Modularity of Generating Functions of Special Cycles on Shimura Varieties

Wei Zhang

Submitted in partial fulfillment of the
Requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2009

Abstract

Modularity of Generating functions of Special Cycles on Shimura Varieties

Wei Zhang

In this thesis we study special cycles on Shimura varieties of orthogonal type. We confirm a conjecture of Kudla in [K2] on the modularity of generating functions of special cycles of any codimension on Shimura varieties of orthogonal type, provided their convergence. This is a generalization of theorems of Hirzebruch-Zagier, Gross-Kohnen-Zagier and Borcherds to high codimensional cycles.

Contents

Acknowledgements

First of all I am very grateful to my advisor Shou-Wu Zhang who introduced me the beautiful subject studied in this thesis. I am also indebted to him for lots of inspiring conversations and encouragement during the five year study at Columbia University.

I also would like to thank Professor Dorian Goldfeld, Hervé Jacquet, Eric Urban from them I have learned a lot on number theory and automorphic forms.

I am also grateful for conversation with many fellow students in the mathematics department: Minglun Hsieh, Zhi Li, Xinyi Yuan, Bei Zhang and etc..

Last but not least, I would like to thank my wife Yan Jiang for all the support she gave me during the preparation of this thesis.

To my parents

Deyin Zhang

and

Daiying Xiong

Chapter 1

Introduction

1.1 Statement of main results

The study of algebraic cycles on algebraic varieties gives rise to various interesting questions in arithmetic and algebraic geometry. On a Shimura variety there is a large supply of algebraic cycles provided by sub-Shimura varieties and their Hecke translations. In this paper, we will study certain special cycles defined by Kudla in [K1] on Shimura varieties of orthogonal type. More precisely, assuming the convergence we will prove the modularity of generating functions of special cycle classes modulo rational equivalence. In codimension one, this is a theorem of Borcherds ([B1]), generalizing Gross-Kohnen-Zagier's theorem ([GKZ]) for modular curves, and Hirzebruch-Zagier's theorem ([HZ]) for Hilbert modular surfaces to high dimensional Shimura varieties of orthogonal type. Kudla and Millson in [KM] also obtained the modularity of generating functions of cohomology classes of special cycles on more general Shimura varieties. We now describe our results more precisely.

Throughout this paper, for a $\mathbb{Z}$-module A and a field k, we define $A_k = A \otimes_{\mathbb{Z}} k$. Let (L, q) be a nondegenerate even lattice (i.e., $q(v)$ is an even integer for every $v \in L$) with signature $(n, 2)$, and let $\langle \cdot, \cdot \rangle$ be the inner product associated to q. Let

$L^\vee$ be its dual (thus $L \subseteq L^\vee$), and let Γ be a congruence subgroup of $O(L)$ that acts trivially on $L^\vee/L$. Let D be the Grassmannian of $L_\mathbb{R}$ with points representing negative definite 2-planes in $L_\mathbb{R}$. In our situation, D is a Hermitian symmetric space of dimension n. When $n = 1, 2, 3$, D is the Poincaré upper half plane, the product of two Poincaré upper half planes, and Siegel upper half space of genus 2, respectively. Let X_Γ be the connected Shimura variety whose complex points are

$$X_\Gamma(\mathbb{C}) = \Gamma\backslash D.$$

When Γ is neat, X_Γ is a smooth quasi-projective variety with a canonical model generally defined over a cyclotomic extension of $\mathbb{Q}$.

For an r-tuple $\underline{v} = (v_1, ..., v_r) \in \underline{\lambda} + L^r$ where $\underline{\lambda} \in (L^\vee/L)^r$, let $Z(\underline{v})$ be the image in $\Gamma\backslash D$ of all negative definite 2-planes perpendicular to all of $v_1, ..., v_r$. $Z(\underline{v})$ is nonempty precisely when $v_1, ..., v_r$ generate a positive definite subspace of $L_\mathbb{Q}$ with dimension denoted by $r(\underline{v})$. Obviously, $Z(\gamma\underline{v}) = Z(\underline{v})$ for $\gamma \in \Gamma$ where Γ acts diagonally $\gamma(v_1, ..., v_r) = (\gamma v_1, ..., \gamma v_r)$.

For an r-tuple $\underline{\lambda} \in (L^\vee/L)^r$ and an $r \times r$ symmetric positive semi-definite matrix $T = (T_{ij}) \in Sym_r(\mathbb{Q})_{\geq 0}$ of rank $r(T)$, let

$$\Omega(T, \underline{\lambda}) = \{\underline{v} \in \underline{\lambda} + L^r | T = Q(\underline{v})\}$$

where the $r \times r$ matrix $Q(\underline{v}) = \frac{1}{2}(\langle v_i, v_j\rangle)$ is called the moment matrix of $\underline{v}$. Kudla defines a cycle ([K1], but there in adelic language) of codimension $r(T)$

$$Z(T, \underline{\lambda}) = Z(T, \underline{\lambda}; \Gamma) = \sum_{\underline{v} \in \Gamma\backslash\Omega(T,\underline{\lambda})} Z(\underline{v}).$$

These cycles are compatible with the nature pull-back map $pr : X_{\Gamma'} \to X_\Gamma$ for

$\Gamma' \subseteq \Gamma$, i.e.,

$$pr^* Z(T, \underline{\lambda}; \Gamma) = Z(T, \underline{\lambda}; \Gamma').$$

Note that $Z(T, \underline{\lambda})$ is empty unless $T_{i,j} \equiv \frac{1}{2}\langle \lambda_i, \lambda_j \rangle \pmod{\mathbb{Z}}$. In this paper, we call all $Z(T, \underline{\lambda})$ *special cycles* of moment T and residue class $\underline{\lambda}$. All such cycles are defined over abelian extension of $\mathbb{Q}$.

As the most interesting examples, for small n our special cycles include:

1. $(n = r = 1)$ CM points (including Heegner points) on a Shimura curve (modular curve included).
2. $(n = 2, r = 1)$ Hecke correspondences on the self-product of a Shimura curve, and Hirzebruch-Zagier cycles on a Hilbert modular surface.
3. $(n = 3, r = 1)$ Humbert surfaces on a Siegel modular variety.
4. $(n = 3, r = 2)$ Shimura curves on a Siegel modular variety.

Let $\mathscr{L} = \mathscr{L}_\Gamma$ be the line bundle on X_Γ that descends from the tautological bundle $\mathscr{L}_D$ on D, cf. Chap. (2.3).

Let $CH^r(X_\Gamma)_\mathbb{C}$ be the Chow group with complex coefficients. Let $\{Z(T, \underline{\lambda})\} \in CH^{r(T)}(X_\Gamma)_\mathbb{C}$ be the cycle class of the codimension-$r(T)$ special cycle $Z(T, \underline{\lambda})$, and let $\{\mathscr{L}^\vee\} \in CH^1(X_\Gamma)_\mathbb{C}$ be the Chern class of the dual bundle of $\mathscr{L}$.

We now define the generating function with coefficients in $CH^r(X_\Gamma)_\mathbb{C}$ by

$$\Theta_{\underline{\lambda}}(\tau) = \sum_{T \geq 0} \{Z(T, \underline{\lambda})\} \cdot \{\mathscr{L}^\vee\}^{r - r(T)} q^T$$

where $q^T = e^{2\pi i tr(T\tau)}$, and

$$\tau \in \mathscr{H}_r = \{\tau \in Sym_r(\mathbb{C}) | Im(\tau) > 0\}$$

is the Siegel upper half space of genus r. Here the product is the intersection product on Chow groups.

For a linear functional ι on $CH^r(X_\Gamma)_{\mathbb{C}}$, we define

$$(\iota, \Theta_{\underline{\lambda}})(\tau) = \sum_{T \geq 0} (\iota, \{Z(T, \underline{\lambda})\} \cdot \{\mathscr{L}^\vee\}^{r-r(T)}) q^T$$

where $(\iota, v) = \iota(v)$ for $v \in CH^r(X_\Gamma)_{\mathbb{C}}$.

Our first result is

Theorem A (Modularity Conjecture of Kudla, [K2]). *For $n \geq 1$, $r \in \{1, 2, ..., n\}$, an even lattice L of signature $(n, 2)$ as above, and $\underline{\lambda} \in (L^\vee/L)^r$, assume that $(\iota, \Theta_{\underline{\lambda}})$ is absolutely convergent on $\mathscr{H}_r$. Then it is a Siegel modular form of genus r and weight $\frac{n}{2} + 1$ for a congruence subgroup in $Sp(2r, \mathbb{Z})$.*

Remark. 1. Note that when $n = r = 1$, our special cycles are precisely CM points on Shimura curves (including modular curves). When L is isotropic, this is a consequence of the Gross-Kohnen-Zagier theorem ([GKZ]). Using his construction of a family of meromorphic automorphic functions in [B0], Borcherds in [B1] reproves their theorem and generalizes it to the situation where the Shimura variety is attached to any signature-$(n, 2)$ lattice L and the special cycles are divisors (i.e., $r = 1$).

2. We would like to point out that in Borcherds' theorem, he also proves that all linear functionals ι on $CH^1(X_\Gamma)_{\mathbb{C}}$ automatically satisfy the convergence assumption above. But for $r \geq 2$, we do not know how to prove, though we expect, that all linear functionals on the Chow groups should satisfy this assumption (see Conjecture 1). By the conjecture of Beilinson-Bloch ([Be]) on the non-degeneracy of the (conditionally defined) height pairing between cohomologically trivial cycles, we only need to check those linear functionals given by height pairings. An particularly interesting class of linear functional is given

by height pairing with special cycles of complementary dimension. We hope to investigate this aspect in the future.

3. In series of papers ([KM]), Kudla and Millson have proven the modularity of generating functions of cohomology classes in a more generality for Shimura varieties attached to orthogonal groups and unitary groups over not only $\mathbb{Q}$ but also totally really fields. In the case of orthogonal groups, it is not hard to check that a linear functional satisfies the convergence assumption above if it factors through the cohomology group via the cycle class map. In fact, for a closed $(n-r, n-r)$ differential form η with compact support on X_Γ, we can define a linear functional ι mapping $Z(\underline{x})$ to the integral $\int_{Z(\underline{x})} \eta$ if the moment matrix $Q(\underline{x})$ is positive definite, and modify the integral by shifting η by a power of the curvature form Ξ of $\mathscr{L}^\vee$ when $Q(\underline{x})$ is only semi-positive. Then, since $|\iota(Z(T,\underline{\lambda}))| \leq C \cdot vol(Z(T,\underline{\lambda}))$ is bounded above by a constant multiple of the volume $vol(Z(T,\underline{\lambda})) = \int_{Z(T,\underline{\lambda})} \Xi^{n-r(T)}$ of the special cycle $Z(T,\underline{\lambda})$, it is not hard to see the absolute convergence of $\iota(\Theta_{\underline{\lambda}})$.

4. One of our main motivations for investigating special cycles is to obtain relations between these cycles and special values or derivatives of certain automorphic L-functions, generalizing the formula of Gross-Zagier and Zhang ([GZ],[Zh]) to higher dimensional varieties. Once we know the modularity of the generating function, we would like to know its spectral decomposition. For example, for a cusp Siegel modular form f of genus r, weight $1+\frac{n}{2}$ and the same level $\Gamma(N)$ as $\Theta_{\underline{0}}$, we can define an "arithmetic theta lifting" ([K2] and [KRY]) using the Petersson inner product

$$\Theta(f) := \int_{\Gamma(N)\backslash\mathscr{H}_r} \Theta_{\underline{0}}(\tau)\overline{f}(\tau)d\mu(\tau) \in CH^r(X_\Gamma)_{\mathbb{C}}.$$

Inspired by the theory of theta lifting of automorphic forms, we would like to ask for a criterion for the (non-)vanishing of this lifting. One would hope for the occurrence of central derivatives of certain automorphic L-functions in this criterion in the same manner as the occurrence of central values of L-functions in Waldspurger's criterion on the non-vanishing of theta lifting for the reductive pair $(\widetilde{SL}(2), O(3))$. This method has been pursued by Kudla, Rapport and Yang in [KRY] for $n = r = 1$. Based on Borcherds' modularity result and other ingredients, they succeed in relating the height pairing of certain special points on Shimura curves to the central derivatives of L-functions of certain new forms of weight 2, and thus establish a criterion of Waldspurger type. Further speculations have been proposed in [K2].

The proof of Theorem A is in Chap. 2.4 (Theorem 2.9) and it turns out to be an application of Borcherds' modularity theorem to a family of subvarieties on X_Γ. In fact, we prove a criterion of modularity for formal power series. More precisely, let L be an even lattice of signature (n, n') with $2|n'$ and let Γ be a congruence subgroup of $O(L)$. For an integer $r \in \{1, 2, ..., n\}$, let F be a function on $\Gamma\backslash L_{\mathbb{Q}}^r$. Define $F(T, \underline{\lambda}) = \sum_{x \in \Gamma\backslash\Omega(T,\underline{\lambda})} F(x)$ for $T \in Sym_r(\mathbb{Q})_{\geq 0}$ and $\underline{\lambda} \in (L^\vee/L)^r$. We define formal q-series for $\underline{\lambda} \in (L^\vee/L)^r$,

$$\Theta_{F,\underline{\lambda}}(\tau) := \sum_{T \geq 0} F(T, \underline{\lambda}) q^T \in \mathbb{C}[[q]].$$

Let $\rho_{L,r}^*$ be a fixed representation of the double covering $\widetilde{Sp}(2r, \mathbb{Z})$ of $Sp(2r, \mathbb{Z})$ acting on the vector space $S_{L,r}^*$ with a basis consisting of $\{\varphi_{\underline{\lambda}}^*\}$ (see Definition 2.2). The representation $\rho_{L,r}^*$ essentially comes from the (dual of) Weil representation attached to $L_{\mathbb{Q}}^r$. Define vector valued q-series

$$\Theta_F = \sum_{\underline{\lambda} \in (L^\vee/L)^r} \Theta_{F,\underline{\lambda}} \varphi_{\underline{\lambda}}^* \in S_{L,r}^*[[q]].$$

For $\underline{x} = (x_1, ..., x_{r-1}) \in L_{\mathbb{Q}}^{r-1}$ let $\mathbb{Q}\underline{x}$ be the subspace $\sum_{1\le i\le r-1} \mathbb{Q}x_i$ of $L_{\mathbb{Q}}$. If $\mathbb{Q}\underline{x}$ is positive definite of dimension denoted by $r(\underline{x})$, let $(\mathbb{Q}\underline{x})^\perp$ be its orthogonal complement. Let $L_{\underline{x}}$ be the lattice $L \cap (\mathbb{Q}\underline{x})^\perp$, and let $\Gamma_{\underline{x}} \subset \Gamma$ be the stabilizer of $\underline{x}$. Assume that $F(x)$ only depends on the space $\mathbb{Q}\underline{x}$ and $F(x) = 0$ if $\mathbb{Q}\underline{x}$ is not positive definite. The restriction $F_{\underline{x}}$ of F on $L_{\underline{x},\mathbb{Q}} \cong L_{\underline{x},\mathbb{Q}} \times \{\underline{x}\} \subseteq L_{\mathbb{Q}}^r$ is $\Gamma_{\underline{x}}$-invariant. Then we have

Theorem B (Criterion of Modularity). *Under the notation above, the q-series* $\Theta_F \in S_{L,r}^*[[q]]$ *is the q-expansion of a Siegel modular form of type* $\rho_{L,r}^*$*, weight* $\frac{n+n'}{2}$ *and genus r (see Definition 2.1) if F satisfies the following two conditions:*

1. Θ_F *is absolutely convergent on* $\mathscr{H}_r$.

2. *For every* $\underline{x} \in L^{\vee, r-1}$ *with positive definite* $\mathbb{Q}\underline{x}$ *of dimension* $r(\underline{x})$*, the q-series* $\Theta_{F_x} \in S_{L_{\underline{x}},1}^*[[q]]$ *is absolutely convergent on* $\mathscr{H} = \mathscr{H}_1$ *and defines a Siegel modular form of type* $\rho_{L_{\underline{x}},1}^*$*, weight* $\frac{n+n'-r(\underline{x})}{2}$ *and genus* 1.

Remark. 1. As the simplest example, we can take a positive definite L of rank n (i.e, $n' = 0$). The the function $F = 1$ on $L_{\mathbb{Q}}^r$ yields the classical theta function.

2. It should also be possible to extend the criterion of modularity to unitary groups.

The proof of criterion of modularity is mainly based on an expansion of Fourier-Jacobi type, and an explicit list of generators of the double covering $\widetilde{Sp}(2r, \mathbb{Z})$ of $Sp(2r, \mathbb{Z})$.

Finally, some development related to the subject of this thesis should also be noted. The author joint with Xinyi Yuan and Shou-Wu Zhang have generalized the Gross–Kohnen–Zagier theorem (i.e., the case of codimension $r = 1$) to totally real fields ([YZZ1]). Hence by the argument of this paper we obtain the modularity of generating functions for any codimension r over totally real fields. Moreover, the modularity result has been used in a crucial way in the proof of a general Gross–Zagier formula for the Rankin L-series ([YZZ2]) and of the central derivative of triple product L-series in terms of heights of Gross–Schoen cycles ([YZZ3]).

1.2 The structure of this article

The organization of this paper is as follows: in Chap. 2.1, we define the Weil representation of $\widetilde{Sp}(2r,\mathbb{Z})$ attached to L, and we describe a simple list of generators of $\widetilde{Sp}(2r,\mathbb{Z})$. We then prove a criterion of modularity for generating functions in Chap. 2.2. In Chap. 2.3, we recall basic facts about our Shimura varieties and special cycles following [K1]. As an application of our criterion of modularity, we prove in Chap. 2.4 the modularity of generating functions of special cycles. In Chap. 3, by an ad hoc method, we prove the finite generation of the subspace of the Chow group generated by special cycles of codimension two. In Chap. 4 we investigate the Hecke action on the space of special cycles from a representation theoretical point of view. In particular, we prove a "multiplicity one" result (Corollary 4.4), generalizing the theorem of Gross–Kohnen–Zagier for Heegner points to special divisors of high dimension.

Chapter 2

Modularity of generating functions

In this chapter we prove the main theorem of this thesis, namely the modularity of generating functions of special cycles assuming the convergence.

2.1 Siegel modular groups $Sp(2r, \mathbb{Z})$ and Weil representations

In this section we want to define the Weil representation of the double covering $\widetilde{Sp}(2r, \mathbb{Z})$ of the Siegel modular group $Sp(2r, \mathbb{Z})$ and vector valued Siegel modular forms. We then give a simple list of generators of $Sp(2r, \mathbb{Z})$ and $\widetilde{Sp}(2r, \mathbb{Z})$, which will be used to check modularity of generating functions.

2.1.1

Let $Sp(2r, \mathbb{Z})$ be the symplectic group of integral matrices, i.e. all $g \in GL(2r, \mathbb{Z})$ satisfying

$$^t g J g = J, \quad J = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$$

and note that $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$.

Let $\mathscr{H}_r$ be the Siegel upper half plane of genus r, i.e.

$$\mathscr{H}_r = \{\tau \in Sym_r(\mathbb{C}) | Im(\tau) > 0\}.$$

The group $Sp(2r, \mathbb{Z})$ acts on $\mathscr{H}_r$ by

$$\gamma\tau = (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $\widetilde{Sp}(2r, \mathbb{Z})$ be the double covering of $Sp(2r, \mathbb{Z})$ ([S2]). For $r = 1$, this is also denoted by $\widetilde{SL}(2, \mathbb{Z})$ defined as in [B1]. It consists of all elements of this form

$$\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \pm\sqrt{det(C\tau + D)} \right), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r, \mathbb{Z}) \tag{2.1}$$

where $\sqrt{det(C\tau + D)}$ is a holomorphic function of τ in the Siegel upper half plane whose square is $det(C\tau + D)$. And the group law is given by

$$(P, f(\cdot))(Q, g(\cdot)) = (PQ, f(Q(\cdot))g(\cdot)).$$

The group $\widetilde{Sp}(2r, \mathbb{Z})$ acts on $\mathscr{H}_r$ through its quotient $Sp(2r, \mathbb{Z})$. And the automorphic factor of the element γ (2.1) is defined to be

$$j\,(\gamma, \tau) = \pm\sqrt{det(C\tau + D)}.$$

2.1.2

Now we give the definition of (vector-valued) Siegel modular form of half integral weight. In the rest of the paper we have the following convention

$$e(x) = e^{2\pi i x}, \quad x \in \mathbb{C}.$$

Let ρ be a representation of $\widetilde{Sp}(2r, \mathbb{Z})$ on a (finite dimensional) complex vector space V_ρ. We assume that ρ factor through a finite quotient and V_ρ is a direct sum of one dimensional eigenspaces under the action of the subgroup

$$\left(\begin{pmatrix} I_r & B \\ 0 & I_r \end{pmatrix}, 1 \right).$$

So we can choose $\{v_i\}$ as a basis of V_ρ consisting of eigenvectors and

$$\rho \left(\begin{pmatrix} I_r & B \\ 0 & I_r \end{pmatrix}, 1 \right) v_i = e(tr(BT_i))v_i$$

where $e(tr(BT_i))$ is the eigenvalue, and entries of $T_i \in Sym_r(\mathbb{Q})$ are well-defined modulo $Sym_r(\mathbb{Z})$.

Definition 2.1. *A Siegel modular form of genus r, weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ for a representation of $\widetilde{Sp}(2r, \mathbb{Z})$ on a (finite dimensional) complex vector space V_ρ is a holomorphic map f from $\mathscr{H}_r$ to V_ρ such that*

$$f(\gamma\tau) = j(\gamma, \tau)^{2k} \rho(\gamma) f(\tau)$$

for all

$$\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \sqrt{det(C\tau + D)} \right) \in \widetilde{Sp}(2r, \mathbb{Z}).$$

and such that f has vanishing Fourier coefficients unless $T \geq 0$ in its Fourier expansion

$$f(\tau) = \sum_i v_i \left(\sum_{T \in Sym_r(\mathbb{Q})_{\geq 0}} a_{T,i} q^T \right) \in V_\rho[[q]]$$

where $\{v_i\}$ is the basis above and $q^T = e(tr(T\tau))$ (so that $a_{T,i} = 0$ unless $T \equiv T_i \,(mod\ \mathbb{Z}\ ))$, and here an element in $V_\rho[[q]]$ is allowed to have fractional exponents. Note that the vanishing condition on Fourier coefficients is automatically verified if $r > 1$ by Koecher principle.

For a given r, the complex vector space $A(k, \rho)$ of Siegel modular form of weight $1 + \frac{n}{2}$ type ρ is finite dimensional by the finiteness result for cohomology of locally free sheaves of finite rank.

2.1.3

One class of representations we will consider is the Weil representation associated to a lattice. Let L be an even lattice of signature (n, n'). Let $L^\vee$ be its dual, thus $L \subseteq L^\vee$. Then we have an induced quadratic pairing on $(L^\vee/L)^r$ valued in $\mathbb{Q}/\mathbb{Z}$ given by

$$\langle \underline{\delta}, \underline{\lambda} \rangle = \sum_{i=1}^{r} \langle \delta_i, \lambda_i \rangle$$

for $\underline{\delta}, \underline{\lambda} \in (L^\vee/L)^r$.

For $r = 1, 2, ...,$ one can associate to L a family of unitary representation $\rho_{L,r}$ of $\widetilde{Sp}(2r, \mathbb{Z})$ on the finite dimensional $\mathbb{C}$-vector space $S_{L,r} = \mathbb{C}[(L^\vee/L)^r]$. In the following, we use $\varphi_{\underline{\lambda}}$ to denote the element in $S_{L,r}$ corresponding to $\underline{\lambda} \in (L^\vee/L)^r$.

Definition 2.2. *The representation $\rho_{L,r}$ of $\widetilde{Sp}(2r, \mathbb{Z})$ on $S_{L,r}$ is given by*

1. $\rho_{L,r}\left(\left(\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \sqrt{det(A)}\right)\right) \varphi_{\underline{\lambda}} = \sqrt{det(A)}^{(n'-n)} \varphi_{\underline{\lambda} A^{-1}}$, *for* $A \in GL(r, \mathbb{Z})$.

2. $\rho_{L,r}\left(\left(\begin{pmatrix} I_r & B \\ 0 & I_r \end{pmatrix}, 1\right)\right)\varphi_{\underline{\lambda}} = e(tr(Q(\underline{\lambda})B))\varphi_{\underline{\lambda}}$.

3. $\rho_{L,r}\left(\left(\begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}, \sqrt{det(\tau)}\right)\right)\varphi_{\underline{\lambda}} = \frac{e(-\frac{r(n-n')}{8})}{|L^\vee/L|^{r/2}} \sum_{\underline{\delta}\in(L^\vee/L)^r} e(-\langle\underline{\delta},\underline{\lambda}\rangle)\varphi_{\underline{\delta}}$.

And we will consider its dual representation $\rho^*_{L,r}$ on the space $S^*_{L,r}$. We fix a basis of $S^*_{L,r}$, i.e., the basis $\{\varphi^*_{\underline{\lambda}}\}$ dual to $\{\varphi_{\underline{\lambda}}\}$.

Remark. 1. To see that $\rho_{L,r}$ essentially comes from a sub-representation of Weil representation on the Schrödinger model, let us consider the standard additive character ψ of the group of adèles $\mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$

$$\psi(x) = e^{2\pi i(x_\infty - \sum_{p<\infty} x'_p)}$$

where $x = (x_p) \in \mathbb{A}$ and for $p < \infty$, $x_p \mapsto x'_p$ is the image under the natural map by taking "partial fraction":

$$\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Let $V = L_{\mathbb{Q}}$, and let the group of finite adèles be $\mathbb{A}_f = \widehat{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $\widehat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$. We denote by ω_f the Weil representation of the double covering $\widetilde{Sp}(2r, \mathbb{A}_f)$ of $Sp(2r, \mathbb{A}_f)$ on the space $\mathscr{S}(V(\widehat{\mathbb{Q}})^r)$ of Schwartz functions. And we consider the subspace $\mathscr{S}_{L,r}$ generated by characteristic functions of closures $\underline{\gamma} + \widehat{L}^r$ in $V(\widehat{\mathbb{Q}})$ of $\underline{\gamma} + L^r$ for all $\underline{\gamma} \in L^{\vee,r}$ where $\widehat{L}^r = L^r \otimes \widehat{\mathbb{Z}}$. Then clearly $S_{L,r}$ is canonically identified with $\mathscr{S}_L$ by identifying $\varphi_{\underline{\gamma}}$ with the characteristic function of the coset $\underline{\gamma} + \widehat{L}^r$. Then we immediately see the relation between Weil representation ω_f and $\rho_{L,r}$ defined above:

$$\rho_{L,r}(g)\varphi_{\underline{\gamma}} = \omega(g_f)\varphi_{\underline{\gamma}}.$$

where $g \in \widetilde{Sp}(2r,\mathbb{Z}) \subseteq \widetilde{Sp}(2r,\mathbb{R})$ and g_f is the unique element in the inverse image in $\widetilde{Sp}(2r,\mathbb{A}_f)$ of $Sp(2r,\widehat{\mathbb{Z}})$ such that $gg_f \in Sp(2r,\mathbb{Q})$. Here $Sp(2r,\mathbb{Q})$ is canonically identified with a subgroup of the double covering $\widetilde{Sp}(2r,\mathbb{A})$ of $Sp(2r,\mathbb{A})$. See also Appendix II of [K2] and section 4.6 of [KRY].

2. If N is an positive integer such that $N\langle\lambda,\mu\rangle$ and $N\langle\lambda,\lambda\rangle/2$ are integers for all $\lambda,\mu \in L^\vee$, then the representation $\rho_{L,1}$ of $\widetilde{SL}(2,\mathbb{Z})$ on $S_{L,1}$ factors through the finite index subgroup $\Gamma'(N)$, the inverse image of the principle congruence subgroup $\Gamma(N)$ under the map $\widetilde{SL}(2,\mathbb{Z}) \to SL(2,\mathbb{Z})$.

2.1.4

In the rest of this section we give a list of generators of $Sp(2r,\mathbb{Z})$ and $\widetilde{Sp}(2r,\mathbb{Z})$ which will make it easy to check modularity of generating functions. Though the list seems simple, we have not seen it before in the literature.

We consider embeddings of the group $Sp(2i,\mathbb{Z})$, $(1 \le i < r)$ into $Sp(2r,\mathbb{Z})$ in $\frac{r!}{i!(r-i)!}$ ways as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_{r-i} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_{r-i} \end{pmatrix}$$

defines an embedding

$$\omega_{i,(1,2,\ldots,i)} : Sp(2i,\mathbb{Z}) \to Sp(2r,\mathbb{Z}).$$

Similarly, by changing the position of entries in I_{r-i}, we can get other embedding $\omega_{i,(j_1,j_2,\ldots,j_i)}$, $1 \le j_1 < j_2 < \ldots < j_i \le r$. Note that these embeddings correspond to different choices of dimension-$2i$ sub-symplectic subspace in the original dimension-$2r$

symplectic space.

Lemma 2.3. *$Sp(2r,\mathbb{Z})$ is generated by*

$$\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}, \quad A \in GL(r,\mathbb{Z}), \quad B \in Sym(r,\mathbb{Z}) \tag{2.2}$$

and ***any one*** *of the following*

$$\omega_{i,(j_1,j_2,...,j_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right), \quad 1 \leq i \leq r,\ 1 \leq j_1 < j_2 < ... < j_i \leq r.$$

Proof. It is a standard fact (see [A]) that $Sp(2r,\mathbb{Z})$ is generated by 2.2 and $\begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$. So it suffices to prove that 2.2 and any $\omega_{i,(j_1,j_2,...,j_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right)$, $1 \leq i < r$ generate $Sp(2r,\mathbb{Z})$. Note also that for a fixed i and two choices of $(j_1,j_2,...,j_i)$ and $(k_1,k_2,...,k_i)$,

$$\omega_{i,(j_1,j_2,...,j_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right)$$

and

$$\omega_{i,(k_1,k_2,...,k_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right)$$

are conjugate by an element in 2.2. It suffices to prove that for a fixed $i < r$, all elements of the form $\omega_{i,(j_1,j_2,...,j_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right)$ and 2.2 generate $Sp(2r,\mathbb{Z})$. Since the standard fact implies that, for any fixed $i < r$ all elements of the form

$\omega_{i,(j_1,j_2,\ldots,j_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right)$ and 2.2 generate all

$$\omega_{1,(j)}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Therefore, it suffices to prove this for $i = 1$. This follows from the following identity

$$\omega_{1,(1)}\omega_{1,(2)}\ldots\omega_{1,(r)}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}.$$

□

Corollary 2.4. *The double covering group* $\widetilde{Sp}(2r,\mathbb{Z})$ *is generated by* $(g, \sqrt{det(C\tau + D)})$ *for* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r,\mathbb{Z})$ *on the list of the previous lemma 2.3.*

Remark. By the above lemma, to check modularity of a generating function, we only need to check the transformation law under elements in the lemma. One element we will use is the following

$$\omega_{1,\{1\}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & I_{r-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r-1} \end{pmatrix}. \tag{2.3}$$

Write an element τ in $\mathscr{H}_r$ as the form $\tau = \begin{pmatrix} \tau_0 & z^t \\ z & \tau' \end{pmatrix}$ where $\tau' \in \mathscr{H}_{r-1}$, $z \in$

$M_{(r-1)\times 1}(\mathbb{C})$ and $\tau_0 \in \mathscr{H} = \mathscr{H}_1$. After an elementary computation, we have

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & I_{r-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r-1} \end{pmatrix} \cdot \begin{pmatrix} \tau_0 & z^t \\ z & \tau' \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau_0} & \frac{z^t}{\tau_0} \\ \frac{z}{\tau_0} & \tau' - \frac{zz^t}{\tau_0} \end{pmatrix}. \tag{2.4}$$

From now on, abusing notation, we denote by $\omega_{i,(j_1,j_2,\ldots,j_i)}$ the element in $\widetilde{Sp}(2r,\mathbb{Z})$:

$$\left(\omega_{i,(j_1,j_2,\ldots,j_i)}\left(\begin{pmatrix} 0 & -I_i \\ I_i & 0 \end{pmatrix}\right), \sqrt{det(\tau_0)}\right).$$

2.2 A criterion of modularity

In this section we will prove a criterion of modularity for formal power series. The inductive nature of Fourier coefficients of Siegel modular forms are reflected in its Fourier-Jacobi expansion and this is the key ingredient of the proof.

2.2.1

Recall that (L,q) is an even lattice of signature (n,n') with n' even. Let $\Gamma \subseteq O(L,q)$ be a congruence subgroup which acts trivially on $L^\vee/L$. For an integer $r \in \{1,2,...,n\}$, let F be a function on $\Gamma\backslash L^r_{\mathbb{Q}}$. Here $O(L,q)$ acts on $L^r_{\mathbb{Q}}$ diagonally.

For $\underline{\lambda} \in (L^\vee/L)^r$ and $T \in Sym_r(\mathbb{Q})_{\geq 0}$, we denote by $\Omega^{L,r}_{T,\underline{\lambda}}$ the set

$$\Omega^{L,r}_{T,\underline{\lambda}} := \{\underline{x} | Q(\underline{x}) = T, \quad \underline{x} \in \underline{\lambda} + L^r\}.$$

For simplicity, we denote by $[\underline{x}]$ the Γ-coset of $\underline{x} = (x_1,...,x_r) \in L^r_{\mathbb{Q}}$. And for a finite set $A \subseteq \Gamma\backslash L^r_{\mathbb{Q}}$, we denote by $F(A) = \sum_{a\in A} F(a)$. Moreover we use $F(T,\underline{\lambda})$ to simplify $F(\Omega^{L,r}_{T,\underline{\lambda}})$.

For $\underline{x} \in L^r_{\mathbb{Q}}$ with $Q(\underline{x}) \geq 0$. Let $r(\underline{x})$ be the dimension of the space $\mathbb{Q}\underline{x} = \sum_{1\leq r \leq r} \mathbb{Q}x_i \subseteq L_{\mathbb{Q}}$. Let $r(Q(\underline{x}))$ be the rank of the moment matrix $Q(\underline{x})$. Obviously one has $r(\underline{x}) \geq r(Q(\underline{x}))$ due to the possible presence of isotropic vectors.

In the following, we will only consider functions F which satisfy:

Hypothesis 1. $F([\underline{x}])$ depends only on the subspace $\mathbb{Q}\underline{x}$ of $L_{\mathbb{Q}}$ and is zero unless this subspace is positive definite (i.e., $r(\underline{x}) = r(Q(\underline{x}))$).

2.2.2

Let $\rho_{L,r}$ be the Weil representation on $S_{L,r}$ of the double covering group $\widetilde{Sp}(2r, \mathbb{Z})$ defined in previous section and $\rho^*_{L,r}$ its dual representation on $S^*_{L,r}$.

Now consider q-series

$$\Theta_F = \sum_{\underline{\lambda} \in (L^\vee/L)^r} \Theta^L_{F,\underline{\lambda}} \varphi^*_{\underline{\lambda}} \in S^*_{L,r}[[q]] \tag{2.5}$$

where

$$\Theta^L_{F,\underline{\lambda}} = \sum_{T\geq 0} F(T, \underline{\lambda}) q^T \in \mathbb{C}[[q]]. \tag{2.6}$$

As formal power series, we can rewrite this as

$$\Theta^L_{F,\underline{\lambda}} = \sum_{[\underline{x}] \in \underline{\lambda} + L^r/\Gamma} F([\underline{x}]) q^{Q(\underline{x})}.$$

This is allowed since the sum in $F(T, \underline{\lambda})$ is actually finite due to the Hypothesis 1 and the finiteness of Γ-orbits ([KM], in the middle of page 132) in $\Omega^{L,r}_{T,\underline{\lambda}}$ for $T > 0$.

For our convenience, here we re-state Theorem B in the introduction. For $\underline{x} = (x_1, ..., , x_{r-1}) \in L^{\vee, r-1}$, with $r(\underline{x}) = r(Q(\underline{x}))$ (i.e., the space $\mathbb{Q}\underline{x}$ is positive definite), we have an orthogonal projector, denoted by $p^{\underline{x}}$, from $L_{\mathbb{Q}}$ to $\mathbb{Q}\underline{x}$. And let $p_{\underline{x}}$ be the projector $id - p^{\underline{x}}$ from $L_{\mathbb{Q}}$ to $(\mathbb{Q}\underline{x})^\perp$. Let $L_{\underline{x}}$ be the lattice $L \cap (\mathbb{Q}\underline{x})^\perp$, and let $\Gamma_{\underline{x}} \subset \Gamma$ be the stabilizer of $\underline{x}$. Note that $L_{\underline{x}}$ is an even lattice of signature $(n - r(\underline{x}), n')$ for the induced quadratic form. The restriction $F_{\underline{x}}$ of F on $L_{\underline{x},\mathbb{Q}} \cong L_{\underline{x},\mathbb{Q}} \times \{\underline{x}\} \subseteq L_{\mathbb{Q}}^r$ is $\Gamma_{\underline{x}}$-invariant. One can therefore associate to $F_{\underline{x}}$ a vector-valued formal q-series $\Theta_{F_{\underline{x}}} \in S^*_{L_{\underline{x}},1}[[q]]$.

Theorem 2.5 (Criterion of Modularity). *Let L be an even lattice of signature (n, n') with $2|n'$ and $r \in \{1, 2, ..., n\}$. For a function F on $\Gamma\backslash L_{\mathbb{Q}}^r$ satisfying Hypothesis 1, the q-series Θ_F is the q-expansion of a Siegel modular form of type $\rho^*_{L,r}$, weight $\frac{n+n'}{2}$ and genus r if F satisfies the following two conditions:*

1. *Θ_F is absolutely convergent on $\mathscr{H}_r$.*
2. *For every $\underline{x} \in L^{\vee,r-1}$ with positive definite $\mathbb{Q}\underline{x}$ of dimension $r(\underline{x})$, the q-series $\Theta_{F_{\underline{x}}}$ is absolutely convergent on $\mathscr{H} = \mathscr{H}_1$ and defines a Siegel modular form of type $\rho^*_{L_{\underline{x}},1}$, weight $\frac{n+n'-r(\underline{x})}{2}$ and genus 1.*

Remark. Theorem 2.5 will still be true if we fix any $r' \leq r-1$ and replace "$\underline{x} \in L^{\vee,r-1}$...on $\mathscr{H} = \mathscr{H}_1$ and defines a Siegel modular form of type $\rho^*_{L_{\underline{x}},1}$, weight $\frac{n+n'-r(\underline{x})}{2}$ and genus 1" by "$\underline{x} \in L^{\vee,r'}$,...,on $\mathscr{H}_{r-r'}$ and defines a Siegel modular form of type $\rho^*_{L_{\underline{x}},r-r'}$, weight $\frac{n+n'-r(\underline{x})}{2}$ and genus $r - r'$". And the proof goes through with trivial modification. Our particular choice comes from the application in the subsequent sections.

2.2.3 Proof of Theorem 2.5

2.2.3.1

By the condition 1, the formal power series Θ_F defines a holomorphic map from $\mathscr{H}_r$ to $S^*_{L,r}$. Thus, it makes sense to talk of $\Theta_F(-\tau^{-1})$ for $\tau \in \mathscr{H}_r$. We then need only to check that Θ_F satisfies the transformation law under elements in Lemma 2.3. The Fourier expansion gives the transformation law under elements in the unipotent group since for $\underline{v} \in \Omega^{L,r}_{T,\underline{\lambda}}$, $B \in Sym_r(\mathbb{Z})$, we have

$$tr(Q(\underline{v})B) \equiv tr(Q(\underline{\lambda})B) \pmod{\mathbb{Z}}.$$

For the transformation law under elements in the Levi subgroup, by Equation 2.6 and Definition of $\rho_{L,r}$ (2.2), we need to check that, for any $A \in GL(r,\mathbb{Z})$,

$$F(T,\underline{\lambda}) = \sqrt{det(A)}^{n'+n}\sqrt{det(A)}^{n'-n}F({}^tATA,\underline{\lambda}A).$$

This follows from $2|n'$ and the following two equalities for $\underline{v} \in L^{\vee,r}$:

$${}^tAQ(\underline{v})A = Q(\underline{v}A),$$

and

$$F([\underline{v}]) = F([\underline{v}A]).$$

The first equality is obvious and the second one follows from the fact that the subspace $\mathbb{Q}\underline{v}$ of $L_{\mathbb{Q}}$ is the same as $\mathbb{Q}(\underline{v}A)$ and the hypothesis that the image $F([\underline{v}])$ depends only on the the subspace $\mathbb{Q}\underline{v}$.

Therefore, by Lemma 2.3 it suffices to check the transformation law under the element $\omega_{1,\{1\}}$.

2.2.3.2

Now we consider the Fourier-Jacobi type expansion of $\Theta^L_{F,(\lambda,\underline{\mu})}$ for $\lambda \in L^\vee/L$ and $\underline{\mu} \in (L^\vee/L)^{r-1}$ defined as follows.

Write $\tau = \begin{pmatrix} \tau_0 & {}^tz \\ z & \tau' \end{pmatrix}$ as in (2.4) and similarly $T = \begin{pmatrix} m & {}^tp/2 \\ p/2 & t \end{pmatrix}$. For $t \in$

$Sym_{r-1}(\mathbb{Q})_{\geq 0}$, we define

$$\theta_{(\lambda,\underline{\mu}),t}(\tau_0, z) = \sum_{m,n} F(\begin{pmatrix} m & {}^tp/2 \\ p/2 & t \end{pmatrix}, (\lambda, \underline{\mu}))e(m\tau_0)e({}^tpz). \tag{2.7}$$

Then, we have a expansion of Fourier-Jacobi type

$$\Theta^L_{F,(\lambda,\underline{\mu})} = \sum_{t \in Sym_{r-1}(\mathbb{Q})_{\geq 0}} \theta_{(\lambda,\underline{\mu}),t}(\tau_0, z)e(tr(t\tau')). \tag{2.8}$$

Therefore, for $\Theta^L_{F,(\lambda,\underline{\mu})}$ to have the right transformation law under the action of the element $\omega_{1,\{1\}}$ (2.3), it suffices to prove the corresponding transformation law for $\theta_{(\lambda,\underline{\mu}),t}$ under $(\tau_0, z) \to (-\frac{1}{\tau_0}, \frac{z}{\tau_0})$ for all $t \in Sym_{r-1}(\mathbb{Q})_{\geq 0}$. Note that $\theta_{(\lambda,\underline{\mu}),t}$ is absolutely convergent on $\mathscr{H} \times \mathbb{C}^{r-1}$, hence it makes sense to talk of $\theta_{(\lambda,\underline{\mu}),t}(-\frac{1}{\tau_0}, \frac{z}{\tau_0})$. After some easy computation, we see that the right transformation law for $\iota(\theta_{\underline{\lambda},x})$ is the following:

$$\theta_{(\lambda,\underline{\mu}),t}\left(-\frac{1}{\tau_0}, \frac{z}{\tau_0}\right) = \sqrt{\tau_0}^{n+2} \frac{e(-\frac{n-n'}{8})}{|L^\vee/L|^{1/2}} e\left(\frac{{}^tztz}{2\tau_0}\right) \sum_{\gamma \in L^\vee/L} e(-\langle\lambda,\gamma\rangle)\theta_{(\lambda,\underline{\mu}),t}(\tau_0, z). \tag{2.9}$$

2.2.3.3

For $\underline{x} \in L^{\vee,r-1}$ satisfying $r(\underline{x}) = r(Q(\underline{x}))$, we define

$$\theta_{\lambda,\underline{x}}(\tau_0, z) = \sum_{m,n} \left(\sum_{v \in \lambda + L/\Gamma_{\underline{x}}, Q(v)=m, \langle v,\underline{x}\rangle = {}^tp} F([v, \underline{x}]) \right) e(m\tau_0)e({}^tpz).$$

By Hypothesis 1 the other $[v, \underline{x}] \in L^{\vee,r}$ will be mapped to zero under F. Therefore we have an equality between formal power series

$$\theta_{(\lambda,\underline{\mu}),t}(\tau_0, z) = \sum_{\underline{x} \in \underline{\mu} + L^{r-1}/\Gamma, Q(\underline{x})=t} \theta_{\lambda,\underline{x}}(\tau_0, z) \tag{2.10}$$

This is allowed since the sum above is finite.

Note that for a functional F such that Θ_F is absolutely convergent, it is not necessarily true that $\theta_{\lambda,\underline{x}}(\tau_0, z)$ is also convergent. However, under the condition 2 that $\Theta_{F_{\underline{x}}}$ is absolutely convergent, we will show that $\theta_{\lambda,\underline{x}}(\tau_0, z)$ is also absolutely convergent.

Proposition 2.6. *For $\lambda \in L^\vee/L$ and any $\underline{x} \in L^{r-1,\vee}$ with $Q(\underline{x}) = t \geq 0$ and $r(\underline{x}) = r(Q(\underline{x}))$, under the conditions 2 of Theorem 2.5, $\theta_{\lambda,\underline{x}}(\tau_0, z)$ is absolutely convergent on $\mathscr{H} \times \mathbb{C}^{r-1}$. Moreover, it satisfies*

$$\begin{aligned}&\theta_{\lambda,\underline{x}}\left(-\frac{1}{\tau_0}, \frac{z}{\tau_0}\right)\\ &= \sqrt{\tau_0}^{n+2}\frac{e(-\frac{n-n'}{8})}{|L^\vee/L|^{1/2}} e\left(\frac{{}^t z t z}{2\tau_0}\right)\sum_{\gamma\in L^\vee/L} e(-\langle\lambda,\gamma\rangle)\theta_{\gamma,\underline{x}}(\tau_0, z).\end{aligned}$$

Remark. For this proposition, we do not need the condition 1 of Theorem 2.5.

Proof of Proposition 2.6. We will express $\theta_{\lambda,\underline{x}}$ as a sum of products of components of $\Theta_{F_{\underline{x}}}$ and certain standard Jacobi forms associated to positive definite lattices. Afterwards, the result will follow from the condition 2 and known facts about those Jacobi forms.

More precisely, let $L^{\underline{x}} = L \cap p^{\underline{x}}(L)$, $L_1 = L_{\underline{x}} \oplus L^{\underline{x}}$. And when we have $i \in (L_{\underline{x}})^\vee, j \in L^{\underline{x},\vee}$, we will denote by $(i, j) \in L_1^\vee$ the corresponding element. Obviously, we have natural embeddings

$$L_1 \subseteq L \subseteq L^\vee \subseteq L_1^\vee. \tag{2.11}$$

Lemma 2.7. *The map $p^{\underline{x}}$ induces an isomorphism of (finite) abelian groups*

$$L/L_1 \simeq p^{\underline{x}}(L)/L^{\underline{x}}.$$

Proof. Clearly $p^{\underline{x}}$ induces a surjective map

$$p : L \to p^{\underline{x}}(L)/L^{\underline{x}}.$$

We need to prove the kernel of p is L_1. Obviously $p(L_1) = 0$. If $p(\alpha) = 0$, i.e., $\beta = p^{\underline{x}}(\alpha) \in L \cap p^{\underline{x}}(L)$. Then $p_{\underline{x}}(\alpha) = \alpha - \beta \in L$. Therefore, $p_{\underline{x}}(\alpha) \in L \cap p_{\underline{x}}(L)$. □

In particular, we have a decomposition

$$L = \coprod_{\alpha' \in p^{\underline{x}}(L)/L^{\underline{x}}} \alpha + L_{\underline{x}} \oplus L^{\underline{x}}$$

where we fix a set of liftings $\alpha \in L$ of $\alpha' \in p^{\underline{x}}(L)/L^{\underline{x}}$.

Note that $F([y, \underline{x}]) = F_{\underline{x}}([y])$ for $y \in L_{\underline{x},\mathbb{Q}} = (\mathbb{Q}\underline{x})^{\perp}$, we have

$$\Theta^{L_{\underline{x}}}_{F_{\underline{x}}, p_{\underline{x}}(\lambda+\alpha)}(\tau_0) = \sum_{m \geq 0} \left(\sum_{y \in \Gamma_{\underline{x}} \backslash p_{\underline{x}}(\lambda+\alpha) + L_{\underline{x}}, Q(y) = m} F([y, \underline{x}]) \right) e(m\tau_0).$$

Then, we have

$$\theta_{\lambda,\underline{x}}(\tau_0, z) = \sum_{\alpha \in L/L_1} \Theta^{L_{\underline{x}}}_{F_{\underline{x}}, p_{\underline{x}}(\lambda+\alpha)}(\tau_0) \theta_{p^{\underline{x}}(\lambda+\alpha)+L^{\underline{x}}, \underline{x}}(\tau_0, z) \tag{2.12}$$

where $\theta_{p^{\underline{x}}(\lambda+\alpha)+L^{\underline{x}}, x}(\tau_0, z)$ is the theta function defined as in the following lem attached to the positive definite lattice $L^{\underline{x}}$.

Lemma 2.8. *Let M be an even positive definite lattice of rank r_1, $M^{\vee}$ be its dual. For any $\lambda \in M^{\vee}/M$ and a fixed $\underline{x} \in M^{\vee, r_2}$ with $Q(\underline{x}) = t \geq 0$, let*

$$\theta_{\lambda+M,\underline{x}}(\tau_0, z) := \sum_{u \in \lambda+M} e(\frac{1}{2}\langle u, u\rangle \tau_0) e(\sum_{i=1}^{r_2} z_i \langle u, x_i \rangle).$$

Then, $\theta_{\lambda+M,\underline{x}}(\tau_0, z)$ absolutely convergent on $\mathscr{H} \times \mathbb{C}^{r_2}$ and it satisfies the transforma-

tion law

$$\theta_{\lambda+M,\underline{x}}(-\frac{1}{\tau_0},\frac{z}{\tau_0}) = \sqrt{\tau_0}^{r_1}\frac{e(-\frac{r_1}{8})}{|M^\vee/M|^{\frac{1}{2}}}e\left(\frac{{}^tztz}{2\tau_0}\right)\sum_{j\in M^\vee/M} e(-\langle\lambda,j\rangle)\theta_{j+M,\underline{x}}(\tau_0,z)).$$

Proof. The proof is standard by applying the Poisson summation formula. □

Remark. From the lemma above we can recognize the transformation law of a Jacobi form.

2.2.3.4

For a proof of Proposition 2.6, it is now easy to see the absolute convergence of $\theta_{\lambda,\underline{x}}$. We then proceed to prove the transformation law of $\theta_{\lambda,\underline{x}}(\tau_0,z)$ under $(\tau_0,z)\to(-\frac{1}{\tau_0},\frac{z}{\tau_0})$.

We have

$$\theta_{\lambda,\underline{x}}(-\frac{1}{\tau_0},\frac{z}{\tau_0}) = \sum_{\alpha\in L/L_1}\Theta^{L_{\underline{x}}}_{F_{\underline{x}},p_{\underline{x}}(\lambda+\alpha)}(-\frac{1}{\tau_0})\theta_{p^{\underline{x}}(\lambda+\alpha)+L^{\underline{x}},\underline{x}}(-\frac{1}{\tau_0},\frac{z}{\tau_0}) \tag{2.13}$$

Since $\Theta_{F_{\underline{x}}}$ is a Siegel form of type $\rho^*_{L_{\underline{x}},1}$ and by Lemma 2.8 for $r_1 = r(\underline{x})$ and $r_2 = r$, we obtain

$$\sum_{\alpha\in L/L_1}\sqrt{\tau_0}^{n-r+2}\frac{e(-\frac{n-r(\underline{x})-n'}{8})}{|\frac{(L_{\underline{x}})^\vee}{L_{\underline{x}}}|^{\frac{1}{2}}}\sum_{i\in(L_{\underline{x}})^\vee/L_x} e\left(-\langle\lambda+\alpha,i\rangle\right)\Theta^{L_{\underline{x}}}_{F_{\underline{x}},i}(\tau_0)$$
$$\times\sqrt{\tau_0}^{r}\frac{e(-\frac{r(\underline{x})}{8})}{|\frac{L^{\underline{x},\vee}}{L^{\underline{x}}}|^{\frac{1}{2}}}e\left(\frac{{}^tztz}{2\tau_0}\right)\sum_{j\in L^{\underline{x},\vee}/L^{\underline{x}}} e(-\langle\lambda+\alpha,j\rangle)\theta_{j+L^{\underline{x}},\underline{x}}(\tau_0,z).$$

Interchange the order of summation,

$$\sqrt{\tau_0}^{n+2}\frac{e(-\frac{n-n'}{8})}{|\frac{(L_{\underline{x}})^\vee}{L_{\underline{x}}}|^{\frac{1}{2}}|\frac{L^{\underline{x},\vee}}{L^{\underline{x}}}|^{\frac{1}{2}}}e\left(\frac{{}^tztz}{2\tau_0}\right)\sum_{i\in(L_{\underline{x}})^\vee/L_{\underline{x}},j\in L^{\underline{x},\vee}/L^{\underline{x}}}$$

$$\left(\sum_{\alpha\in L/L_1}e(-\langle\alpha,(i,j)\rangle)\times e(-\langle\lambda,(i,j)\rangle)\Theta^{L_{\underline{x}}}_{F_{\underline{x}},i}(\tau_0)\theta_{j+L^{\underline{x}},\underline{x}}(\tau_0,z)\right).$$

Note that for any $\underline{\delta}\in(L_1^\vee/L_1)^r$, we have

$$\sum_{\alpha\in L/L_1}e(\langle\alpha,\delta\rangle)=\begin{cases}|L/L_1| & \text{if } \delta\in L^\vee/L_1,\\ 0 & \text{otherwise.}\end{cases}$$

By the following equality between discriminants of various lattices,

$$\left|\frac{(L_{\underline{x}})^\vee}{L_{\underline{x}}}\right|\left|\frac{L^{\underline{x},\vee}}{L^{\underline{x}}}\right|=\left|\frac{L_1^\vee}{L_1}\right|=\left|\frac{L}{L_1}\right|^2\left|\frac{L^\vee}{L}\right|,$$

we get

$$\sqrt{\tau_0}^{n+2}\frac{e(-\frac{n-n'}{8})}{|L^\vee/L|^{\frac{r-1}{2}}}e\left(\frac{{}^tztz}{2\tau_0}\right)$$
$$\times\sum_{\delta\in L^\vee/L_1}e(-\langle\lambda,\delta\rangle)\Theta^{L_{\underline{x}}}_{F_{\underline{x}},p_{\underline{x}}(\delta)}(\tau_0)\theta_{p^{\underline{x}}(\delta)+L^{\underline{x}},\underline{x}}(\tau_0,z).$$

Note that $e(-\langle\lambda,\delta\rangle)$ depends only on the coset $\gamma=\delta+L\in L^\vee/L$. By the equation (2.12) we can group terms in the above equation and end up with a sum of $\theta_{\gamma,x}(\tau,z)$ where γ run over all elements in $L^\vee/L$

$$\sqrt{\tau_0}^{n+2}\frac{e(-\frac{n-n'}{8})}{|L^\vee/L|^{1/2}}e\left(\frac{{}^tztz}{2\tau_0}\right)\sum_{\gamma\in L^\vee/L}e(-\langle\lambda,\gamma\rangle)\theta_{\gamma,\underline{x}}(\tau_0,z).$$

Thus we have proven the desired formula

$$\theta_{\lambda,\underline{x}}\left(-\frac{1}{\tau_0},\frac{z}{\tau_0}\right)$$
$$=\sqrt{\tau_0}^{n+2}\frac{e(-\frac{n-n'}{8})}{|L^\vee/L|^{1/2}}e\left(\frac{{}^tztz}{2\tau_0}\right)\sum_{\gamma\in L^\vee/L}e(-\langle\lambda,\gamma\rangle)\theta_{\gamma,\underline{x}}(\tau_0,z).$$

And this completes the proof of Proposition 2.6. □

2.2.3.5

By the equation 2.10, the proposition above immediately implies the transformation law 2.9. Putting all together, by Lemma 2.3 we have proved Theorem 2.5.

2.3 Special cycles and their generating functions

In this section we briefly recall Kudla's definitions of special cycles on Shimura varieties of orthogonal type and their generating functions. We refer to [K1] and [K2] for more details.

Let (V,q) be a quadratic space of signature $(n,2)$ defined over $\mathbb{Q}$ with induced inner product $\langle\cdot,\cdot\rangle$. Let G be the similitude spin group $GSpin(V)$. And let D be the associated Hermitian symmetric domain, i.e. the Grassmannian of oriented negative definite 2-planes. This can be identified with the open subset of a quadric in $\mathbb{P}(V\otimes\mathbb{C})$

$$D=\{v\in V\otimes\mathbb{C}|\langle v,v\rangle=0,\ \langle v,\overline{v}\rangle<0\}/\mathbb{C}^*,\quad dim_{\mathbb{C}}D=n.$$

The above data defines a Shimura variety $Sh(G,D)=\varprojlim_K X_K$ with a canonical model over $\mathbb{Q}$, where, for $K\subseteq G(\mathbb{A}_f)$ a compact open subgroup,

$$X_K(\mathbb{C})=G(\mathbb{Q})\setminus D\times G(\mathbb{A}_f)/K.$$

If we write

$$G(\mathbb{A}) = \coprod_j G(\mathbb{Q})G(\mathbb{R})g_j K$$

then we can write $X_K(\mathbb{C})$ as disjoint union of connected components

$$X_K(\mathbb{C}) = \coprod_j \Gamma_j \backslash D, \quad \Gamma_j = G(\mathbb{Q}) \cap g_j K g_j^{-1}.$$

We can pick a distinguished component $X_\Gamma = X_{\Gamma,K}$ corresponding to $g_j = 1$,

$$X_\Gamma(\mathbb{C}) = \Gamma \backslash D, \quad \Gamma = G(\mathbb{Q}) \cap K$$

where we omit K without confusion in the context. X_Γ is defined over a cyclotomic extension E_K of $\mathbb{Q}$ depending on K. The Galois group $Gal(E_K/\mathbb{Q})$ acts simply transitively on the group of connected components ([K1], (1.10))

$$X_{\Gamma_j} \simeq X_\Gamma^{\sigma_j}$$

where $\sigma_j \in Gal(E_K/\mathbb{Q})$ is associated to $g_j \in G(\widehat{\mathbb{Q}})$ under the reciprocity map, so that ([K1], (1.11))

$$X_K = \coprod_j X_{\Gamma_j} = \coprod_{\sigma \in Gal(E_K/\mathbb{Q})} X_\Gamma^\sigma . \tag{2.14}$$

Let $\mathscr{L}_D$ be the tautological bundle of lines corresponding to points on D, or equivalently, the restriction to D of the line bundle $\mathscr{O}(-1)$ on $\mathbb{P}(V \otimes \mathbb{C})$. The group $G(\mathbb{Q})$ acts equivariantly on $\mathscr{L}_D$ so that $\mathscr{L}_D$ descends to a line bundle $\mathscr{L}_K$ on X_K. We will use $\mathscr{L}$ to denote the line bundle without confusion. It also has a canonical model over $\mathbb{Q}$ ([K1] and references therein).

For a positive definite subspace W of V, and $g \in G(\mathbb{A}_f)$, we define a cycle

$Z(W, g; K)$, at level K,

$$Z(W, g; K) := G_W(\mathbb{Q}) \setminus D_W \times G_W(\mathbb{A}_f)/K_W^g \to G(\mathbb{Q}) \setminus D \times G(\mathbb{A}_f)/K$$

$$G_W(\mathbb{Q})(z, h)K_W^g \mapsto G(\mathbb{Q})(z, hg)K$$

where G_W the spin group associated to the quadratic space $W^{\perp}$, i.e. the stabilizer of W and

$$D_W = \{v \in D | v \perp W\}, \quad K_W^g = G_W(\mathbb{A}_f) \cap gKg^{-1}.$$

This cycle is again rational over $\mathbb{Q}$. For an integer r, $1 \leq r \leq n$, and an r-tuple $\underline{x} = (x_1, ..., x_r) \in V^r$, we define $Z(\underline{x}, g; K)$ as $Z(W, g; K)$ if the space $W = \sum_i \mathbb{Q}x_i$ generated by $x_i, i = 1, ..., r$ is positively definite, 0 otherwise. Note that the tautological bundle $\mathscr{L}_{W,g}$ on $Z(W, g; K)$ is naturally isomorphic to the restriction of $\mathscr{L}$.

For a K-invariant Schwartz function $\varphi \in \mathscr{S}(V(\widehat{\mathbb{Q}})^r)$, and an $r \times r$ symmetric positive semi-definite matrix $T \in Sym_r(\mathbb{Q})_{\geq 0}$ of rank $r(T)$, let

$$\Omega_T = \{\underline{x} \in V^r | Q(\underline{x}) = T, \, r(T) = r(\underline{x})\}.$$

Decompose

$$\Omega_T(\mathbb{A}_f) \cap supp(\varphi) = \coprod_j K g_i^{-1} x$$

for $x \in \Omega_T(\mathbb{Q})$ and finitely many $g_j \in G(\mathbb{A}_f)$. Then we define a cycle of codimension $r(T)$

$$Z(T, \varphi; K) = \sum_j \varphi(g_j^{-1}x) Z(x, g_j; K).$$

The cycle $Z(T, \varphi; K)$ is compatible with the pull back map $X_{K'} \to X_K$ for $K' \subseteq K$ ([K1], proposition (5.10)). Therefore, we can drop K from the notation and write them as $Z(T, \varphi)$. The compatibility also holds for the tautological line bundle $\mathscr{L}$.

One can also consider special cycles on the connected Shimura variety X_Γ (2.14).

If we denote the restriction of $Z(T,\varphi;K)$ on X_Γ by $Z(T,\varphi;\Gamma)$, we have the following decomposition

$$Z(T,\varphi;K) = \sum_{\sigma\in Gal(E_K/\mathbb{Q})} Z(T,\varphi;\Gamma)^\sigma. \tag{2.15}$$

We call all cycles $Z(T,\varphi;K)$ defined above *special cycles*. They are all defined over $\mathbb{Q}$ and their components lying on the connected Shimura variety are defined over E_K.

The generating function with coefficients in $CH^r(X_K)_\mathbb{C}$ is defined to be

$$\Theta_\varphi(\tau) := \sum_{T\in Sym_r(\mathbb{Q})_{\geq 0}} \{Z(T,\varphi)\}\cdot\{\mathscr{L}^\vee\}^{r-r(T)} q^T \tag{2.16}$$

where $q^T = e^{2i\pi tr(T\tau)}$, and

$$\tau \in \mathscr{H}_r = \{\tau \in Sym_r(\mathbb{C}) | Im(\tau) > 0\}$$

is the Siegel upper half plane of genus r. And the intersection product is taken according to the intersection pairing between Chow groups. We use $\{Z(T,\varphi)\}$ to denote its class in the Chow group.

For a linear functional ι on $CH^r(X_K)_\mathbb{C}$, let $\Theta_{\varphi,\iota}$ be the complex valued generating function

$$\Theta_{\varphi,\iota} = \sum_{T\in Sym_r(\mathbb{Q})_{\geq 0}} \iota(\{Z(T,\varphi)\}\cdot\{\mathscr{L}^\vee\}^{r-r(T)}) q^T.$$

2.4 Modularity of generating functions of special cycles

In this section, as an application of the criterion of modularity proved in section 2.2, we will prove the modularity of generating functions of special cycles. In the first half of the section we will work in the classical setting as in [B1], while in the second half we will re-formulate the result in adelic language as in [K1]. The reason for this treatment is that we want to work with connected Shimura varieties whose geometry is more convenient to describe. Though they are generally defined over a cyclotomic extension of $\mathbb{Q}$, connected components are linked together via Galois action, and special cycles on different components are Galois conjugate to each other.

2.4.1

Suppose we are in the situation of the introduction. Let (L, q) be a nondegenerate even lattice of signature $(n, 2)$. For a neat Γ, let X_Γ be the connected Shimura variety. X_Γ is a smooth quasi-projective variety with a canonical model generally defined over a cyclotomic extension of $\mathbb{Q}$ depending on Γ. Let $Z(T, \underline{\lambda})$ be the special cycle of moment $T \in Sym_r(\mathbb{Q})_{\geq 0}$ and residue class $\lambda \in (L^\vee/L)^r$ defined in Introduction 1. All such cycles are defined over cyclotomic extension of $\mathbb{Q}$. Note that the cycle $Z(T, \underline{\lambda})$ is of codimension $r(T)$.

Consider $CH^r_L = CH^r(X_\Gamma)_\mathbb{C}$ where we write down L to emphasize the dependence on L. For a linear functional ι on CH^r_L, we associate a function F_ι on $\Gamma\backslash L^r_\mathbb{Q}$ as follows. For $\underline{x} = (x_1, ..., x_r) \in L^{\vee,r}$ with $T = Q(\underline{x}) \geq 0$, F_ι maps the Γ-coset $[\underline{x}]$ to $\iota(\{\mathscr{L}\}^{\vee, r-r(T)} \cdot \{Z(\underline{x})\}) \in \mathbb{C}$ if the space $\mathbb{Q}\underline{x}$ generated by $x_1, ..., x_r$ is positive definite, and maps the other $[\underline{x}]$ to zero. The function F_ι obviously satisfies Hypothesis 1.

Theorem 2.9. *(Modularity Conjecture of Kudla, [K2]) For any $n \geq 1, r \in \{1, 2, ..., n\}$, and any $\iota \in Hom(CH^r_L, \mathbb{C})$, if Θ_{F_ι} is absolutely convergent, then it is a Siegel modular form of type $\rho^*_{L,r}$, weight $1 + \frac{n}{2}$ and genus r.*

Proof. To apply Theorem (2.5), we need to verify the condition 2 that for $\underline{x} \in L^{\vee, r-1}$ with $Q(x) \geq 0$ and $r(\underline{x}) = r(Q(\underline{x}))$, the generating function $\Theta_{F_{\iota,\underline{x}}} \in S^*_{L_{\underline{x}},1}[[q]]$ is absolutely convergent and defines a modular form of type $\rho^*_{L,1}$ and weight $\frac{n+2-r(\underline{x})}{2}$.

Note that we have a morphism between Shimura varieties $f_{\underline{x}} : X_{\Gamma_{\underline{x}}} \to X_\Gamma$. The morphism $f_{\underline{x}}$ is a closed immersion (Theorem 5.16 in [Mi]), hence proper. This induces a push-forward map

$$(f_{\underline{x}})_* : CH^{*-r(\underline{x})}_{L\underline{x}} \to CH^*_L.$$

Shifting by a power of $\mathscr{L}^\vee$, we define

$$i_{\underline{x}} = (f_x)_* \cdot \{\mathscr{L}^\vee\}^{r-1-r(\underline{x})} : CH^1_{L_{\underline{x}}} \to CH^r_L.$$

Then $\iota \circ i_{\underline{x}} \in Hom(CH^1_{L\underline{x}}, \mathbb{C})$ and we can associate a generating function $\Theta_{F_{\iota \circ i_{\underline{x}}}} \in S^*_{L_{\underline{x}},1}[[q]]$. It is a theorem of Borcherds ([B1]) that $\Theta_{F_{\iota\circ i_{\underline{x}}}}$ is absolutely convergent and defines a modular form of type $\rho^*_{L,1}$ and weight $\frac{n+2-r(\underline{x})}{2}$. For a sketch of the proof, we refer to [K2], Theorem 3.2.

Thus, to verify the condition 2, it suffices to prove that $\Theta_{F_{\iota,\underline{x}}} = \Theta_{F_{\iota\circ i_{\underline{x}}}} \in S^*_{L_{\underline{x}},1}[[q]]$. To simplify notations, let $y' = p_{\underline{x}}(y) \in L_{\underline{x},\mathbb{Q}}$. Let $Z_{\underline{x}}(y')$ be the special cycle on $X_{\Gamma_{\underline{x}}}$ and let $\mathscr{L}_{\underline{x}}$ be the tautological line bundle on $X_{\Gamma_{\underline{x}}}$. Abusing notation, $Z(y, \underline{x}), Z_{\underline{x}}(y')$ etc. will also mean their cycles classes in the Chow group. Then, we only need to prove that for any $y \in L_{\mathbb{Q}}$ such that $\mathbb{Q}y + \mathbb{Q}\underline{x}$ is positive definite, we have an equality

$$Z(y, \underline{x}) \cdot \mathscr{L}^{\vee, r-r(y,\underline{x})} = i_{\underline{x}}(Z_{\underline{x}}(y') \cdot \mathscr{L}_{\underline{x}}^{\vee,1-r(y')}) \in CH^r_L. \tag{2.17}$$

If y is in $\mathbb{Q}\underline{x}$, then $y' = y$ and we have $Z(y, \underline{x}) = Z(\underline{x}) \in CH^{r(\underline{x})}_L$, and $Z_{\underline{x}}(y') \cdot \mathscr{L}_{\underline{x}}^{\vee,1-r(y')} = Z_{\underline{x}}(0) \cdot \mathscr{L}^\vee_{\underline{x}} = \mathscr{L}^\vee_{\underline{x}} \in CH^1_{L_{\underline{x}}}$. But the tautological line bundle $\mathscr{L}_{\underline{x}}$ on $X_{\Gamma_{\underline{x}}}$ is isomorphic to the restriction of $\mathscr{L}$. Equation 2.17 now follows easily. If y is not in the subspace $\mathbb{Q}\underline{x}$, the proof of Equation 2.17 is similar and even more straightforward.

Therefore, by Theorem 2.5, we complete the proof of Theorem 2.9. □

Remark. The convergence assumption for the linear functional ι in the case $r \geq 2$ is necessary and it seems that we can not deduce it from Borcherds' theorem for $r = 1$. In fact, we can only obtain the convergence of each individual Fourier-Jacobi coefficient. Heuristically, we may look at the following question. Consider a generating function $\Theta = \sum_{T \in Sym_2(\mathbb{Z})_{\geq 0}} a_T q^T$, $a_T \in \mathbb{C}$, with the property that

1. The T-th coefficient a_T is invariant when we switch $T = \begin{pmatrix} m & n/2 \\ n/2 & t \end{pmatrix}$ to $T' = \begin{pmatrix} t & n/2 \\ n/2 & m \end{pmatrix}$.
2. The m-th "Fourier-Jacobi" coefficient θ_m is absolutely convergent and defines a Jacobi form of index m ([EZ]).

But only these information does not imply the convergence of the series Θ since the property 1) is not sufficient to yield a uniform control for the convergence of all θ_m.

Nevertheless, we would like to propose the following conjecture whose proof may need some new ideas.

Conjecture 1. *For any linear functional ι on $CH^r(X_\Gamma)_\mathbb{C}$, the generating function Θ_{F_ι} is absolutely convergent on $\mathscr{H}_r$.*

Let $SC^r(L)$ be the subspace of $CH^r(X_\Gamma)_\mathbb{C}$ generated by codimension-r cycle classes $\{Z(T, \underline{\lambda})\} \cdot \{\mathscr{L}^\vee\}^{r-r(T)}$ for all $\underline{\lambda} \in (L^\vee/L)^r$ and $T \in Sym_r(\mathbb{Q})_{\geq 0}$. Let $SC_0^r(L)$ be the maximal quotient of $SC^r(L)$ on which any linear functional is of convergent type (i.e., such that Θ_{F_ι} is absolutely convergent). Then the dual space $SC_0^r(L)^\vee = Hom(SC_0^r(L), \mathbb{C})$ is canonically identified with the space of all linear functionals of convergent type on $SC^r(L)$. The conjecture above amounts to say $SC^r(L) = SC_0^r(L)$.

Corollary 2.10. *For $r = 1, 2, ..., n$, the space $SC_0^r(L)$ is finite dimensional.*

Proof. The assignment $\iota \mapsto \Theta_{F_\iota}$ defines a linear map from $Hom(SC_0^r(L), \mathbb{C})$ to the complex vector space $A(1+\frac{n}{2}, \rho_{L,r}^*)$ of Siegel modular form of weight $1+\frac{n}{2}$, type $\rho_{L,r}^*$. The map is injective since the value of ι on the generator $\{Z(T, \underline{\lambda})\} \cdot \{\mathscr{L}^\vee\}^{r-r(T)}$ of $SC_0^r(L)$ can be recovered as the Fourier coefficient $a_{T,\underline{\lambda}}$ of the modular form Θ_{F_ι}. Since the space $A(1+\frac{n}{2}, \rho_{L,r}^*)$ is finite dimensional, so are $Hom(SC_0^r(L), \mathbb{C})$ and $SC_0^r(L)$. □

2.4.2

We now re-formulate our results in adelic language. We proceed from the previous section where we have defined special cycles etc. on Shimura varieties $Sh(G,D) = \varprojlim_K X_K$ associated to a rational quadratic space V. Recall that ω_f is the Weil representation of $\widetilde{Sp}_{2r}(\widehat{\mathbb{Q}})$ on $\mathscr{S}(V(\widehat{\mathbb{Q}})^r)$ (see Remark 1 after the definition of $\rho_{L,r}$).

Theorem 2.11. *Let $n \geq 1, r \in \{1, 2, ..., n\}$, and $\varphi \in \mathscr{S}(V(\widehat{\mathbb{Q}})^r)^K$ for a compact subgroup $K \in G(\mathbb{A}_f)$. Let Θ_φ be the generating function (2.16). Let $\iota \in Hom(CH^r(X_K)_{\mathbb{C}})$ Then, assuming Conjecture 1 for all lattices L, we have*

$$\Theta_{\varphi,\iota}(\gamma\tau) = j(\gamma,\tau)^{n+2}\Theta_{\omega_f(\gamma_f^{-1})\varphi,\iota}(\tau), \quad \gamma \in \widetilde{Sp}(2r,\mathbb{Z}).$$

for any linear functional ι. Here γ_f is the unique element in $\widetilde{Sp}(2r,\widehat{\mathbb{Q}})$ such that $\gamma\gamma_f \in Sp(2r,\mathbb{Q})$.

Proof. Any function φ in $\mathscr{S}(V(\widehat{\mathbb{Q}})^r)$ is a linear combination of characteristic functions of $\underline{\lambda} + \widehat{L}^r$ for $\underline{\lambda} \in V(\mathbb{Q})$ and a certain $\mathbb{Z}$-lattice $L \subset V$ (we require that all lattices have full rank).

When $\underline{\lambda} \in V^r$, we can assume $\underline{\lambda} \in L^\vee$ by replacing a smaller lattice. Then, if φ is the characteristic function of $\underline{\lambda} + \widehat{L}^r$, $\underline{\lambda} \in L^\vee$, by the relation 2.15 between special cycles on X_K and special cycles on its connected component X_Γ , we have

$$\Theta_\varphi = \sum_{\sigma \in Gal(E_K/\mathbb{Q})} \Theta_{\underline{\lambda}}^\sigma$$

where the Galois group $Gal(E_K/\mathbb{Q})$ acts on $\Theta_{\underline{\lambda}}$ via acting on its coefficients.

Now the assertion follows from Conjecture 1, Theorem 2.9 and the relation between adelic Weil representation ω_f and our $\rho_{L,r}^*$. □

Now it is routine that we can extend the generating function as a function defined over $\widetilde{Sp}(2r,\mathbb{A})$. Note that we have identified $Sp(2r,\mathbb{Q})$ with a subgroup of $\widetilde{Sp}(2r,\mathbb{A})$.

For $g \in \widetilde{Sp}(2r, \mathbb{A})$ and $g = \gamma(g_\infty, k)$ where $\gamma \in Sp(2r, \mathbb{Q})$ and $k \in \widetilde{Sp}(2r, \widehat{\mathbb{Z}})$, we define

$$\widetilde{\Theta}_{\varphi,\iota}(g) := j(g_\infty, \sqrt{-1}I_r)^{-\frac{n+2}{2}} \Theta_{\omega_f(k)\varphi,\iota}(g_\infty(\sqrt{-1}I_r)). \tag{2.18}$$

Then, by the above corollary, this is independent of the choice of decomposition $g = \gamma(g_\infty, k)$ and is invariant under $Sp(2r, \mathbb{Q})$, i.e.:

Corollary 2.12. *Let n, r, φ and the linear functional ι be as in Theorem 2.11. Then, assuming Conjecture 1 for all lattices L, we have*

$$\widetilde{\Theta}_{\varphi,\iota}(\gamma g) = \widetilde{\Theta}_{\varphi,\iota}(g), \qquad \gamma \in Sp(2r, \mathbb{Q}),\ g \in \widetilde{Sp}(2r, \mathbb{A})$$

for any linear functional ι. Further, for $g' \in \widetilde{Sp}(2r, \mathbb{A}_f)$,

$$\widetilde{\Theta}_{\varphi,\iota}(gg') = \widetilde{\Theta}_{\omega_f(g')\varphi,\iota}(g).$$

Proof. The first half is obvious. For the second half, the proof is completely parallel to the proof for the case $n = r = 1$ in section 4.7 of [KRY]. □

Chapter 3

Finite generation of special cycles of codimension two

3.1 The result

In this chapter we prove the finite generation of the subspace $SC^2(L)$ of $CH^2(X_\Gamma)_\mathbb{C}$ generated by special cycles of codimension two (without assuming Conjecture 1). Unfortunately our method for codimension two cycles is ad hoc and does not generalize to higher codimension cycles.

Theorem 3.1. *The space $SC^2(L)$ is finite dimensional.*

Remark. It is generally very hard to detect rational equivalence for cycles other than divisors. This can be illustrated by the example of Mumford ([Mum]) for zero-cycles on a surface with a non-trivial holomorphic 2-form. According to the conjecture of Beilinson-Bloch, it is expected for a smooth variety defined over a number field, the Chow group should have finite rank. In the case of divisors, the finite generation essentially follows from Mordell-Weil theorem for abelian variety. Little has been proved for cycles of codimension larger than one. We can think of our result as a piece of evidence for their conjecture.

The proof relies the following simple observation. From Borcherds' theorem, we can not only obtain the finite generation of the space $SC^1(L)$ of special divisors but also find a basis of $SC^1(L)$. In fact the first several terms already form the basis. Otherwise, we would find a modular form with very high vanishing orders at all cusps. This would contradict Riemann-Roch !

3.2 Preliminary on modular forms

Before we prove Theorem 3.1, we need some preparation. Recall that we have q-series with coefficients in SC^2_L for $\lambda, \mu \in L^\vee/L$,

$$\Theta_{\lambda,\mu} = \sum_{T \geq 0} Z(T; \lambda, \mu) q^T.$$

Fix $t > 0$ and consider the t-the Fourier-Jacobi coefficients of $\Theta_{\lambda,\mu}$,

$$\theta_{\lambda,\mu}(\tau, z) = \theta_{\lambda,\mu;t}(\tau, z) = \sum_{m,p} Z_{\lambda,\mu}(m,p) q^m \xi^p, \quad q = e(\tau), \xi = e(z)$$

where the sum runs over (m,p) such that $mt - \frac{1}{4}p^2 \geq 0$ and $Z_{\lambda,\mu}(m,p) = Z(T; \lambda, \mu)$ for $T = \begin{pmatrix} m & \frac{1}{2}p \\ \frac{1}{2}p & t \end{pmatrix}$ (cf. Equation 2.7, we change the letter τ_0 to τ). Since the theorem of Borcherds verifies the condition 2 of Theorem 2.5, by Equation 2.10 and Proposition 2.6 we know that $\iota(\theta_{\lambda,\mu})$ is convergent for all linear functionals ι on $SC^2(L)$. Thus for simplicity we can drop ι keeping in mind that all equalities hold when we apply linear functionals. Now we see that $\theta_{\lambda,\mu}$ obviously satisfies the following equations: for $a, b, c \in \mathbb{Z}$,

1) $\theta_{\lambda,\mu}(-\frac{1}{\tau}, \frac{z}{\tau}) = \sqrt{\tau}^{n+2} \frac{e(-\frac{n-2}{8})}{|L^\vee/L|^{1/2}} e(\frac{{}^t z t z}{\tau}) \sum_\gamma e(\langle \lambda, \gamma \rangle) \theta_{\gamma,\mu}(\tau, z)$.

2) $\theta_{\lambda,\mu}(\tau + a, z) = e(\frac{1}{2}a\langle \lambda, \lambda \rangle) \theta_{\lambda,\mu}(\tau, z)$.

3) $\theta_{\lambda,\mu}(\tau, z+b) = e(b\langle\lambda,\mu\rangle)\theta_{\lambda,\mu}(\tau,z)$.

4) $\theta_{\lambda,\mu}(\tau, z+c\tau) = e(-2tcz - tc^2\tau)\theta_{\lambda-c\mu,\mu}(\tau,z)$.

Note that 4) is equivalent to

$$Z_{\lambda,\mu}(m,p) = Z_{\lambda-c\mu,\mu}(m+pc+tc^2, p+2tc). \tag{3.1}$$

Let N be a positive integer such that $\frac{N}{2}$ kills $L^\vee/L$. Then, if $N|c$ we have $Z_{\lambda,\mu}(m,p) = Z_{\lambda,\mu}(m+pc+tc^2, p+2tc)$. We can therefore define $Z_{\lambda,\mu;w}(d) = Z_{\lambda,\mu}(m,p)$ for one (hence for every) choice of representative p of $w\,(mod\,2Nt)$ and m such that $d = m - \frac{p^2}{4t}$. We now have

$$\theta_{\lambda,\mu}(\tau,z) = \sum_{w\,(mod\,2Nt)} \theta_{\lambda,\mu;w}(\tau)\theta_{w,2Nt}(\tau,z).$$

where

$$\theta_{\lambda,\mu;w}(\tau) = \sum_{d\geq 0} Z_{\lambda,\mu;w}(d)q^d$$

and

$$\theta_{w,2Nt}(\tau,z) = \sum_{p\equiv w\,(mod\,2Nt)} q^{\frac{p^2}{4t}}\xi^p.$$

Lemma 3.2. *For $\lambda,\mu \in L^\vee/L$ and $w\,(mod\,2Nt)$, $\theta_{\lambda,\mu;w}(\tau)$ is a modular form for $\Gamma'(2N^2t)$ of weight $k = \frac{1+n}{2}$. Further, $\theta_{\lambda,\mu;w}(-\frac{1}{\tau})$ is a linear combination of $\theta_{\lambda',\mu';w'}$ for $\lambda',\mu' \in L^\vee/L$ and $w'\,(mod\,2Nt)$.*

Proof. By our choice of N, the representation $\rho^*_{L,1}$ factors through $\Gamma'(N)$ (cf. Remark 2). Consider the one dimensional lattice $\mathbb{Z}e$ with a quadratic form $q(e) = 2N^2t$. Then, applying Lemma 2.8 to $x = \frac{1}{N}e$ we know that $\theta_{w,2Nt}(\tau,z)$ is a Jacobi form of weight 1/2 and index t on $\Gamma'(2N^2t)$ (also by Remark 2). It follows that $\theta_{\lambda,\mu;w}(\tau)$ is a modular form for $\Gamma'(2N^2t)$ of weight $k = \frac{1+n}{2}$. For the second statement, we just compare the formula of $\theta_{\lambda,\mu}(-\frac{1}{\tau},\frac{z}{\tau})$ with that of $\theta_{w,2Nt}(-\frac{1}{\tau},\frac{z}{\tau})$ using Equation 2.9 and Lemma 2.8. This completes the proof. □

3.3 Proof of Theorem 3.1

We claim that $SC^2(L)$ is generated by $Z(T;\lambda,\mu)$ for all $\lambda,\mu \in L^\vee/L$ and all $T \geq 0$ of the form

$$T = \begin{pmatrix} m & \frac{1}{2}p \\ \frac{1}{2}p & t \end{pmatrix},\ m \leq \frac{4(k+1)}{3},\ t \leq \frac{4(k+1)}{3}.$$

It follows from this claim that $SC^2(L)$ is finite dimensional.

For $t > 0$, let S_t be the space spanned by cycle classes of $Z(T;\lambda,\mu)$ for $T = \begin{pmatrix} m & \frac{1}{2}p \\ \frac{1}{2}p & t \end{pmatrix}$ and all $\lambda,\mu \in L^\vee/L$. Let S'_t be the subspace spanned by $Z(T;\lambda,\mu)$ for $\lambda,\mu \in L^\vee/L$ and T with $det(T) \leq (k+1)t$. We claim that $S_t = S'_t$. In fact, if $S'_t \neq S_t$, one could construct a nontrivial linear functional ι on S_t that vanishes on S'_t. Thus at least one of $\theta_{\lambda,\mu;w}$ is a nonzero modular form. Since $\iota(Z_{\lambda,\mu}(m,p))$ vanish for all λ,μ and m,p with $m - \frac{p^2}{4t} = \frac{det(T)}{t} \leq k+1$, it follows from Lemma 3.2 that, at all cusps, the Fourier expansion of $\iota(\theta_{\lambda,\mu;w})$ vanishes up to $(k+1)$-th coefficients. But this is impossible due to the following lemma applied to the group $\Gamma'(2N^2t)$.

Lemma 3.3. *For any integer $N > 0$, there does not exist a nonzero modular form f of weight k on $\Gamma'(N)$ with vanishing Fourier coefficient $a_{n,i}$ for $n \leq k+1$ at all cusps $i \in \Sigma_N$. Here Σ_N is the set of all inequivalent cusps.*

Proof. If such f exists, one can assume that f is actually a form on $\Gamma(N)$ after taking square of f. At the cusp $i \in \Sigma_N$, since the uniformizer for $\Gamma(N)$ is $q^{\frac{1}{N}} = e(\frac{\tau}{N})$, the vanishing order of f is at least $N(k+1)$. Comparing with the degree of the line bundle of modular form of weight k on $\Gamma(N)$ (see [S1], page 23), we have an inequality:

$$k\mu_N \geq \sum_{i\in\Sigma_N} N(k+1)$$

where $\mu_N = [SL_2(\mathbb{Z}) : \Gamma(N)]$. But $\Gamma(N)$ has exactly $\frac{\mu_N}{N}$ cusps. Contradiction! This completes the proof. $\square$

Moreover, by Equation 3.1, we can assume that S_t is generated by $Z(T;\lambda,\mu)$ for all $\lambda,\mu\in L^\vee/L$ and $T=\begin{pmatrix} m & \frac{1}{2}p \\ \frac{1}{2}p & t \end{pmatrix}$ with $det(T)\leq t(k+1)$ and $-t\leq p<t$. Thus we have $m\leq k+1+\frac{t}{4}$. If $t>\frac{4(k+1)}{3}$, we have $m<t$. Since t,m are symmetric, we can switch them and repeat the argument above. We thus prove that $SC^2(L)$ is generated by $Z(\lambda,\mu;T)$ for all $\lambda,\mu\in L^\vee/L$ and $T=\begin{pmatrix} m & \frac{1}{2}p \\ \frac{1}{2}p & t \end{pmatrix}\geq 0$ with $m,t\leq\frac{4(k+1)}{3}$. This completes the proof of Theorem 3.1.

Remark.

1. Lemma 3.3 also implies that the dimension of $SC^1(L)$ is at most $\frac{(n+3)}{2}|L^\vee/L|$ and moreover that $SC^1(L)$ is generated by the first $\frac{(n+3)}{2}|L^\vee/L|$ terms.

2. We give an explicit example. Consider the self-product $S=X_0(N)\times X_0(N)$ of the level-N modular curve $X_0(N)$ defined over $\mathbb{Q}$. A point x on $X_0(N)$ represents a pair (E,G) consisting of an elliptic curve E and a cyclic N-subgroup G of E. For $T\in Sym_2(\mathbb{Z})_{>0}$, let Z_T be the zero cycle on S defined summing all points (x_1,x_2) such that $Hom(x_1,x_2)$ represents T. Here $Hom(x_1,x_2)$ is endowed with the quadratic form defined by the degree of morphism from x_1 to x_2. Then it follows from Theorem 3.1 that for all $T>0$, the rational equivalence classes of zero-cycles Z_T generate a finite dimensional subspace of $CH^2(S)_{\mathbb{Q}}$.

Chapter 4

Hecke action on the space of special cycles

In this chapter, we investigate the Hecke action on the space of special cycles from a representation theoretical point of view.

4.1 Hecke action

Recall that V is a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$ and we have the $\mathbb{Q}$-algebraic group $G = GSpin(V)$. Let $G' = \widetilde{Sp}_{2r}$ be the double covering of Sp_{2r}. Although G' is not an algebraic group, we still denote by $G'(\mathbb{Q}_p)$ ($G'(\mathbb{A}_f)$, respectively) the double covering of $Sp_{2r}(\mathbb{Q}_p)$ ($Sp_{2r}(\mathbb{A}_f)$, respectively). Let ω_f be the restriction of Weil representation of $G(\mathbb{A}) \times G'(\mathbb{A})$ to $G(\mathbb{A}_f) \times G'(\mathbb{A}_f))$ on $\mathscr{S}(V(\mathbb{A}_f)^r)$.

Let CH^r be the inductive limit $\varinjlim_K CH^r(X_K)_{\mathbb{C}}$ of Chow groups with respect to pull-back maps. Let $SC^r(X_K)$ be the subspace of $CH(X_K)_{\mathbb{C}}$ generated by classes of special cycles $Z(T, \varphi)$ for all $T \in Sym_r(\mathbb{Q})_{\geq 0}$ and $\varphi \in \mathscr{S}(V(\mathbb{A}_f)^r)^K$. Similarly, we form the inductive limit

$$SC^r = \varinjlim_K SC^r(X_K).$$

The group $G(\mathbb{A}_f)$ acts on CH^r via Hecke correspondences. More precisely, for $g \in G(\mathbb{A}_f)$, and $K_g = gKg^{-1}$, there are two morphisms $p_1, p_2 : X_{K\cap K_g} \to X_K$ with p_1 induced by the inclusion $K \cap K_g \subseteq K$ and p_2 the composition

$$X_{K\cap K_g} \longrightarrow X_{K_g} \xrightarrow{\cdot g} X_K \ .$$

Then g acts on the Chow group via $p_{1*}p_2^*$. One can check that SC^r is $G(\mathbb{A}_f)$-stable and we have a $G(\mathbb{A}_f)$-equivariant map from $\mathscr{S}(V(\mathbb{A}_f)^r)$ to SC^r (see Corollary 5.11 of [K1]), i.e.,

$$gZ(T, \varphi) = Z(T, \omega_f(g)\varphi). \tag{4.1}$$

And this induces an action of $G(\mathbb{A}_f)$ on the smooth dual space

$$SC^{r,\vee} = Hom(SC^r, \mathbb{C}).$$

Here a linear functional is smooth if it is fixed by some open compact subgroup K of $G(\mathbb{A}_f)$.

4.2 Multiplicity one

Let $\mathscr{A}(G')$ be the space of automorphic forms on $G'(\mathbb{A})$ on which $G'(\mathbb{A}_f) \times (\mathscr{G}, K'_\infty)$ acts by translation from the right. Let the $(\mathscr{G}, K'_\infty)$-module $\sigma_{\infty,1+n/2}$ be the holomorphic discrete series of weight $\frac{n}{2}+1$. Define a $G'(\mathbb{A}_f)$-module

$$\mathscr{A}_0(\frac{n}{2}+1, G') = Hom_{(\mathscr{G},K'_\infty)}(\sigma_{\infty,1+n/2}, \mathscr{A}(G')).$$

Proposition 4.1. *Assume Conjecture 1. Then there is a $G(\mathbb{A}_f)$-equivariant injective map*

$$SC^{r,\vee} \hookrightarrow Hom_{G'(\mathbb{A}_f)}(\omega_f, \mathscr{A}_0(\frac{n}{2}+1, G')).$$

Proof. Firstly we have

$$Hom_{G'(\mathbb{A}_f)}(\omega_f, \mathscr{A}_0(\frac{n}{2}+1, G')) = Hom_{G'(\mathbb{A}_f)\times(\mathscr{G},K'_\infty)}(\omega_f \otimes \sigma_{\infty,1+n/2}, \mathscr{A}(G')).$$

For $\iota \in SC_0^{r,\vee}$, we can define a linear map $\Theta(\iota)$ from $\mathscr{S}(V(\mathbb{A}_f)^r)\otimes\sigma_{\infty,1+n/2}$ to $\mathscr{A}(G')$ as follows. Let u be the lowest weight vector of $\sigma_{\infty,1+n/2}$ (unique up to scalars). For $\varphi \in \mathscr{S}(V(\mathbb{A}_f)^r)$, let $\Theta(\iota)(\varphi\otimes u)$ be $\widetilde{\Theta}_{\varphi,\iota}$ as defined in Equation 2.18. By Corollary 2.12, this defines an element in $\mathscr{A}(G')$. Since u generates $\sigma_{\infty,1+n/2}$ under the action of $(\mathscr{G}, K'_\infty)$, we can extend the map $(\mathscr{G}, K'_\infty)$-equivariantly to $\mathscr{S}(V(\mathbb{A}_f)^r)\otimes\sigma_{\infty,1+n/2}$. By Corollary 2.12, $\Theta(\iota)$ is $G'(\mathbb{A}_f)$-equivariant.

This gives rise to a map from $SC^{r,\vee}$ to $Hom_{G'(\mathbb{A}_f)}(\omega_f, \mathscr{A}_0(\frac{n}{2}+1, G'))$. It is injective since the generators of SC^r can be recovered from Fourier coefficients of generating functions. By the equation (4.1), the map $\iota \mapsto \Theta(\iota)$ is $G(\mathbb{A}_f)$-equivariant. This completes the proof. □

We now consider an irreducible $G(\mathbb{A}_f)$-module $\pi_f = \prod_{p<\infty}\pi_p$. We assume that the local Howe duality conjecture holds for the reductive pair $(O(V), G')$ for all non-archimedean places. Recall that the local Howe duality conjecture asserts that for any irreducible representation π_v of G_v, there is at most one irreducible representation σ_v of G'_v such that $Hom(\omega_v\otimes\pi_v, \sigma_v) \neq 0$ and the Hom is at most one dimensional. Note that our assumption does not hurt too much since the local Howe duality conjecture is proved for reductive pairs over p-adic field with $p \neq 2$ by Waldspurger ([W2]), and for any p and π_v supercuspidal by Kudla ([K0]).

Note that $G = GSpin(V)$ and the action of $G(\mathbb{A}_f)$ on $\mathscr{S}(V(\mathbb{A}_f)^r)$ factors through the special orthogonal group $SO(V)(\mathbb{A}_f)$.

Lemma 4.2. *Let V be a quadratic space over a non-archimedean local field with $dimV \geq r+1$. Assuming the local Howe duality conjecture for $(O(V), G')$, then for* *an irreducible representation π (σ, respectively) of $SO(V)$ (G', respectively) we have*

$$dim\, Hom_{SO(V)\times G'}(\omega, \pi\otimes\sigma) \leq 1.$$

And for each π, there exists at most one σ such that the dimension of Hom is one.

Proof. Recall that $SO(V)$ is the kernel of the determinant homomorphism det : $O(V) \to \{\pm 1\}$. Take any element τ with determinant -1 and let π^τ be the twist of π by τ. Let ω_σ be the $O(V)$-module $Hom_{G'}(\omega, \sigma)$. Then we have $Hom_{SO(V)\times G'}(\omega, \pi \otimes \sigma) = Hom_{SO(V)}(\omega_\sigma|_{SO(V)}, \pi)$. We distinguish two cases.

1. When $\pi \neq \pi^\tau$, the induced representation $Ind_{SO(V)}^{O(V)}\pi$ is irreducible. And the space $Hom_{SO(V)}(\omega_\sigma|_{SO(V)}, \pi)$ is isomorphic to $Hom_{O(V)}(\omega_\sigma, Ind_{SO(V)}^{O(V)}\pi)$. By the local Howe duality for the reductive pair $(O(V), G')$, the latter is at most one dimensional and it is one dimensional for at most one σ.

2. When $\pi = \pi^\tau$, the representation $Ind_{SO(V)}^{O(V)}\pi$ split into two components. In this case, we can extend π to an irreducible representation of $O(V)$ in exactly two ways. We denote them by π^+ and π^-. Then, $\pi^+ = \pi^- \otimes det$ and we have

$$Hom_{SO(V)}(\omega_\sigma|_{SO(V)}, \pi) = Hom_{O(V)}(\omega_\sigma, \pi^+) + Hom_{O(V)}(\omega_\sigma, \pi^-).$$

   Now note that $dim\, V \geq r+1$. By a result of Rallis (appendix of [R], see also [P] sec. 5, page 282), at most one of π^+ and $\pi^- \otimes det$ appears in the local Howe duality. Then it follows that $dim\, Hom_{SO(V)}(\omega_\sigma|_{SO(V)}, \pi) \leq 1$ and for at most one σ the dimension is one.

This completes the proof. □

Now Lemma 4.2 implies that, for an irreducible $G(\mathbb{A}_f)$-module $\pi_f = \prod_{p<\infty} \pi_p$, there exists at most one $G'(\mathbb{A}_f)$-module, denoted by $\theta(\pi_f) = \prod_{p<\infty} \theta_p(\pi_p)$, such that

$$dim_{\mathbb{C}}\, Hom_{G(\mathbb{A}_f)\times G'(\mathbb{A}_f)}(\pi_f \otimes \omega_f, \theta(\pi_f)) = 1.$$

If such $\theta(\pi_f)$ exists, we only focus on the situation that the $G'(\mathbb{A}_f)\times(\mathscr{G}, K_\infty)$-module $\theta(\pi_f)\otimes\sigma_{\infty,1+n/2}$ is cuspidal automorphic. For such π_f, we define

$$m(\pi_f) = dim_{\mathbb{C}}\, Hom_{G'(\mathbb{A}_f)}(\theta(\pi_f), \mathscr{A}_0(\frac{n}{2}+1, G')).$$

Theorem 4.3. *Assume Conjecture 1 and that the local Howe duality conjecture holds for the reductive pair $(O(V), G')$ for all non-archimedean places. Then for an irreducible $G(\mathbb{A}_f)$-module π_f, we have $dim_{\mathbb{C}}\, Hom_{G(\mathbb{A}_f)}(\pi_f, SC_0^{r,\vee}) = 0$ if $\theta(\pi_f)$ does not exist and $dim_{\mathbb{C}}\, Hom_{G(\mathbb{A}_f)}(\pi_f, SC_0^{r,\vee}) \leq m(\pi_f)$ if $\theta(\pi_f)\otimes\sigma_{\infty,1+n/2}$ is a cuspidal automorphic representation.*

Proof. By Proposition 4.1, we have

$$Hom_{G(\mathbb{A}_f)}(\pi_f, SC_0^{r,\vee}) \subseteq Hom_{G(\mathbb{A}_f)}(\pi_f, Hom_{G'(\mathbb{A}_f)}(\omega_f, \mathscr{A}_0(\frac{n}{2}+1, G'))).$$

The latter is isomorphic to

$$Hom_{G(\mathbb{A}_f)\times G'(\mathbb{A}_f)}(\pi_f \otimes \omega_f, \mathscr{A}_0(\frac{n}{2}+1, G')).$$

If l is a non-trivial element in this space, let σ_f be an irreducible $G'(\mathbb{A}_f)$-invariant subspace of the image of l. By Lemma 4.2, $Hom_{G(\mathbb{A}_f)\times G'(\mathbb{A}_f)}(\pi_f\otimes\omega_f, \sigma_f) = 0$ unless $\sigma_f \simeq \theta(\pi_f)$. This proves the first assertion.

Since the space of cuspidal automorphic forms decomposes discretely and each irreducible cuspidal automorphic representation has finite multiplicity, the argument above shows that the image of any non-trivial linear functional l is actually a direct sum of finitely many of $\theta(\pi_f)$. The local Howe duality also implies that the image of l is actually irreducible. Therefore, any l factors through $\theta(\pi_f)$ and it follows that the dimension of $Hom_{G(\mathbb{A}_f)\times G'(\mathbb{A}_f)}(\pi_f \otimes \omega_f, \mathscr{A}_0(\frac{n}{2}+1, G'))$ is the same as $m(\pi_f)$. This completes the proof. □

In particular, when $r = 1$, we get the following unconditional result.

Corollary 4.4. *Suppose $r = 1$, i.e. $G' = \widetilde{SL}_2$. And suppose that $\theta(\pi_f) \otimes \sigma_{\infty,1+n/2}$ is a cuspidal automorphic representation. Then we have*

$$dim_{\mathbb{C}}\, Hom_{G(\mathbb{A}_f)}(\pi_f, SC^{1,\vee}) \leq 1.$$

Proof. In this case, Conjecture 1 holds. And the local Howe duality conjecture is known for $(O(V), \widetilde{SL}_2)$ for all non-archimedean places. The multiplicity one for cuspidal automorphic representations of $\widetilde{SL}_2$ is proved by Waldspurger ([W1]). Thus under the assumption that $\theta(\pi_f) \otimes \sigma_{\infty,1+n/2}$ is a cuspidal automorphic representation, we have $m(\pi_f) = 1$. This completes the proof. □

Remark.

1. One expects that the multiplicity one holds for cuspidal automorphic representations on the group $\widetilde{Sp}_{2r}(\mathbb{A})$ with $r > 1$. But it seems to be unproven at this moment.

2. One of the most interesting questions is to find a criterion when the space $Hom_{G(\mathbb{A}_f)}(\pi, SC^{r,\vee})$ is non-trivial. In the classical case of Heegner points, the answer is given by Gross-Zagier's formula in terms of central derivative of certain L-functions. We expect that the non-triviality of $Hom_{G(\mathbb{A}_f)}(\pi, SC^{r,\vee})$ is also controlled by a formula of Gross-Zagier type. For more discussion, we refer to [K2].

3. Gross-Kohnen-Zagier in [GKZ] proves that Heegner points with different discriminants contribute at most one-dimension to the Mordell-Weil group of the

elliptic curve attached to a newform of level N. Using the corollary above, one can recover this result. This avoids computing Neron-Tate height pairing between pairwise Heegner points.

Bibliography

[A] Andrianov, A. N.; Zhuravlëv, V. G. *Modular forms and Hecke operators.* Translated from the 1990 Russian original by Neal Koblitz. Translations of Mathematical Monographs, 145. American Mathematical Society, Providence, RI, 1995. viii+334 pp.

[Be] Beilinson, A. A. *Height pairing between algebraic cycles.* K-theory, arithmetic and geometry (Moscow, 1984–1986), 1–25, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[B0] Borcherds, Richard E. *Automorphic forms with singularities on Grassmannians,* Invent. Math. 132, 491-562 (1998).

[B1] Borcherds, Richard E. *The Gross-Kohnen-Zagier theorem in higher dimensions.* Duke Math. J. 97 (1999), no. 2, 219–233.

[EZ] Eichler, Martin; Zagier, Don. *The theory of Jacobi forms.* Progress in Mathematics, 55. Birkhäuser Boston, Inc., Boston, MA, 1985. v+148 pp.

[GZ] Gross, Benedict H.; Zagier, Don B. *Heegner points and derivatives of L-series.* Invent. Math. 84 (1986), no. 2, 225–320.

[GKZ] Gross, B.; Kohnen, W.; Zagier, Don. *Heegner points and derivatives of L-series. II.* Math. Ann. 278 (1987), no. 1-4, 497–562.

[HZ] Hirzebruch, F.; Zagier, D. *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus.* Invent. Math. 36 (1976), 57–113.

[K0] Kudla, Stephen S. *On the local theta-correspondence.* Invent. Math. 83 (1986), no. 2, 229–255.

[K1] Kudla, S. *Algebraic cycles on Shimura varieties of orthogonal type.* Duke Math. J. 86 (1997), no. 1, 39–78.

[K2] Kudla, S. *Special cycles and derivatives of Eisenstein series.* in *Heegner points and Rankin L-series,* 243–270, Math. Sci. Res. Inst. Publ., 49, Cambridge Univ. Press, Cambridge, 2004.

[KM] Kudla, Stephen S.; Millson, John J. *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables.* Inst. Hautes tudes Sci. Publ. Math. No. 71 (1990), 121–172.

[KRY] Kudla, Stephen S.; Rapoport, Michael; Yang, Tonghai. *Modular forms and special cycles on Shimura curves.* Annals of Mathematics Studies, 161. Princeton University Press, Princeton, NJ, 2006. x+373 pp.

[Mi] Milne, J. S. *Introduction to Shimura varieties.* in *Harmonic analysis, the trace formula, and Shimura varieties,* 265–378, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.

[Mum] Mumford, D. *Rational equivalence of* 0*-cycles on surfaces.* J. Math. Kyoto Univ. 9 1968 195–204.

[P] Prasad, Dipendra. *On the local Howe duality correspondence.* Internat. Math. Res. Notices 1993, no. 11, 279–287.

[R] Rallis, S. *On the Howe duality conjecture.* Compositio Math. 51 (1984), no. 3, 333–399.

[S1] Shimura, Goro. *Introduction to the arithmetic theory of automorphic functions.* Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. xiv+267 pp.

[S2] Shimura, Goro. *On certain reciprocity-laws for theta functions and modular forms.* Acta Math. 141 (1978), no. 1-2, 35–71.

[W1] Waldspurger, J.-L. *Correspondance de Shimura.* J. Math. Pures Appl. (9) 59 (1980), no. 1, 1–132.

[W2] Waldspurger, J.-L. *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$.* Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), 267–324, Israel Math. Conf. Proc., 2, Weizmann, Jerusalem, 1990.

[YZZ1] X. Yuan, S. Zhang, W. Zhang. *The Gross-Kohnen-Zagier theorem over totally real fields.* to appear in Compositio Math.

[YZZ2] X. Yuan, S. Zhang, W. Zhang. *Heights of CM points (I): Gross–Zagier formula.* preprint.

[YZZ3] X. Yuan, S. Zhang, W. Zhang. *Triple product L-series and Gross–Schoen cycles.* preprint.

[Zh] Zhang, Shou-Wu. *Gross-Zagier formula for* GL_2. Asian J. Math. 5 (2001), no. 2, 183–290.