3.0.14. Lemma 3.0.15 is a variant of parts (ii) and (iii) of (McDuff (2010), Lemma 4.2). This is the one point in our argument where we require that $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda$. This variant, and thus the stronger condition on $\mathcal{S}(\gamma)$, is used at a key point in Lemma 3.0.19. Note that this version of the lemma eliminates the requirement on the minimal Chern number.

Lemma 3.0.15 *Suppose that $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda$ and let $\sigma = \sigma_0 - B$ for some $B \in H_2(X)$ where $\omega(B) > 0$. Then for all $a \in H_*(X)$:*

1. $\langle a,b\rangle_\sigma^P = 0, \forall b \in H_*(X)$.

2. $\forall w \in H_*(P), \langle a,w\rangle_\sigma^P$ *depends only on* $w \cap X$.

Proof : Begin with the proof of (3.0.15.i). Assume, without loss of generality, that b is a basis element ξ_i. Such an invariant is determined by the Seidel representation. Namely, as in (McDuff and Salamon (2004), (11.4.4)),

$$\mathcal{S}(\gamma) * a = \sum_{i,\sigma} \langle a, \xi_i\rangle_\sigma^{P_\gamma} \xi_i^* \otimes q^{-c_1^{\mathrm{vert}}(\sigma)} t^{-u_\gamma(\sigma)}. \tag{3.0.2}$$

Thus $\langle a,\xi_i\rangle_\sigma^P$ is the coefficient of $\xi_i^{*X} \otimes q^{-c_1^{\mathrm{vert}}(\sigma)} t^{-u_\gamma(\sigma)}$ in

$$\mathcal{S}(\gamma) * a = (\mathbb{1} \otimes \lambda) * a = a \otimes \lambda. \tag{3.0.3}$$

Since $-u_\gamma(\sigma) > -u_\gamma(\sigma_0) = \nu(\mathbb{1} \otimes \lambda)$, this invariant vanishes. Since any two classes w, w' with $w \cap X = w' \cap X$ differ by a fiber class, (3.0.15.ii) immediately follows from (3.0.15.i). □

We will also need two lemmas from (McDuff (2010). The first is a special case of (McDuff (2010), Lemma 4.5).

Lemma 3.0.16 *Suppose that* $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda$. *Then:*

1. $s(pt) = \sigma_0$.

2. $s(a) \cap X = a, \forall a \in H_*(X,\mathbb{R})$.

The second is (McDuff (2010), Lemma 4.1).

Lemma 3.0.17 *Suppose that* $a, b \in H_*(X), v, w \in H_*(P_\gamma, \mathbb{R})$ *and* $B \in H_2(X,\mathbb{Z}) \subset H_2(P_\gamma, \mathbb{Z})$. *Then*

1. $\langle a,b,v\rangle_B^P = 0$.

2. $\langle a,v,w\rangle = \langle a, v\cap X, w \cap X\rangle_B^X$.

For proofs of these two lemmas, see (McDuff (2010)). Lemma 3.0.16 follows from simple computations using the definition of s:

$$s(pt) \cdot v = \sigma_0 \cdot v,$$

for any divisor class v). Lemma 3.0.17 essentially depends on the fact that the two fiber constraints can be located in different fibers. If J is compatible with the fibration, the J-holomorphic curve will reside entirely in one fiber, and thus can intersect at most one of the fiber constraints. The second part follows similarly, since the B curve must lie in the same fiber as a.

The two hypotheses for the results in this chapter will be that $H_*(X)$ satisfies property $\mathcal{D}$ (or that the two point invariants $\langle a, b\rangle_A^X$ vanish on X) and that $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda$. If the two point invariants vanish, Lemma 3.0.18 is not needed for the proof of Proposition 1.0.3. However, if this is not the case, we have property $\mathcal{D}$ and we use Lemma 3.0.18 at exactly two points in the proof of Proposition 1.0.3 (specifically, in the proofs of Lemma 3.0.19 and Lemma 3.0.21). The other results (and the main result, Proposition 1.0.3) thus require these conditions only so that they can use results of Lemma 3.0.19 and Lemma 3.0.21.

Lemma 3.0.18 *Let (X, ω) be a symplectic manifold, $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$, and P the associated bundle. Assume $\mathcal{D}$ is the part of $H_{ev}(P, \mathbb{R})$ generated by divisors $\{D_i, X\}$ and $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda$. Let $a \in H_*(X, \mathbb{R})$ be a fiber class, $v \in \mathcal{D}$, and $B \in H_2(X, \mathbb{Z})$ such that $\omega(B) > 0$. Then the Gromov-Witten invariant $\langle v, a\rangle^P_{\sigma_0 - B}$ vanishes.*

Proof : Suppose not. Take a section of minimal energy such that some invariant of this form does not vanish and call it σ'. v is a product of divisors, and we claim we may assume that each of these divisors D_i satisfies $D_i \cdot \sigma' = 0$. First, we can show that none of the $D_i = X$. If any of them did, then v would be a fiber class, and $\langle v, a\rangle^P_{\sigma_0 - B}$ would vanish by Lemma 3.0.15. Then to any other D_i, we can add a multiple of X to obtain a new class D_i' which differs from D_i by a fiber class and has $D_i' \cdot \sigma' = D_i \cdot \sigma' + kX \cdot \sigma' = 0$, for appropriate choice of k. Lemma 3.0.15 shows that adding a fiber class to v does not change our Gromov-Witten invariant. Now consider the set

$$\{v_i | \langle v_i, a \rangle^P_{\sigma'} = 0\}.$$

Each of these v_i is a linear combination of products of k divisors. We will assume, without loss of generality, that v is one of these v_i and it is exactly a product of k divisors. We will perform induction on k.

If $k = 1$, the invariant vanishes by the divisor axiom, which says that $\langle D_1, a \rangle^P_{\sigma'} = \langle a \rangle^P_{\sigma'} (D_1 \cdot \sigma') = 0$. If $k > 1$, we use Theorem 1 of Lee-Pandharipande from (Lee and Pandharipande (2004)), as restated in (McDuff (2010), (4.2)). This identity is stated as follows. Take a basis ξ_i of $H_*(X)$ and extend it to a basis of $H_*(P)$ by adding classes ξ_i^* such that $\xi_i \cdot \xi_j^* = \delta_{ij}$ and $\xi_i^* \cdot \xi_j^* = 0$. Note that the ξ_i here are fiber classes, but the ξ_i^* cannot be fiber classes. Now take classes $u, v, w \in H_*(P_\gamma), H \in H_{2N}(P)$ a divisor, and $\alpha \in H_2(P)$. Then Lee and Pandharipande show that

$$\begin{aligned}\langle Hu, v, w \rangle^P_\alpha = \langle u, Hv, w \rangle^P_\alpha + (\alpha \cdot H) \langle u, \tau v, w \rangle^P_\alpha \\ - \sum_{i, \alpha 1 + \alpha 2 = \alpha} (\alpha 1 \cdot H) \langle u, \xi_i, \ldots \rangle^P_{\alpha_1} \langle \xi_i^*, v, \ldots \rangle^P_{\alpha_2}\end{aligned} \quad (3.0.4)$$

where τ is a descendant constraint and "..." indicates that the w term may appear in either factor.

Now, assume that the statement is true for all $v \in \mathcal{D}$ of codimension $2k - 2$. Let $v = D_1 \cdots D_{k-1} \cdot D_k$ (where, as above, we can assume that all of these divisors have $D_i \cdot \sigma' = 0$). Given any section class σ, McDuff shows in Lemma 2.9 of (McDuff (2000)) that in the above sum, a section class can only decompose into $\sigma - \alpha$ and α where either α is a fiber class or $\sigma - \alpha$ is a fiber class. In both cases, the other element of the decomposition will be a section class by necessity. This follows from considering J-holomorphic curves where J is compatible with the fibration. By combining equation 3.0.4 and this decomposition into divisors, one sees that (we take $w = D_1 \cdots D_{k-1}$ and $D = D_k$, to simplify our notation)

$$\begin{aligned}
\langle w \cdot D, a\rangle^P_{\sigma'} &= \langle w \cdot D, a, X\rangle^P_{\sigma'} && (3.0.5)\\
&= \langle w, D \cdot a, X\rangle^P_{\sigma'} + (D \cdot \sigma')\langle w, \tau a, X\rangle^P_{\sigma'} && (3.0.6)\\
&- \sum_{c \in H_*(X),i} ((\sigma' - c) \cdot D)\langle w, \xi_i, \ldots\rangle^P_{\sigma'-c}\langle \xi_i^*, a, \ldots\rangle^P_c && (3.0.7)\\
&- \sum_{c \in H_*(X),i} ((\sigma' - c) \cdot D)\langle w, \xi_i^*, \ldots\rangle^P_{\sigma'-c}\langle \xi_i, a, \ldots\rangle^P_c && (3.0.8)\\
&- \sum_{c \in H_*(X),i} (c \cdot D)\langle w, \xi_i, \ldots\rangle^P_c\langle \xi_i^*, a, \ldots\rangle^P_{\sigma'-c} && (3.0.9)\\
&- \sum_{c \in H_*(X),i} (c \cdot D)\langle w, \xi_i^*, \ldots\rangle^P_c\langle \xi_i, a, \ldots\rangle^P_{\sigma'-c}. && (3.0.10)
\end{aligned}$$

We will go through the right hand side of this equation line by line and show that each of them must vanish. Line (3.0.6) has two terms - the first one vanishes because w is of codimension $2k-2$ and the second one vanishes because $D \cdot \sigma' = 0$. If line (3.0.7) does not vanish then we must have either $c = 0$, or $\omega(c) > 0$ and X in the first factor (otherwise the second factor would vanish by Lemma 3.0.17). If $c = 0$, then $(\sigma' - c) \cdot D = \sigma' \cdot D = 0$ and line (3.0.7) vanishes. Thus our first factor is

$$\langle w, \xi_i, X\rangle^P_{\sigma'-c}$$

with $\omega(c) > 0$ which vanishes by the minimality of σ'. Line (3.0.8) must vanish by Lemma 3.0.17 because the second factor is a fiber invariant with two fiber constraints. In line (3.0.9, the X must insert into the second term by Lemma 3.0.17), and thus we have invariants of the form

$$\langle w, \xi_i\rangle^P_c\langle \xi_i^*, a, X\rangle^P_{\sigma'-c}.$$

Note that $c = 0$, so this vanishes by minimality of σ'.

Finally, line 3.0.10 must vanish because the second factor is of the form $\langle \xi_i, a, X\rangle^P_{\sigma'-c} = \langle \xi_i, a\rangle^P_{\sigma'-c}$ which vanishes because this invariant has two fiber constraints. Assume it does not vanish. Then it would contribute to $\mathcal{S}(\gamma) * a$, as in (3.0.3). But since it doesn't vanish, $\omega(c) > 0$, and thus $\sigma' - c$ has less energy than σ_0, which contradicts the definition of σ_0. Therefore, the entire invariant vanishes, and by induction, all such invariants vanish. $\square$

Lemma 3.0.19 *Assume the conditions of Theorem 1.0.4. Then* $\langle h, \sigma_0\rangle^P_{\sigma_0} = 0$ *where* $h \in H_*(X, \mathbb{R})$ *is the Poincaré dual of the symplectic form in* X.

Proof : We can take any divisor class D in P such that $D \cap X = h$ and add copies of X to get a class K such that $K \cap X = h$ and $K \cdot \sigma_0 = 0$. Then the identity of Lee-Pandharipande gives us

$$\begin{aligned}
\langle h, \sigma_0\rangle^P_{\sigma_0} &= \langle h, \sigma_0, X\rangle^P_{\sigma_0} \\
&= \langle X, K\sigma_0, X\rangle^P_{\sigma_0} + (\sigma_0 \cdot K)\langle X, \tau\sigma_0, X\rangle^P_{\sigma_0} \\
&- \sum_{\alpha \in H_2(P)} ((\sigma_0 - \alpha)\cdot K)\langle X, \xi_i, \ldots\rangle^P_{\sigma_0-\alpha}\langle \xi_i^*, \sigma_0, \ldots\rangle^P_\alpha \\
&- \sum_{\alpha \in H_2(P)} ((\sigma_0 - \alpha)\cdot K)\langle X, \xi_i^*, \ldots\rangle^P_{\sigma_0-\alpha}\langle \xi_i, \sigma_0, \ldots\rangle^P_\alpha
\end{aligned}$$

Note that, since $K \cdot \sigma_0 = 0$, the first two terms on the right hand side vanish. Now, assume the some term in the first sum does not vanish. Since the first factor has two fiber constraints, $\sigma_0 - \alpha$ must be a section class. Also, $\omega(\alpha) > 0$ since otherwise, $(\sigma_0 - \alpha) \cdot K = \sigma_0 \cdot K = 0$. But then $\langle X, \xi_i, X\rangle_{\sigma_0-\alpha} = \langle X, \xi_i\rangle_{\sigma_0-\alpha}$ vanishes by Lemma 3.0.15.i. Now, we assume that some term in the second sum does not vanish. Note that $\sigma_0 - \alpha$ is a nonzero class (since $(\sigma_0 - \alpha)\cdot K = 0$). If it is a fiber class, then by Lemma 3.0.17, we have the first factor (either $\langle X, \xi_i^*\rangle^P_{\sigma_0-\alpha}$ or $\langle X, \xi_i^*, X\rangle^P_{\sigma_0-\alpha}$) is either $\langle X, \xi_i^* \cap X\rangle^X_{\sigma_0-\alpha}$ or $\langle X, \xi_i^* \cap X, X \cap X\rangle^X_{\sigma_0-\alpha}$. Both of these vanish because $\sigma_0 - \alpha \neq 0$ and the first term in the invariant is unconstrained. Therefore, $\sigma_0 - \alpha$ must be a section class. The invariant is thus of the form:

$$\langle X, \xi_i^*, X\rangle^P_{\sigma_0-\alpha}\langle \xi_i, pt\rangle^X_\alpha$$

with $\alpha = 0$ and $\omega(\alpha) > 0$ since $((\sigma - \alpha)\cdot K) = 0$. Note that the second factor is a two point invariant on the fiber. By our assumption, either this second factor vanishes or we have property $\mathcal{D}$. If the second factor does not vanish, then property $\mathcal{D}$ tells us that the class ξ_i is generated by divisors, because $pt \in \mathcal{D}$. If ξ_i is generated by divisors, ξ_i^* must be generated by divisors, and Lemma 3.0.18 tells us that the first factor must vanish.

Corollary 3.0.20 *Assuming condition (1) of Theorem 1.0.4, we have $H \cdot \sigma_0 = 0$.*

Proof : By the definition of s, we have $s(h) \cdot \sigma_0 = \frac{1}{\langle pt\rangle^P_{\sigma_0}}\langle h, \sigma_0\rangle^P_{\sigma_0}$, which is 0 by Lemma 3.0.19.

Lemma 3.0.21 *Assuming condition (1) of Theorem 1.0.4,*

$$\langle H^{N+1-k}, XH^k\rangle^P_{\sigma_0} = 0$$

for all k.

The proof of this statement follows exactly as Lemma 4.8 in (McDuff (2010)). We sketch the proof briefly.

Proof: We prove the statement using induction on k. For $k = N$, this invariant vanishes by Corollary 3.0.20 and the divisor axiom. Now assume that this invariant does not vanish for some k, and choose the maximal such k. Using equation 3.0.4, we have

$$\begin{aligned}\langle H^{N+1-k}, X \cap H^k\rangle^P_{\sigma_0} &= \langle H^{N+1-k}, X \cap H^k, X\rangle^P_{\sigma_0} \\ &= \langle H^{N-k}, H \cdot XH^{k+1}, X\rangle^P_{\sigma_0} \\ &+ (\sigma_0 \cdot H)\langle H^{N-k}, \tau XHk, X\rangle^P_{\sigma_0} \\ &- \sum_{\alpha \in H_2(P)} (\alpha \cdot H)H^{n-k}, \xi_i, \ldots\rangle^P_\alpha \langle \xi_i^I, XH^k, \ldots\rangle^P_{\sigma_0-\alpha} \\ &- \sum_{\alpha \in H_2(P)} (\alpha \cdot H)\langle H^{n-k}, \xi_i^*, \ldots\rangle^P_\alpha \langle \xi_i, XH^k, \ldots\rangle^P_{\sigma_0-\alpha}\end{aligned}$$

The first two terms vanish by the inductive hypothesis and Corollary 3.0.20. Now examine the first sum. If α is a section class, then it cannot be σ_0, since $\alpha \cdot H = 0$. Thus, $\alpha = \sigma_0 - \beta, \beta \neq 0$. Then the term in the first sum becomes

$$\begin{aligned}&(-\omega(\beta))\langle H^{n-k}, \xi_i, \ldots\rangle^P_\alpha \langle \xi_i^*, XH^k, \ldots\rangle^P_\beta \\ &\qquad = (-\omega(\beta))\langle H^{n-k}, \xi_i, \ldots\rangle^P_\alpha \langle X\xi_i^*, XH^k, \ldots\rangle^X_\beta .\end{aligned}$$

If $\mathrm{QH}_*(X, \Lambda)$ has property $\mathcal{D}$, the second factor can only be nonzero if ξ_i^* is generated by divisors. But then ξ_i is generated by divisors, and the first factor vanishes by Lemma 3.0.18. If $\mathrm{QH}_*(X, \Lambda)$ does not have property $\mathcal{D}$, then the second term vanishes by the other assumption because it is a two-point invariant. If α is a fiber class then X must be in the second factor. Therefore, the first factor is a two point invariant:

$$\langle H^{n-k} \cap X, \xi_i\rangle^X_\alpha$$

By hypothesis, either these invariants vanish, or $\mathrm{QH}_*(X, \Lambda)$ has property $\mathcal{D}$. In this case, for the first factor to be nonzero, $\xi_i \in \mathcal{D}$, and thus ξ_i^* is generated by divisors. Then the second factor must vanish by Lemma 3.0.18. Therefore, the first sum is 0. Now we examine the second sum. The second factor has two fiber constraints, so $\sigma_0 - \alpha$ must be a section class. $\sigma_0 - \alpha$ must have lower energy than σ_0 because $\alpha \cdot H = 0$. Therefore, this vanishes by Lemma 3.0.15.ii. $\square$

The following corollary is identical to Corollary 4.9 in (McDuff (2010)) and the proof is the same.

Corollary 3.0.22 *Assuming condition (1) of Theorem 1.0.4, $H^{N+1} = 0$.*

Proof : Taking Lemma 3.0.21 with $k = 1$, one sees that

$$0 = \langle H^N, h\rangle^P_{\sigma_0} = \langle pt\rangle^P_{\sigma_0} s(h) \cdot H^N.$$

But $s(h) = H$, so $H^{N+1} = 0$. □

The proof of Proposition 1.0.3 now follows from these results.

Proof of Proposition 1.0.3: X satisfies the conditions of Lemma 3.0.19. Then by Corollary 3.0.20, $H_{P_\gamma} \cdot \sigma_0 = 0$. Similarly, Lemma 3.0.22 shows that $H^{m+n+1} = 0$. Thus, H satisfies the conditions of the Lemma 3.0.14. □

Proof of Theorem 1.0.4: Since $S(n\gamma)$ is of the form $\mathbb{1} \otimes \lambda$, Proposition 1.0.3 implies that $\nu(S(n\gamma)) = 0$. This implies that there exists a section class σ_0 such that $c_1^{\text{vert}}(\sigma_0) = u_{n\gamma}(\sigma_0) = 0$. Therefore, by Equation 2.0.7, $I(n\gamma) = 0$. But $I(n\gamma) = nI(\gamma)$, so $I(\gamma) = 0$. □

Chapter 4

Properties of Monotone Symplectic Manifolds

Chapter 3 establishes that under certain conditions on X we can show $I = 0$ by showing that the Seidel element of some power is $\mathbb{1} \otimes \lambda$ for all γ. Monotone symplectic manifolds have several unique properties that allow us to make progress on this question.

4.1 Constraints on $\mathcal{S}(\gamma)$

Lemma 4.1.1 *If (X, ω) is monotone and $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$, then the Seidel element $\mathcal{S}(\gamma, \sigma')$ (and also the Seidel element $\mathcal{S}(\gamma)$) will have coefficients in $\mathbb{Z}$ and a finite number of terms.*

Proof : Assume that the sum is not finite - that is, an infinite number of sections σ_i contribute to $S(\gamma, \sigma')$. The difference of any two section classes is a fiber class, so $\sigma_i - \sigma' = C$ for $C \in H_2(X)$. But the definition of the Seidel element implies that $0 \le c_1(C) \le 2N$. For a monotone symplectic manifold, the bound on $c_1(C)$ gives a bound on energy: $0 \le \omega(C) \le \kappa 2N$, where $\omega = \kappa c_1(X)$. By Corollary 5.3.2

in (McDuff and Salamon (2004)), there are only finitely many homotopy classes which can be represented by J-holomorphic curves with energy $0 \leq \omega(C) \leq \kappa 2N$. Therefore, the sum must be finite.

It remains to show that the Seidel element has integral coefficients. Theorem 7.3.1 in (McDuff and Salamon (2004)) establishes that the genus 0 Gromov-Witten invariants are integral for closed semipositive symplectic manifolds. This condition is equivalent to showing that the evaluation map is a pseudocycle with bordism class independent of the almost complex structure J. Of course, for a general γ, P_γ is not a semipositive manifold. However, for the Seidel element to have integral coefficients, one only needs two-pointed Gromov-Witten invariants where A is a section class to satisfy this condition. The proof will be similar to that of Theorem 6.6.1 in (McDuff and Salamon (2004)), with appropriate modifications for this case (6.6.1 addresses semipositive manifolds). In a semipositive manifold, one can show that, while multiply covered curves might appear in the boundary of $\mathcal{M}_{0,A}(J)$, they have codimension at least 2 and do not affect the evaluation map. Therefore, the Gromov-Witten invariants will be integral.

Although P_γ is not necessarily semipositive, even for monotone X, a similar argument holds for the space $\mathcal{M}_{0,\sigma}(J)$ where σ is a section class. Lemma 2.9 in (McDuff (2000)) says that if J is a fibered almost complex structure, then each element of the limit set can be divided into a unique component called the stem which represents a section class and a finite number of connected pieces each lying in a different fiber. The curves in the boundary of the moduli space will also have a stem class, which cannot be multiply covered, because it represents a section class. The curves lying in the fibers could contain multiply covered curves, but the monotonicity condition forces these to be codimension at least 2. Thus, they do not affect the evaluation map, and the Gromov-Witten invariants of the form $\langle a, b, c\rangle_\sigma^{P_\gamma}$ will be integral. Since the definition of the Seidel element only involves invariants of this form, the Seidel element will have integral coefficients. Therefore, the evaluation map is a pseudocycle. □

Therefore, the Seidel element actually lies in a subring of $\mathrm{QH}_*(X, \Lambda_{\mathrm{enr}})$ where elements have integral coefficients and only a finite number of terms. Since we will be working in this subring of $\mathrm{QH}_*(X, \Lambda_{\mathrm{enr}})$, we will give it a name.

Definition 4.1.2 *Let $\Lambda_{\mathrm{enr},\mathbb{Z}}$ be the subring of Λ_{enr} with integral coefficients. Then define $Q_{\mathrm{enr}}(X) = \mathrm{QH}_*(X, \Lambda_{\mathrm{enr},\mathbb{Z}})$ to be the subring of $\mathrm{QH}_*(X, \Lambda_{\mathrm{enr}})$ which consists of finite sums of elements with coefficients in $\mathbb{Z}$. Thus, a typical element is*

$$\sum_{i=0}^{n} x_i e^{-C_i}$$

where $x_i \in H_*(X, \mathbb{Z})$ and $C_i \in H_2^S(X, \mathbb{Z})$.

4.2 Products and Property $\mathcal{D}$

If X and Y both have homology generated by divisors, it is clear that the homology of $X \times Y$ will be generated by divisors by the Kunneth formula. We can make a similar statement about property $\mathcal{D}$, but we need to add an additional condition on our manifolds; they must be monotone with the same constant of monotonicity so that we can use the monotone version of the quantum Kunneth formula given in section 11.1 of (McDuff and Salamon (2004)). A more general version of the quantum Kunneth formula exists (see (Kaufmann (1996))) but it seems unlikely to imply that property $\mathcal{D}$ holds.

Lemma 4.2.1 *Assume that (X', ω) and (X'', η) are monotone symplectic manifolds with the same constant of monotonicity: $c_1(X') = \kappa[\omega]$ and $c_1(X'') = \kappa[\eta]$. Then if $\mathrm{QH}_*(X', \Lambda_{univ})$ and $\mathrm{QH}_*(X'', \Lambda_{univ})$ have property $\mathcal{D}$, $\mathrm{QH}_*(X' \times X'', \Lambda_{univ})$ also has property $\mathcal{D}$.*

Proof : Let $H_*(X') = \mathcal{D}' \oplus \mathcal{V}'$ and $H_*(X'') = \mathcal{D}'' \oplus \mathcal{V}''$ be the decompositions given by property $\mathcal{D}$. By the classical Kunneth formula,

$$H_k(X' \times X'') = \bigoplus_{i+j=k} H_i(X') \otimes_{\mathbb{Q}} H_j(X'')$$

Let

$$\mathcal{D}_k = \bigoplus_{i+j=k} \mathcal{D}'_i \otimes_{\mathbb{Q}} \mathcal{D}''_j$$
$$\mathcal{V}_k = \bigoplus_{i+j=k} \left((\mathcal{D}'_i \otimes_{\mathbb{Q}} \mathcal{V}''_j) \oplus (\mathcal{V}'_i \otimes_{\mathbb{Q}} \mathcal{D}''_j) \oplus (\mathcal{V}'_i \otimes_{\mathbb{Q}} \mathcal{V}''_j) \right).$$

By the classical Kunneth formula, $\mathcal{V}$ is an additive complement for $\mathcal{D}$, and $\mathcal{D}$ is the part of $H_*(X' \times X'')$ generated by divisors. It remains to show that $d \cdot v = 0$ and $\langle d, v \rangle = 0$. Let $d = d'_1 \otimes d''_1$. $\mathcal{V}$ is generated by elements of the form $v = d'_2 \otimes v''_1 + v'_1 \otimes d''_2 + v'_2 \otimes v''_2$ (note that v'_2 and v''_2 are either both of even degree or both of odd degree). First we show that $d \cdot v = 0$, using the classical Kunneth formula:

$$\begin{aligned} d \cdot v &= (d'_1 \otimes d''_1) \cdot (d'_2 \otimes v''_1 + v'_1 \otimes d''_2 + v'_2 \otimes v''_2) \\ &= (d'_1 \cdot d'_2) \otimes (d''_1 \cdot v''_1) + (d'_1 \cdot v'_1) \otimes (d''_1 \cdot d''_2) + (d'_1 \cdot v'_2) \otimes (d''_1 \cdot v''_2) \\ &= (d'_1 \cdot d'_2) \otimes 0 + 0 \otimes (d''_1 \cdot d''_2) + 0 \otimes 0 \\ &= 0. \end{aligned}$$

Note that the intersections in the second line vanish because of property $\mathcal{D}$ if the v_i are of even degree and for dimensional reasons if the v_i are of odd degree. Now we will show that $\langle d, v \rangle = 0$. This will follow from the monotone quantum Kunneth formula. The monotone quantum Kunneth formula says that if X' and X'' are two monotone symplectic manifolds with the same constant of monotonicity,

$$QH_*(X' \otimes X'', \Lambda_{\text{univ}}) = QH_*(X', \Lambda_{\text{univ}}) \otimes_{\Lambda_{\text{univ}}} QH_*(X'', \Lambda_{\text{univ}}).$$

Then we have that

$$\begin{aligned}
d * v &= (d_1' \otimes d_1'') * (d_2' \otimes v_1'' + v_1' \otimes d_2'' + v_2' \otimes v_2'') \\
&= (d_1' * d_2') \otimes (d_1'' * v_1'') + (d_1' * v_1') \otimes (d_1'' * d_2'') + (d_1' * v_2') \otimes (d_1'' * v_2'') \\
&= (d_1' * d_2') \otimes 0 + 0 \otimes (d_1'' * d_2'') + 0 \otimes 0 \\
&= 0.
\end{aligned}$$

As above, the quantum products vanish either because of property $\mathcal{D}$ or for dimensional reasons. □

Chapter 5

Manifolds with $\mathcal{S}(k\gamma) = \mathbb{1} \otimes \lambda$ and Property $\mathcal{D}$)

Now we will discuss several monotone symplectic manifolds which we can show have $I = 0$, and several others where we can prove partial results. We do this by showing results about property $\mathcal{D}$ and the form of the Seidel element. Note that we need to show that the Seidel element $S(\gamma k) = \mathbb{1} \otimes \lambda$. In some cases, it may be easier to show that the enriched Seidel element $S(k\gamma, k\sigma)$ has this form. Then Lemma 2.0.8 implies that $S(k\gamma)$ has the form $\mathbb{1} \otimes \lambda$.

5.1 $\mathbb{C}P^m \times \mathbb{C}P^n$

Let X be $\mathbb{C}P^m \times \mathbb{C}P^n$ with the monotone symplectic form, and let $N = m + n$.

Note 5.1.1 *In fact, all of the results in this section are true for products of an arbitrary number of projective spaces with the monotone symplectic form. For simplicity of notation, though, we will prove them for $\mathbb{C}P^m \times \mathbb{C}P^n$ only.*

In order to show that $I = 0$, we need to show that some power of the Seidel element is of the form $\mathbb{1} \otimes \lambda$. This is a consequence of the algebraic structure of the quantum homology: namely, that the subring Q_{enr} is a group ring over an ordered group.

Definition 5.1.2 *An ordered group is a group G equipped with a total order $\leq$ which is translation invariant: $g \leq h \Rightarrow g \cdot a \leq h \cdot a$ and $a \cdot g \leq a \cdot h$ $\forall g, h, a \in G$.*

Theorem 5.1.3 *If X is $\mathbb{C}P^m \times \mathbb{C}P^n$ with the monotone symplectic form, then $Q_{\mathrm{enr}}(X)$ is an $\mathbb{Z}$ group ring over an ordered group.*

Theorem 5.1.3 follows directly from Lemmas 5.1.4 and 5.1.5.

Lemma 5.1.4 *If $Q_{\mathrm{enr}}(X', \Lambda'_{\mathrm{enr},\mathbb{Z}})$ and $Q_{\mathrm{enr}}(X'', \Lambda''_{\mathrm{enr},\mathbb{Z}})$ are both group rings over ordered groups, then $Q_{\mathrm{enr}}(X' \times X'', \Lambda'_{\mathrm{enr},\mathbb{Z}} \otimes \Lambda''_{\mathrm{enr},\mathbb{Z}})$ is also a group ring over an ordered group.*

Proof : $Q_{\mathrm{enr}}(X' \times X'', \Lambda'_{\mathrm{enr},\mathbb{Z}} \otimes \Lambda''_{\mathrm{enr},\mathbb{Z}}) \cong H_*(X' \times X'') \otimes_{\mathbb{Z}} \Lambda'_{\mathrm{enr},\mathbb{Z}} \otimes \Lambda''_{\mathrm{enr},\mathbb{Z}}$ as additive groups. Thus, by the classical Künneth formula, $Q_{\mathrm{enr}}(X' \times X'', \Lambda'_{\mathrm{enr},\mathbb{Z}} \otimes \Lambda''_{\mathrm{enr},\mathbb{Z}}) \cong Q_{\mathrm{enr}}(X', \Lambda_{\mathrm{enr},\mathbb{Z}}) \otimes_{\mathbb{Z}} QH_*(X'', \Lambda''_{\mathrm{enr},\mathbb{Z}})$ as additive groups. By the quantum Kunneth formula (Section 11.1, McDuff and Salamon (2004)), that this is actually a ring isomorphism. Therefore, $Q_{\mathrm{enr}}(X' \times X'', \Lambda'_{\mathrm{enr},\mathbb{Z}} \otimes \Lambda''_{\mathrm{enr},\mathbb{Z}}) \cong Q_{\mathrm{enr}}(X', \Lambda_{\mathrm{enr},\mathbb{Z}}) \otimes_{\mathbb{Z}} Q_{\mathrm{enr}}(X'', \Lambda''_{\mathrm{enr},\mathbb{Z}})$.

We assumed that both of these subrings were group rings over ordered groups. Therefore, we have that $Q_{\mathrm{enr}}(X' \times X'', \Lambda'_{\mathrm{enr},\mathbb{Z}} \otimes \Lambda''_{\mathrm{enr},\mathbb{Z}}) \cong \mathbb{Z}(G') \otimes_{\mathbb{Z}} \mathbb{Z}(G'')$. But this is isomorphic to $\mathbb{Z}(G' \times G'')$. Give $G' \times G''$ the lexicographic ordering. The product of two ordered groups with the lexicographic ordering is still an ordered group: $(g', g'') \leq (h', h'') \Rightarrow (g' \cdot a', g'' \cdot a'') \leq (h' \cdot a', h'' \cdot a'') and (a' \cdot g', a'' \cdot g'') \leq (a' \cdot h', a'' \cdot h'')$. $\square$

Lemma 5.1.5 $Q_{\mathrm{enr}}(\mathbb{C}P^n)$ *is a group ring over an ordered group.*

Proof: Recall that

$$Q_{\text{enr}}(\mathbb{C}P^n) \cong \frac{\Lambda_{\text{enr},\mathbb{Z}}[x]}{\langle x^{n+1} = e^{-A}\rangle},$$

where A is the class of the generator in $H_2(X, \mathbb{Z})$. Let $q = e^A$ and then let G be the group generated by x and q with relation $x^{n+1} = q^{-1}$ (note that $G = \mathbb{Z}$). This group can be ordered by using the mapping $\phi : (G, \cdot) \to (\mathbb{Q}, +)$ where $\phi(x) = \frac{1}{n+1}$ and $\phi(q) = -1$. Then G is ordered by the pullback of the ordering on $\mathbb{Q}$. Clearly, $Q_{\text{enr}}(\mathbb{C}P^n)$ is just the $\mathbb{Z}$ group ring of G. $\square$

Now we can combine Theorem 5.1.3 and an algebraic lemma to determine the units of $Q_{\text{enr}}(X)$.

Lemma 5.1.6 *If G is an ordered group, then the units of $\mathbb{Z}(G)$ are $\pm G$.*

Lemma 5.1.6 is proved as Lemma 45.3 of (Sehgal (1993)). The proof is provided here for convenience of the reader:

Proof (Sehgal): Take a nonmonomial unit$p = \sum_{i=1}^{t} u_i * g_i$ of the group ring and its inverse (which must also then be nonmonomial) $p^{-1} = \sum_{i=1}^{\ell} v_i * h_i$, with $g_1 < g_2 < \ldots < g_t$ and $h_1 < h_2 < \ldots < h_\ell$. If we multiply these two elements, we get $1_G = u_1v_1 * g_1h_1 + \ldots + u_tv_\ell * g_th_\ell$. Then, for this equation to be true, the group element in any term on the right hand side must be 1_G or cancel with the group element from another term. However $g_1h_1 < g_ih_j$, for $i = 1, j = 1$ and $g_ih_j < g_th_\ell$ for $i = t, j = \ell$, so these group elements cannot cancel with other terms. Thus, we must have $g_1h_1 = 1_G = g_th_\ell$ and thus $g_1^{-1} = h_1$ and $g_t^{-1} = h_\ell$. But we have $g_1 \le g_t \Rightarrow g_1^{-1} > g_t^{-1} \Rightarrow h_1 > h_\ell$, which is a contradiction. Therefore, p must be monomial. $\square$

Corollary 5.1.7 *Let $X = \mathbb{C}P^m \times \mathbb{C}P^n$ with the monotone symplectic form. Then the only units in $Q_{\text{enr}}(X)$ are the monomial units - those of the form $\pm a^i b^j \otimes e^C$, $C \in H_2(X, \mathbb{Z})$.*

Proof: Theorem 5.1.3 shows that $Q_{\text{enr}}(X)$ is isomorphic to an integral group ring over the group generated by a, b, and e^A (where A is a generator of $H_2(X, \mathbb{Z})$). Since this group is ordered, all of its units are monomial by Lemma 5.1.6. $\square$

Theorem 5.1.8 *For $\mathbb{C}P^m \times \mathbb{C}P^n$ with the monotone symplectic form and for any loop $\gamma \in \pi_1(\text{Ham}(\mathbb{C}P^m \times \mathbb{C}P^n, \omega))$, $\mathcal{S}(\gamma)$ has finite order.*

Proof : Let σ be a section class in $H_2(P_\gamma)$. Corollary 5.1.7 shows that $\mathcal{S}(\gamma, \sigma)$ must be of the form $\pm a^f b^g \otimes e^{-C}$. Let $k = (m+1)(n+1)$. Then $S(k\gamma, k\sigma) = a^{kf} b^{kg} \otimes e^{-C} = \mathbb{1} \otimes e^{(kh-(n+1)f)A} e^{-C}$. By Lemma 2.0.8, the same k also works for $\mathcal{S}(\gamma k) = \mathbb{1} \otimes \lambda$. $\square$

Proof of Theorem 1.0.1: The first condition of Theorem 1.0.4 is satisfied because all classes in $H_*(\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_k})$ are generated by divisors. Therefore, $\text{QH}_*(X, \Lambda)$ satisfies property $\mathcal{D}$. Theorem 5.1.8 shows that the second condition is satisfied for a product of two projective spaces, and thus Theorem 1.0.4 shows that $I = 0$. By using Lemma 5.1.4 $(k-1)$ times, one can show Theorem 5.1.8 for the product of more than two projective spaces. The result follows. $\square$

5.2 Grassmannians

The next class of manifolds which we will discuss are the complex Grassmannians. Let $X = G(k, n)$ be the space of k-dimensional subspaces in $\mathbb{C}^n$ with the standard symplectic form ω (note that $[\omega] = c_1(X)$. Then Witten showed in (Witten (1995)) (further details were worked out by Siebert and Tian in (Siebert and Tian (1997))) that the quantum homology could be completely described. Consider the two bundles E and F over V, where the fibre $E_V = V$ and $F_V = \mathbb{C}^n/V$. Consider the

Chern classes of their dual bundles:

$$x_i = c_i(E_*)$$
$$y_j = c_j(F_*)$$

These classes generate the cohomology (and thus their Poincaré duals, which we will represent by the same notation, generate homology). Since E is of rank k and F is of rank $n-k$, we have that $x_i = 0 = y_j$ for all $i > k$ and $j > n-k$. Also, since $E \oplus F$ is isomorphic to the trivial bundle,

$$\sum_{i=0}^{\ell} x_i y_{\ell-i} = 0$$

for $\ell = 1, \ldots, n$. This determines the y_j inductively in terms of the x_i :

$$y_j = -x_1 y_{j-1} - \cdots - x_{j-1} y_1 - x_j.$$

Classically, the relations $y_j = 0$ for $j > n-k$ are known to be the only relations on $H_*(X, \mathbb{Q})$. Thus,

$$H_*(X, \mathbb{Q}) = \frac{\mathbb{Q}[x_1, \ldots, x_k]}{\langle y_{n-k+1}, \ldots, y_n \rangle}.$$

The results of Witten and Siebert-Tian show that the quantum homology is in fact very similar:

$$QH_*(X, \Lambda_{\mathrm{univ}}) = \frac{\Lambda[x_1, \ldots, x_k]}{\langle y_{n-k+1}, \ldots, y_{n-1}, y_n + (-1)^{n-k} q^n \rangle}$$

Our main technique for dealing with the quantum homology will be a generalization of these results called quantum Schubert calculus.

5.3 An Introduction to Quantum Schubert Calculus

To study the quantum homology of the Grassmannian, we will need an understanding of quantum Schubert calculus. Bertram has defined quantum versions of the

Giambelli and Pieri relations from classical Schubert calculus in (Bertram (1997)). Recall that the Schubert cells are indexed by $(n-k)$-tuples of integers a with $k \geq a_1 \geq a_2 \geq \cdots \geq a_{n-k} \geq 0$. The codegree of a is $|a| = \sum_{i=1}^{n-k} a_i$. Let x_i be the Poincaré dual to the i-th Chern class of $G(k,n)$ as above. Giambelli's formula states that σ_a, the Schubert cell associated to a, is given by the determinant of the following matrix:

$$\sigma_a = \begin{vmatrix} x_{a_1} & x_{a_1+1} & x_{a_1+2} & \cdots & x_{a_1+n-k-1} \\ x_{a_2-1} & x_{a_2} & x_{a_2+1} & \cdots & x_{a_2+n-k-2} \\ x_{a_3-2} & s_{a_3-1} & x_{a_3} & \cdots & x_{a_3+n-k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{a_{n-k}-(n-k)+1} & \cdots & \cdots & \cdots & x_{a_{n-k}} \end{vmatrix}$$

Here, $x_0 = 1$ and $x_\ell = 0$ for all $\ell > k$. The quantum version of Giambelli's formula is identical, using the quantum product instead of the intersection product. However, all of the quantum multiplications involved are undeformed, so the quantum Schubert cells are the same as the ordinary Schubert cells.

The other classical formula from Schubert calculus, Pieri's formula, is given by

$$x_i \cdot \sigma_a = \sum_b \sigma_b \tag{5.3.1}$$

where the sum is taken over all b such that $|b| = |a|+i$ and $k \geq b_1 \geq a_1 \geq b_2 \geq \cdots \geq b_{n-k} \geq a_{n-k} \geq 0$. For example, if $X = G(3,7)$, we have $x_1 \cdot \sigma_{3,2,1,1} = \sigma_{3,3,1,1} + \sigma_{3,2,2,1}$.

In the quantum version of the formula, there is an additional quantum term:

$$x_i \cdot \sigma_a = \sum_b \sigma_b + q \sum_c \sigma_c \tag{5.3.2}$$

where $|c| = |a| + i - n$ and $a_1 - 1 \geq c_1 \geq a_2 - 1 \geq \cdots \geq a_{n-k} - 1 \geq c_{n-k} \geq 0$, where q is of degree $2n$. Therefore, our example from $G(3,7)$ above would become $x_1 * \sigma_{3,2,1,1} = \sigma_{3,3,1,1} + \sigma_{3,2,2,1} + q\sigma_{1,0,0,0}$. Recall that in classical Schubert calculus,

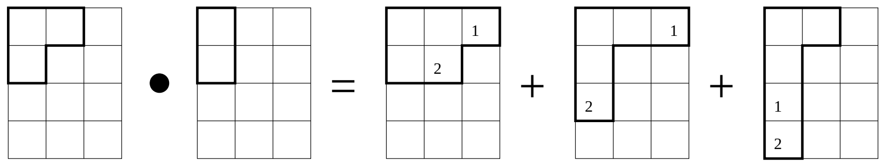

Figure 5.3.1: The multiplication $\sigma_{2,1,0,0} \cdot \sigma_{1,1,0,0}$ in $G(3,7)$ with words labeled.

one can identify each Schubert cell with a corresponding Young tableau. The Young tableau corresponding to $\sigma_{a_1,\ldots,a_{n-k}}$ consists of n rows of boxes with a_i boxes in each row, where $k \geq a_1 \geq \cdots \geq a_{n-k} \geq 0$ (we will refer to tableaux and their partitions interchangeably). In classical Schubert calculus, the product of two Schubert cells represented by Young tableaux is given by the Littlewood-Richardson rule. It says that

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu N^\nu_{\lambda\mu} \sigma_\nu$$

where $N^\nu_{\lambda\mu}$ is the number of tableaux on the skew shape ν/λ of content μ whose word is a reverse lattice word. Tableaux on the *skew shape* of ν/λ of content μ means that we look at tableaux ν where we can label the complement of λ with μ_1 1s, μ_2 2s, $\ldots$, and μ_{n-k} $(n-k)$s, so that the numbering is weakly increasing across rows and strictly increasing down columns. The *word* of this numbering is the list of entries read left-to-right, bottom-to-top. It is a *reverse lattice word* if, at any point along its length, the number of 1s remaining is greater than or equal to the number of 2s remaining, which is greater than or equal to the number of 3s remaining, etc. The product of $\sigma_{2,1,0,0} \cdot \sigma_{1,1,0,0}$ in $G(3,7)$ is shown in Figure 5.3 for illustration. We see that $\sigma_{2,1,0,0} \cdot \sigma_{1,1,0,0} = \sigma_{3,2,0,0} + \sigma_{3,1,1,0} + \sigma_{2,1,1,1}$.

Bertram, Ciocan-Fonatanine, and Fulton prove a quantum version of this rule using the quantum Giambelli and Pieri formulas given above in (Bertram et al. (1999)). In this rule, the quantum terms come from removing n-rim hooks. An *n-rim hook* is a collection of n contiguous boxes along the right edge (or rim), starting at the bottom of one of the columns, which ends in the last box of a row. If an n-rim hook does not end in the last box of a row, it is called an *illegal n-rim hook*. The width of the rim hook w is the number of columns that its boxes are contained in. Examples of n-rim hooks and quantum Young tableaux appear in figures 5.7 and 5.7. Bertram, Ciocan-Fontanine, and Fulton prove the quantum Littlewood-Richardson rule, given here:

Lemma 5.3.1 *Let $\lambda = \lambda_1, \ldots, \lambda_s$ be a Young tableau with $s \geq (n-k), \lambda_i \leq k$. Then this tableau represents a quantum Schubert class. Specifically, it represents the class:*

1. *0, if λ contains an illegal n-rim hook, or if $s > n-k$ and λ does not contain an n-rim hook.*
2. *$(-1)^{k-w} q\sigma_\mu$, where μ is the tableau which results from removing an n-rim hook of width w.*

Then the quantum Littlewood-Richardson rule is exactly the same as the classical Littlewood-Richardson rule, except quantum classes are represented by tableaux with more than $n-k$ rows.

5.4 $G(2,4)$

We begin with our main result on Grassmannians, Theorem 1.0.2. This result states that the action-Maslov homomorphism vanishes for the Grassmannian of 2-planes in $\mathbb{C}^4$. We need to show two things to prove this statement: that $\mathcal{S}(k\gamma) = \mathbb{1} \otimes \lambda$ and that $G(2,4)$ satisfies property $\mathcal{D}$. First, we will show that the Seidel element must be cyclic. Unlike the products of projective spaces, here we do not even need to use enriched coefficients. Instead of $Q_{\text{enr}}(X)$, we will look at the analogous subring

$Q(X)$ of finite sums with integral coefficients in $\mathrm{QH}_*(X,\Lambda)$. This ring can be stated using the results of Siebert-Tian as

$$Q(X) = \frac{\Lambda[x_1,x_2]}{\langle x_1^3 - 2x_1x_2, x_1^2x_2 - x_2^2 - t^4\rangle}$$

Because $\dim(X) = 8$ and the minimal Chern number is 4, the terms which can appear in the Seidel element are sharply limited. Assume that a section σ contributes to the Seidel element. Then any other contributing sections are of the form $\sigma' = \sigma + kL$, where $k \in \mathbb{Z}$ and $L = x_1x_2$ is the class of a line in X). Since contributing sections must have $-8 \leq c_1^{\text{vert}}(\sigma') \leq 0$, clearly another section can only exist if $\sigma' = \sigma \pm L$ and $c_1^{\text{vert}}(\sigma) = 0$ or -8. $H_*(X)$ has generators organized by degree as follows:

0	x_2^2
2	x_1x_2
4	x_1^2, x_2
6	x_1
8	$\mathbb{1}$

The Seidel elements form a subgroup of the units - the product of two Seidel elements is a Seidel element, and so is the inverse. All of these elements have degree equal to the dimension of X, which is 8. Thus, the Seidel element can only be of the form:

$$a\mathbb{1}t^{\epsilon} + bx_2^2q^4t^{4+\epsilon} \tag{5.4.1}$$

$$ax_1q^1t^{\epsilon} \tag{5.4.2}$$

$$ax_2q^2t^{\epsilon} + bx_2q^2t^{\epsilon} \tag{5.4.3}$$

$$ax_1x_2q^3t^{\epsilon} \tag{5.4.4}$$

Since these elements are of degree 8, we will work in $\mathrm{QH}_*(X,\Lambda)$ (note that the Schubert calculations will retain qs, since the terms are not all in fixed degrees).

Similarly, the exponent of t is determined up to a constant multiple $\lambda = t^{\epsilon}$ so we will also suppress t. These elements must be units in the quantum homology (with inverses of the same form), and since the symplectic form is monotone, a and b must be integers.

Lemma 5.4.1 *The Seidel element, up to appropriate powers of q and t, is either $\mathbb{1}, x2, x_1^2 - x_2$, or x_2^2.*

Proof : If $\mathcal{S}(\gamma)$ has the form given in (5.4.2), then $\mathcal{S}(\gamma^{-1})$ is of the form given in (5.4.4). By the quantum Giambelli's formula, $x_1 = \sigma_{1,0} = \sigma_1$ and $x_1 x_2 = \sigma_{2,1}$. By quantum Pieri, we have

$$\begin{aligned}\sigma_1 * \sigma_{2,1} &= \sigma_{2,2} + q^4 \sigma_{0,0} \\ &= x_2^2 + q^4 \mathbb{1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{1} &= (a x_1) * (b x_1 x_2) \\ &= ab x_1^2 x_2 \\ &= ab(\mathbb{1} + x_2^2).\end{aligned}$$

This implies that $ab = 0$ and $ab = 1$, which is impossible. Thus, no such elements can be Seidel elements. Now we look at (5.4.1). First, note that by Giambelli's and Pieri's formula,

$$\begin{aligned}x_2^4 &= \sigma_2^2 * \sigma_{2,2} \\ &= \sigma_2 * (q^4 \sigma_{1,1}) \\ &= q^8 \sigma_{0,0} \\ &= \mathbb{1} q^8.\end{aligned}$$

Then we have

$$\mathbb{1} = (a\mathbb{1} + bx_2^2) * (c\mathbb{1} + dx_2^2)$$
$$= (ac + bd)\mathbb{1} + (ad + bc)x_2^2.$$

Hence, $(ad + bc) = 0$ and $(ac + bd) = 1$. Then, either $d = 0$ or $a = \frac{-bc}{d}$. By substituting this for a, one obtains that $b = \frac{d}{d^2-c^2}$ and $a = \frac{-c}{d^2-c^2}$. Since a and b are both integers, this means that $d^2 - c^2$ divides both c and d. There are only two cases when this occurs - when $\{a, b, c, d\} = \{0, \pm1, 0, \pm1\}$ or $\{\pm1, 0, \pm1, 0\}$. Therefore, the Seidel element must be either $\mathbb{1}$ or x_2^2 multiplied by some λ.

Finally, we look at (5.4.3). For this, we will use

$$\begin{aligned} x_1^4 &= \sigma_1^3 * \sigma_{1,0} \\ &= \sigma_1^2 * (\sigma_{1,1} + \sigma_{2,0}) \\ &= \sigma_1 * (2\sigma_{2,1}) \\ &= 2\sigma_{2,2} + 2q^4\sigma_{0,0} \\ &= 2x_2^2 + 2q^4\mathbb{1} \\ x_1^2x_2 &= \sigma_1 * \sigma_{2,1} \\ &= \sigma_{2,2} + q^4\sigma_{0,0} \\ &= x_2^2 + q^4\mathbb{1}. \end{aligned}$$

This implies that

$$\mathbb{1} = (ax_1^2 + bx_2) * (cx_1^2 + dx_2)$$
$$= acx_1^4 + (bc + ad)x_1^2x_2 + bdx_2^2$$
$$= (2ac + bc + ad + bd)x_2^2 + (2ac + bc + ad)\mathbb{1}.$$

This will be true if and only if $(2ac + bc + ad) = -bd = 1$. Thus we have $b = -d = \pm1$. If $b = -d = 1$, then $c = \frac{1-a}{-1-2a}$, which is only integral if $\{a, b, c, d\} =$

$\{0, 1, -1, -1\}$ or $\{-1, 1, 0, -1\}$. Similarly, if $b = -d = -1$, then $c = \frac{1+a}{1-2a}$, which is only integral if $\{a, b, c, d\} = \{0, -1, 1, 1\}$ or $\{1, -1, 0, 1\}$. Therefore, we have either $\pm(x_1^2 - x_2)$ or $\pm x_2$. This completes the proof. $\square$

Lemma 5.4.2 *Let $\mathcal{S}(\gamma)$ be the Seidel element of $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$. Then $S(4\gamma) = \mathbb{1} \otimes \lambda$.*

Proof: $\mathcal{S}(\gamma)$ must be of a form listed in Lemma 5.4.1. Clearly, since Λ is a field, the coefficient λ does not affect invertibility, and we only need to concern ourselves with the homology terms. Since $\mathbb{1}^4 = \mathbb{1}$ and $x_2^4 = q^8$, this is obvious for the first two cases and the third case, when $c = 0$. This leaves only the case where $\mathcal{S}(\gamma) = x_1^2 - x_2$. But $(x_1^2 - x_2)^2 = x_2^2$, so the statement also holds in this case. $\square$

In order to show that the action-Maslov homomorphism vanishes on $G(2, 4)$, we also need to show that it satisfies Property $\mathcal{D}$. This is slightly weaker than requiring that the quantum homology be generated by divisors. $\mathrm{QH}_*(G(2, 4), \Lambda)$ is not generated by divisors, but does satisfy property $\mathcal{D}$.

Lemma 5.4.3 *The quantum homology of $G(2, 4)$ with coefficients in Λ satisfies property $\mathcal{D}$.*

Proof: First, note that the homology is generated (over $\mathbb{Q}$) by x_1 in every degree except 4 ($x_1x_2 = \frac{1}{2}x_1^3$ and $x_2^2 = x_1^2x_2 = \frac{1}{2}x_1^4$). Therefore, $\mathcal{V}$ must be generated by some class $ax_1^2 + bx_2$. But by the dimension formula for genus 0 Gromov-Witten invariants, if

$$\langle d, v\rangle_A \neq 0,$$

then the codegrees of d and v must add up to $8 + 2c_1(A) - 2$. If $A \neq 0$, we have $c_1(A) \geq 4$, so the sum of the codegrees must be at least 14. But if v is $ax_1^2 + bx_2$, it has codegree 4 and d must have codegree 10. But $G(2, 4)$ is 8 dimensional, so this cannot happen. Therefore, $\langle d, ax_1^2 + bx_2 \rangle_A = 0$ for all $d \in \mathcal{D}$ and $G(2,4)$ satisfies property $\mathcal{D}$. $\square$

Lemmata 5.4.2 and 5.4.3 are sufficient to show that the action-Maslov homomorphism vanishes on $G(2,4)$. This completes the proof of Theorem 1.0.2

5.5 G(2, 5)

In the case of $X = G(2,5)$, though, neither of these conditions are satisfied.

Proposition 5.5.1 *The element $x_1 = PD(c_1(G(2,5))$ is a unit in $\mathrm{QH}_*(X,\Lambda)$ of infinite order.*

Proposition 5.5.2 *$G(2,5)$ does not satisfy property $\mathcal{D}$.*

Proof: Using Giambelli's formula, we see that:

$$
\begin{array}{ll}
\sigma(0,0,0) = 1 & \sigma(1,0,0) = x_1 \\
\sigma(2,0,0) = x_2 & \sigma(1,1,0) = x_1^2 - x_2 \\
\sigma(2,1,0) = x_1 x_2 & \sigma(1,1,1) = x_1^3 - 2x_1 x_2 \\
\sigma(2,2,0) = x_2^2 & \sigma(1,1,1) = x_1^3 - 2x_1 x_2 \\
\sigma(2,2,1) = x_1 x_2^2 & \sigma(2,2,2) = x_2^3.
\end{array}
$$

Proof of Proposition 5.5.1: We will show that $\sigma_{(1,0,0)} = x_1$ is a unit with infinite order.

$$
\begin{aligned}
x_1 * (x_1^2 - 2x_2^2) &= x_1 * (\sigma_{(2,1,1)} - \sigma_{(2,2,0)}) \\
&= q\sigma_{(0,0,0)} + \sigma_{(2,2,1)} - \sigma_{(2,2,1)} \\
&= q\mathbb{1}
\end{aligned}
$$

Therefore, x_1 is a unit. Now we will show that it has infinite order. Consider the sequences of quantum Schubert cells given by

$$a_i = \{\sigma_{(1,0,0)}, \sigma_{(1,1,0)}, \sigma_{(2,1,0)}, \sigma_{(2,1,1)}, \sigma_{(2,2,1)}, q\sigma_{(1,0,0)}, q\sigma_{(1,1,0)}, q\sigma_{(1,1,1)}, \ldots\}$$
$$b_j = \{\sigma_{(2,0,0)}, \sigma_{(1,1,1)}, \sigma_{(2,2,0)}, q\sigma_{(0,0,0)}, \sigma_{(2,2,2)}, q\sigma_{(2,0,0)}, q\sigma_{(2,0,0)}, q\sigma_{(1,1,1)}, \ldots\}$$

Using quantum Pieri, we can see that $x_1 * a_i = a_{i+1} + b_i$ and that $x_1 * b_j = a_{j+2}$. Because of this recursive relation, we obtain that

$$x_1^i = f_i a_i + f_{i-1} b_{i-1}$$

where f_i is the ith Fibonacci number. Clearly $x_1^i \neq \mathbb{1} \otimes \lambda$ for any $i > 0$. Therefore, it has infinite order. $\square$

Proof of Proposition 5.5.2: Now we wish to show that $G(2,5)$ does not satisfy property $\mathcal{D}$. We will show that there does not exist an additive complement $\mathcal{V}$ to the subring $\mathcal{D}$ generated by the divisors which has $d \cdot v = 0$ and $\langle d, v\rangle_\beta = 0$ for all $d \in \mathcal{D}, v \in \mathcal{V}$. We will take $v = x_1^2 x_2 - 2x_2^2$. Then, by classical Giambelli,

$$\begin{aligned} x_1^2 \cdot (x_1^2 x_2 - 2x_2^2) &= x_1^2 \cdot (\sigma_{(2,1,1)} - \sigma_{(2,2,0)}) \\ &= x_1 \cdot (\sigma_{(2,2,1)} - \sigma_{(2,2,1)}) \\ &= 0. \end{aligned}$$

However, there is a nontrivial two point invariant involving v:

$$\begin{aligned} pt * v &= x_2^3 * (\sigma_{(2,1,1)} - \sigma_{(2,2,0)}) \\ &= x_2^2 * (q\sigma_{(1,0,0)} - \sigma_{(2,2,2)}) \\ &= x_2 * (q\sigma_{(2,1,0)} - q\sigma_{(1,1,1)}) \\ &= q\sigma_{(2,2,1)} - q^2\sigma_{(0,0,0)} \\ &= q x_1 x_2^2 - q^2 \mathbb{1}. \end{aligned}$$

$\langle pt, v\rangle_\ell$ is the coefficient of the $x_1 x_2^2$ term, so $\langle pt, v\rangle_\ell = 1$. Therefore, no such additive complement exists, and property $\mathcal{D}$ is not satisfied. $\square$

5.6 $G(2,6)$

Let $X = G(2,6)$. Then we can show using the same techniques that all Seidel elements have finite order.

Proof of Proposition 1.0.5: The homology of X is generated by the Poincaré dual of the first and second Chern classes - we'll denote these homology classes by x_1 and x_2 respectively, with relations

$$x_1^5 - 4x_1^3x_2 + 3x_1x_2^2 = 0$$
$$x_1^4x_2 - 3x_1^2x_2^2 + x_2^3 = 0.$$

Then the homology is additively generated by these classes:

$$H_0 = x_1^2x_2^3 \qquad H_2 = x_1^3x_2^2$$
$$H_4 = x_1^2x_2^2, x_2^3 \qquad H_6 = x_1^5, x_1^3x_2$$
$$H_8 = x_1^4, x_1^2x_2, x_2^2 \qquad H_{10} = x_1^3, x_1x_2$$
$$H_{12} = x_1^2, x_2 \qquad H_{14} = x_1$$
$$H_{16} = \mathbb{1}.$$

The choice of generators in degrees 0 through 6 is arbitrary if we consider homology over $\mathbb{Q}$. However, this particular choice of generators generates the subring of homology with $\mathbb{Z}$ coefficients. The quantum homology is generated by the same classes along with an additional divisor class q. The relations on quantum homology are:

$$x_1^5 - 4x_1^3x_2 + 3x_1x_2^2 = 0$$
$$x_1^4x_2 - 3x_1^2x_2^2 + x_2^3 = -q^6.$$

Recall that because X is monotone, the Seidel element will have integral coefficients. Additionally, since the minimal Chern number is $n = 6$, we are rather limited in the what the Seidel element can be. The possible Seidel elements are

$$
\begin{aligned}
&a\mathbb{1} + bx_1^2x_2^2q^{-6} + cx_2^3q^{-6} && ax_1q^{-1} + bx_1^3x_2^2q^{-7} \\
&ax_1^2q^{-2} + bx_2q^{-2} + cx_1^2x_2^3q^{-8} && ax_1^3q^{-3} + bx_1x_2q^{-3} \\
&ax_1^4q^{-4} + bx_1^2x_2q^{-4} + cx_2^2q^{-4} && ax_1^5q^{-5} + bx_1^3x_2q^{-5}
\end{aligned}
$$

up to some multiple of q, where all coefficients are integral. As with $G(2,4)$ and $G(2,5)$, we use $\mathrm{QH}_*(X,\Lambda)$, ignoring the q terms. An element of the first type will have an inverse of the same type:

$$
\begin{aligned}
\mathbb{1} &= (a\mathbb{1} + bx_1^2x_2^2 + cx_2^3) * (d\mathbb{1} + ex_1^2x_2^2 + fx_2^3) \\
&= ad\mathbb{1} + (ae+bd)x_1^2x_2^2 + (af+cd)x_2^3 \\
&\qquad + (bf+ce)x_1^2x_2^5 + bex_1^4x_2^4 + cfx_2^6
\end{aligned}
\tag{5.6.1}
$$

Using quantum Schubert calculus, we see that

$$
\begin{aligned}
x_1^2x_2^5 &= x_1^2x_2 * \sigma_{(2,2,2,2)} \\
&= x_1^2 * (q\sigma_{(1,1,1,1)} \\
&= x_1 * (q\sigma_{(2,1,1,1)}) \\
&= q\sigma_{(2,2,1,1)} + q^2\sigma_{(0,0,0,0)} \\
&= q^2\mathbb{1} + qx_1^2x_2^2 - qx_2^3
\end{aligned}
$$

$$
\begin{aligned}
x_1^4x_2^4 &= x_1^4 * \sigma_{(2,2,2,2)} \\
&= x_1^3 * (q\sigma_{(1,1,1,0)}) \\
&= x_1^2 * (q\sigma_{(1,1,1,1)} + q\sigma_{(2,1,1,0)}) \\
&= x_1 * (2q\sigma_{(2,1,1,1)} + q\sigma_{(2,2,1,0)}) \\
&= 3q\sigma_{(2,2,1,1)} + q\sigma_{(2,2,2,0)} + 2q^2\sigma_{(0,0,0,0)} \\
&= 3qx_1^2x_2^2 - 2qx_2^3 + 2q^2\mathbb{1}
\end{aligned}
$$

$$
\begin{aligned}
x_2^6 &= x_2^2 * \sigma_{(2,2,2,2)} \\
&= x_2 * (q\sigma_{(1,1,1,1)}) \\
&= q^2 \sigma_{(0,0,0,0)} \\
&= q^2 \mathbb{1}
\end{aligned}
$$

Therefore, 5.6.1 reduces to

$$\mathbb{1} = (ad+bf+ce+be+cf)\mathbb{1} + (ae+bd+bf+ce+2be)x_1^2x_2^2 + (af+cd-bf-ce-be)x_2^3.$$

Therefore, we have

$$ad + bf + ce + be + cf = 1 \tag{5.6.2}$$

$$ae + bd + bf + ce + 2be = 0 \tag{5.6.3}$$

$$af + cd - bf - ce - be = 0. \tag{5.6.4}$$

Adding the first and third equation, we see that $ad + af + cd + cf = (a + c)(d + f) = 1$. Since these are integers, $(a + c) = (d + f) = \pm 1$. Note that the second equation is

$$(a + c)e + (d + f)b + 2be = (\pm 1)(b + e) + 2be \tag{5.6.5}$$

$$e = \frac{b}{\pm 2b - 1}. \tag{5.6.6}$$

But e can only be integral if $b = 0$ or $b = e = \pm 1$. If $b = e = 0$, the equations reduce to

$$
\begin{aligned}
ad + cf &= 1 \\
af + cd &= 0.
\end{aligned}
$$

Then either $f = c = 0$, $a = d = \pm 1$, or $a = \frac{-cd}{f}$. Then we have

$$\begin{aligned}
\frac{-cd^2}{f} + cf &= 1 \\
c(f^2 - d^2) &= f \\
c(f-d)(f+d) &= f \\
\pm c(f-d) &= f \\
\pm c(2f \mp 1) &= f \\
f &= \frac{-c}{1 \mp 2c}.
\end{aligned}$$

Since f is integral, either $c = 0$ and we have either the same result as above or $c = f = \pm 1$, $a = d = 0$. In these cases, we have that our invertible element is either $\pm\mathbb{1}$ or $\pm x_2^3$. If $b = e = \pm 1$, then 5.6.4 becomes

$$\begin{aligned}
af + cd \mp f \mp c &= 1 \\
af + cd \pm f(a+c) \pm c(d+f) &= 1 (\text{because } a + c = f + d = \pm 1) \\
2af + 2cd + 2cf &= 1 \text{ or } -2cf = 1.
\end{aligned}$$

However, neither of these has integral solutions, so we must have $b = e = 0$ as above.

We move on to elements of the second type. In this case, the potential inverse is of the sixth type:

$$\begin{aligned}
\mathbb{1} &= (ax_1 + bx_1^3x_2^2) * (cx_1^5 + dx_1^3x_2) \\
&= acx_1^6 + bcx_1^8x_2^2 + adx_1^4x_2 + bdx_1^6x_2^3 \quad (5.6.7)
\end{aligned}$$

We perform quantum Schubert calculus again:

$$\begin{aligned}
x_1^6 &= x_1^2 * (\sigma_{(1,1,1,1)} + 3\sigma_{(2,1,1,0)} + 2\sigma_{(2,2,0,0)}) \\
&= x_1 * (4\sigma_{(2,1,1,1)} + 5\sigma_{(2,2,1,0)}) \\
&= 9\sigma_{(2,2,1,1)} + 5\sigma_{(2,2,2,0)} + 4q\sigma_{(0,0,0,0)} \\
&= 9x_1^2x_2^2 - 4x_2^3 + 4q\mathbb{1}
\end{aligned}$$

$$\begin{aligned}
x_1^8x_2^2 &= x_1^7 * (\sigma_{(2,2,1,0)}) \\
&= x_1^6 * (\sigma_{(2,2,2,0)} + \sigma_{(2,2,1,1)}) \\
&= x_1^5 * (2\sigma_{(2,2,2,1)} + q\sigma_{(1,0,0,0)}) \\
&= x_1^4 * (2\sigma_{(2,2,2,2)} + q\sigma_{(2,0,0,0)} + 2q\sigma_{(1,1,0,0)}) \\
&= x_1^3 * (4q\sigma_{(1,1,1,0)} + 3q\sigma_{(2,1,0,0)}) \\
&= x_1^2 * (4q\sigma_{(1,1,1,1)} + 7q\sigma_{(2,1,1,0)} + 3q\sigma_{(2,2,0,0)}) \\
&= x_1 * (11q\sigma_{(2,1,1,1)} + 10q\sigma_{(2,2,1,0)}) \\
&= 21q\sigma_{(2,2,1,1)} + 10q\sigma_{(2,2,2,0)} + 11q^2\sigma_{(0,0,0,0)} \\
&= 21qx_1^2x_2^2 - 11qx_2^2 + 11q^2\mathbb{1}
\end{aligned}$$

$$\begin{aligned}
x_1^4x_2 &= x_1^3 * \sigma_{(2,1,0,0)} \\
&= x_1^2 * (\sigma_{(2,2,0,0)} + \sigma_{(2,1,1,0)}) \\
&= x_1 * (2\sigma_{(2,2,1,0)} + \sigma_{(2,1,1,1)}) \\
&= 3\sigma_{(2,2,1,1)} + 2\sigma_{(2,2,2,0)} + q\sigma_{(0,0,0,0)} \\
&= 3x_1^2x_2^2 - x_2^3 + q\mathbb{1}
\end{aligned}$$

$$
\begin{aligned}
x_1^6 x_2^3 &= x_1^5 * \sigma_{(2,2,2,1)} \\
&= x_1^4 * (\sigma_{(2,2,2,2)} + q\sigma_{(1,1,0,0)}) \\
&= x_1^3 * (2q\sigma_{(1,1,1,0)} + q\sigma_{(2,1,0,0)}) \\
&= x_1^2 * (3q\sigma_{(2,1,1,0)} + 2q\sigma_{(1,1,1,1)} + q\sigma_{(2,2,0,0)}) \\
&= x_1 * (5q\sigma_{(2,1,1,1)} + 4q\sigma_{(2,2,1,0)}) \\
&= 9q\sigma_{(2,2,1,1)} + 4q\sigma_{(2,2,2,0)} + 5q^2\sigma_{(0,0,0,0)} \\
&= 9x_1^2x_2^2 - 5x_2^3 + 5q^2\mathbb{1}
\end{aligned}
$$

Therefore, 5.6.7 reduces to

$$\mathbb{1} = (4ac+11bc+ad+5bd)\mathbb{1}+(-4ac-11bc-ad-5bd)x_2^3+(9ac+21bc+ad+9bd)x_1^2x_2^2.$$

The first coefficient is equal to 1 and the second is equal to 0, but they are the same up to sign. So there are no solutions (not even rational) to this set of equations. So there are no units of this type.

The same phenomenon occurs when we have an element of the fourth type:

$$
\begin{aligned}
\mathbb{1} &= (ax_1^3 + bx_1x_2) * (cx_1^3 + dx_1x_2) \\
&= acx_1^6 + bdx_1^2x_2^2 + (ad+bc)x_1^4x_2 \\
&= (4ac + ad + bc)\mathbb{1} + (-4ac - ad - bc)x_2^3 \\
&\qquad + (9ac + bd + ad + bc)x_1^2x_2^2
\end{aligned}
$$

So there are no solutions to this equation, and thus no units of this type.

Finally, we examine units of the third and fifth types:

$$\begin{aligned}
\mathbb{1} &= (ax_1^2 + bx_2 + cx_1^2x_2^3) * (dx_1^4 + ex_1^2x_2 + fx_2^2) \\
&= adx_1^6 + (bd+ae)x_1^4x_2 + cdx_1^6x_2^3 + (be+af)x_1^2x_2^2 + cex_1^4x_2^4 + bfx_2^3 + cfx_1^2x_2^5 \\
&= (4ad + bd + ae + 5cd + 2ce + cf)\mathbb{1} \\
&+ (-4ad - bd - ae - 5cd - 2ce + bf - cf)x_2^3 \\
&\qquad + (9ad + 3bd + 3ae + 9cd + be + af + 3ce + cf)x_1^2x_2^2.
\end{aligned}$$

Therefore, we have the equations

$$4ad + bd + ae + 5cd + 2ce + cf = 1 \tag{5.6.8}$$

$$-4ad - bd - ae - 5cd - 2ce + bf - cf = 0 \tag{5.6.9}$$

$$9ad + 3bd + 3ae + 9cd + be + af + 3ce + cf = 0 \tag{5.6.10}$$

Summing the first two equations, we have $bf = 1$, and thus $b = f = \pm 1$. If $b = f = 1$, we subtract twice the first equation from the third equation to obtain

$$\begin{aligned}
ad + d + ae - cd + e + a - ce - c + 2 &= 0 \\
(a - c + 1)(d + e + 1) &= -1 \\
(c - a - 1)(d + e + 1) &= 1 \\
c - a - 1 = d + e + 1 &= \pm 1.
\end{aligned}$$

If $c - a - 1 = d + e + 1 = 1$, we have $c = 2 + a$ and $d = -e$. Then 5.2.14 becomes

$$\begin{aligned}
-6ae - 7e + 2 + a &= 1 \\
e &= \frac{1+a}{7+6a}.
\end{aligned}$$

This is only integral if $a = -1$, $e = 0$. Then $c = 1$ and $d = 0$, and we have the units $-x_1^2 + x_2 + x_1^2x_2^3$ and x_2. However, by Schubert calculus,

$$\begin{aligned} -x_1^2q + x_2q + x_1^2x_2^3 &= -x_1^2q + x_2q + x_1 * \sigma_{(2,2,2,1)} \\ &= -x_1^2q + x_2q + \sigma_{(2,2,2,2)} + q\sigma_{(1,1,0,0)} \\ &= -x_1^2q + x_2q + x_2^4 + q(x_1^2 - x_2) \\ &= x_2^4. \end{aligned}$$

If $c - a - 1 = d + e + 1 = -1$, then we have $c = a$ and $d = -2 - e$. Then 5.6.8 becomes

$$\begin{aligned} -6ae - 17a - 2 - e = 1 \\ a = \frac{3+e}{-17-6e} \end{aligned}$$

This is only integral if $e = -3$, $a = 0$. Then $c = 0$ and $d = 1$, and we have the units x_2 and $x_1^4 - 3x_1^2x_2 + x_2^2$. As above, we can see that

$$\begin{aligned} x_2^5 &= x_2 * \sigma_{(2,2,2,2)} \\ &= q\sigma_{(1,1,1,1)} \\ &= q(x_1^4 - 3x_1^2x_2 + x_2^2) \end{aligned}$$

Thus, all of the units found here are equal to $x_2^iq^j$. If we take $b = f = -1$, we can follow the same process to obtain these same units multiplied by -1. Therefore, all of the potential Seidel elements in all cases have the form $x_2^iq^j$. Since $x_2^6 = \mathbb{1} \otimes q^2$, this completes our proof. $\square$

Unfortunately, $G(2,6)$ does not satisfy property $\mathcal{D}$.

Proposition 5.6.1 $G(2,6)$ *does not satisfy property* D.

Proof: To show that the quantum homology satisfies property $\mathcal{D}$, we need to show that the subring $\mathcal{D}$ generated by the divisors has additive complement $\mathcal{V}$ such that $d \cdot v = 0$ and $\langle d, v\rangle_\beta^X = 0$ for all $d \in \mathcal{D}$, $v \in \mathcal{V}$.

In order for $\langle d,\, v\rangle_\beta = 0$, we need $\mathrm{codim}(d) + \mathrm{codim}(v) + 2 = 16 + 12$ and $\beta = \ell$, the class of the line (this is $x_1^3x_2^2$ in our notation above). Then we have either $\mathrm{codim}(d) = 16$, $\mathrm{codim}(v) = 10$ or $\mathrm{codim}(d) = 14$, $\mathrm{codim}(v) = 12$ (because H_0 and H_2 are both generated by divisors, we cannot reverse these codegrees). If $\mathrm{codim}(v) = 10$, then $v = rx_1^5 + sx_1^3x_2$ for some $r, s \in \mathbb{Q}$. In order for $v \cdot d = 0$ for all $d \in \mathcal{D}$, we would need

$$\begin{aligned}
0 &= x_1^3 \cdot (rx_1^5 + sx_1^3x_2)\\
&= x_1^3 \cdot ((4r+s)\sigma_{(2,1,1,1)} + (5r+2s)\sigma_{(2,2,1,0)})\\
&= x_1^2 \cdot ((9r+3s)\sigma_{(2,2,1,1)} + (5r+2s)\sigma_{(2,2,2,0)})\\
&= x_1 \cdot (14r+5s)\sigma_{(2,2,2,1)}\\
&= (14r+5s)\sigma_{(2,2,2,2)}\\
&= (14r+5s)pt.
\end{aligned}$$

Here, all calculations are carried out using classical Pieri and Giambelli. In order for this to be 0, we need $14r+5s = 0$, and thus $(r, s) = k(5, -14)$ (we'll take $k = 1$). Unfortunately, $\langle pt, v\rangle_\ell \neq 0$. We look at

$$\begin{aligned}
pt * (5x_1^5 - 14x_1^3x_2) &= x_2^4 * (6\sigma_{(2,1,1,1)} - 3\sigma_{(2,2,1,0)})\\
&= x_2^3 * (6q\sigma_{(1,0,0,0)} - 3\sigma_{(2,2,2,1)})\\
&= x_2^2 * (6q\sigma_{(2,1,0,0)} - 3q\sigma_{(1,1,1,0)})\\
&= x_2 * (6q\sigma_{(2,2,1,0)} - 3q\sigma_{(2,1,1,1)})\\
&= 6q\sigma_{(2,2,2,1)} - 3q^2\sigma_{(1,0,0,0)}\\
&= 6qx_1x_2^3 - 3q^2x_1.
\end{aligned}$$

$\langle pt, v\rangle_\ell$ is the coefficient of the $x_1x_2^3$ term in this product, so we have $\langle pt, v\rangle_\ell = 6$. Therefore, $G(2, 6)$ does not satisfy property $\mathcal{D}$. $\square$

5.7 Grassmannians do not satisfy property $\mathcal{D}$

The methods that we have used in the proofs of Propositions 5.5.2 and 5.6.1 to show that $G(2,5)$ and $G(2,6)$ do not satisfy property $\mathcal{D}$ generalize to other Grassmannians.

Proof of Proposition 1.0.6 : We will use the quantum Schubert calculus techniques from above, as expanded in (Bertram et al. (1999)). We will consider the classes $d = pt = \sigma_{k,\ldots,k}$ and $v = r\sigma_{k,1,\ldots,1} - s\sigma_{k,2,1,\ldots,1,0}$ for some $r, s > 0$. The classical Pieri's formula tells us that v is not in $\mathcal{D}$, where $\mathcal{D}$ is the subring of $\mathrm{QH}_*(X, \Lambda)$ generated by divisors (see the definition of Property $\mathcal{D}$ in section 2). Specifically, it tells us that

$$\begin{aligned}
\sigma_1^{n-1} &= \sigma_1^{n-2} \cdot \sigma_{1,0,\ldots,0} \\
&= \sigma_1^{n-3} \cdot (\sigma_{1,1,0,\ldots,0} + \sigma_{2,0,\ldots,0}) \\
&= \sigma_1^{n-4} \cdot (\sigma_{1,1,1,0,\ldots,0} + \sigma_{3,0,\ldots,0} + 2\sigma_{2,1,0,\ldots,0}) \\
&= \ldots \\
&= \sum_{|a|=n-1} b_a \sigma_a
\end{aligned}$$

Here, b_a is the number of paths from $\sigma_{0,\ldots,0}$ to σ_a in the adjacency graph of the acceptable partitions. The adjacency graph is formed by taking all partitions as vertices and placing a directed edge from a to b if σ_b appears in the product $\sigma_1 \cdot \sigma_a$. Figure 5.7 shows the adjacency graph for $G(3,7)$, where one can see by counting paths that $\sigma_1^6 = 5\sigma_{3,1,1,1} + 9\sigma_{2,2,1,1} + 5\sigma_{2,2,2,0} + 2\sigma_{3,2,1,0} + \sigma_{3,3,0,0}$.

The actual numbers b_a determined by the adjacency graph are unimportant. Since σ_1^{n-1} has positive coefficients for each term in degree $n-1$ and v has a negative coefficient in one of those classes, $v \notin \mathcal{D}$.

We can use the classical Pieri's formula to find r and s such that

$$\sigma_1^{(k-1)*(n-k-1)} \cdot v = 0.$$

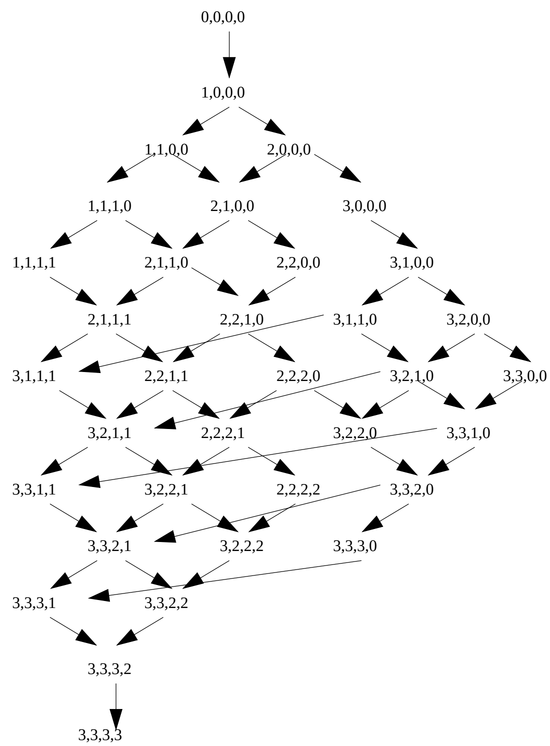

Figure 5.7.1: Adjacency graph for the Schubert cells of $G(3,7)$.

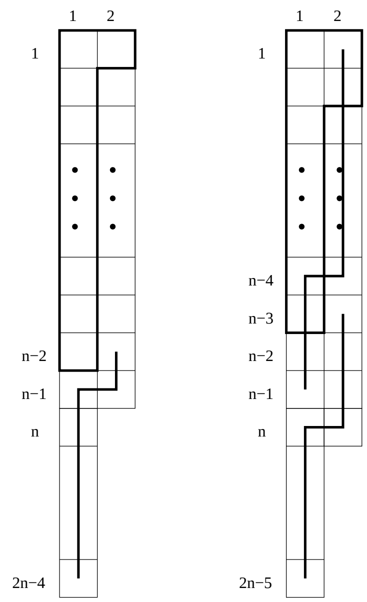

Figure 5.7.2: The Young tableaux for the quantum product $d * v$ when $k = 2$. The original tableaux are shown outlined in bold, and the n-rim hooks are represented as bold lines.

We can do this because

$$\sigma_1^{(k-1)*(n-k-1)} \cdot \sigma_{k,1,\ldots,1} = s\sigma_{k,\ldots,k}$$
$$\sigma_1^{(k-1)*(n-k-1)} \cdot \sigma_{k,2,1,\ldots,1,0} = r\sigma_{k,\ldots,k}$$

with $r, s > 0$. These numbers can be found by using the classical Pieri's formula, and count the number of paths from the partition $(k, 1, \ldots, 1)$ (respectively, the partition $(k, 2, 1, \ldots, 1, 0)$) to the partition $(k, \ldots, k)$ in the adjacency graph of Schubert cells, as in Figure 5.7. For $G(3, 7)$ the number of paths from $\sigma_{3,1,1,1}$ to $\sigma_{3,3,3,3}$ is 14, and the number of paths from $\sigma_{3,2,1,0}$ to $\sigma_{3,3,3,3}$ is 16.

Unfortunately, this number does not have a simple closed form. However, it is clearly positive and nonzero, which is sufficient for our purposes. We have thus shown that for some positive r and $s, d \cdot v = 0$.

Now we will show, using quantum Littlewood-Richardson, that $d * v \neq 0$ for

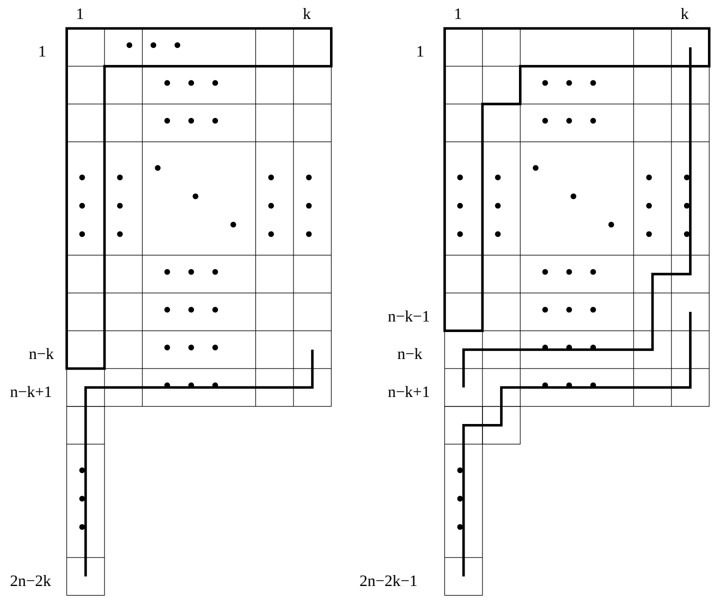

Figure 5.7.3: The Young tableaux for the quantum product $d * v$ when $k > 2$. The original tableaux are shown outlined in bold, and the n-rim hooks are represented as bold lines.

some $d \in \mathcal{D}$. The only possible class which could have nontrivial quantum product with v is the class of a point, $\sigma_{k,\dots,k}$, for dimensional reasons. The class of a point is in $\mathcal{D}$. There are two cases - when $k = 2$ and when $k > 2$. Let $k = 2$. The only possible tableau which results from multiplying any tableau α by $\sigma_{k,\dots,k}$ is the tableau with $n - k$ additional boxes in each column. Therefore, if $k = 2$, the two elements in $d * v$ have the tableau given by Figure 5.7. Note that these tableaux are only valid if $n \geq 7$. If $n \geq 7$, we have

$$\sigma_{2,\dots,2} * (r\sigma_{2,1,\dots,1} - s\sigma_{2,2,1,\dots,1,0}) = r\sigma_{2,\dots,2,1} - sq^2\sigma_{2,2,1,\dots,1,0,0,0}.$$

Note that we have already calculated this result above for $n = 5$ and $n = 6$. This completes the case when $k = 2$.

If $k > 2$, a very similar result holds, with slightly different n-rim hooks. The quantum calculation is shown in Figure 5.7. We thus have that

$$\sigma_{k,\dots,k} * (r\sigma_{k,1,\dots,1} - s\sigma_{k,2,1,\dots,1,0}) = rq\sigma_{k,\dots,k,k-1} - sq^2\sigma_{k-1,\dots,k-1,k-2,k-2,0}$$

Note that this only holds if $n-k \geq 4$. If we assume that $k \leq n$, this leaves exactly the case $G(3,6)$. For $G(3,6)$, we have:

$$\sigma_{3,3,3} * (r\sigma_{3,1,1} - s\sigma_{3,2,0}) = rq\sigma_{3,3,2} - sq^2\sigma_{1,1,0}$$

In all of these cases, since $r, s > 0$, the quantum product is nontrivial, and thus $\langle d, v\rangle = 0$. Therefore, the Grassmannians other than the projective spaces and $G(2,4)$ do not have property $\mathcal{D}$. $\square$

5.8 $G(2, 2n+1)$ has units of infinite order

For $G(2,5)$, we showed that there exist units which have infinite order. In fact, this is merely the first case of a more general result, which we stated as Proposition 1.0.7. The proof uses quantum Schubert calculus.

Proof of 1.0.7: Let $u = \sigma_{1,0,\dots,0} = \sigma_1 = x_1$, and take

$$v = \sigma_{2,1,\dots,1} - \sigma_{2,2,1,\dots,1,0} + \sigma_{2,2,2,1,\dots,1,0,0} + \cdots + (-1)^{n-1}\sigma_{2,\dots,2,0,\dots,0}.$$

Then by the quantum Pieri's formula, we have

$$\begin{aligned}\sigma 1 * v &= q\sigma_{0,\dots,0} + \sigma_{2,2,1,\dots,1} - \sigma_{2,2,1,\dots,1} - \sigma_{2,2,2,1,\dots,1,0} \\ &\quad + \sigma_{2,2,2,1,\dots,1,0} + \cdots + (-1)^{n-2}\sigma_{2,\dots,2,1,0,\dots,0} + (-1)^{n-1}\sigma_{2,\dots,2,1,0,\dots,0} \\ &= q\sigma_{0,\dots,0} \\ &= \mathbb{1} \otimes q\end{aligned}$$

Therefore, vq^{-1} is the inverse of σ_1, and σ_1 is a unit. However, σ_1 has infinite order.

Lemma 5.8.1 $\sigma_1^{\ell(2n+1)} \neq \mathbb{1} \otimes q^\ell$ *for all* $\ell > 0$.

Proof: Assume not. Then

$$\begin{aligned}
\sigma_1^{\ell(2n+1)} &= \mathbb{1} \otimes q^{\ell} \\
\sigma_1^{\ell(2n+1)} q^{-\ell+1} &= \mathbb{1} \otimes q \\
\sigma_1 * (\sigma_1^{\ell(2n+1)-1} q^{-\ell+1}) &= \mathbb{1} \otimes q \\
\sigma_1^{\ell(2n+1)-1} q^{-\ell+1} &= v
\end{aligned}$$

But by the quantum Pieri's formula, any power of σ_1 will only have positive coefficients in the Schubert cells. But v has negative coefficients, so this is a contradiction. $\square$

The lemma completes the proof that x_1 has infinite order. Therefore, any power of x_1 is a unit of infinite order in $\mathrm{QH}_*(G(2, 2n+1), \Lambda)$. $\square$

5.9 Monotone Toric 4-manifolds

The monotone toric 4-manifolds are $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, and $\mathbb{C}P^2$ blown up at 1, 2, and 3 points. $I = 0$ for $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ in the monotone case (from the results of this paper, but also as previously shown in McDuff (2010), Entov and Polterovich (2008), Ostrover (2006)). However, McDuff shows in (McDuff (2002)) that $I \neq 0$ on $\mathbb{C}P^2$ blown up at one point, even in the monotone case. A similar argument can be carried out to show that $I \neq 0$ for $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.

This leaves one monotone toric 4-manifold, $X = \mathbb{C}P^2 \# 3\mathbb{C}\bar{P}^2$. But we can easily show $I = 0$ using a result independently proven by Pinsonnault (Pinsonnault (2009)) and Evans (Evans (2009)). They show that $\pi_1(\mathrm{Ham}(X, \omega)) = \mathbb{Z}^2$. This implies that the toric actions on X generate $\pi_1(\mathrm{Ham}(X, \omega))$. We can combine this with another result (see the final digression in section 1.4 of (Entov and Polterovich (2009)) which states that for a Fano toric manifold, I vanishes on toric loops if and only if the special point of the moment polytope corresponds to the barycenter. In

4 dimensions, these points are aligned for $\mathbb{C}P^2$ blown up at 3 points. Therefore, I vanishes on the toric loops, and thus on the entirety of $\pi_1(\mathrm{Ham}(X, \omega))$. Therefore, $I = 0$ for $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, and $\mathbb{C}P^2$ blown up at 3 points.

Therefore $I(\gamma) = 0$ when γ is a circle action is equivalent to $I(\gamma) = 0$ for all γ if X is a monotone toric 4-manifold or if X is a monotone product of projective spaces. It would be interesting to see if this result extends to all monotone toric manifolds - namely, if X is monotone, toric, and has $I(\gamma) = 0$ for all circle actions γ, does $I(\gamma) = 0$ for all γ? The answer is currently unknown, but these results provide evidence that it is true.

Chapter 6

Consequences

The action-Maslov homomorphism is related to a number of other important properties of symplectic manifolds. These include the spectral invariants, the Hofer diameter, and quasimorphisms.

6.1 Spectral Invariants

Let a be a class in $\mathrm{QH}(X, \Lambda)$ and $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(X, \omega)$. The spectral invariant $c(a, \tilde{\phi})$was defined by Schwarz in (Schwarz (2000)) and extended to all symplectic manifolds by Oh and Usher (Oh (2005), Usher (2008)). If $H : X \to \mathbb{R}$ is the Hamiltonian corresponding to $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(X, \omega)$, then

$$c(a, \tilde{\phi}) = \inf_{\alpha \in \mathbb{R}/\mathrm{Spec}(H)} \{a \in \mathrm{im}(i_\alpha)\}$$

where $i_\alpha : \mathrm{HF}_{\alpha *}(X, \tilde{\phi}) \to \mathrm{QH}_*(X, \Lambda)$ is the map from the filtered Floer homology at level α to the quantum homology (see McDuff and Salamon (2004)). These $c(a, \tilde{\phi})$ are the spectral invariants, and the asymptotic spectral invariants are given by

$$\bar{c}(a, \tilde{\phi}) \lim_{k \to \infty} \frac{c(a, \tilde{\phi}^k)}{k}.$$

The spectral invariants always exist and have the following properties:

$$-\|\tilde{\phi}\| \leq c(a,\tilde{\phi}) = c(a,\tilde{\psi}\tilde{\phi}\tilde{\psi}^{-1}) \leq \|\tilde{\phi}\| \ \forall a \in H_*(X), \tilde{\psi} \in \widetilde{\mathrm{Ham}}(X,\omega) \tag{6.1.1}$$

$$c(\lambda a,\tilde{\phi}) = c(a,\tilde{\phi}) + \nu(\lambda) \text{ for all } \lambda \in \Lambda \tag{6.1.2}$$

$$c(a,\tilde{\phi}\circ\gamma) = c(\mathcal{S}(\gamma) * a,\tilde{\phi}) \text{ for all } \gamma \in \pi_1(\mathrm{Ham}(X,\omega)) \tag{6.1.3}$$

$$c(a * b,\tilde{\phi}\circ\tilde{\psi}) \leq c(a,\tilde{\phi}) + c(b,\tilde{\psi}) \tag{6.1.4}$$

Property 6.1.1 implies that $c(a,\mathrm{id}) = 0$ for all $a \in H_*(X)$, where id denotes the constant loop at the identity. Then the other two properties imply that

$$\begin{aligned} c(\mathbb{1},\mathrm{id}) = c(\mathcal{S}(\gamma),\mathrm{id}) & \\ &= \nu(\mathcal{S}(\gamma))c(\mathbb{1},\mathrm{id}) \\ &= \lim_{k\to\infty} \frac{\nu(S(k\gamma))}{k}. \end{aligned}$$

Recall that we showed that for X equal to a monotone product of projective spaces or $G(2,4)$, for all $\gamma \in \pi_1(\mathrm{Ham}(X,\omega))$, we have $\mathcal{S}(k\gamma) = \mathbb{1}\otimes\lambda$, with $\nu(\lambda) = 0$. Therefore, we have:

Corollary 6.1.1 *Let (X,ω) be a monotone product $\mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k}$ or $G(2,4)$. Then the asymptotic spectral invariants vanish on $\pi_1(\mathrm{Ham}(X,\omega)) \subset \widetilde{\mathrm{Ham}}(X,\omega)$ and are well defined on $\mathrm{Ham}(X,\omega)$.*

6.2 Hofer diameter

A well known result, stated in (McDuff (2010)), states that vanishing of the asymptotic spectral invariants implies that $\mathrm{Ham}(X,\omega)$ has infinite Hofer diameter. We restate the argument as presented in (McDuff (2010)) for completeness.

Lemma 6.2.1 *Let (X,ω) be a monotone product $\mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k}$ or $G(2,4)$. Then there exist Hamiltonian diffeomorphisms ψ_s such that $\lim_{s\to\infty}\|\psi_s\| = \infty$.*

Proof: The definition of these maps is originally due to Ostrover (Ostrover (2003)). Without loss of generality, assume that $\int_X \omega^n = 1$. Let H be a small mean-normalized Morse function and U an open set displaced by its time 1 map ϕ_H. Now let F be a Hamiltonian function with support on U such that $F - \int_X F\omega^n$ is mean normalized. Let f_t be the flow of F. Now, take

$$\tilde{\psi}_s = \{f_{ts}\phi_H\}_{t\in[0,1]}$$

Each $\tilde{\psi}_s$ is an element of $\mathrm{Ham}(X,\omega)$. $\tilde{\psi}_s$ is generated by the mean-normalized Hamiltonian $sF + H \circ f_{ts} - s\int_X F\omega^n$. Since $\mathrm{supp}(F)$ is displaced by ϕ_1^H, $f_s\phi_1^H$ will have the same fixed points as ϕ_1^H. Therefore, the fixed point p_a whose critical value is $c(a,\tilde{\psi}_s)$ will remain unchanged as $s \to \infty$. However, it's value does change. When $s = 0$, there exists p_a such that

$$c(a,\tilde{\psi}_0) = -H(p_a).$$

Therefore, we have

$$c(a,\tilde{\psi}_s) = -H(p_a) + s\int_X F\omega^n. \tag{6.2.1}$$

Now we need to show that if the asymptotic spectral invariants descend, then $\|\pi(\tilde{psi}_s)\| = \|\psi s\| \to \infty$ as s goes to ∞. First, we will show that there exist g_i conjugate to $\tilde{\phi}_1^H$ such that

$$\tilde{\psi}_{sk}\tilde{g}_1 \cdots \tilde{g}_{k-1} = (\psi_s)^k.$$

But this follows from simple algebra: let $a = \tilde{f}_s$ and $b = \tilde{\phi}_1^H$. Then $\tilde{\psi}_{sk} = a^k b$ and $(\tilde{\psi}_s)^k = (ab)^k$. Then we have

$$\begin{aligned}(ab)k = abab\cdots ab &= (a^k bb^{-1}a^{-k+1})(b)(a^{k-1}a^{-k+2})(b)\cdots(a^2a^{-1})(b)(a)(b)\\ &= (a^kb)(b^{-1})(a^{-k+1}ba^{k-1})(a^{-k+2}ba^{k-2})\cdots(a^{-1}ba)b\\ &= (a^kb)b^{-1}(a^{-k+1}ba^{k-1})(a^{-k+2}ba^{k-2})\cdots(a^{-1}ba)b.\end{aligned}$$

Now let e be an idempotent. By 6.1.4, we have $c(e,(\tilde{\psi})^k) \leq kc(e,\tilde{\psi})$ for all $k > 1$. Then for all $\gamma \in \pi_1(\mathrm{Ham}(X,\omega))$,

$$c(e,\tilde{\psi}_s \circ \gamma) = \lim_{k\to\infty} c(e,(\tilde{\psi}_s \circ \gamma)^k) \leq c(e,\tilde{\psi}_s \circ \gamma) \leq \|\psi_s \circ \gamma\|.$$

Since the invariants descend, we have

$$\begin{aligned}
\bar{c}(e,\tilde{\psi}_s \circ \gamma) &= \bar{c}(e,\tilde{\psi}_s) \\
&= \lim_{k\to\infty} \frac{1}{k} c(e,(\tilde{\psi}_s)^k) \\
&= \lim_{k\to\infty} \frac{1}{k} c(e,\tilde{\psi}_{sk}\tilde{g}_1 \cdots \tilde{g}_{k-1}) \\
&\geq \lim_{k\to\infty} \frac{1}{k} \left(c(e,\tilde{\psi}_{sk}) - \sum_{i=1}^{k-1} c(e,\tilde{g}_i) \right) \\
&\geq s \int_X F\omega^n - \|\tilde{\phi}_1^H\|.
\end{aligned}$$

Here, we use (6.1.4) and (6.2.1). Therefore, we have

$$\begin{aligned}
\|\psi_s\| &= \inf_{\gamma\in\pi_1(\mathrm{Ham}(X,\omega))} \{\|\tilde{\psi}_s \circ \gamma\|\} \\
&\geq \inf_{\gamma\in\pi_1(\mathrm{Ham}(X,\omega))} \{c(a,\tilde{\psi}_s \circ \gamma)\} \\
&\geq s \int_X F\omega^n - \|\tilde{\phi}_1^H\|
\end{aligned}$$

and the Hofer diameter of $\mathrm{Ham}(X,\omega)$ is infinite. $\square$

6.3 Quasimorphisms

Banyaga showed in (Banyaga (1978)) that if X is a compact symplectic manifold, $\mathrm{Ham}(X,\omega)$ is a simple group, and thus there is no nontrivial homomorphism from $\mathrm{Ham}(X,\omega) \to \mathbb{R}$. However, there may exist nontrivial quasimorphisms. A quasimorphism is a map $f : X \to Y$ such that

$$|f(x) + f(y) - f(x+y)| \leq Cf.$$

In his thesis (Py (2008)), Py defines a quasimorphism $\mathfrak{G} : \widetilde{\mathrm{Ham}}(X, \omega) \to \mathbb{R}$. This quasimorphism is constructed as follows.

Let (V, ω) be a monotone symplectic manifold, and let $\pi : M \to V$ be an S^1 bundle over V with Euler class equal to $2c_1(V)$. Let X be the vector field on M generated by the S^1 action ψ. Then there exists a 1-form α on M with $\alpha(X) = 1$ and $d\alpha = \pi_*(s\omega)$.

Choose an almost complex structure J on V compatible with ω. This makes TV a hermitian vector bundle, and we can choose a trivialization over a covering $\{U_\gamma\}$ with unitary transition maps $g_{\beta\gamma} : U_\beta \cap U_\gamma \to U(n)$. The family of maps $(\det^2 g_{\beta\gamma})$ determines an S^1 bundle $E \to V$ which is isomorphic to M. Now take $\Lambda(V)$ to be the lagrangian Grassmannian bundle over V. Over any trivialization $U_\gamma \times \mathbb{C}^n$, an element $L \in \Lambda(V)$ is $(x, u_\gamma(\mathbb{R}^n))$ for some unitary u_γ. Therefore, there is a map $det^2 : \Lambda(V) \to E$ given by taking $(x, u_\gamma(\mathbb{R}^n)) \to (x, det^2(u_\gamma))$. If we choose an isomorphism from E to M, then we obtain a map $\phi : \Lambda(V) \to M$. This map depends on the isomorphism, but restricted to the fiber, induces an isomorphism on fundamental groups. Therefore, for another such map $\phi' : \Lambda(V) \to \mathbb{R}$ we have

$$\phi'(L) = \psi(\chi(\pi(L))) \cdot \psi(e^{2\pi i \kappa(L)}) \cdot \phi(L)$$

for some maps $\chi : V \to S^1$ and $\kappa : \Lambda(V) \to \mathbb{R}$.

Now we are ready to construct our quasimorphism. Let $H_t : S^1 \times V \to \mathbb{R}$ be a mean-normalized time-dependent 1-periodic Hamiltonian with vector field Z_t and flow $f_t \in \widetilde{\mathrm{Ham}}(X, \omega)$. Z_t can be lifted to a vector field $\hat{Z}_t$ on M such that $\alpha(\hat{Z}_t) = 0$. Let $\Theta(f_t)$ be the flow of the vector field

$$\hat{Z}_t - (H_t \circ \pi)X.$$

Then by our discussion above, we have

$$\phi(df_t \cdot L) = e^{2\pi i \vartheta(t)} \Theta(f_t)(\phi(L)).$$

Then we can define a map angle : $\Lambda(V) \times \widetilde{\text{Ham}}(X,\omega) \to \mathbb{R}$ by $\text{angle}(L, f_t) = \vartheta(1) - \vartheta(0)$. This map in turn, allows us to define a map which we will also call $\text{angle} : V \to \mathbb{R}$ by

$$\text{angle}(x, f_t) = \inf_{L \in \Lambda(V)_x} angle(L, f_t).$$

This allows us to define a quasimorphism

$$G(f_t) = \int_V \text{angle}(x, f_t)\omega^n.$$

The homogenization of this quasimorphism

$$\mathfrak{G}(f_t) = lim_{p\to\infty} \frac{1}{p} G(f_t^p)$$

is what we refer to as Py's quasimorphism. It does not depend on ϕ or J, and Py shows in Proposition 2.3.1 of his thesis (Py (2008)) that the restriction of $\mathfrak{G}$ to $\pi_1(\text{Ham}(X,\omega))$ is

$$\mathfrak{G}(f_t) = vol(V) \cdot I(f_t)$$

Therefore, if the action-Maslov homomorphism vanishes, Py's quasimorphism is well-defined on $\text{Ham}(X,\omega)$. We thus have the following corollary:

Corollary 6.3.1 *Py's quasimorphism* $\mathfrak{G}$ *descends to a quasimorphism* $\mathfrak{G} : \text{Ham}(X,\omega) \to \mathbb{R}$ *when* (X,ω) *is a monotone product of projective spaces or* $G(2,4)$.

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