The Hilbert-Chow Morphism and the Incidence Divisor

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Abstract

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In this thesis, we study the locus of intersecting cycles inside a product of Chow varieties of a smooth projective variety X. The main case considered is that of pairs of 1-cycles on a threefold. In this situation we construct a Cartier divisor supported on the incidence locus $\mathscr{I} \hookrightarrow \mathscr{C}_1(X) \times \mathscr{C}_1(X)$. We also study the case $\mathscr{I} \hookrightarrow \mathscr{C}_0(X) \times \mathscr{C}_{\dim(X)-1}(X)$, and here we make use of explicit descriptions of both Chow varieties.

In both cases we proceed by defining an incidence line bundle $\mathcal{L}$ on a product of Hilbert schemes mapping to the corresponding Chow varieties. The essential ingredients of the incidence bundle are the universal families over the Hilbert schemes and the determinant line bundle of a perfect complex. We are thus led to problems of descent: to define an isomorphism between two pullbacks of $\mathcal{L}$, satisfying the cocycle condition; and then to show the effectiveness of the descent datum thus obtained.

The first step towards defining the descent datum is a characterization of functions on a seminormal scheme as pointwise functions compatible with specialization. Along with a straightforward K-theoretic interpretation of the Hilbert-Chow morphism, this characterization converts the problem of defining the descent datum to understanding how K-theory behaves under specialization. As for the effectiveness, the seminormality of the Chow variety produces a criterion for effective descent, and the explicitness of the descent datum allows us to verify it in our situation.

Contents

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To my parents

Chapter 1

Introduction

1.1 Sketch of the problem

Let $(X, \mathcal{O}_X(1))$ be a smooth projective variety over an algebraically closed field k. The *Chow variety* $\mathscr{C}_{d,d'}(X)$ parameterizes algebraic cycles on X. In particular, set-theoretically,

$$\mathscr{C}_{d,d'}(X)(k) = \{\text{effective algebraic cycles on } X \text{ of dimension } d \text{ and degree } d'\}.$$

An element of this set is an expression $\sum_i n_i Z_i$, with $n_i \in \mathbb{Z}_{\geq 0}$ and $Z_i \subset X$ d-dimensional integral closed subschemes such that $\sum_i n_i \deg_{\mathcal{O}(1)} Z_i = d'$. We use the definition(s) of a family of cycles from [24]. Cycles in positive characteristic have pathological tendencies, but in all characteristics one has a seminormal scheme $\mathscr{C}_{d,d'}(X)$ which coarsely represents a reasonable Chow functor. A general line of inquiry is to understand how the geometry of X is reflected in the geometry of its associated moduli spaces. More specifically, the following question has been asked by Mazur [26] and studied directly in [33] and [3].

Question. Let a, b be nonnegative integers such that $a + b + 1 = \dim(X)$. What is the structure of the incidence locus $\{(A, B) | A \cap B \neq \emptyset\} \hookrightarrow \mathscr{C}_a \times \mathscr{C}_b$?

Over $\mathbb{C}$, Mazur constructs a Weil divisor supported on the incidence locus as follows. Consider the diagram of schemes:

$$\begin{tikzcd} X \times \mathscr{C}_a(X) \times \mathscr{C}_b(X) \arrow[r, "\Delta"] \arrow[d, "pr_{23}"] & X \times X \times \mathscr{C}_a(X) \times \mathscr{C}_b(X) \\ \mathscr{C}_a(X) \times \mathscr{C}_b(X) & \end{tikzcd}$$

Let U_a, U_b denote the universal cycles on $X \times \mathscr{C}_a(X)$, $X \times \mathscr{C}_b(X)$ respectively (these exist in characteristic zero). Then since Δ is a local complete intersection morphism, via intersection theory [11] one has $D := pr_{23*}\Delta^!(U_a \boxtimes U_b)$, a cycle of codimension 1 on $\mathscr{C}_a(X) \times \mathscr{C}_b(X)$. Among other things, Mazur [26] asks whether D is Cartier. This thesis gives an affirmative answer in some cases.

1.2 Wang's result on the Archimedean height pairing

In an attempt to answer Mazur's question, Wang [33] embarks on a study of the geometry of Chow varieties (over $\mathbb{C}$) using the Archimedean height pairing $\langle A, B\rangle$ on algebraic cycles. Given disjoint cycles A, B on X as above, one has the pairing $\langle A, B\rangle := \int_A [G_B]$ defined by integrating a normalized Green's current for B over A. Wang views $\langle A, B\rangle$ as a function on the open set U in $\mathscr{C}_a(X) \times \mathscr{C}_b(X)$ consisting of disjoint cycles, then studies the behavior of the function as the cycles collide. He obtains the following result [33, Thm. 1.1.2].

Theorem. *There exist a metrized line bundle L on $\overline{U} \subset \mathscr{C}_a(X) \times \mathscr{C}_b(X)$ and a rational section s that is regular and nowhere zero on U, such that:*

$$\log ||s(A, B)||^2 = (\dim(X) - 1)!\langle A, B\rangle.$$

We explain briefly some of the geometric ideas behind Wang's construction. Since the height pairing is well-behaved under products and correspondences, this approach is amenable to reduction to the diagonal $\Delta \hookrightarrow X \times X$. In particular, one is led to seek a correspondence:

$$
\begin{tikzcd}
 & \Gamma \subset X \times X \times Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
X \times X & & Y
\end{tikzcd}
$$

with the following properties:

(1) there is a point $y_\Delta \in Y$ such that ${}^t\Gamma_* y_\Delta = \Delta$ as cycles on $X \times X$,

(2) $\Gamma_*(A \times B)$ is a divisor in Y, and

(3) the height pairings satisfy

$$\langle A \times B, {}^t\Gamma_* y_\Delta \rangle = \langle \Gamma_*(A \times B), y_\Delta \rangle + c(y_\Delta, A \times B)$$

with c a continuous function in both variables.

Suppose such a Γ exists. Then letting L be the line bundle on Y corresponding to the divisor $\Gamma_*(A \times B)$, one obtains a section s of the canonical $\mathcal{O}(1)$ on $\mathbb{P}(\Gamma(Y, L))$ (the linear system containing $\Gamma_*(A \times B)$) by evaluating a section defining $\Gamma_*(A \times B)$ at y_Δ. Then using the Poincaré-Lelong formula and multiplying s by $\exp(-c)$, one obtains

$$\langle A, B \rangle = \log ||s(\Gamma_*(A \times B))||^2.$$

So the correspondence Γ induces a morphism $\mathscr{C}_a(X) \times \mathscr{C}_b(X) \to \mathbb{P}(\Gamma(Y, L))$ via $(A, B) \mapsto \Gamma_*(A \times B)$. The desired line bundle and section are then the pullback of $\mathcal{O}(1)$ and s.

A natural way to find such a Γ is to realize Δ as the intersection cycle of $X \times X$ and some subvariety of projective space (for example, as a proper intersection), in some projective embedding. Unfortunately, such an embedding does not generally exist, and Wang is unable to find a suitable Γ by other means. His solution is to use a twisted embedding of $X \times X$ into some Grassmannian. Then one can move $(\dim(X) - 1)!\Delta$ within its rational equivalence class to a cycle of the form $(X \times X \cap \sigma) - r(X \times X \cap L)$ with $r \in \mathbb{Z}$, σ a special Schubert cycle in the Grassmannian, and $L \subset \mathbb{P}^n$ a linear space. The technical heart of [33] is the identity analogous to the one in (3) for the cycle σ; the (presumably easier) cycle L is analyzed in [34].

1.3 Barlet-Kaddar via Deligne cohomology

Using rather different techniques, Barlet and Kaddar [3] associate to a family of cycles over a base S a Cartier (incidence) divisor on S (again over $\mathbb{C}$). Specifically, suppose we are given:

(1) X a complex manifold of dimension $n = a + b + 1$,

(2) $(A_s)_{s \in S}$ an analytic family of a-cycles on X over a reduced complex base space S, and

(3) $B \subset X$ a b-cycle,

such that the incidence has expected codimension and is generically finite over the base. More specifically, we require:

(1) the analytic set $(\mathrm{Supp}(B) \times S) \cap \mathrm{Supp}(A)$ has codimension $n + 1 = \mathrm{codim}(A) + \mathrm{codim}(B)$ in $X \times S$; and

(2) the analytic set $(\mathrm{Supp}(B) \times S) \cap \mathrm{Supp}(A) \subset X \times S$ is proper over S and generically finite onto its image $\mathrm{Supp}(\Sigma_B)$, which is nowhere dense in S.

To such data they associate an effective Cartier divisor Σ_B in S with support equal to $\mathrm{Supp}(\Sigma_B)$. Their construction enjoys the following properties:

(1) in case of the universal family 0-cycles of degree k on $\mathbb{C}$ (so $S = \mathrm{Sym}^k(\mathbb{C})$) and $B = 1\{0\}$, we have $\Sigma_B = \{$zero locus of the k-th elementary symmetric function$\}$;

(2) compatibility with base change $S' \to S$ preserving the hypotheses;

(3) localization on X (in which case the hypotheses are automatically preserved);

(4) direct and inverse images of cycles via holomorphic maps $X' \to X$ preserving the hypotheses (in particular, the maps are such that pushforward or pullback families are defined); and

(5) moving B in a family of cycles (over a reduced base) preserving the hypotheses fiberwise.

In fact, their construction is characterized by these properties. Their strategy is to study properties of relative fundamental classes in Deligne cohomology, then use special cohomology classes to "build" the Cartier divisor.

1.4 Statement of results and techniques

In this thesis, we will pursue a new approach to Mazur's question. The main idea is to define the incidence line bundle $\mathcal{L}$ on a product of Hilbert schemes mapping to the corresponding Chow varieties; construct a descent datum (identification of pullbacks satisfying the cocycle condition) on $\mathcal{L}$; and demonstrate its effectiveness, i.e. that $\mathcal{L}$ is induced by a line bundle on the Chow varieties.

The main case of interest in which we are able to carry out this program is $a = b = 1$, so X is a smooth projective threefold and $\mathscr{C}_1 := \mathscr{C}_a(X) = \mathscr{C}_b(X)$ is the Chow variety of 1-cycles on X. Let $\mathscr{H}_1$ denote the seminormalization of the Hilbert scheme of 1-dimensional subschemes of X (i.e. the disjoint union of $\mathscr{H}^P(X)^{sn}$ as P ranges over numerical polynomials of degree one). Then we have the Hilbert-Chow morphism $\pi : Y_0 := \mathscr{H}_1 \times \mathscr{H}_1 \to \mathscr{C}_1 \times \mathscr{C}_1 =: C$.

The structure sheaf of the universal flat family $U \hookrightarrow X \times \mathscr{H}_1$ is a perfect complex on $X \times \mathscr{H}_1$, and the projection morphism $pr_{23} : X \times \mathscr{H}_1 \times \mathscr{H}_1 \to \mathscr{H}_1 \times \mathscr{H}_1$ is smooth and proper. Hence we can define the line bundle $\mathcal{L} := \det \mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}} \otimes^{\mathbf{L}} \mathcal{O}_{\mathscr{U}})$ on Y_0. Of course, we may define $\mathcal{L}$ on any pair of Hilbert schemes, and we will use this fact. The morphism π gives rise to a proper hypercovering $Y_\bullet$ augmented over C whose i-th term Y_i is the seminormalization of $Y_0 \times_C \ldots \times_C Y_0$ ($i+1$ factors), with the canonical morphisms. Now we can make precise our result regarding the existence of a descent datum.

Theorem 1 (5.3.13)**.** *The sheaf $\mathcal{L}$ lifts to an invertible sheaf on $Y_\bullet$, i.e. there is an isomorphism $\phi : {p_1}^*\mathcal{L} \cong {p_2}^*\mathcal{L}$ on Y_1 satisfying the cocycle condition on Y_2.*

The effectiveness of $(\mathcal{L}, \phi)$ is a subtle question. To state the result we need slightly more notation. Let $\mathscr{H}_1' \hookrightarrow \mathscr{H}_1$ denote the seminormalization of the subscheme consisting of subschemes $T \subset X$ such that all irreducible components of T are 1-dimensional. Set $Y_0' := \mathscr{H}_1' \times \mathscr{H}_1'$. We may form $Y_\bullet'$ with i-th term the seminormalization of $Y_0' \times_C \ldots \times_C Y_0'$ ($i+1$ factors), also augmented over C. On $Y_\bullet'$ we have the restriction of $(\mathcal{L}, \phi)$. We expect $\mathcal{L}$ descends via π, but in any event we prove the following result.

Theorem 2 (5.4.12)**.** *The restriction of $\mathcal{L}$ to $Y_\bullet'$ descends to an invertible sheaf $\mathcal{M}$ on C.*

This work is done in Chapter 5, relying on a result proved in Chapter 3. We outline the argument momentarily.

We begin Chapter 2 with background material on determinants and perfect complexes. Then we discuss in detail how to obtain the incidence divisor $\mathscr{I} \hookrightarrow \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{G}$ directly from the construction of $\mathscr{C}_d(\mathbb{P}^n)$ when restricting to the Grassmannian $\mathscr{G}$ of $(n-d-1)$-planes in one factor. In this situation, we show in Proposition 2.4.3 that the line bundle corresponding to $\mathscr{I}$ pulls back via the Hilbert-Chow morphism to the incidence bundle $\mathcal{L}$ defined above. Additionally, we show how the ruled join construction, together with the divisor $\mathscr{I}$, solves the problem for $\mathbb{P}^n$.

Chapter 3 is devoted to the following result, which characterizes functions on a seminormal scheme.

Theorem 3 (3.1.5; see also Definition 3.1.1)**.** *A Noetherian ring A is seminormal if and only if every pointwise function on Spec A which varies algebraically along complete DVRs is induced by an element $f \in A$.*

We deduce consequences tailored for our descent problem, as described below. We phrase certain aspects of the descent problem as questions about morphisms of Picard schemes, so this chapter also collects needed facts about Picard schemes.

We discuss the case of zero-cycles and divisors in Chapter 4; in particular the incidence line bundle $\mathcal{L}$ on $\mathscr{H}^m(X) \times \mathscr{H}^q(X)$, where $m \in \mathbb{Z}_{\geq 1}$ and $\deg(q) = \dim(X) - 1$, descends to $\mathscr{C}_{0,m}(X) \times \mathscr{C}_{\dim(X)-1}(X)$. The problem of defining the descent datum is essentially algebraic, and we use [21] for an explicit algebraic description of the Hilbert-Chow morphism $\mathscr{H}^m(X) \to \mathscr{C}_{0,m}(X)$ for zero-cycles. To show effectiveness, we use descent criteria as described below in the $a = b = 1$ case, though the argument here is considerably simpler. Certain calculations done in this chapter (summarized in Proposition 4.2.3) play a role in the definition of the descent datum for the case $a = b = 1$. That the determinant formula and the algebraic description of $\mathscr{H}^m(X) \to \mathscr{C}_{0,m}(X)$ mesh so well can be seen as motivation for pursuing the determinant formula in general.

As a consequence of Theorem 3, it suffices to define the descent datum ϕ of Theorem 1 on field and DVR points, compatibly. In Chapter 5 we complete the proof of Theorem 1 by constructing such a compatible system of isomorphisms. The main advantage of having to work over at worst a DVR R is that given R-flat subschemes $C_1, C_2, D_1, D_2 \subset X_R$ of relative dimension 1 such that $[C_1] = [C_2] = \sum n_i C_i$ and $[D_1] = [D_2] = \sum m_j D_j$, one can define $\det \mathbf{R}\pi_*(\mathcal{O}_{C_1} \otimes^{\mathbf{L}} \mathcal{O}_{D_1}) \cong \det \mathbf{R}\pi_*(\mathcal{O}_{C_2} \otimes^{\mathbf{L}} \mathcal{O}_{D_2})$

by defining $\det \mathbf{R}\pi_*(\mathcal{O}_{C_1} \otimes^{\mathbf{L}} \mathcal{O}_{D_1}) \cong \otimes_{i,j} (\det \mathbf{R}\pi_*(\mathcal{O}_{C_i} \otimes^{\mathbf{L}} \mathcal{O}_{D_j}))^{n_i m_j}$ (and similarly for C_2, D_2). Over an arbitrary base, the components of a flat family do not in general vary flatly, so the "determinant of the cycle" is not generally defined.

We construct the isomorphism $\det \mathbf{R}\pi_*(\mathcal{O}_{C_1} \otimes^{\mathbf{L}} \mathcal{O}_{D_1}) \cong \otimes_{i,j} (\det \mathbf{R}\pi_*(\mathcal{O}_{C_i} \otimes^{\mathbf{L}} \mathcal{O}_{D_j}))^{n_i m_j}$ by expressing $\mathcal{O}_{C_1} = \sum n_i \mathcal{O}_{C_i} + Z$ and $\mathcal{O}_{D_1} = \sum m_j \mathcal{O}_{D_j} + W$ in $K_0(X_R)$ with Z, W supported in dimension zero over the generic point. Then, using properties of the determinant functor, we are left to trivialize (in some sufficiently canonical manner) the factors of the form $\det \mathbf{R}\pi_*(Z \otimes^{\mathbf{L}} \bullet)$ and $\det \mathbf{R}\pi_*(\bullet \otimes^{\mathbf{L}} W)$.

As for Theorem 2, i.e. the effectiveness of $(\mathcal{L}, \phi)$, an outgrowth of Theorem 3 is a criterion for effective descent (Corollary 3.2.4): $\mathcal{L} \in \mathrm{Pic}(Y_\bullet)$ descends to $\mathcal{M} \in \mathrm{Pic}(C)$ if it can be trivialized locally on C, compatibly with the descent datum ϕ. Compatibility with ϕ is delicate, so we work pointwise on one factor $\mathscr{C}_1$ of C. A moving lemma is the main tool needed to show $\mathcal{L}|_{y \times \mathscr{H}_1'}$ admits such a trivialization for every $y \in \mathscr{H}_1'$, and as a consequence we obtain pointwise descent. Another consequence of Theorem 3 is a criterion for pointwise lifts of a morphism to glue (Proposition 3.1.14), and the pointwise descents are shown to glue by showing the corresponding morphism of Picard schemes satisfies the criterion. This gives us line bundles on $\mathscr{C}_1 \times \mathscr{H}_1'$ and $\mathscr{H}_1' \times \mathscr{C}_1$. To finish the argument, we show they each extend to line bundles on the proper hypercovering formed in the $\mathscr{H}_1'$ variable. Then faithfully flat descent and another application of Proposition 3.1.14 produce for us the desired line bundle on C.

Chapter 2

Background

2.1 Determinant functor: definition and properties

The main reference for this section is [23]. See also [10] and [22]. All schemes in this chapter are assumed locally Noetherian.

Let $D(X)$ denote the derived category of the abelian category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules. We denote by $D^+(X) \subset D(X)$ the full subcategory of bounded below complexes of $\mathcal{O}_X$-modules, and similarly we have $D^-(X)$ and $D^b(X) = D^+(X) \cap D^-(X)$.

Definition 2.1.1. A complex $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be *pseudo-(quasi)coherent* if its cohomology sheaves $\mathcal{H}^q(\mathcal{F})$ are (quasi)coherent. A complex $\mathcal{F}$ is said to be of *finite Tor-amplitude* if locally on X, $\mathcal{F}$ is quasi-isomorphic to a bounded complex of flat sheaves $\mathcal{E}$ such that $\mathcal{E}^q = 0$ for $q \notin [a, b]$, some $a \leq b \in \mathbb{Z}$; if we wish to be more precise we say $\mathcal{F}$ is of *Tor-amplitude* in $[a, b]$. A complex $\mathcal{F}$ is said to be *perfect* if it is both pseudo-coherent and of finite Tor-amplitude, and is said to be of *perfect amplitude* in $[a, b]$ if it is pseudo-coherent and of Tor-amplitude in $[a, b]$.

We denote by $D_{(q)coh}(X)$ the full triangulated subcategory of $D(X)$ consisting of pseudo-(quasi)coherent complexes, and by $D^*_{(q)coh}(X)$ the corresponding bounded category for $* = +, -, b$. We denote by $\mathrm{Parf}(X) \subset D^b(X)$ the full triangulated subcategory consisting of perfect complexes, and by $\mathrm{Parf}^0(X) \subset \mathrm{Parf}(X)$ those perfect complexes with perfect cohomology sheaves.

Let $\text{Parf-is}(X)$ denote the category whose objects are perfect complexes on X, with morphisms isomorphisms in $D(X)$. Let $\mathrm{Pic}(X)$ denote the (Picard) category whose objects are invertible sheaves on X, and whose morphisms are isomorphisms.

Given an abelian category $\mathscr{A}$, we recall the definition of the category $VT(\mathscr{A})$ of true triangles of $D(\mathscr{A})$ [23, Defn. 2]. First form the abelian category $\mathscr{A}_3$ whose objects are $\mathscr{A}$-sequences $A'' \xrightarrow{\alpha} A \xrightarrow{\beta} A'$ such that $\beta \circ \alpha = 0$, and whose morphisms are triples of $\mathscr{A}$-maps making the resulting diagram commute. Then $VT(\mathscr{A})$ is the subcategory of $D(\mathscr{A}_3)$ whose objects are short exact sequences of complexes.

For convenience we now record the main result of [23, Thm. 2].

Definition-Theorem 2.1.2. There exists (up to canonical isomorphism) a unique *determinant functor* $\det_X : \text{Parf-is}(X) \to \mathrm{Pic}(X)$ satisfying the following axioms.

I. $\det_X(0) = 1 = \mathcal{O}_X$; we call $\nu : \det_X(0) \cong \mathcal{O}_X$ the *normalization* isomorphism.

II. For every true triangle of complexes $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ in $\text{Parf-is}(X)$, we have an isomorphism $i_X(\alpha, \beta) : \det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \xrightarrow{\sim} \det(\mathcal{F}_2)$. The isomorphisms i are required to satisfy some axioms.

   A. On true triangles of the form $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{F} \to 0 \to 0$, we have $i_X(\alpha, 0) = \mathit{mult} \circ (1 \otimes \nu) : \det_X(\mathcal{F}) \otimes \det_X(0) \cong \det_X(\mathcal{F})$; same for those of the form $0 \to 0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{F} \to 0$.

   B. For every isomorphism of true triangles (all squares commute as $\mathcal{O}_X$-modules, not just up to homotopy):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] & \mathcal{F}_2 \arrow[r, "\beta"] \arrow[d, "f_2"] & \mathcal{F}_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & \mathcal{F}_1{}' \arrow[r, "\alpha'"] & \mathcal{F}_2{}' \arrow[r, "\beta'"] & \mathcal{F}_3{}' \arrow[r] & 0
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} \det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \arrow[r, "i(\alpha,\beta)"] \arrow[d, "\det(f_1)\otimes\det(f_3)"] & \det(\mathcal{F}_2) \arrow[d, "\det(f_2)"] \\ \det(\mathcal{F}_1{}') \otimes \det(\mathcal{F}_3{}') \arrow[r, "i(\alpha',\beta')"] & \det(\mathcal{F}_2{}') \end{tikzcd}$$

C. Given a true triangle of true triangles (an exact square commuting on the nose; we omit the rows and columns of zeroes):

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] & \mathcal{F}_2 \arrow[r, "\beta"] \arrow[d, "f_2"] & \mathcal{F}_3 \arrow[d, "f_3"] \\ \mathcal{F}_1{}' \arrow[r, "\alpha'"] \arrow[d, "g_1"] & \mathcal{F}_2{}' \arrow[r, "\beta'"] \arrow[d, "g_2"] & \mathcal{F}_3{}' \arrow[d, "g_3"] \\ \mathcal{F}_1{}'' \arrow[r, "\alpha''"] & \mathcal{F}_2{}'' \arrow[r, "\beta''"] & \mathcal{F}_3{}'' \end{tikzcd}$$

the following diagram commutes (here $s(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d$):

$$\begin{tikzcd} \det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \otimes \det(\mathcal{F}_1{}'') \otimes \det(\mathcal{F}_3{}'') \arrow[r, "i(\alpha,\beta)\otimes i(\alpha'',\beta'')"] \arrow[d, "(i(f_1,g_1)\otimes i(f_3,g_3))\circ s"] & \det(\mathcal{F}_2) \otimes \det(\mathcal{F}_2{}'') \arrow[d, "i(f_2,g_2)"] \\ \det(\mathcal{F}_1{}') \otimes \det(\mathcal{F}_3{}') \arrow[r, "i(\alpha',\beta')"] & \det(\mathcal{F}_2{}') \end{tikzcd}$$

III. The functor $\det_X$ and the isomorphisms i agree with the usual determinant (top exterior power) and usual i on the complexes consisting of a single locally free sheaf of finite rank (in degree zero) and the short exact sequences of such.

IV. The collection of such functors is compatible with base change. If $f : X \to Y$ is a morphism of schemes, we have an isomorphism of functors (from Parf-is(Y) to Pic(X)) $\eta_f : f^* \circ \det_Y \xrightarrow{\sim} \det_X \mathbf{L}f^*$. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a true triangle on Y, then the following diagram commutes:

$$\begin{tikzcd} f^*(\det_Y(\mathcal{F}_1)) \otimes f^*(\det_Y(\mathcal{F}_3)) \arrow[r, "f^*(i)"] \arrow[d, "\eta\otimes\eta"] & f^*(\det_Y(\mathcal{F}_2)) \arrow[d, "\eta"] \\ \det_X(\mathbf{L}f^*\mathcal{F}_1) \otimes \det_X(\mathbf{L}f^*\mathcal{F}_3) \arrow[r, "i(\mathbf{L}f^*)"] & \det_X(\mathbf{L}f^*\mathcal{F}_2) \end{tikzcd}$$

Note our η is η^{-1} in [23].

Additionally if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of schemes, and $\theta : \mathbf{L}f^* \circ \mathbf{L}g^* \cong \mathbf{L}(gf)^*$ denotes the canonical isomorphism, then the following diagram commutes:

$$\begin{tikzcd}
f^*(g^* \det_Z) \arrow[rr, "\text{can}"] \arrow[d, "f^*(\eta_g)"] & & (gf)^* \det_Z \arrow[d, "\eta_{gf}"] \\
f^*(\det_Y \mathbf{L}g^*) \arrow[r, "\eta_f"] & \det_X \mathbf{L}f^*\mathbf{L}g^* \arrow[r, "\det_X(\theta)"] & \det_X \mathbf{L}(gf)^*
\end{tikzcd}$$

Remark 2.1.3. When X is reduced, i extends to distinguished triangles, and commutativity in $D(X)$ suffices. We then have Axiom II.B. for any isomorphism of distinguished triangles (in Parf(X)), and Axiom II.C. for any distinguished triangle of distinguished triangles [23, Prop. 7].

Lemma 2.1.4. *Let X be a normal (in particular, reduced) Noetherian scheme. Suppose $\mathcal{G} \in$ Parf(X) satisfies Supp($\mathcal{H}^q(\mathcal{G})$) has depth ≥ 2 for all q. (In particular, $\mathcal{G}|_\xi$ is exact for any ξ of depth 0.) Then:*

- *There is a unique isomorphism $\gamma : \det_X(\mathcal{G}) \cong \mathcal{O}_X$ such that $\gamma|_\xi = \det(0) : \det_\xi(\mathcal{G}|_\xi) \cong \det_\xi(0) = \mathcal{O}_{X,\xi}$ for any ξ of depth 0.*

- *If $q : \mathcal{G}_1 \to \mathcal{G}_2$ is a quasi-isomorphism of complexes satisfying the hypotheses on $\mathcal{G}$ above, the following diagram commutes:*

$$\begin{tikzcd}
\det_X(\mathcal{G}_1) \arrow[r, "\gamma_1"] \arrow[d, "\det(q)"] & \mathcal{O}_X \arrow[d, "="] \\
\det_X(\mathcal{G}_2) \arrow[r, "\gamma_2"] & \mathcal{O}_X
\end{tikzcd}$$

- *If $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to^{+1}$ is a distinguished triangle in Parf(X) such that each $\mathcal{G}_i$ satisfies the hypotheses on $\mathcal{G}$ above, the following diagram commutes:*

$$\begin{tikzcd}
\det_X \mathcal{G}_1 \otimes \det_X \mathcal{G}_3 \arrow[r] \arrow[d, "\gamma_1 \otimes \gamma_3"] & \det_X \mathcal{G}_2 \arrow[d, "\gamma_2"] \\
\mathcal{O}_X \otimes \mathcal{O}_X \arrow[r, "mult"] & \mathcal{O}_X
\end{tikzcd}$$

- *If $\mathcal{G} \in$ Coh(X) has finite Tor-dimension, γ agrees with $\bigtriangledown_{\mathcal{G}}$ in [10, Thm. 2.2].*

Proof. The isomorphism $\det_\xi(\mathcal{G}|_\xi) \cong \otimes_q(\det_\xi(\mathcal{H}^q(\mathcal{G}|_\xi)))^{(-1)^q} = \det_\xi(0)$ at the depth 0 points extends away from a set of depth ≥ 2 by hypothesis, hence it extends to all of X by normality. Since an isomorphism of line bundles is determined by its restriction to points of depth 0, the uniqueness is clear.
Similarly the second item follows from $\det(q|_\xi) = \det(0) : \det_X(\mathcal{G}_1|_\xi) \cong \det_X(\mathcal{G}_2|_\xi)$. To see the third item, it suffices to show the diagram commutes at each ξ of depth 0. Since the trivializations of acyclic complexes are additive on triangles [23, p.32], the commutativity follows.
The property of agreeing with the identity at points of depth 0 characterizes the homomorphism $\bigtriangledown_{\mathcal{G}}$, which is an isomorphism in our case. $\square$

We call the isomorphism $\det_X(\mathcal{G}) \cong \mathcal{O}_X$ of the preceding lemma the *canonical* γ. In the application, X will also be Cohen-Macaulay, so there will be no distinction between depth and codimension.

Cartier divisors associated to sheaves, complexes of sheaves, and morphisms. Here we review the definition of the Div of a morphism following [28, Ch. 5 S. 3].
First we define the Div of a sheaf. Let X be a Noetherian scheme and suppose $\mathcal{F} \in \mathrm{Coh}(X)$ satisfies:

- $\mathcal{F}$ is of finite cohomological dimension, i.e. $\mathcal{F}$ is perfect; and
- $\mathrm{Supp}(\mathcal{F})$ does not contain any points of depth 0.

Then we can define $\mathrm{Div}(\mathcal{F})$, a Cartier divisor on X. Given $x \in X$, a local equation for $\mathrm{Div}(\mathcal{F})$ on a neighborhood U of x is obtained as follows. Possibly after shrinking U, choose a resolution of $\mathcal{F}$ by coherent free $\mathcal{O}_U$-modules: $0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0$. Since $0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to 0$ is exact on $U - \mathrm{Supp}(\mathcal{F})$, this sequence determines a canonical isomorphism $\sigma : \mathcal{O}_{U-\mathrm{Supp}(\mathcal{F})} \to \otimes_i \det(\mathcal{E}_i)^{(-1)^i}$. Since the $\mathcal{E}_i$ are free on U, there is an isomorphism $\tau : \otimes_i \det(\mathcal{E}_i)^{(-1)^i} \to \mathcal{O}_U$ unique up to unit. The composition $\tau\sigma$ gives a nonzero section f of $\Gamma(U - \mathrm{Supp}(\mathcal{F}), \mathcal{O})$, and f defines $\mathrm{Div}(\mathcal{F})$ on U. One checks f is not a zero divisor, independence of choice of resolution, and that the equations agree on overlaps. We have $\mathrm{Supp}(\mathrm{Div}(\mathcal{F})) \subseteq \mathrm{Supp}(\mathcal{F})$. In [23] the construction is extended to perfect complexes supported at points of depth ≥ 1, and morphisms between perfect complexes which are quasi-isomorphisms at every point of depth 0.

The construction can also be extended to define the Div of certain morphisms of schemes. For now suppose $f : X \to Y$ is a projective morphism of Noetherian schemes of finite Tor-dimension satisfying:

- $f^{-1}(y)$ is empty for $y \in Y$ of depth 0, and
- $f^{-1}(y)$ is finite for $y \in Y$ of depth 1.

Then we define $\mathrm{Div}(f)$ to be $\mathrm{Div}(f_*\mathcal{M})$ for any invertible sheaf $\mathcal{M}$ on X such that $R^i f_*(\mathcal{M}) = 0$ for $i > 0$. Such an $\mathcal{M}$ exists because f is projective, and one checks $f_*\mathcal{M}$ has finite Tor-dimension and is not supported on any depth 0 points, and that the divisor thus obtained is independent of the choice of $\mathcal{M}$. We have $\mathrm{Supp}(\mathrm{Div}(f)) \subseteq f(X)$.

More generally, [23, Prop. 9] proves that if $f : X \to Y$ is a proper morphism of finite Tor-dimension, and $\mathcal{F}$ is a perfect complex on X satisfying the support conditions above (there conditions on $\mathcal{O}_X$), then for any $\mathcal{M} \in \mathrm{Pic}(X)$, we have

$$\mathrm{Div}(\mathbf{R}f_*(\mathcal{F})) = \mathrm{Div}(\mathbf{R}f_*(\mathcal{F} \otimes \mathcal{M})).$$

Therefore in the situation above, $\mathrm{Div}(f) = \mathrm{Div}\mathbf{R}f_*\mathcal{O}_X$.

2.2 Perfect complexes: operations and invariants

We begin with some very general facts about the behavior of perfect complexes under various operations. The standard reference is [14, Exp. I,II,III]; for a translation of some parts see [20].

Proposition 2.2.1. *Let $f : X \to Y$ be a smooth morphism of schemes. Suppose $\mathcal{F} \in Coh(X)$ has finite Tor-dimension (e.g. is flat) over Y. Then $\mathcal{F}$ is perfect on X, i.e $\mathcal{F} \in Parf(X)$.*

Proof. See [14, Exp. III. Prop. 3.6]. □

Lemma 2.2.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose $\mathcal{F} \in Parf(Y)$. Then $\boldsymbol{L}f^*\mathcal{F} \in Parf(X)$.*

Proof. See [14, Exp. I. Cor. 4.19.1(a)]. □

Proposition 2.2.3. *Let X be a scheme, and suppose $\mathcal{F}, \mathcal{G} \in Parf(X)$. Then $\mathcal{F} \otimes^{\boldsymbol{L}} \mathcal{G} \in Parf(X)$.*

Proof. See [14, Exp. I. Cor. 4.19.1(b)]. □

Proposition 2.2.4. *Let $f : X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) \in Parf(Y)$ (e.g. f is smooth). Suppose Y is Noetherian. Then for $\mathcal{F} \in Parf(X)$, we have $\boldsymbol{R}f_*\mathcal{F} \in Parf(Y)$.*

Proof. See [14, Exp. III. Cor. 4.8.1]. □

Lemma 2.2.5. *Let $i : Z \to X$ be a closed immersion of schemes such that $i_*(\mathcal{O}_Z) \in Parf(X)$, let $f : X \to Y$ be a morphism of schemes as in Proposition 2.2.4, and $p = f \circ i : Z \to Y$ the composition. Then there is a canonical isomorphism of functors*

$$\boldsymbol{R}f_*(i_*(\mathcal{O}_Z) \otimes^{\boldsymbol{L}} -) \cong \boldsymbol{R}p_*(\boldsymbol{L}i^* -)$$

from $Parf(X)$ to $Parf(Y)$.

Proof. For any $\mathcal{F} \in \mathrm{Parf}(X)$, we have $i_*(\mathcal{O}_Z) \otimes^{\mathbf{L}} \mathcal{F} = i_*(\mathbf{L}i^*\mathcal{F})$. Since i_* is exact, we have $\mathbf{R}p_* \cong \mathbf{R}f_* \circ \mathbf{R}i_* = \mathbf{R}f_* \circ i_*$. The claim follows. $\square$

Lemma 2.2.6. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}^\bullet \in D^+(X)$ such that $R^i f_* \mathcal{F}^j = 0$ for all j, all $i > 0$.*
Then $\boldsymbol{R}f_*\mathcal{F}^\bullet$ is quasi-isomorphic to the complex $f_*\mathcal{F}^\bullet$ (with the induced differential).

Proof. This is the Leray acyclicity lemma. $\square$

Lemma 2.2.7. *Let $f : X \to Y$ be an affine morphism of schemes. Then there is a canonical isomorphism of functors*

$$\boldsymbol{R}f_*(-) \cong f_*(-)$$

from $D^+_{qcoh}(X)$ to $D^+_{qcoh}(Y)$. (The right hand side means apply f_* termwise.)
If f is finite locally free and Y is Noetherian, the isomorphism is also as functors from Parf(X) to Parf(Y).

Proof. This is a consequence of the cohomological characterization of affine morphisms [13, II. Cor. 5.2.2] together with Lemma 2.2.6. That $D^+_{qcoh}(X)$ lands in $D^+_{qcoh}(Y)$ is a special case of [17, Ch. II Prop. 2.1]. For the final statement, we use Proposition 2.2.4 and finite $\Rightarrow$ proper. $\square$

Definition 2.2.8. Let $\pi : X \to T$ be a morphism of schemes. The relative dimension of π is the maximal dimension of any fiber. If $\mathcal{F} \in \mathrm{Coh}(X)$, the relative dimension of $\mathrm{Supp}(\mathcal{F})$ is the maximal dimension of the support of any fiber $\mathcal{F} \otimes_T \kappa(t) \in \mathrm{Coh}(X_t)$. We denote this number by $\dim(\mathrm{Supp}(\mathcal{F}))$. If $\mathcal{F} \in \mathrm{Parf}(X)$, the relative dimension of $\mathrm{Supp}(\mathcal{F})$ is the maximal dimension of the support of any (derived) fiber $\mathcal{F} \otimes^{\mathbf{L}}_T \kappa(t) \in \mathrm{Parf}(X_t)$.

Remark 2.2.9. By [19, 3.29] we have $\mathrm{Supp}(\mathcal{F}) \cap X_t = \mathrm{Supp}(\mathcal{F} \otimes^{\mathbf{L}} \kappa(t))$, so $\mathrm{Supp}(\mathcal{F}) = \max_{q,t} \dim(\mathrm{Supp}(\mathcal{H}^q(\mathcal{F})_t))$. Furthermore, given a base change $f : T' \to T$, let $f' : X \times_T T' \to X$ denote the canonical morphism. For $\mathcal{F} \in \mathrm{Parf}(X)$ such that

$\dim(\mathrm{Supp}(\mathcal{F})) \leq d$, we have $\dim(\mathrm{Supp}(\mathbf{L}f'^*(\mathcal{F}))) \leq d$. (Write $\mathbf{L}f'^*(\mathcal{F})\otimes^{\mathbf{L}}_{T'}\kappa(t') = \mathcal{F}\otimes^{\mathbf{L}}_{T}\kappa(f(t')) \otimes \kappa(t')$ and use the flatness of $\kappa(f(t')) \to \kappa(t')$.)

Let X be a variety (an integral separated scheme of finite type over a field, not necessarily geometrically integral) of dimension d, and let $V, W \subset X$ be subvarieties of dimensions a, b respectively. If $a + b < d$, then V and W are not expected to meet. The next result is a cohomological interpretation of this idea. First we recall that if X is a proper scheme over a field and $\mathcal{F} \in \mathrm{Coh}(X)$, $\chi(\mathcal{F}) := \sum_i (-1)^i h^i(X, \mathcal{F})$. If T is an integral scheme with generic point η, and X is a proper T-scheme, for $\mathcal{F} \in \mathrm{Coh}(X)$ we define $\chi_T(\mathcal{F}) := \chi(\mathcal{F}_\eta)$. We will often abuse notation and write $\chi(\mathcal{F})$ for $\chi_T(\mathcal{F})$ if no confusion seems likely to result.

Lemma 2.2.10. *Let T be an integral scheme, and let $\pi : X \to T$ be a smooth and proper morphism of relative dimension d. Suppose also X is irreducible. Let $\mathcal{F}, \mathcal{G} \in$ Parf(X) satisfy:*

$$\dim(\mathit{Supp}(\mathcal{F})) + \dim(\mathit{Supp}(\mathcal{G})) < d.$$

Then $\chi(\mathcal{F}\otimes^{\mathbf{L}}\mathcal{G}) = 0$.

Proof. Since the Euler characteristic is calculated at the generic point of T, we reduce to the case: X is irreducible, and smooth and proper over a field. Since the Euler characteristic is defined on K-theory, and because the group generated by coherent sheaves supported in dimension $\leq a$ is generated by $[\mathcal{O}_V]$, $V \subset X$ a subvariety of dimension $\leq a$ [11, Ex. 15.1.5], we may assume $\mathcal{F}, \mathcal{G}$ are structure sheaves of subvarieties of dimensions a, b respectively, with $a + b < d$.

Now since X is smooth, any coherent sheaf has a finite length resolution by finite rank locally free sheaves, so we may apply [11, 18.3.1 (c)] to the closed immersion $i : V \to X$ with $\beta = \mathcal{O}_V$. This gives, in the Chow group $A_*(X)$:

$$i_*(\mathrm{ch}(\mathcal{O}_V) \cap \mathrm{Td}(V)) = \mathrm{ch}(i_*\mathcal{O}_V) \cap \mathrm{Td}(X).$$

As $\mathrm{ch}(\mathcal{O}_V) = 1$ and $\mathrm{Td}(V) = [V] + r_V$ with $r_V \in A_{<a}(V)$, the left hand side lies in $A_{\leq a}(X)$.

Since X is smooth, $A^p(X) \cap A_q(X) \subset A_{q-p}(X)$ by [11, 8.3 (b)]. As $\mathrm{Td}(X) = [X] + r_X$ with $r_X \in A_{<d}(X)$, by equating terms in each degree, we find $\mathrm{ch}(i_*\mathcal{O}_V) \in A^{\geq d-a}(X)$. In summary:

$$\dim(\mathrm{Supp}(\mathcal{F})) \leq a \Rightarrow \mathrm{ch}_i(\mathcal{F}) = 0 \text{ for } i < d - a.$$

By Grothendieck-Riemann-Roch (for the smooth X, as in [11, 15.2.1]) and the action of ch on $\otimes$, $\chi(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) = \int_X \mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{G}) \cdot \mathrm{Td}_X$. Here $\cdot$ means intersection product of cycle classes. The first possible nonzero term in $\mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{G})$ would come from $\mathrm{ch}_{d-a}(\mathcal{F}) \cdot \mathrm{ch}_{d-b}(\mathcal{G})$, but this term is zero for degree reasons. □

Lemma 2.2.11. *With $\pi : X \to T$ as in Lemma 2.2.10, suppose in addition $d = 3$. Let $\mathcal{F}, \mathcal{G} \in Parf(X)$ satisfy:*

- $\dim(Supp(\mathcal{F})) \leq 1$*;*
- $rk(\mathcal{G}) = 0$*; and*
- $\det(\mathcal{G}) \cong \mathcal{O}_X$*.*

Then $\chi(\mathcal{F} \otimes^{\boldsymbol{L}} \mathcal{G}) = 0$.

Proof. The hypotheses on $\mathcal{G}$ imply $\mathrm{ch}_0(\mathcal{G}) = \mathrm{ch}_1(\mathcal{G}) = 0$, so we apply Grothendieck-Riemann-Roch as in the proof of Lemma 2.2.10. □

Corollary 2.2.12. *Let $\pi : X \to T$ be as in Lemma 2.2.10. Suppose $\mathcal{F} \in Coh(X)$ is perfect and $\dim(Supp(\mathcal{F})) \leq a$. Suppose $\mathcal{G} \in Parf(X)$ satisfies $\dim(Supp(\mathcal{G})) < d - a$. Let $u : \mathcal{F} \to \mathcal{F}$ be an automorphism. Then $\det_T \boldsymbol{R}\pi_*(u \otimes^{\boldsymbol{L}} 1)$ is the identity on $\det_T \boldsymbol{R}\pi_*(\mathcal{F} \otimes^{\boldsymbol{L}} \mathcal{G})$.*

Proof. It suffices to show $\det_T \mathbf{R}\pi_*(u \otimes^{\mathbf{L}} 1)$ is the identity at the generic point of T, so we reduce to the case T is the spectrum of a field K. We can base change to an algebraic closure $\overline{K}$ and check the result there, so assume $K = \overline{K}$. Now $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$ is a finite K-algebra. Therefore u satisfies a monic equation P with coefficients in K, and since K is algebraically closed, the equation splits: $P(X) = \prod_i (X - \lambda_i)^{e_i}$.
We proceed by induction on $\deg(P)$. If $\deg(P) = 1$, u is multiplication by some element (also called u) in K. Then $\det_T \mathbf{R}\pi_*(u \otimes^{\mathbf{L}} 1)$ acts by $u^{\chi(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G})} = 1$.
For the induction step, we use the following isomorphism of short exact sequences (u' denotes the restriction).

$$\begin{tikzcd}
0 \arrow[r] & (u-\lambda_i)\mathcal{F} \arrow[r] \arrow[d, "u'"] & \mathcal{F} \arrow[r] \arrow[d, "u"] & \overline{\mathcal{F}} \arrow[r] \arrow[d, "\overline{u}=\lambda_i"] & 0 \\
0 \arrow[r] & (u-\lambda_i)\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \overline{\mathcal{F}} \arrow[r] & 0
\end{tikzcd}$$

Now $\det_T \mathbf{R}\pi_*(u \otimes^{\mathbf{L}} 1) = \det_T \mathbf{R}\pi_*(u' \otimes^{\mathbf{L}} 1) \det_T \mathbf{R}\pi_*(\overline{u} \otimes^{\mathbf{L}} 1)$ follows from Axiom II.B. of Definition-Theorem 2.1.2. By the induction hypothesis and the case $\deg(P) = 1$, each of the factors on the right hand side is the identity. $\square$

We conclude this section with two consequences of the moving lemma. We use the notation $T_R := T \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$ for a k-scheme T and a k-algebra R. We use the subscript $_0$ to denote the closed fiber of an object over a DVR.

Lemma 2.2.13. *Let X be a smooth projective variety over a field k, and let $R \supset k$ be a DVR. Let $Z \subset X_R$ be a subscheme which is finite over R. Let $L \in Pic(X)$. Then $({\pi_1}^* L)|_Z \cong \mathcal{O}_Z$.*

Proof. Write $L = \mathcal{O}_X(D)$ for a Cartier divisor $D \subset X$. By the moving lemma [31, Thm.], we can find a divisor D' on X such that $D \sim D'$ and $(D'_R)_0 \cap Z_0 = \emptyset$. The set $D'_R \cap Z$ is closed in Z; since $(D'_R \cap Z)_0 = \emptyset$, we have $D'_R \cap Z = \emptyset$. The bundle $\mathcal{O}_{D'_R}$ has a (canonical) trivialization on $X_R - \mathrm{Supp}(D'_R)$, in particular on Z. We have a global isomorphism $\mathcal{O}(D_R) \cong \mathcal{O}(D'_R)$ on X_R, so the result follows. $\square$

Now we state a K-theoretic consequence of the moving lemma.

Lemma 2.2.14. *Let X be a smooth projective variety of dimension $n \geq a+b+1$ over a (not necessarily algebraically closed) field. Let $C \subset X$ be a closed subset of dimension $\leq a$, and let $Y \subset X$ be a subscheme of dimension $\leq b$. Then there exist $T_i \in Coh(X)$ and $n_i \in \mathbb{Z}$ such that:*

(1) $[\mathcal{O}_Y] = \sum n_i[T_i]$ in $K(X)$,

(2) $\dim(Supp(T_i)) \leq b$ for all i, and

(3) $C \cap Supp(T_i) = \emptyset$ for all i.

Proof. We use the classical moving lemma (which is valid over any field, see [31, Thm.]) and the surjective homomorphism of groups [11, Ex. 15.1.5]:

$$A_k(X) \to \mathrm{Gr}_k K(X) := F_k(K(X))/F_{k-1}(K(X)).$$

Move $[Y]$ to a b-cycle Y' such that $C \cap \mathrm{Supp}(Y') = \emptyset$. This gives, in $K(X)$, $[\mathcal{O}_Y] = \sum n_i T_i + \sum_j m_j Z_j$ where the T_i satisfy the second and third conditions of the lemma, and $\dim(\mathrm{Supp}(Z_j)) \leq b-1$ for all j. Now move the Z_j away from C to obtain $\sum_j m_j Z_j = \sum_k n_k T_k$ where the T_k also satisfy the second and third conditions of the lemma and are supported in dimension $\leq b-2$. This process terminates with $T_i \in \mathrm{Coh}(X)$ with the stated properties. □

2.3 The Chow variety and incidence divisor for $\mathbb{P}^n$

In this section we define the Chow variety and the Hilbert-Chow morphism, and construct the incidence divisor for $\mathbb{P}^n$.

Definition-Theorem 2.3.1 (Existence of the Chow variety)**.** Let P be a smooth projective variety over a field k. The Chow variety $\mathscr{C}_{d,d'}$ of P is a k-scheme with the following properties.

(1) It is projective over k.

(2) It is seminormal.

(3) For every point $w \in \mathscr{C}_{d,d'}$ there exist purely inseparable field extensions $\kappa(w) \subset L_i$ and cycles Z_i on P_{L_i} such that:

(a) Z_i and Z_j are essentially equivalent [24, I.3.8]: they agree as cycles over the perfection $\kappa(w)^{perf}$ of $\kappa(w)$;

(b) the intersection of the fields L_i is $\kappa(w)$, which is the Chow field (field of definition of the Chow form in any projective embedding of P; see 2.3.6) of any of the Z_i [24, I.3.24.1]; and

(c) for any cycle Z on P_M defined over a subfield $k \subset M \subset \kappa(w)^{perf}$ which agrees with the Z_i over $\kappa(w)^{perf}$ (equivalently, agrees with one Z_i), we have $\kappa(w) \subset M$ (the Chow field is the intersection of all fields of definition of the cycle).

(4) Points w of $\mathscr{C}_{d,d'}$ are in bijective correspondence with systems $(k \subset \kappa(w), \{\kappa(w) \subset L_i, Z_i\}_{i\in I})$ up to an obvious equivalence relation.

(5) For any DVR $R \supset k$ and any cycle Z on P_R of relative dimension d and degree d' in the generic fiber, we obtain a morphism $g : \operatorname{Spec} R \to \mathscr{C}_{d,d'}$ such that the generic fiber Z_η and the special fiber Z_s agree with the systems of cycles of the previous property at $g(\eta)$ and $g(s)$.

(6) For any numerical polynomial q of degree d and with leading coefficient $d'/(d!)$, we obtain a morphism (the Hilbert-Chow morphism)

$$FC : (\mathscr{H}^q)^{sn}_{red} \to \mathscr{C}_{d,d'}$$

by taking the fundamental cycle of the components of maximal relative dimension $(= d)$[24, I.6.3.1]. A finite number of $(\mathscr{H}^q)^{sn}_{red}$'s surject onto $\mathscr{C}_{d,d'}$.

(7) Let $\eta \in \mathscr{C}_{d,d'}$ be a generic point. Then either $\dim \overline{\{\eta\}} = 0$ or there exists a cycle Z_η on P_η defined over $\kappa(\eta)$. In particular, if k is perfect then there exists a Z_η for every generic point η of $\mathscr{C}_{d,d'}$ [24, I.4.14].

Remark 2.3.2. Properties (1)-(4) and (6) characterize $\mathscr{C}_{d,d'}$. Suppose given two solutions $\mathscr{C}^1$ and $\mathscr{C}^2$. Now consider $T \subset \mathscr{C}^1 \times_k \mathscr{C}^2$, the image of $FC_1 \times FC_2 : (\mathscr{H})^{sn}_{red} \times_k (\mathscr{H})^{sn}_{red} \to \mathscr{C}^1 \times_k \mathscr{C}^2$. Then $T \to \mathscr{C}^i$ is a proper bijection inducing an isomorphism on all residue fields, hence is an isomorphism by the seminormality of $\mathscr{C}^i$.

Remark 2.3.3. Properties (1)-(5) characterize $\mathscr{C}_{d,d'}$. This follows from the characterization of seminormal schemes (Theorem 3.1.5).

Remark 2.3.4. If $k \to k'$ is a field extension, then the Chow variety of $P_{k'}$ over k' is the seminormalization of the reduction of $\mathscr{C}_{d,d'} \times_k k'$. If k is perfect, since $\mathscr{C}_{d,d'}$ is finite type, $\mathscr{C}_{d,d'} \times_k k'$ is already reduced and seminormal [12, 5.9].

Definition 2.3.5. Let P be a smooth projective variety of dimension n over a field k. Then the incidence locus $\mathscr{I} \subset \mathscr{C}_d \times_k \mathscr{C}_{n-d-1}$ is defined (set-theoretically) as $\{(Z, W) | Z \cap W \neq \emptyset\}$. This is a closed subscheme.

The main point of this thesis is to construct a Cartier divisor supported on the incidence locus. Now we explain how one obtains the incidence divisor from the construction of the Chow variety for $\mathbb{P}^n_k$, k an algebraically closed field.

Construction 2.3.6. Let $\mathscr{G}$ denote the Grassmannian of $(n-d-1)$-planes in $\mathbb{P}^n$. Let $\mathscr{D} \hookrightarrow \mathrm{CDiv}(\mathscr{G}) \times \mathscr{G}$ denote the universal Cartier divisor (consisting of pairs (D, L) such that $L \in D$). Identify a d-cycle V in $\mathbb{P}^n$ with its "Chow point" F_V, the codimension 1 set of linear spaces of dimension $(n-d-1)$ which meet V. (The defining equation of F_V is called the Chow form of V; see e.g. [24, I.3.23.4].) The assignment $V \mapsto F_V$ is injective [24, I.3.24.5] and works in families [24, I.3.23.1.1]. Classically the Chow variety was defined as the reduced closed subset $\mathscr{C}' \subset \mathrm{CDiv}(\mathscr{G})$ of forms arising as the

Chow form of some cycle. For us the Chow variety $\mathscr{C}_d(\mathbb{P}^n)$ is the seminormalization of $\mathscr{C}'$.

Proposition 2.3.7. *The incidence locus* $\mathscr{I} \hookrightarrow \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{G}$ *of Definition* 2.3.5 *has the structure of an effective Cartier divisor.*

Proof. For this discussion we can use the classical construction (the subscheme $\mathscr{C}' \subset \mathrm{CDiv}(\mathscr{G})$) and seminormalize at the end. Taking the product with $\mathscr{G}$ yields a closed immersion:

$$\mathscr{C}' \times \mathscr{G} \xrightarrow{i} \mathrm{CDiv}(\mathscr{G}) \times \mathscr{G}.$$

Note that in every component $C \subset \mathscr{C}'$, we can find $(n-d-1)$-planes missing a general member of C. Hence the pullback Cartier divisor $i^*\mathscr{D}$ is defined. Now $i^*\mathscr{D}$ consists of pairs (V, L) such that $L \in F_V$, which means exactly that $L \cap V \neq \emptyset$. Therefore the incidence locus $\mathscr{I} \hookrightarrow \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{G}$ consisting of pairs (V, L) such that $L \cap V \neq \emptyset$, is the Cartier divisor $(i^*\mathscr{D})^{sn}$. □

Corollary 2.3.8. *For any fixed linear subspace* $L_0 \in \mathscr{G}$*, the set of d-cycles meeting* L_0 *has the structure of an effective Cartier divisor in* $\mathscr{C}_d(\mathbb{P}^n)$*.*

Construction 2.3.9 (Ruled join)**.** [11, 8.4.5] Let V and W be subvarieties of $\mathbb{P}^n$ of dimensions k and l respectively. We construct a variety $J(V, W)$, the ruled join of V and W, of dimension $k + l + 1$, inside $\mathbb{P}^{2n+1}$.

Define embeddings i_1, i_2: $\mathbb{P}^n \longrightarrow \mathbb{P}^{2n+1}$ as follows:

$i_1([X_0 : X_1 : \ldots : X_n]) = [X_0 : X_1 : \ldots : X_n : 0 : 0 : \ldots : 0]$

$i_2([X_0 : X_1 : \ldots : X_n]) = [0 : 0 : \ldots : 0 : X_0 : X_1 : \ldots : X_n]$

The join $J(V, W)$ consists of all lines in $\mathbb{P}^{2n+1}$ connecting a point of $i_1(V)$ to a point of $i_2(W)$; $J(V, W)$ is a subvariety of dimension $k + l + 1$ and degree = (deg V)(deg W). In coordinates, $J(V, W) = \{[aX_0 : aX_1 : \ldots : aX_n : bY_0 : \ldots : bY_n] \mid X \in V; Y \in W; [a : b] \in \mathbb{P}^1\}$. Algebraically, if $[X_0 : \ldots : X_n : Y_0 : \ldots : Y_n]$ are coordinates on

$\mathbb{P}^{2n+1}$, and $\{F_i\}$ generate the ideal of V and $\{G_j\}$ generate the ideal of W, then the ideal of $J(V,W)$ is generated by $\{F_i(X); G_j(Y)\}$.

Now let $\mathbb{P}^n \cong L \subset \mathbb{P}^{2n+1}$ be the subvariety cut out by the equations $\{X_i = Y_i\}$. We have an isomorphism of schemes: $V \cap W \cong L \cap J(V,W)$. In particular, $V \cap W \neq \emptyset \iff L \cap J(V,W) \neq \emptyset$.

We omit the proof of the following result, which is straightforward. Anyway we only use it to show the ruled join induces a morphism of Chow varieties, which is already known over $\mathbb{C}$ (see the proof of Theorem 2.3.11 below).

Lemma 2.3.10. *The ruled join is compatible with specialization. In other words, if R is a DVR with residue field k_0, and $V, W \in Z_*(\mathbb{P}^n_R)$ are cycles on $\mathbb{P}^n_R$ with cycle-theoretic fibers ([24, I.3.9]) $V_0, W_0 \in Z_*(\mathbb{P}^n_{k_0})$, then we have an equality of cycles:*

$$J(V_0, W_0) = (J(V,W))_0 \in Z_*(\mathbb{P}^{2n+1}_{k_0}).$$

Theorem 2.3.11. *The incidence locus $\mathscr{I} \hookrightarrow \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{C}_{n-d-1}(\mathbb{P}^n)$ has the structure of an effective Cartier divisor.*

Proof. Consider the map $RJ : \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{C}_{n-d-1}(\mathbb{P}^n) \to \mathscr{C}_n(\mathbb{P}^{2n+1})$ defined by

$$(\sum_i d_i V_i, \sum_j e_j W_j) \mapsto \sum_{i,j} d_i e_j J(V_i, W_j).$$

This is a morphism of algebraic varieties: using seminormality (Corollary 3.1.13), this follows from the compatibility of the ruled join with specialization (Lemma 2.3.10). Over $\mathbb{C}$, a direct algebraic proof using Chow forms is given in [29, Thm. 2.3].

By Corollary 2.3.8 the set of n-cycles in $\mathbb{P}^{2n+1}$ meeting the $(2n+1) - n - 1 = n$-dimensional linear subspace L (defined above) is a Cartier divisor D_{inc} in $\mathscr{C}_n(\mathbb{P}^{2n+1})$. Since there exist pairs of nonintersecting cycles in every pair of irreducible components, $\mathscr{C}_d(\mathbb{P}^n) \times \mathscr{C}_{n-d-1}(\mathbb{P}^n) \not\subset (RJ)^{-1}(D_{inc})$. Therefore the pullback Cartier divisor $(RJ)^*(D_{inc})$ is defined. As $V \cap W \neq \emptyset \iff L \cap J(V,W) \neq \emptyset$, $(RJ)^*(D_{inc})$ consists exactly of those pairs (V,W) such that $V \cap W \neq \emptyset$. $\square$

2.4 The Hilbert-Chow morphism and the incidence locus

In this section we define an invertible sheaf (the "incidence bundle") on a product of Hilbert schemes, and show the incidence bundle is pulled back from a product of Chow varieties in the case of $\mathbb{P}^n$.

Construction 2.4.1. Let P be a smooth projective variety over any base scheme B (over which we take all fiber products), and let $\mathscr{H}_1, \mathscr{H}_2$ denote the Hilbert schemes corresponding to numerical polynomials q_1, q_2. Over each $\mathscr{H}_i$ we have a universal flat family (a closed subscheme of $P \times \mathscr{H}_i$); denote by $\mathscr{U}_i$ its pullback to $P \times \mathscr{H}_1 \times \mathscr{H}_2$. Then the first four results from Section 2.2 imply $\mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathbf{L}} \mathcal{O}_{\mathscr{U}_2})$ is a perfect complex on $\mathscr{H}_1 \times \mathscr{H}_2$. The incidence bundle $\mathcal{L}$ is defined to be its determinant:

$$\mathcal{L} := \det{}_{\mathscr{H}_1 \times \mathscr{H}_2} \mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathbf{L}} \mathcal{O}_{\mathscr{U}_2}).$$

Remark 2.4.2. We are interested in the case B is the spectrum of a field and $\deg(q_1) + \deg(q_2) + 1 = \dim(P)$, but the definition presented here makes sense with no restrictions on the base B or the degrees of the q_i.

Given a morphism $f : S \to \mathscr{H}_1 \times \mathscr{H}_2$, we now describe the pullback line bundle $f^*\mathcal{L}$ on S. Consider the cartesian square:

$$\begin{tikzcd}
P \times S \arrow[r, "f'"] \arrow[d, "p"] & P \times \mathscr{H}_1 \times \mathscr{H}_2 \arrow[d, "pr_{23}"] \\
S \arrow[r, "f"] & \mathscr{H}_1 \times \mathscr{H}_2
\end{tikzcd}$$

Using the compatibility of $\det \mathbf{R}p_*(-)$ with base change (this holds for p proper of finite Tor-dimension [23, p.46]) and the identification

$$\mathbf{L}f'^*(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathbf{L}} \mathcal{O}_{\mathscr{U}_2}) \cong \mathbf{L}f'^*(\mathcal{O}_{\mathscr{U}_1}) \otimes^{\mathbf{L}} \mathbf{L}f'^*(\mathcal{O}_{\mathscr{U}_2}) \cong f'^*(\mathcal{O}_{\mathscr{U}_1}) \otimes^{\mathbf{L}} f'^*(\mathcal{O}_{\mathscr{U}_2}),$$

we obtain a canonical identification:

$$f^*\mathcal{L} \cong \det{}_S \mathbf{R} p_*(\mathcal{O}_{\mathscr{U}_{1S}} \otimes^{\mathbf{L}} \mathcal{O}_{\mathscr{U}_{2S}}).$$

In particular, if $S = s = \operatorname{Spec} k$ and f corresponds to the pair of subschemes Z_1, Z_2, then the fiber of $\mathcal{L}$ at s is $\mathcal{L}|_s = \det_k \mathbb{H}(\mathcal{O}_{Z_1} \otimes^{\mathbf{L}} \mathcal{O}_{Z_2})$.

Now we analyze the relationship between the determinant formula and the Hilbert-Chow morphism as defined in [24, I.6.3.1] and [28, Ch. 5 S. 4] in case $P = \mathbb{P}^n$. We use the notation from Construction 2.3.6.

Let $\mathscr{H}$ denote the Hilbert scheme of $\mathbb{P}^n$ corresponding to a numerical polynomial of degree d and with leading coefficient $d'/(d!)$. Since seminormalization is a functor, constructing $\mathscr{H} \to \operatorname{CDiv}(\mathscr{G})$ which factors through $\mathscr{C}'$ will determine the morphism $FC : \mathscr{H}^{sn}_{red} \to \mathscr{C}$ of Definition-Theorem 2.3.1.

The Hilbert-Chow morphism (for $\mathbb{P}^n$) is defined by producing (functorially) for every S-point of $\mathscr{H}$ an S-point of $\operatorname{CDiv}(\mathscr{G})$ which factors through $\mathscr{C}' \hookrightarrow \operatorname{CDiv}(\mathscr{G})$. Let $\mathscr{U}$ denote the universal hyperplane in $\mathbb{P}^n \times \mathscr{G}$ (and its pullbacks), and pr_1, pr_2 the projections from $\mathscr{U}$. Let $i : V \hookrightarrow \mathbb{P}^n \times S$ define an S-point of $\mathscr{H}$. Consider the diagram:

$$\begin{tikzcd} V' \arrow[r, "i'"] \arrow[d] & \mathscr{U} \arrow[d, "pr_1"] \arrow[dr, "pr_2"] & \\ V \arrow[r, "i"] & \mathbb{P}^n \times S & \mathscr{G} \times S \end{tikzcd} \tag{2.4.1}$$

where the square on the left is a fiber square. Then $\operatorname{Div}(pr_2 \circ i')$ is an S-point of $\operatorname{CDiv}(\mathscr{G})$ which factors through $\mathscr{C}'$ [28, Ch. 5 S. 4 5.10].

The next result is basically contained in [9, Thms. 1.2, 1.4].

Proposition 2.4.3. *With the notation as in Construction* 2.4.1, *suppose in addition* $P = \mathbb{P}^n$ *and* $\mathscr{H}_2 = \mathscr{G}$, *a Grassmannian. Let* $F : \mathscr{H} \to \mathscr{C}$ *denote the Hilbert-Chow morphism (and its pullbacks), and let* $\mathscr{I} \hookrightarrow \mathscr{C} \times \mathscr{G}$ *denote the incidence divisor of*

Proposition 2.3.7. *Then there is a canonical isomorphism*

$$\mathcal{L} \cong F^*\mathcal{O}(\mathscr{I})$$

of invertible sheaves on $\mathscr{H} \times \mathscr{G}$.

Proof. The basic point is that the universal family of $(n-d-1)$-planes is flat over $\mathbb{P}^n$, so the derived tensor product reduces to a usual tensor product. In the diagram (2.4.1) put $S = \mathscr{H}$, so V is the universal flat family. Consider the diagram:

$$\begin{tikzcd}
V'' \arrow[r, "i''"] \arrow[d, "r"] & \mathscr{U} \arrow[d, "h"] \arrow[dr, "f"] & \\
V' \arrow[r, "i'"] \arrow[d, "s"] & \mathbb{P}^n \times \mathscr{H} \times \mathscr{G} \arrow[r, "pr_{23}"] \arrow[d, "pr_{12}"] & \mathscr{H} \times \mathscr{G} \\
V \arrow[r, "i"] & \mathbb{P}^n \times \mathscr{H} &
\end{tikzcd}$$

in which the left squares are fiber squares. Then the procedure described above gives a Cartier divisor in $\mathscr{H} \times \mathscr{G}$, namely $\mathrm{Div}(f \circ i'') = \mathrm{Div}\mathbf{R}(fi'')_*\mathcal{O}_{V''}$.

Since h is a closed immersion, by Lemma 2.2.5 we have an identification $h_*\mathbf{L}h^*i'_*\mathcal{O}_{V'} \cong i'_*\mathcal{O}_{V'}\otimes^{\mathbf{L}}h_*\mathcal{O}_{\mathscr{U}}$. Note also that $\mathcal{O}_{V'} = s^*\mathcal{O}_V$. By considering the other ways of traversing the square from V to $\mathscr{U}$, we obtain canonical isomorphisms:

$$\mathbf{L}h^*i'_*\mathcal{O}_{V'} \cong \mathbf{L}(pr_{12} \circ h)^*i_*\mathcal{O}_V \cong i''_*\mathbf{L}(sr)^*\mathcal{O}_V = i''_*(sr)^*\mathcal{O}_V = i''_*\mathcal{O}_{V''}.$$

From this we extract $i'_*\mathcal{O}_{V'}\otimes^{\mathbf{L}}h_*\mathcal{O}_{\mathscr{U}} \cong h_*i''_*\mathcal{O}_{V''}$.

Therefore $\mathbf{R}pr_{23*}(i'_*\mathcal{O}_{V'}\otimes^{\mathbf{L}}h_*\mathcal{O}_{\mathscr{U}}) \cong \mathbf{R}pr_{23*}(h_*i''_*\mathcal{O}_{V''}) \cong \mathbf{R}f_*(i''_*\mathcal{O}_{V''})$.

Hence $\mathcal{L} \cong \det \mathbf{R}f_*(i''_*\mathcal{O}_{V''}) \cong \mathcal{O}(\mathrm{Div}(f \circ i''))$, canonically.

The diagram of divisors on the left is obtained by pulling back along the diagram of schemes on the right:

$$\begin{tikzcd}
\mathrm{Div}(f \circ i'') \arrow[rr] \arrow[dr] & & \mathscr{D} \\
& \mathscr{I} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{H} \times \mathscr{G} \arrow[rr] \arrow[dr, "F"] & & \mathrm{CDiv}(\mathscr{G}) \times \mathscr{G} \\
& \mathscr{C} \times \mathscr{G} \arrow[ur] &
\end{tikzcd}$$

Therefore $\mathrm{Div}(f \circ i'') = F^*\mathscr{I}$, and we obtain $\mathcal{L} \cong F^*\mathcal{O}(\mathscr{I})$ as desired. □

Remark 2.4.4. We expect the ruled join to be compatible with the Hilbert-Chow morphism, hence we expect the natural generalization of Proposition 2.4.3: the pull-back of the line bundle constructed in Theorem 2.3.11 via the morphism $\mathscr{H}^d(\mathbb{P}^n) \times \mathscr{H}^{n-d-1}(\mathbb{P}^n) \to \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{C}_{n-d-1}(\mathbb{P}^n)$ is the incidence bundle defined in Construction 2.4.1.

Chapter 3

Seminormal schemes

3.1 A characterization of seminormal schemes

Notation. If R is a ring and $\mathfrak{p} \subset R$ is a prime ideal, let $\kappa(\mathfrak{p})$ denote $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$, the residue field of $\mathfrak{p}$. If $f : R \to S$ is a ring homomorphism and $\mathfrak{q} \subset S$ is a prime ideal with $\mathfrak{p} := f^{-1}\mathfrak{q}$, let $\overline{f_\mathfrak{p}} : \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ denote the induced map on residue fields (we may write $\overline{f}$ where the prime is clear). If R is a discrete valuation ring, let K denote its field of fractions and k_0 its residue field; if $r \in R$, let $\overline{r}$ denote the class of r in k_0.

We take the following definitions from [12]: a ring R is a *Mori ring* if it is reduced and its integral closure R^ν (in its total quotient ring Q) is finite over it; if R is a Mori ring, R^{sn} denotes its seminormalization, the largest subring $R \subset R^{sn} \subset R^\nu$ such that Spec $R^{sn} \to$ Spec R is bijective and all maps on residue fields are isomorphisms. The seminormalization is described elementwise in [24, I.7.2.3]. We say R is seminormal if $R = R^{sn}$ (so we only define seminormality for Mori rings). For a Mori ring R, it is a theorem of Hamann [16] that R is seminormal if and only if for all $a \in Q$, $a^2, a^3 \in R \Rightarrow a \in R$. Brewer and Nichols [5] observe that R is seminormal if and only if for all $a \in Q$, $a^n, a^{n+1}, \ldots \in R$ for some $n > 0$ implies $a \in R$.

Definition 3.1.1. Let A be a ring, and let $S = \{f_y \in \kappa(y) | y \in \text{Spec } A\}$ be a collection of elements, one in each residue field. Then we say S *is a pointwise function* *on (Spec) A.* We say the pointwise function S *varies algebraically along (complete) DVRs* if it has the following property: for every specialization $\mathfrak{p}_1 \subset \mathfrak{p}_2$ in A and every (complete) discrete valuation ring R covering that specialization via a ring homomorphism $g : A \to R$, there exists a (necessarily) unique $f_R \in R$ such that $\overline{g_{\mathfrak{p}_1}}(f_{\mathfrak{p}_1}) = f_R$ and $\overline{g_{\mathfrak{p}_2}}(f_{\mathfrak{p}_2}) = \overline{f_R}$.

Remark 3.1.2. Any specialization is covered by a DVR by [13, 7.1.4], in fact by a complete DVR with algebraically closed residue field.

Now we show that a pointwise function which varies algebraically along DVRs pushes forward (functorially) via a ring homomorphism.

Lemma 3.1.3. *Let $\varphi : A \to B$ be a ring homomorphism, and let $S = \{f_y \in \kappa(y) | y \in \text{Spec } A\}$ be a pointwise function which varies algebraically along DVRs. Then B naturally inherits a pointwise function $S^\varphi = \{f_z \in \kappa(z) | z \in \text{Spec } B\}$ which varies algebraically along DVRs.*

Proof. For $\mathfrak{p} \in \text{Spec } B$, define $f_\mathfrak{p} = \overline{\varphi}(f_{\varphi^{-1}\mathfrak{p}})$; this gives the collection S^φ. Now suppose given a specialization $\mathfrak{p}_1 \subset \mathfrak{p}_2$ in B and a DVR R covering this specialization via $g : B \to R$. Consider the diagram (whose vertical arrows are the canonical maps):

$$\begin{tikzcd}
\kappa(\varphi^{-1}\mathfrak{p}_1) \arrow[r, "\overline{\varphi}"] & \kappa(\mathfrak{p}_1) \arrow[r, "\overline{g}"] & K \\
A \arrow[u] \arrow[d] \arrow[r, "\varphi"] & B \arrow[u] \arrow[d] \arrow[r, "g"] & R \arrow[u] \arrow[d] \\
\kappa(\varphi^{-1}\mathfrak{p}_2) \arrow[r, "\overline{\varphi}"] & \kappa(\mathfrak{p}_2) \arrow[r, "\overline{g}"] & k_0
\end{tikzcd}$$

Since taking the preimage is functorial, $g \circ \varphi : A \to R$ covers the specialization $\varphi^{-1}\mathfrak{p}_1 \subset \varphi^{-1}\mathfrak{p}_2$. By the assumption that S varies algebraically along DVRs, we have $f_R \in R$ which agrees with $\overline{g\varphi}(f_{\varphi^{-1}\mathfrak{p}_1})$ in K and with $\overline{g\varphi}(f_{\varphi^{-1}\mathfrak{p}_2})$ in k_0. Since the map on residue fields is functorial, and by the definition of the pointwise function S^φ, the same f_R works for $g : B \to R$. Thus S^φ varies algebraically along DVRs. $\square$

Remark 3.1.4. It is clear from the proof that the construction is functorial: if $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ are ring homomorphisms, and S is a pointwise function on A, then $S^{\psi\circ\varphi} = (S^{\varphi})^{\psi}$ as pointwise functions on C.

Theorem 3.1.5. *Let A be a seminormal (in particular, Mori) ring which is Noetherian. Let $\{f_y \in \kappa(y) | y \in Spec\ A\}$ be a pointwise function on A which varies algebraically along DVRs. Then there exists a unique $f \in A$ whose image in $\kappa(y)$ is f_y for all $y \in Spec\ A$.*

Remark 3.1.6. Once we show existence, uniqueness follows from reducedness.

Our strategy is to first prove the theorem in the case A is an integrally closed Noetherian domain, and use this to prove the theorem in the case A is any integrally closed Noetherian ring. Next we apply the functoriality just described to the natural inclusion $A \to A^{\nu}$. Finally we show the element $f \in A^{\nu}$ obtained by applying the theorem to the collection S^{ν} for A^{ν} actually lies in the seminormalization of A.
Having proved the main theorem, we show it suffices to have the condition only on complete DVRs, and we prove also the converse.

Lemma 3.1.7. *Let A be an integrally closed Noetherian domain, and let S be a pointwise function on A which varies algebraically along DVRs. Then there exists a unique $f \in A$ whose image in $\kappa(y)$ is f_y for all $y \in Spec\ A$.*

Proof. Actually, we will only need to use certain DVRs. Let $\mathfrak{p} \subset A$ be a height 1 prime. The ring $A_{\mathfrak{p}}$ is a DVR, and the localization $g : A \to A_{\mathfrak{p}}$ covers the specialization $(0) \subset \mathfrak{p}$. By the hypothesis on S, there exists $f_{A_{\mathfrak{p}}} \in A_{\mathfrak{p}}$ agreeing with f_y for $y = (0), \mathfrak{p}$. In the diagram (whose vertical arrows are the canonical maps):

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & A_{\mathfrak{p}} \arrow[d] \\ \kappa((0)) \arrow[r, "\overline{g_{(0)}}"] & K \end{tikzcd}$$

all of the arrows are inclusions, and $\overline{g_{(0)}}$ is the identity on the fraction field of A. Therefore we know the element $f_{(0)} \in S$ is $f_{A_{\mathfrak{p}}}$. In particular, we see $f_{(0)} \in A_{\mathfrak{p}}$ for all

$\mathfrak{p}$ of height 1. But for an integrally closed Noetherian domain we have $A = \cap_{\mathfrak{p} \text{ ht } 1} A_{\mathfrak{p}}$, so in fact $f_{(0)} \in A$. The element $f_{(0)}$ is part of the data of S, so it is clearly unique. To see $f_{(0)}$ has the correct value at every $\mathfrak{p} \in \text{Spec } A$, not necessarily of height 1, we use the fact that any specialization in A can be covered by a DVR. So for arbitrary $\mathfrak{p} \in \text{Spec } A$, consider $g : A \to R$ covering the specialization $(0) \subset \mathfrak{p}$. We use the diagram whose vertical arrows are the canonical maps:

$$\begin{tikzcd}
\kappa((0)) \arrow[r, "\overline{g_{(0)}}"] & K \\
A \arrow[u] \arrow[r, "g"] \arrow[d] & R \arrow[u] \arrow[d] \\
\kappa(\mathfrak{p}) \arrow[r, "\overline{g_{\mathfrak{p}}}"] & k_0
\end{tikzcd}$$

We have $f_R \in R$ agreeing with $f_{(0)}$ in K. Since we know $f_{(0)} \in A$, we have $g(f_{(0)}) = f_R$. Since the bottom square is commutative and $\overline{g_{\mathfrak{p}}}$ is injective, $\overline{g(f_{(0)})} = \overline{f_R} = \overline{g_{\mathfrak{p}}}(f_{\mathfrak{p}})$ implies the image of $f_{(0)}$ in $\kappa(\mathfrak{p})$ is $f_{\mathfrak{p}}$. □

Lemma 3.1.8. *Let A be an integrally closed Noetherian ring, and let S be a pointwise function on A which varies algebraically along DVRs. Then there exists a unique $f \in A$ whose image in $\kappa(y)$ is f_y for all $y \in$ Spec A.*

Proof. If $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ denote the minimal primes of A, then we have an identification of rings $A \cong A/\mathfrak{p}_1 \times \ldots \times A/\mathfrak{p}_r$, where the $A/\mathfrak{p}_i$ are integrally closed Noetherian domains. Next we observe that for any DVR R, any map $g : A \to R$ must factor through one of the projections $\pi_i : A \to A/\mathfrak{p}_i$. For if we had $a_i \in A/\mathfrak{p}_i$ and $a_j \in A/\mathfrak{p}_j$, $i \neq j$, with $g(a_i) \neq 0$ and $g(a_j) \neq 0$, we would have $g(a_i)g(a_j) = g(a_i a_j) = g(0) = 0$, which contradicts the fact that R is a domain. (Geometrically, this just says that under a morphism of Noetherian topological spaces, the image of an irreducible set is irreducible.) Finally we observe that any $\mathfrak{q} \in \text{Spec } A$ has the form $A/\mathfrak{p}_1 \times \ldots \times \mathfrak{p} \times \ldots \times A/\mathfrak{p}_r$, where $\mathfrak{p} \in \text{Spec } (A/\mathfrak{p}_i)$, so that $\kappa(\mathfrak{q}) \cong \kappa(\mathfrak{p})$. We conclude that a pointwise function S on A which varies algebraically along DVRs is exactly the same thing as pointwise functions S_1 on $A/\mathfrak{p}_1, \ldots, S_r$ on $A/\mathfrak{p}_r$, each varying algebraically along DVRs. Lemma 3.1.7 implies $f = f_{\mathfrak{p}_1} \times \ldots \times f_{\mathfrak{p}_r}$ is the unique solution. $\square$

Proof of Theorem 3.1.5. Let S be a pointwise function on $A = A^{sn}$ varying algebraically along DVRs, and let $\nu : A \to A^\nu$ denote the normalization. By Lemma 3.1.3, we obtain S^ν varying algebraically along DVRs. Lemma 3.1.8 implies there is a unique $f \in A^\nu$ agreeing with all $f_y \in \kappa(y)$ in the collection S^ν. By [24, I.7.2.3], A consists of those $h \in A^\nu$ such that for all $y \in \text{Spec } A^\nu$, the image of h in $\kappa(y)$ lies in $\kappa(\nu^{-1}y)$. By construction our f has this property, hence $f \in A$. $\square$

In fact it suffices to know the pointwise function varies algebraically along *complete* DVRs.

Theorem 3.1.9. *Let A be a seminormal ring which is Noetherian. Let $\{f_y \in \kappa(y) | y \in \text{Spec } A\}$ be a pointwise function on A which varies algebraically along complete DVRs. Then there exists a unique $f \in A$ whose image in $\kappa(y)$ is f_y for all $y \in \text{Spec } A$.*

Proof. We will show that a pointwise function S which varies algebraically along complete DVRs automatically varies algebraically along arbitrary DVRs. So suppose given a specialization $\mathfrak{p}_1 \subset \mathfrak{p}_2$ in A and a DVR R covering the specialization via $g : A \to R$. Since R is a Noetherian local ring, the canonical map $R \to \hat{R}$ is injective. Now consider the composite map $\hat{g} : A \to R \hookrightarrow \hat{R}$. Since $\hat{g}$ also covers $\mathfrak{p}_1 \subset \mathfrak{p}_2$, by the hypothesis we have $\hat{f} \in \hat{R}$ agreeing with S. Consider the diagram:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & R \arrow[r] \arrow[d] & \hat{R} \arrow[d] \\ \kappa(\mathfrak{p}_1) \arrow[r] & K \arrow[r] & \hat{K} \end{tikzcd}$$

Since $\hat{f}$ agrees with $f_{\mathfrak{p}_1}$, in fact $\hat{f} \in K$. Since $R = \hat{R} \cap K$ (by comparing the valuations; more generally see [27, Ch. 3 S. 9 "III. $\Rightarrow$ I."]), we are done. $\square$

Additionally we have the converse, so a characterization of seminormal rings.

Proposition 3.1.10. *Let A be ring which is Mori and Noetherian. Suppose that any pointwise function varying algebraically along (complete) DVRs comes from a unique $f \in A$. Then A is seminormal.*

Proof. Since A and A^{sn} have the same spectra and residue fields, a pointwise function on A is equivalent to one on A^{sn}. That this bijection also identifies those pointwise functions varying algebraically along DVRs follows from the universal property of the seminormalization [24, I.7.2.3.3]:

$$\mathrm{Hom}_{\mathrm{rings}}(A, B) = \mathrm{Hom}_{\mathrm{rings}}(A^{sn}, B)$$

for any seminormal ring B, for example a DVR. Of course $A^{sn} \to R$ and $A \to A^{sn} \to R$ cover the same specialization.
Now any $f \in A^{sn}$ determines a pointwise function on A varying algebraically along DVRs. By the hypothesis we have $f \in A$, whence $A = A^{sn}$. □

The following corollary translates the preceding commutative algebra into a global result. A locally Noetherian scheme X is Mori if and only if it has an affine cover by Noetherian Mori rings [12, Def. 3.1].

Corollary 3.1.11. *Let X be a seminormal locally Noetherian (in particular, Mori) scheme, and let $L, M \in Pic(X)$. Then an isomorphism $L \cong M$ is equivalent to an "identification of fibers varying algebraically along DVRs," that is:*

> *for any field or DVR R, any Spec $R \xrightarrow{f} X$, an identification $\beta_f : f^*L \cong f^*M$ compatible with restriction to the closed and generic points: if $s \xrightarrow{i}$ Spec $R, \eta \xrightarrow{j}$ Spec R denote the inclusions, then $\beta_{fi} = i^*\beta_f$ and $\beta_{fj} = j^*\beta_f$.*

Proof. Fix an open cover $X = \cup_i \mathrm{Spec}\ S_i$ with S_i a seminormal (Noetherian and Mori) ring which trivializes both L and M, and fix trivializations $\varphi_i : L_i := L|_{\mathrm{Spec}\ S_i} \cong \mathcal{O}_{\mathrm{Spec}\ S_i}$, $\psi_i : M_i := M|_{\mathrm{Spec}\ S_i} \cong \mathcal{O}_{\mathrm{Spec}\ S_i}$. Then defining $L \cong M$ is equivalent to identifying $\Gamma(\text{Spec } S_i, L_i) \cong \Gamma(\text{Spec } S_i, M_i)$ as S_i-modules (for all i), compatibly with restrictions. Then considering the diagram:

$$\begin{tikzcd} \Gamma(L_i) \arrow[r, "\cong"] \arrow[d, "\varphi_i"] & \Gamma(M_i) \arrow[d, "\psi_i"] \\ S_i \arrow[r] & S_i \end{tikzcd}$$

and its pullbacks to spectra of fields and DVRs, we see that relative to the fixed φ_i, ψ_i, a family β_f as in the statement is equivalent to an invertible pointwise function on each S_i varying algebraically along DVRs. By Theorem 3.1.5 this is equivalent to a family of elements $f_i \in S_i{}^\times = \text{Isom}_{S_i}(S_i, S_i)$. The f_i thus obtained agree on overlaps by the uniqueness statement in Theorem 3.1.5. Then using the above diagram again we see that relative to the fixed trivializations, the family f_i is equivalent to a family of isomorphisms $\Gamma(L_i) \cong \Gamma(M_i)$ compatible with restrictions. □

Remark 3.1.12. Theorem 3.1.9 implies the preceding result remains true if we replace everywhere "DVR" with "complete DVR."

Additionally we have a characterization of morphisms from a seminormal scheme.

Corollary 3.1.13. *Let X and Y be locally Noetherian and Mori schemes. Suppose X is seminormal. Then to define a morphism $X \to Y$ is equivalent to specifying a compatible system of set maps between R-points, for R any field or DVR, i.e. $\{X(R) \to Y(R) | R \text{ field or DVR}\}$, compatible with base change to the closed and generic fibers.*

Proof. Defining a morphism is local on the target, so we may assume Y is affine. Then we may cover X by affines and define the morphism on each affine open (agreeing on overlaps), so we assume X is affine as well. Then a morphism $X \to Y$ is the same as a morphism $X \to \mathbb{A}^n$ set-theoretically factoring through Y. Hence we may assume $Y = \mathbb{A}^n$. But a morphism $X \to \mathbb{A}^n$ is simply a collection of n elements of $\Gamma(X, \mathcal{O}_X)$, and these are characterized pointwise by Theorem 3.1.5. □

Now we state a circumstance under which pointwise lifts of a morphism glue.

Proposition 3.1.14. *Let T be a seminormal scheme of finite type over a field, and let $f : X \to Y$ be a proper radiciel morphism of locally finite type schemes over the same field. Let $g : T \to Y$ be a morphism, and suppose for all field points of T, the morphism g lifts to X, i.e. we have a commutative diagram:*

$$\begin{tikzcd} t \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & Y \end{tikzcd}$$

(Here t means Spec $\kappa(t)$.)

Then g lifts to a morphism $T \to X$.

Proof. Let $p_2 : (X \times_Y T)_{red} \to T$ be the natural morphism. Since all points of T factor through X, the image of f set-theoretically contains the image of g. As a radiciel morphism is univerally a bijection onto its image, we conclude p_2 is a bijection. We claim p_2 induces an isomorphism on all residue fields. Any $t \in T$ admits a lift to X, so maps to $X \times_Y T$. By the universal property of the reduction, t (being a reduced scheme) lifts to $(X \times_Y T)_{red}$. This means $\kappa(t)$ contains $\kappa(s)$, where $s \in (X \times_Y T)_{red}$ is the unique point lying over t. Whence $\kappa(t) \cong \kappa(s)$.

Finally we observe p_2 is proper. Now since T is seminormal we conclude p_2 is an isomorphism. The composition $p_1 \circ (p_2)^{-1} : T \to X$ is the desired lift. □

3.2 Proper hypercoverings and Picard schemes

Let X be a proper variety over a field k. Let $\pi : X_\bullet \to X$ be a proper hypercovering. We want to study the map

$$\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\bullet).$$

The goal is to show $\pi^* : \underline{\mathrm{Pic}}_X \to \underline{\mathrm{Pic}}_{X_\bullet}$ is a proper radiciel morphism of schemes when X is seminormal and k is perfect.

For details on simplicial schemes and proper hypercoverings, see [6].

Proposition 3.2.1. *Suppose $\pi : Y \to X$ is proper surjective with X seminormal of finite type over a field. Then there is a canonical algebra homomorphism splitting the injective map*

$$\mathcal{O}_X \to Ker(\pi_* \mathcal{O}_Y \to (\pi \times \pi)_* \mathcal{O}_{Y \times_X Y}).$$

The splitting is an isomorphism if Y is reduced.

Proof. Let Z be the normalization of the reduction of Y. Then there is a canonical map

$$\mathrm{Ker}(\pi_* \mathcal{O}_Y \to (\pi \times \pi)_* \mathcal{O}_{Y \times_X Y}) \to \mathrm{Ker}(\pi_* \mathcal{O}_Z \to (\pi \times \pi)_* \mathcal{O}_{Z \times_X Z}).$$

This map is injective when Y is reduced, so it suffices to show the map of the lemma is an isomorphism in the case Y is normal.

Now assuming Y normal, we show how an element of the RHS gives rise to a pointwise function on X. Then we show the function thus obtained varies algebraically along DVRs. This suffices since both X and Y are locally Noetherian and Mori.

Since Y is normal, we may think of an element f of the RHS as a pointwise function $\{f_y\}$ varying algebraically along DVRs. For any $x \in X$, choose some $y \in \pi^{-1}x \subset Y$ such that $\kappa(x) \hookrightarrow \kappa(y)$ is finite (proper implies finite type). Then since $f_y \otimes 1 = 1 \otimes f_y \in \kappa(y) \otimes_{\kappa(x)} \kappa(y)$, we have $f_y \in \kappa(x)$. To see this element is independent of the choice of point in the fiber, note that for any other $y' \in \pi^{-1}x$, there exists $z \in Y \times_X Y$ such that $p_1(z) = y$ and $p_2(z) = y'$. In the diagram of field extensions:

$$\begin{tikzcd} \kappa(x) \arrow[r] \arrow[d] & \kappa(y) \arrow[d] \\ \kappa(y') \arrow[r] & \kappa(z) \end{tikzcd}$$

we have $f_y = f_{y'}$ in $\kappa(z)$, whence they agree in $\kappa(x)$.

Now we show that $\{f_y\}$ varying algebraically along DVRs (as a function on Y) implies the same for X. So let $x_1 \rightsquigarrow x_0$ be a specialization in X and Spec $R \to X$ a covering

DVR. Since π is surjective we can find a specialization $y_1 \rightsquigarrow y_0$ in Y covering $x_1 \rightsquigarrow x_0$. There exists a DVR R' covering $y_1 \rightsquigarrow y_0$, a surjective morphism Spec $R' \to$ Spec R, and a commutative diagram [32, 3.3.4]:

$$\begin{tikzcd}
\operatorname{Spec} R' \arrow[r] \arrow[d] & Y \arrow[d, "\pi"] \\
\operatorname{Spec} R \arrow[r] & X
\end{tikzcd}$$

By localizing we obtain a commutative diagram of ring homomorphisms:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "\pi^{\#}"'] & & \kappa(x_1) \arrow[dr] \arrow[dd] & \\
& R \arrow[rr] \arrow[dd] & & K \arrow[dd] \\
B \arrow[rr] \arrow[dr] & & \kappa(y_1) \arrow[dr] & \\
& R' \arrow[rr] & & K'
\end{tikzcd}$$

We have $g_{R'} \in R'$ agreeing with f_{y_1} and f_{y_0}. Since $g_{R'}$ agrees with f_{y_1} in K' and f_{y_1} lies in $k(x_1)$, we can think of $g_{R'}$ as an element of K. Now since $v_{R'}(g_{R'}) \geq 0$, $v_K(g_{R'}) \geq 0$, we conclude $g_{R'} \in R$. Then by construction of the pointwise function on X, $g_{R'}$ agrees with f_{x_1} and f_{x_0}. $\square$

Remark 3.2.2. In the proof we may replace $Y \times_X Y$ with any Z admitting a proper surjective morphism to $Y \times_X Y$. Then we may interpret Proposition 3.2.1 as saying for X seminormal of finite type over a field and $X_\bullet \to X$ a proper hypercovering, $\mathcal{O}_X = \pi_*(\mathcal{O}_{X_\bullet})$. In this equality the units of each side are identified, and π_* preserves units, so we deduce $\mathcal{O}_X^* = \pi_*(\mathcal{O}_{X_\bullet}^*)$.

Lemma 3.2.3. *Let X be a proper variety over a field. Let $\pi : X_\bullet \to X$ be a proper hypercovering with all X_i reduced. If X is seminormal, then $\pi^* : Pic(X) \to Pic(X_\bullet)$ is injective.*

Proof. We use the Leray spectral sequence:

$$H^p(X, R^q\pi_*\mathcal{O}^*_{X_\bullet}) \Rightarrow H^{p+q}(X_\bullet, \mathcal{O}^*_{X_\bullet}).$$

If V denotes the kernel of $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$, we have an exact sequence:

$$0 \to H^1(X, \pi_*\mathcal{O}^*_{X_\bullet}) \to H^1(X_\bullet, \mathcal{O}^*_{X_\bullet}) \to V \to 0$$

Proposition 3.2.1 asserts $\mathcal{O}^*_X = \pi_*(\mathcal{O}^*_{X_\bullet})$, so we find $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X_\bullet)$.
(The validity of this proof in this generality is presumably known, see [1, Lemma 9 (v)].) □

Corollary 3.2.4. *Suppose $\pi : Y \to X$ is proper surjective with X seminormal of finite type over a field and Y reduced. Suppose $Z \to Y \times_X Y$ is proper surjective with Z reduced, let $p_1, p_2 : Z \to Y$ denote the natural maps, and let $\pi_1 = \pi \circ p_1 = \pi \circ p_2 : Z \to X$. Suppose $\mathcal{L} \in Pic(Y), \phi : {p_1}^*\mathcal{L} \xrightarrow{\sim} {p_2}^*\mathcal{L}$ satisfies the following property:*
For every $x \in X$, there exists an open $U \subset X$ containing x and a trivialization $T_x : \mathcal{L}|_{\pi^{-1}(U)} \xrightarrow{\sim} \mathcal{O}_{\pi^{-1}(U)}$ compatible with ϕ in the sense that the diagram

$$\begin{tikzcd}
{p_1}^*(\mathcal{L}|_{\pi^{-1}(U)}) \arrow[r, "{p_1}^*T"] \arrow[d, "\phi"] & \mathcal{O}_{(p_1)^{-1}(\pi^{-1}(U))} \arrow[d, "="] \\
{p_2}^*(\mathcal{L}|_{\pi^{-1}(U)}) \arrow[r, "{p_2}^*T"] & \mathcal{O}_{(p_2)^{-1}(\pi^{-1}(U))}
\end{tikzcd}$$

commutes.
Let $a_i : \mathcal{L} \to p_{i*}{p_i}^*\mathcal{L}$ denote the canonical maps. Then $\mathcal{M} := Ker(\pi_{1*}(\phi) \circ \pi_*(a_1) - \pi_*(a_2) : \pi_*\mathcal{L} \to \pi_{1*}{p_2}^*\mathcal{L})$ is the unique invertible sheaf on X satisfying $(\pi^*\mathcal{M}, can) \cong (\mathcal{L}, \phi)$.

Proof. The hypothesis guarantees that locally on X (!) the following diagram, in which the vertical isomorphisms are induced by T, commutes.

$$\begin{tikzcd}
& \pi_{1*}{p_1}^*\mathcal{L} \arrow[dr, "\pi_{1*}\phi"] \arrow[dd] & \\
\pi_*\mathcal{L} \arrow[ur] \arrow[rr] \arrow[dd] & & \pi_{1*}{p_2}^*\mathcal{L} \arrow[dd] \\
& \pi_{1*}{p_1}^*\mathcal{O} \arrow[dr, "\pi_{1*}(1)"] & \\
\pi_*\mathcal{O} \arrow[ur] \arrow[rr] & & \pi_{1*}{p_2}^*\mathcal{O}
\end{tikzcd}$$

That is, locally on X, $\mathcal{M} \cong \mathrm{Ker}(\pi_*\mathcal{O}_Y \to \pi_{1*}\mathcal{O}_Z)$. Now apply Proposition 3.2.1. Uniqueness follows from Lemma 3.2.3. □

We endeavor to establish a scheme-theoretic version of Lemma 3.2.3.

Definition 3.2.5. Let $X_\bullet$ be a simplicial k-scheme. Then the simplicial Picard group $\mathrm{Pic}(X_\bullet)$ is the group of invertible $\mathcal{O}_{X_\bullet}$-modules; for the definition of a simplicial sheaf (of sets) see [6, Def. 6.1]. The Picard functor of $X_\bullet$ is the *fppf*-sheafification of the functor which assigns to a k-scheme T the Picard group $\mathrm{Pic}(X_\bullet \times T)$ modulo pullbacks of invertible sheaves on T.

Lemma 3.2.6. *Let k be a perfect field, and let $X_\bullet$ be a seminormal and proper simplicial k-scheme. (This means all X_i are seminormal.) Then the Picard functor $Pic(X_\bullet)$ is representable by a group scheme $\underline{Pic}_{X_\bullet}$ which is locally of finite type over k and separated.*

Proof. The proof of representability in [2, A.2] carries over to our setting. For example, we can reduce to the case k is algebraically closed since X_i seminormal implies $X_i \times_k \overline{k}$ is seminormal by [12, 5.9]. Let $\{X_{0,i}\}$ denote the connected components of X_0. If there are $x_{0,1} \in X_{0,1}, x_{0,2} \in X_{0,2}$ such that there exists no $y \in X_1$ with $p_1(y) = x_{0,1}$ and $p_2(y) = x_{0,2}$, then $X_\bullet$ can be written as the direct sum of proper simplicial subschemes. In this way, and using the properness of the components $X_{0,i}$, we reduce to the case $\Gamma(X_\bullet, \mathcal{O}_{X_\bullet}) = k$.

Now we verify $\underline{\mathrm{Pic}}_{X_\bullet}$ is separated. If S is locally Noetherian and $X \to S$ is locally of finite type, then the diagonal $\Delta : X \to X \times_S X$ is quasi-compact. A quasi-compact monomorphism of locally finite type group schemes is automatically a closed immersion by [8, Exp. $\mathrm{VI_B}$ Cor. 1.4.2], hence any locally finite type group scheme is automatically separated. $\square$

Lemma 3.2.7. *Let X be a proper variety over a perfect field k, and let $\pi : X_\bullet \to X$ be a seminormal proper hypercovering. Then the morphism $\pi^* : \underline{Pic}_X \to \underline{Pic}_{X_\bullet}$ is affine.*

Proof. The composition $\underline{\mathrm{Pic}}_X \to \underline{\mathrm{Pic}}_{X_\bullet} \to \underline{\mathrm{Pic}}_{X_0}$ is affine by [30, Cor. 1.5(b)]. The scheme $\underline{\mathrm{Pic}}_{X_\bullet}$ is separated, hence the morphism $\underline{\mathrm{Pic}}_{X_\bullet} \to \underline{\mathrm{Pic}}_{X_0}$ is separated. Now by [18, II. Ex. 4.8], we conclude $\underline{\mathrm{Pic}}_X \to \underline{\mathrm{Pic}}_{X_\bullet}$ is affine. $\square$

Lemma 3.2.8. *Keep the notation and hypotheses as in Lemma 3.2.7. If X is seminormal and k is perfect, the morphism $\pi^* : \underline{Pic}_X \to \underline{Pic}_{X_\bullet}$ is radiciel, i.e. for any field K, $\underline{Pic}_X(K) \to \underline{Pic}_{X_\bullet}(K)$ is injective.*

Proof. The product of seminormal schemes (over a perfect field k) is again seminormal if one factor is locally of finite type over k [12, 5.9]. In particular X_K is seminormal and of finite type over K. It suffices to show injectivity for algebraically closed fields K [15, Ch. I 3.5.5], in which case we know $\underline{\mathrm{Pic}}_{X_{\bullet K}}$ exists by Lemma 3.2.6. Injectivity follows from the equality $\mathrm{Pic}(X_K) = \underline{\mathrm{Pic}}_{X_K}(K)$, the inclusion $\mathrm{Pic}(X_{\bullet K}) \to \underline{\mathrm{Pic}}_{X_{\bullet K}}(K)$, and Lemma 3.2.3. $\square$

Lemma 3.2.9. *Let G_1, G_2 be group schemes locally of finite type over a field, and let $f : G_1 \to G_2$ be a quasi-compact and radiciel morphism. Then f is finite.*

Proof. Since f is radiciel, its kernel is discrete. Since f is quasi-compact, by [8, Exp. $\mathrm{VI_B}$ 1.4.1(v)], having discrete kernel is equivalent to being finite. $\square$

For convenience we record a combination of the previous results.

Theorem 3.2.10. *Let X be a seminormal proper variety over a perfect field k, and let $\pi : X_\bullet \to X$ be a proper hypercovering. Let T be a seminormal k-scheme of finite type, and let $g : T \to \underline{Pic}_{X_\bullet}$ be a morphism. Suppose that for all field points of T, the morphism g lifts to $\underline{Pic}_X$. Then g lifts to a morphism $T \to \underline{Pic}_X$.*

Proof. By Proposition 3.1.14 it suffices to show $\pi^* : \underline{\mathrm{Pic}}_X \to \underline{\mathrm{Pic}}_{X_\bullet}$ is proper and radiciel. Since X is seminormal and k is perfect, Lemma 3.2.8 implies π^* is radiciel. Lemma 3.2.7 implies π^* is quasi-compact. Then Lemma 3.2.9 implies π^* is finite, in particular proper. $\square$

We conclude this chapter with a general fact which will be used towards the end of the argument in Chapter 5.

Proposition 3.2.11. *Let k be a perfect field. For $i = 1, 2$, let $Y^i_\bullet$ be a seminormal proper hypercovering of a seminormal k-scheme X^i. Let $\mathcal{L} \in Pic(Y^1_\bullet \times Y^2_\bullet)$. Suppose $\mathcal{L}$ descends to $\mathcal{M} \in Pic(Y^1_0 \times X^2)$ and $\mathcal{N} \in Pic(X^1 \times Y^2_0)$. Then $\mathcal{M}$ extends to an element of $Pic(Y^1_\bullet \times X^2)$ and $\mathcal{N}$ extends to an element of $Pic(X^1 \times Y^2_\bullet)$. Furthermore, $\mathcal{L}$ and the pullbacks of $\mathcal{M}$ and $\mathcal{N}$ agree on $Y^1_\bullet \times Y^2_\bullet$.*

Proof. Consider the product simplicial object (we omit the arrows in the other direction):

$$\begin{tikzcd}
Y^1_2 \times X^2 \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & Y^1_2 \times Y^2_0 \arrow[l, "h_{20}"'] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \cdots & \\
Y^1_1 \times X^2 \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & Y^1_1 \times Y^2_0 \arrow[l, "h_{10}"'] \arrow[d, shift right, "v^1_{10}"'] \arrow[d, shift left, "v^2_{10}"] & Y^1_1 \times Y^2_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift right] \arrow[d, shift left] & \\
Y^1_0 \times X^2 \arrow[d] & Y^1_0 \times Y^2_0 \arrow[l, "h_{00}"'] \arrow[d] & Y^1_0 \times Y^2_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \\
X^1 \times X^2 & X^1 \times Y^2_0 \arrow[l] & X^1 \times Y^2_1 \arrow[l, shift left] \arrow[l, shift right] & X^1 \times Y^2_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

We want to extend $\mathcal{M}$ along the leftmost column, i.e. define $a^*\mathcal{M} \cong b^*\mathcal{M}$ satisfying the cocycle condition. Lemma 3.2.3 implies $h^*_{10} : \mathrm{Pic}(Y^1_1 \times X^2) \hookrightarrow \mathrm{Pic}(Y^1_1 \times Y^2_\bullet)$. To show $a^*\mathcal{M} \cong b^*\mathcal{M}$, then, it suffices to show they determine the same element of

$\mathrm{Pic}(Y_1^1 \times Y_\bullet^2)$. This means there is an isomorphism $h_{10}^* a^* \mathcal{M} \cong h_{10}^* b^* \mathcal{M}$ compatible with the simplicial structure on $Y_1^1 \times Y_\bullet^2$. The commutativity of the squares in the product simplicial object gives the isomorphism, and the hypothesis $(h_{00}^* \mathcal{M}, can) = \mathcal{L}|_{Y_0^1 \times Y_\bullet^2}$ guarantees the isomorphism is compatible with $Y_1^1 \times Y_\bullet^2$.

Notice also we obtain a *unique* $a^* \mathcal{M} \cong b^* \mathcal{M}$ in this way: the one pulling back via h_{10} "correctly," as dictated by the other way of traversing the square. (Since h_{10} is surjective, $\mathrm{Aut}_{Y_1^1 \times X^2}(b^* \mathcal{M}) \to \mathrm{Aut}_{Y_1^1 \times Y_0^2}(h_{10}^* b^* \mathcal{M})$ is injective.)

We may check $a^* \mathcal{M} \cong b^* \mathcal{M}$ satisfies the cocycle condition after pulling back by the surjective morphism h_{20}. By construction of $a^* \mathcal{M} \cong b^* \mathcal{M}$, this follows from the cocycle condition on ${v_{10}^1}^* \mathcal{L} \cong {v_{10}^2}^* \mathcal{L}$. $\square$

Chapter 4

Zero-cycles and Divisors

4.1 The Hilbert-Chow morphism for zero-cycles

Notation. Let X be a quasi-projective scheme over a field k. Let $\mathscr{H}^m(X)$ denote the Hilbert scheme of m points on X, and let $S^m(X)$ denote the mth symmetric product of X. If X is seminormal (e.g. smooth), $\mathscr{C}_{0,m}(X) = S^m(X)$ [24, I.3.22].

Let $\mathscr{H}^{\leq \dim(X)-1}(X)$ denote the disjoint union of the (reduced) Hilbert schemes $\mathscr{H}^P(X)$ as P ranges over polynomials of degree $\leq \dim(X)-1$. Let $\mathrm{CDiv}(X)$ denote the space of relative effective Cartier divisors on X [24, I.1.11, I.1.12]; this is likewise a disjoint union indexed by discrete invariants. We work one discrete invariant at a time and suppress it from the notation. If X is projective over k, $\mathrm{CDiv}(X) \subset \mathscr{H}^{\leq \dim(X)-1}(X)$ is an open subscheme; if in addition X is smooth over k, $\mathrm{CDiv}(X) \subset \mathscr{H}^{\leq \dim(X)-1}(X)$ is universally closed [24, I.1.13]. We have also $\mathscr{C}_{n-1}(X) = \mathrm{CDiv}(X)^{sn}$ by [24, I.3.4.2] and [24, I.3.23.2.2].

Let $i_Z : Z \hookrightarrow X \times \mathscr{H}^m(X)$ the universal closed subscheme with (constant) Hilbert polynomial m, and let $i_D : D \hookrightarrow X \times \mathrm{CDiv}(X)$ denote the universal Cartier divisor. By abuse of notation let Z, D also refer to the closed subschemes of $X \times \mathscr{H}^m(X) \times \mathrm{CDiv}(X)$ corresponding to the projection morphisms from $\mathscr{H}^m(X) \times \mathrm{CDiv}(X)$; likewise the inclusions. Let $p = pr_2 \circ i_Z : Z \to \mathscr{H}^m(X)$ denote the universal finite flat morphism of degree m (or its product with $\mathrm{CDiv}(X)$), and let $pr_1 : Z \to X$ denote the projection.

We start with an algebraic fact which will allow us to "compute" the Hilbert-Chow morphism at a point.

Lemma 4.1.1. *Let k be an algebraically closed field, and let A be an Artinian local k-algebra with maximal ideal $\mathfrak{m}$. Let $u \in A^*$, and denote by $m_u : A \to A$ multiplication by u. Then $det(m_u) = (u \mod \mathfrak{m})^n$, where $n = \dim_k A$.*

Proof. Choose a k-basis for A as follows. Let 1 be a basis for $A/\mathfrak{m}$. Given a basis $\{1, e_2, \dots, e_l\}$ for $A/\mathfrak{m}^q$, choose a basis $\{1, e_2, \dots, e_l, e_{l+1}, \dots e_s\}$ for $A/\mathfrak{m}^{q+1}$, in other words, put the nonzero elements of $\mathfrak{m}^q/\mathfrak{m}^{q+1}$ at the end.
Since eventually $\mathfrak{m}^N = (0)$, we obtain a basis $\{1 = e_1, e_2, \dots, e_n\}$ for A and a sequence of integers $1 = r_1 < r_2 \cdots < r_N = n$ with the following property:

$$\sum a_i e_i \in \mathfrak{m}^q \iff a_i = 0 \text{ for all } i \leq r_q.$$

Write $u = \sum b_i e_i$. Suppose $e_j \in \mathfrak{m}^q - \mathfrak{m}^{q+1}$. Then $u \cdot e_j = b_0 e_j + \sum c_l e_l$ with $l \geq r_{q+1}$ for every l appearing in the sum. With respect to the basis $\{1 = e_1, e_2, \dots, e_n\}$, m_u is represented by a lower triangular matrix M with b_0 in every diagonal entry. In fact, setting $r_0 = 0$, there are diagonal matrices of size $(r_i - r_{i-1}) \times (r_i - r_{i-1})$ along the diagonal of M. Therefore $\det(m_u) = (b_0)^n = (u \mod \mathfrak{m})^n$, as desired. □

Lemma 4.1.2. *If $X \to Y$ is a closed immersion of quasi-projective schemes, then the morphism $S^m(X) \to S^m(Y)$ is a closed immersion as well.*

Proof. This is a local question. Since X is quasi-projective, every finite set of points is contained in an affine. Hence we may assume that X and Y are affine. Thus it suffices to show that

$$(A \otimes \ldots \otimes A)^{S_m} \longrightarrow (B \otimes \ldots \otimes B)^{S_m}$$

is surjective whenever the k-algebra map $A \to B$ is surjective. But this is just a question about k-vector spaces. So suppose that $m = n_1 + n_2 + \ldots + n_k$ is a partition of m, and $b_1, \ldots, b_k$ are elements of B. To the data $(n_i, b_i)_{i=1\ldots k}$ we will associate an element $\bar{b} = \bar{b}(\{n_i, b_i\}) \in (B \otimes \ldots \otimes B)^{S_m}$ as follows:

$$\bar{b}(\{n_i, b_i\}) := \sum\nolimits_{\phi} b_{\phi(1)} \otimes b_{\phi(2)} \otimes \ldots \otimes b_{\phi(m)}$$

where the sum is over all surjective maps $\phi : \{1, \ldots, m\} \to \{1, \ldots, k\}$ such that $\#\{i \mid \phi(i) = j\} = n_j$ for all $j = 1, \ldots, k$. (In the extreme case where $k = 1$, and $m = n_1$ we have one term. In the extreme case where $k = m$, and $m = 1 + \ldots + 1$, we have $m!$ terms.) Now it is easy to see that these elements generate $(B \otimes \ldots \otimes B)^{S_m}$ as a k-vector space. By lifting each b_i to an element $a_i \in A$ we see that $\bar{b}(\{n_i, b_i\})$ is the image of the corresponding element $\bar{a}(\{n_i, a_i\})$. □

Theorem 4.1.3. *Let X be a quasi-projective scheme over a field k. There is a canonical morphism*

$$FC_X : \mathscr{H}^m(X) \longrightarrow S^m(X)$$

with the following properties:

- *If X is affine and k is infinite, FC_X is described by (the unique) k-algebra map $\Gamma(S^m(X), \mathcal{O}) \to \Gamma(\mathscr{H}^m(X), \mathcal{O})$ with the following property: for any $f \in \Gamma(X, \mathcal{O})$,*

$$FC_X^{\#} : f^{\otimes m} \mapsto det_{m,X}(f).$$

- *The morphism FC_X is functorial for immersions and compatible with ground field extensions.*

- *If k is algebraically closed and $Z \subset X$ is a closed subscheme of length m (determining the point $m_Z \in \mathscr{H}^m(X)(k)$), then $FC_X(m_Z) = [Z] := \sum_p \ell(\mathcal{O}_{Z,p})[p]$.*

Proof. We explain the notation in the first property. We denote $\mathcal{O}_{m,X} := p_*\mathcal{O}_Z$; this is a finite locally free sheaf of $\mathcal{O}_{\mathscr{H}^m(X)}$-algebras of rank m. For any global section $f \in \Gamma(X, \mathcal{O}_X)$, we get a global section $(pr_1)^{\#}(f)$ of the structure sheaf of Z and hence a global section of $\mathcal{O}_{m,X}$. We denote

$$\det_{m,X}(f) \in \Gamma(\mathscr{H}^m(X), \mathcal{O}_{\mathscr{H}^m(X)})$$

the determinant of multiplication by $(pr_1)^{\#}(f)$ on the locally free sheaf $\mathcal{O}_{m,X}$.

Now we simply unpack the construction of the morphism in [21]. Since $X \to \operatorname{Spec} k$ is quasi-projective, it is a flat morphism satisfying Condition 1.1 of [21, II.1]: for all $s \in \operatorname{Spec} k$, any finite set of points of the fiber X_s is contained in an open affine $U \subset X$ whose image is contained in an open affine of Spec k. The functor of m-fold sections $F^m_{X/\operatorname{Spec} k}$ is equal to the Hilbert functor $\mathscr{H}^m_X$. Then [21, II.3.2] constructs a natural transformation of functors:

$$FC_X : \mathscr{H}^m_X \to \operatorname{Hom}_{Sch/k}(-, S^m(X))$$

which we now describe. Let T be a k-scheme, and suppose $i_T : Z_T \hookrightarrow X \times T$ belongs to $\mathscr{H}^m_X(T)$. Then FC_X maps Z_T to the following composition.

$$T \xrightarrow{\theta_{Z/T}} S^m(Z_T \times_T \ldots \times_T Z_T) \xrightarrow{S^m(i_T)} S^m((X \times T) \times_T \ldots \times_T (X \times T)) \to S^m(X) \quad (4.1.1)$$

Note that the composition of the second and third arrows is $S^m(pr_1)$.

To see the transformation FC_X has the first asserted property, let X be affine, and let $f \in \Gamma(X, \mathcal{O})$. Then we have

$$S^m(pr_1)^{\#} : \Gamma(S^m(X), \mathcal{O}) \to \Gamma(S^m(\times^m_T Z_T), \mathcal{O})$$

$$f^{\otimes m} \mapsto (pr_1^{\#}(f))^{\otimes m}.$$

Suppose $Z_T \to T$ corresponds locally to the finite locally free map of k-algebras $A \to B$, so that B is rank m as A-module. Then $\theta_{Z/T}$ corresponds to the ring map

$(B \otimes_A \ldots \otimes_A B)^{S_m} \to A$ which sends $b^{\otimes m}$ to $\det_A(b : B \to B) \in A$, the determinant of multiplication by b [21, I.2.3, I.4.3].

Combining this with the formula for $S^m(pr_1)^{\#}$, we see $FC_X^{\#}(f^{\otimes m}) = \det_{m,X}(f)$, as claimed.

Uniqueness of $FC_X^{\#}$. To show uniqueness it suffices to show that the elements $f^{\otimes m}$ generate the k-vector space $(A \otimes \ldots \otimes A)^{S_m}$. Let $W \subset (A \otimes \ldots \otimes A)^{S_m}$ be the k-sub vector space spanned by the elements $f^{\otimes m}$. In the proof of Lemma 4.1.2 we showed that the elements $\bar{a}(\{n_i, a_i\})$ generate $(A \otimes \ldots \otimes A)^{S_m}$. We will show $\bar{a}(\{n_i, a_i\}) \in W$. Given $a_i \in A$, $i = 1, \ldots, m$, and $t_i \in k$, $i = 1, \ldots, m$ note that

$$(t_1 a_1 + \ldots + t_m a_m)^{\otimes m} = \sum\nolimits_{n_1 + \ldots + n_m = m,\ n_i \geq 0} \bar{a}(\{n_i, a_i\}) \prod t_i^{n_i}.$$

Here the sum is over all ways of writing m as a sum of m nonnegative integers n_i. If some of the indices $n_j = 0$, then, by abuse of notation, we denote $\bar{a}(\{n_i, a_i\})$ the expression where we drop these entries from the list $\{n_i, a_i\}$. Now, since the monomials $\prod t_i^{n_i}$ are linearly independent in the polynomial ring, and since k is infinite, we see that the all the expressions $\bar{a}(\{n_i, a_i\})$ can be written in terms of expressions $(t_1 a_1 + \ldots + t_m a_m)^{\otimes m}$ for suitable choices of $t_i \in k$. In other words $\bar{a}(\{n_i, a_i\}) \in W$.

Compatibility with immersions and field extensions. Let $X' \to X$ be an immersion. Let f be a regular function on X, f' its restriction to X'. Functoriality means that $\det_{m,X}(f)$ restricts to $\det_{m,X'}(f')$ under the morphism $\mathscr{H}^m(X') \to \mathscr{H}^m(X)$. Consider the following commutative diagram.

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X \\
\mathscr{H}^m(X') \arrow[r] & \mathscr{H}^m(X) &
\end{tikzcd}$$

The left square is a fiber product, i.e. the universal family over $\mathscr{H}^m(X')$ is obtained by pulling back the universal family over $\mathscr{H}^m(X)$. Therefore $\mathcal{O}_{m,X'}$ is the pullback of $\mathcal{O}_{m,X}$. Via this isomorphism, $(pr_1 \circ i_{Z'})^{\#}(f')$ comes from $(pr_1 \circ i_Z)^{\#}(f)$. Hence the pullback of $\det_{m,X}(f)$ agrees with $\det_{m,X'}(f')$.

The invariance under base field extension is similar: in the above argument use $X' = X_{k'} := X \times_k k'$. Note that when k is finite, there is a map which is characterized by doing the correct thing after base change to any infinite extension of k. Consider a field extension $k \subset k'$. We have

$$\Gamma(S^m(X_{k'}), \mathcal{O}) = \Gamma(S^m(X), \mathcal{O}) \otimes_k k'$$

and similarly for $\mathscr{H}$. Let $R = \Gamma(S^m(X), \mathcal{O})$ and let $S = \Gamma(\mathscr{H}^m(X), \mathcal{O})$. By the above we have k'-algebra maps $R \otimes_k k' \to S \otimes_k k'$ functorially for all *infinite* field extensions k'/k. Since $k \to k'$ is faithfully flat, there is at most one k-algebra map $R \to S$ giving rise to these maps, and the construction of $FC_X^{\#}$ shows there is at least one.

Pointwise calculation of fundamental cycle. Let k be algebraically closed and keep the notation as in the theorem. We claim $FC_X(m_Z) = [Z] = \sum_{x \in X(k)} \ell(\mathcal{O}_{Z,x})$. We may assume X is affine. Let $f \in \Gamma(X, \mathcal{O})$. The map FC_X is characterized by mapping $f^{\otimes m}$ to $\det_{m,X}(f)$. Hence it is enough to show that

$$\det(f|_{\Gamma(Z,\mathcal{O}_Z)}) = \prod_{x \in X(k)} f(x)^{\ell(\mathcal{O}_{Z,x})}.$$

Since $\Gamma(Z, \mathcal{O}_Z) = \oplus_i R_i$, with R_i an Artinian local k-algebra, and both sides are multiplicative on $\oplus$, it suffices to consider the case Z is supported at a single point. This is the content of Lemma 4.1.1. □

Now we prove an amplification of the first property.

Proposition 4.1.4. *Let X and Y be affine of finite type over an infinite field k. Then there is a unique k-algebra map*

$$\Gamma(S^m(X) \times Y, \mathcal{O}) \to \Gamma(\mathscr{H}^m(X) \times Y, \mathcal{O})$$

with the following property: for any $f \in \Gamma(X, \mathcal{O}), b \in \Gamma(Y, \mathcal{O})$,

$$f^{\otimes m} \otimes b \mapsto det_{m,X}(fb).$$

Proof. The morphism $X \times Y \to Y$ satisfies Condition 1.1 of [21, II.1]. We have a canonical isomorphism $S^m(\times_Y^m(X \times Y)) \cong S^m(X) \times Y$ [21, I.1.2]. The sections θ are compatible with base change by [21, I.4.5].

Let T be an affine k-scheme of finite type. The observations in the preceding paragraph imply we have a commutative diagram whose bottom row is the product of (4.1.1) with Y:

$$\begin{tikzcd}
T \times Y \arrow[r, "\theta_{Z\times Y/T\times Y}"] \arrow[d, "="] & S^m(\times^m_{T\times Y}(Z_T \times Y)) \arrow[r, "S^m(i_T)"] \arrow[d, "\cong"] & S^m(\times^m_{T\times Y}(X \times T \times Y)) \arrow[r] \arrow[d, "\cong"] & S^m(\times^m_Y(X \times Y)) \arrow[d, "\cong"] \\
T \times Y \arrow[r, "\theta_{Z/T}\times 1"] & S^m(\times^m_T(Z_T)) \times Y \arrow[r, "S^m(i_T)"] & S^m(\times^m_T(X \times T)) \times Y \arrow[r] & S^m(X) \times Y
\end{tikzcd}$$

□

The incidence bundle. Keeping the notation as above, for the remainder of this section assume in addition X is smooth and projective over k. By pulling back Construction 2.4.1 along the morphism $\mathscr{H}^m(X) \times \mathrm{CDiv}(X) \hookrightarrow \mathscr{H}^m(X) \times \mathscr{H}^{\leq \dim(X)-1}(X)$, we obtain an invertible sheaf on $\mathscr{H}^m(X) \times \mathrm{CDiv}(X)$:

$$\mathcal{L} = \det \mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z) \otimes^{\mathbf{L}} i_{D*}(\mathcal{O}_D)).$$

We give a convenient description of $\mathcal{L}$ which takes into account the vanishing of the higher direct images.

Lemma 4.1.5. *With $X, \mathcal{L}$ as above, there is a canonical isomorphism:*

$$\mathcal{L} = (\det pr_{23*}(\mathcal{I}_D \otimes i_{Z*}(\mathcal{O}_Z)))^{-1} \otimes \det pr_{23*}(i_{Z*}(\mathcal{O}_Z)).$$

Proof. First we claim there is a canonical quasi-isomorphism of perfect complexes:

$$\mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z) \otimes^{\mathbf{L}} i_{D*}(\mathcal{O}_D)) \cong [pr_{23*}(\mathcal{I}_D \otimes i_{Z*}(\mathcal{O}_Z)) \to pr_{23*}(i_{Z*}(\mathcal{O}_Z))] \tag{4.1.2}$$

with $pr_{23*}(i_{Z*}(\mathcal{O}_Z))$ in degree 0.

To see (4.1.2), first notice there is a quasi-isomorphism

$$i_{Z*}(\mathcal{O}_Z)\otimes^{\mathbf{L}} i_{D*}(\mathcal{O}_D) \cong [\mathcal{I}_D \otimes i_{Z*}(\mathcal{O}_Z) \to i_{Z*}(\mathcal{O}_Z)] \tag{4.1.3}$$

with $\mathcal{I}_D \otimes i_{Z*}(\mathcal{O}_Z)$ in degree -1 and $i_{Z*}(\mathcal{O}_Z)$ in degree 0: use the following canonical resolution of $i_{D*}(\mathcal{O}_D)$ by locally free $\mathcal{O}_{X\times \mathrm{CDiv}(X)}$-modules.

$$0 \to \mathcal{I}_D \to \mathcal{O}_{X\times \mathrm{CDiv}(X)} \to i_*(\mathcal{O}_D) \to 0$$

Now pullback to $X \times \mathscr{H}^m(X) \times \mathrm{CDiv}(X)$ and tensor with $i_{Z*}(\mathcal{O}_Z)$ to obtain (4.1.3). To see (4.1.2), use the quasi-isomorphisms in Lemmas 2.2.5 and 2.2.7. Taking the determinant of (4.1.2) finishes the proof. □

Now we combine the algebra from Theorem 4.1.3 and the preceding lemma into a description of $\mathcal{L}$ on sufficiently small affines.

Lemma 4.1.6. *With $X, \mathcal{L}$ as above, let $f : S = \mathit{Spec}\ A \to \mathscr{H}^m(X) \times \mathit{CDiv}(X)$ be a morphism, and let Z_S, D_S denote the corresponding closed subschemes of $X \times S$. Then, possibly after shrinking S, we have:*

(1) There is an isomorphism of A-modules $\beta : \Gamma(Z_S, \mathcal{O}_{Z_S}) \cong \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S})$.

(2) Given two such isomorphisms β_r, β_s, write $\beta_r = \beta_s \circ u$, where u denotes multiplication by $u \in \Gamma(Z_S, \mathcal{O}_{Z_S})^*$. Then the isomorphisms $f^*\mathcal{L} \cong A$ induced by β_r, β_s differ by $\det_S(u)$.

Proof. To see such an isomorphism of A-modules exists, we make the following observations:

- Such an isomorphism exists when A is a field, as the line bundle $\mathcal{I}_D \otimes \mathcal{O}_Z$ on the zero-dimensional scheme Z admits a trivialization.

- If $(A, \mathfrak{m})$ is a local ring, and Z_0, D_0 denote the closed fibers, then $\Gamma(Z, \mathcal{O}_Z) \otimes_A A/\mathfrak{m} \cong \Gamma(Z_0, \mathcal{O}_{Z_0})$ as $A/\mathfrak{m}$-modules. Similarly for $\mathcal{I}_D \otimes \mathcal{O}_Z$.

- The A-modules $\Gamma(Z, \mathcal{O}_Z)$ and $\Gamma(Z, \mathcal{I}_D \otimes \mathcal{O}_Z)$ are coherent.

Then by Nakayama's lemma, the isomorphism $\Gamma(Z, \mathcal{O}_Z) \otimes_A A/\mathfrak{m} \cong \Gamma(Z, \mathcal{I}_D \otimes \mathcal{O}_Z) \otimes_A A/\mathfrak{m}$ lifts to an isomorphism $\Gamma(Z, \mathcal{O}_Z) \cong \Gamma(Z, \mathcal{I}_D \otimes \mathcal{O}_Z)$ on some open U, with Spec $(A/\mathfrak{m}) \in U \subset$ Spec (A). After shrinking, we may assume U is affine.

Now we flesh out the second claim. We think of an isomorphism $\beta_r : \Gamma(Z_S, \mathcal{O}_{Z_S}) \cong \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S})$ as a choice of generator $r \in \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S})$.
Now we define $\psi_r : f^*\mathcal{L} \cong A$ as follows. Let $\lambda \in (\det_A \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S}))^{-1}$ and $v \in \det_A \Gamma(Z_S, \mathcal{O}_{Z_S})$. Then we set:

$$\psi_r(\lambda \otimes v) = \lambda(\det_S \beta_r(v))$$

and extend linearly over general tensors.
Given another choice of generator $s \in \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S})$, there exists a unique $u \in \Gamma(Z_S, \mathcal{O}_{Z_S})^*$ such that we have a commutative diagram of $\Gamma(Z_S, \mathcal{O}_{Z_S})$-module isomorphisms:

$$\begin{tikzcd} & \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S}) \\ \Gamma(Z_S, \mathcal{O}_{Z_S}) \arrow[ur, "\beta_r"] \arrow[dr, "\cdot u"'] & \\ & \Gamma(Z_S, \mathcal{O}_{Z_S}) \arrow[uu, "\beta_s"'] \end{tikzcd}$$

This shows $\det_S(\beta_r) = \det_S(u) \cdot \det_S(\beta_s)$. Therefore:

$$\psi_r(\lambda \otimes v) = \lambda(\det_S \beta_r(v)) = \lambda(\det_S(u) \det_S \beta_s(v)) = \det_S(u) \psi_s(\lambda \otimes v).$$

□

Now we return to the problem of defining a descent datum on the incidence bundle $\mathcal{L}$. The morphism $FC : \mathscr{H}^m(X) \to S^m(X)$ gives rise to the fiber square:

$$\begin{tikzcd} \mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathscr{H}^m(X) \arrow[d, "FC"] \\ \mathscr{H}^m(X) \arrow[r, "FC"] & S^m(X) \end{tikzcd}$$

Taking the product with $\mathrm{CDiv}(X)$ yields a pullback morphism for $i = 1, 2$ (set $p_i := p_i \times \mathrm{Id}$):

$$\begin{array}{c} \mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times \mathrm{CDiv}(X) \\ \Big\downarrow {\scriptstyle p_i} \\ \mathscr{H}^m(X) \times \mathrm{CDiv}(X) \end{array}$$

Additionally we may form the triple fiber product $\mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X)$, which admits three canonical maps q_{12}, q_{13}, q_{23} to $\mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X)$. The purpose of this section is to construct the isomorphisms in the following theorem.

Theorem 4.1.7. *Keep $X, \mathcal{L}$ as above and use the notation immediately above. There is an isomorphism $\phi : {p_1}^*\mathcal{L} \cong {p_2}^*\mathcal{L}$ of invertible sheaves on $\mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times$ CDiv(X) with the following properties:*

- *The isomorphism ϕ satisfies the cocycle condition: $q_{12}^*(\phi) \circ q_{23}^*(\phi) = q_{13}^*(\phi)$ on $\mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times CDiv(X)$.*

- *Given any $m_{Z,D} := ([Z], [D]) \in S^m(X) \times CDiv(X)$, there exists an open $U \subset S^m(X)$ containing m_Z, and a trivialization $T_{m_{Z,D}} : \mathcal{L}|_{FC^{-1}U} \cong \mathcal{O}_{FC^{-1}U}$ such that the following diagram commutes:*

$$\begin{array}{ccc} {p_1}^*(\mathcal{L}|_{FC^{-1}U}) & \xrightarrow{{p_1}^*T} & \mathcal{O}_{(p_1)^{-1}(FC^{-1}U)} \\ \Big\downarrow {\scriptstyle \phi} & & \Big\downarrow {\scriptstyle =} \\ {p_2}^*(\mathcal{L}|_{FC^{-1}U}) & \xrightarrow{{p_2}^*T} & \mathcal{O}_{(p_2)^{-1}(FC^{-1}U)} \end{array}$$

Proof. First we reformulate algebraically the problem of defining ϕ. Then we use the algebraic characterization of the morphism FC_X obtained in the previous section, as expressed in Lemma 4.1.6.

To construct ϕ requires, for an affine scheme S and a morphism $f : S \to \mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times \mathrm{CDiv}(X)$ (for every (f, S) in a sufficiently large set), an isomorphism

$\phi_f : {p_1}^*\mathcal{L}|_S \cong {p_2}^*\mathcal{L}|_S$, compatible with localization. For example, it would suffice to define ϕ_f as f ranges over the inclusions of affine opens (forming a cover of $\mathscr{H}^m(X) \times_{S^m(X)} \mathscr{H}^m(X) \times \mathrm{CDiv}(X)$), compatibly on overlaps. We will do a little better than this: we will construct a system of isomorphisms ϕ_f for S a sufficiently small affine scheme, compatible with arbitrary base change among (sufficiently small) affine schemes.

Suppose such an f is given. For $i = 1, 2$, set $f_i = p_i \circ f : S = \mathrm{Spec}\ A \to \mathscr{H}^m(X) \times \mathrm{CDiv}(X)$, and let Z_{iS}, D_S denote the corresponding subschemes of $X \times S$. Lemma 4.1.6 gives us an isomorphism $\psi^i_r : {f_i}^*\mathcal{L} \cong A$ depending on a choice of generator $r \in \Gamma(Z_{iS}, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_{iS}})$. We claim $(\psi^2_r)^{-1} \circ \psi^1_r$ is independent of the choice of generator r. So let s be another generator, say $r = su$, and denote by $\det^i_S(u)$ the determinant of multiplication by $u : \Gamma(Z_{iS}, \mathcal{O}_{Z_{iS}}) \to \Gamma(Z_{iS}, \mathcal{O}_{Z_{iS}})$. Then by Lemma 4.1.6:

$$(\psi^2_r)^{-1} \circ \psi^1_r = [\det^2_S(u)\psi^2_s]^{-1} \circ \det^1_S(u)\psi^1_s = [\det^2_S(u)]^{-1}\det^1_S(u)(\psi^2_s)^{-1} \circ \psi^1_s.$$

The equality $\det^1_S(u) = \det^2_S(u)$ follows from Proposition 4.1.4. Therefore $\phi_f = (\psi^2_r)^{-1} \circ \psi^1_r = (\psi^2_s)^{-1} \circ \psi^1_s$ is well-defined.

Compatibility with base change. Now suppose given a morphism $g : T \to S$ from an affine scheme $T = \mathrm{Spec}\ B$. We aim to show $g^*\phi_f = \phi_{fg}$. In fact we'll show the ψ's are compatible with base change; this follows from the observation that a generator of $\Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S})$ determines a generator of $\Gamma(Z_T, \mathcal{I}_{D_T} \otimes \mathcal{O}_{Z_T})$.

Set $M_R := \Gamma(Z_R, \mathcal{I}_{D_R} \otimes \mathcal{O}_{Z_R}), F_R := \Gamma(Z_R, \mathcal{O}_{Z_R})$. Using base change isomorphisms and the compatibility among them (θ in Axiom IV. of Definition-Theorem 2.1.2), and the quasi-isomorphism (4.1.2), we obtain a commutative diagram:

$$\begin{tikzcd}
g^*(f^* \det \mathbf{R}pr_{23*}(\mathcal{O}_Z \otimes^{\mathbf{L}} \mathcal{O}_D)) \arrow[r, "\mathrm{can}"] \arrow[d] & (fg)^* \det \mathbf{R}pr_{23*}(\mathcal{O}_Z \otimes^{\mathbf{L}} \mathcal{O}_D) \arrow[d] \\
g^*((\det_S M_S)^{-1} \otimes \det_S F_S) \arrow[r, "\eta_g"] \arrow[d, "g^*(\psi_r)"] & (\det_T M_T)^{-1} \otimes \det_T F_T \arrow[d, "\psi_{r\otimes B}"] \\
A \otimes_A B \arrow[r, "\mathrm{can}"] & B
\end{tikzcd}$$

Given an A-module isomorphism $\beta_r : \Gamma(Z_S, \mathcal{O}_{Z_S}) \cong \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S})$, we obtain naturally a B-module isomorphism:

$$\Gamma(Z_T, \mathcal{O}_{Z_T}) \cong \Gamma(Z_S, \mathcal{O}_{Z_S}) \otimes_A B \xrightarrow{\beta_r \otimes B} \Gamma(Z_S, \mathcal{I}_{D_S} \otimes \mathcal{O}_{Z_S}) \otimes_A B \cong \Gamma(Z_T, \mathcal{I}_{D_T} \otimes \mathcal{O}_{Z_T}).$$

Thus $\psi_{r\otimes B}$ is defined. The equality $g^*\phi_f = \phi_{fg}$ follows.

Cocycle condition. We may check the equality $q_{12}^*(\phi) \circ q_{23}^*(\phi) = q_{13}^*(\phi)$ on field points, but it is no more difficult to check this on the ϕ_f just defined on certain affine schemes. We claim the following diagram commutes

$$\begin{tikzcd}
 & & f_2^*\mathcal{L} \arrow[dr, "\psi^2"] & & \\
 & A \arrow[ur, "(\psi^2)^{-1}"] & & A \arrow[dr, "(\psi^3)^{-1}"] & \\
f_1^*\mathcal{L} \arrow[ur, "\psi^1"] \arrow[rr, "\psi^1"] & & A \arrow[rr, "(\psi^3)^{-1}"] & & f_3^*\mathcal{L}
\end{tikzcd}$$

We suppress the choice of generator since $(\psi^i)^{-1} \circ \psi^j$ is independent of that choice. Since we could have chosen the same generator to define all ψ^i's, the commutativity is clear.

Descent property. This follows from the construction of ϕ. Namely, given $m_{Z,D}$, we find an affine open Spec $A \subset X$ with the following properties:

- $\mathrm{Supp}(Z) \subset$ Spec A (use X projective); and

- $\mathcal{I}_D$ admits a trivialization on Spec A.

For fixed Spec A, the first property is an open condition on $[Z]$, so we choose any open $U \ni m_Z$ on which the first property holds. Set $U' := FC^{-1}U$. Choose a generator $r \in \Gamma(Z_{U'}, \mathcal{I}_D \otimes \mathcal{O}_{Z_{U'}})$. Now we define $T_{m_{Z,D}}$ to be ψ_r:

$$\mathcal{L}|_{U'} = \det \Gamma(Z_{U'}, \mathcal{O}_{Z_{U'}}) \otimes (\det \Gamma(Z_{U'}, \mathcal{I}_D \otimes \mathcal{O}_{Z_{U'}}))^{-1} \xrightarrow{\psi_r} \mathcal{O}_{U'}.$$

We can use this r to define ϕ, and the equality $p_1^*T = p_2^*T \circ \phi$ then follows from the commutativity of the following diagram:

$$\begin{tikzcd}
p_1^*(\mathcal{L}|_{U'}) \arrow[d, "\psi_r^1"] \arrow[dr, "p_1^*T=\psi_r^1"] \arrow[dd, bend right=60, "\phi"'] & \\
\mathcal{O}_{U'} & \mathcal{O}_{U'} \\
p_2^*(\mathcal{L}|_{U'}) \arrow[u, "\psi_r^2"'] \arrow[ur, "p_2^*T=\psi_r^2"'] &
\end{tikzcd}$$

□

Now we show effectiveness of the descent datum just defined, i.e. that $(\mathcal{L}, \phi)$ is induced by an element of $\mathrm{Pic}(S^m(X) \times \mathrm{CDiv}(X)^{sn})$. By abuse of notation we do not distinguish between $\mathcal{L}, \phi$ and their pullbacks to $\mathrm{CDiv}(X)^{sn} = \mathscr{C}_{n-1}(X)$.

Theorem 4.1.8. *Keeping the notation as in Theorem* 4.1.7*, suppose in addition k is algebraically closed. The line bundle $\mathcal{L}$ on $\mathscr{H}^m(X) \times \mathscr{C}_{n-1}(X)$ descends to $\mathscr{C}_{0,m}(X) \times \mathscr{C}_{n-1}(X)$.*

Proof. We form a proper hypercovering $\mathcal{X}_\bullet$ augmented over the seminormal scheme $\mathscr{C}_{0,m}(X)$ as follows

$\mathcal{X}_0 := \mathscr{H}^m(X)$

$\mathcal{X}_1 := \mathscr{H}^m(X) \times_{\mathscr{C}_{0,m}(X)} \mathscr{H}^m(X)$

$\mathcal{X}_n := \mathscr{H}^m(X) \times_{\mathscr{C}_{0,m}(X)} \ldots \times_{\mathscr{C}_{0,m}(X)} \mathscr{H}^m(X) \quad (n+1 \text{ factors})$

with the obvious maps. The incidence bundle $\mathcal{L}$ on $\mathcal{X}_0 \times \mathscr{C}_{n-1}(X)$ has the descent datum ϕ on $\mathcal{X}_1 \times \mathscr{C}_{n-1}(X)$ satisfying the cocycle condition on $\mathcal{X}_2 \times \mathscr{C}_{n-1}(X)$. We think of these data as a morphism $\mathscr{C}_{n-1}(X) \to \underline{\mathrm{Pic}}_{\mathcal{X}_\bullet}$. The descent criterion in Corollary 3.2.4 and the descent property in Theorem 4.1.7 say that for every $[D] \in \mathscr{C}_{n-1}(X)$, the morphism lifts to $\underline{\mathrm{Pic}}_{\mathscr{C}_{0,m}(X)}$. By Theorem 3.2.10, the pointwise lifts glue to a morphism $\mathscr{C}_{n-1}(X) \to \underline{\mathrm{Pic}}_{\mathscr{C}_{0,m}(X)}$. To see this morphism determines the desired line bundle, consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(\mathscr{C}_{n-1}) \arrow[r] \arrow[d, "="] & \mathrm{Pic}(\mathscr{C}_{0,m} \times \mathscr{C}_{n-1}) \arrow[r] \arrow[d] & \underline{\mathrm{Pic}}_{\mathscr{C}_{0,m}}(\mathscr{C}_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}(\mathscr{C}_{n-1}) \arrow[r] & \mathrm{Pic}(\mathcal{X}_\bullet \times \mathscr{C}_{n-1}) \arrow[r] & \underline{\mathrm{Pic}}_{\mathcal{X}_\bullet}(\mathscr{C}_{n-1}) &
\end{tikzcd}$$

We claim the top row is exact, i.e. that the morphism $\mathscr{C}_{n-1}(X) \to \underline{\mathrm{Pic}}_{\mathscr{C}_{0,m}(X)}$ is induced by a line bundle on $\mathscr{C}_{0,m} \times \mathscr{C}_{n-1}$. Since $\mathscr{C}_{0,m}(X)$ is connected and k is algebraically closed, the morphism $\pi : \mathscr{C}_{0,m}(X) \to \mathrm{Spec}\ k$ satisfies $\pi_*(\mathcal{O}_{\mathscr{C}_{0,m}(X)}) = \mathcal{O}_{\mathrm{Spec}\ k}$ and has a section. Hence by [4, 8.1 Prop. 4], the top row is exact.

Since the outer maps are monomorphisms, even though we do not have right exactness in the bottom row, so is the middle one. Hence our $\mathcal{L}$ determines a unique line bundle on $\mathscr{C}_{0,m} \times \mathscr{C}_{n-1}$. □

Computation of fibers of $\mathcal{L}$. Keep the notation and hypotheses as in Theorem 4.1.8: let X be smooth and projective over $k = \overline{k}$, and $\mathcal{L}$ the incidence bundle. Given $Z \subset X$ a zero-dimensional subscheme of length m, and $D \subset X$ a Cartier divisor, we will "compute" the one-dimensional vector space $\mathcal{L}|_{m_{Z,D}}$, which by (4.1.2) is equal to:

$$\det H^0(X, \mathcal{O}_Z) \otimes \left(\det H^0(X, \mathcal{I}_D \otimes \mathcal{O}_Z)\right)^{-1}.$$

The result suggests the intersecting cycles are moving away from each other.

Proposition 4.1.9. *With $X, Z, D, \mathcal{L}$ as above, suppose $Z = \sum m_i P_i$, $P_i \in X(k)$. There is a canonical isomorphism*

$$\mathcal{L}|_{m_{Z,D}} \cong \bigotimes_{P_i} \left(Hom(\mathcal{I}_D/\mathcal{I}_D^2, k)|_{P_i}\right)^{\otimes m_i} =: \bigotimes_{P_i \in Supp(Z \cap D)} \left(\mathcal{N}_{D\subset X}|_{P_i}\right)^{\otimes m_i}.$$

(If $Z \cap D = \emptyset$, we understand the RHS to mean the trivial one-dimensional vector space.)

Proof. Since X is projective, we may choose an open affine $\operatorname{Spec} A \subset X$ containing the support of Z and on which $\mathcal{I}_{\mathcal{D}}$ admits a trivialization. The factors of $\mathcal{L}|_{m_{Z,D}}$ consist in:

- $H^0(X, \mathcal{O}_Z) = (A/J)$, an Artinian k-algebra of dimension m as k-vector space; and
- $H^0(X, \mathcal{I}_D \otimes \mathcal{O}_Z) = B \otimes_A (A/J)$, with B an ideal of A such that $B \cong A$ as A-modules.

First, since A/J is a direct sum of Artinian local k-algebras, and both sides are multiplicative over sums, we may assume A/J is local with maximal ideal $\mathfrak{m}$. Then our claim is that $\det_k(A/J) \otimes (B/B^2 \otimes_A A/\mathfrak{m})^{\otimes m} \otimes (\det_k(B \otimes_A A/J))^{-1}$ has a canonical generator.

First assume $B \subset \mathfrak{m}$ and choose a generator f for B, i.e. an A-module isomorphism $f : A \cong B$. We claim $v \otimes f^{\otimes m} \otimes ((\det f)(v))^{-1}$ is a canonical element. Changing v to $w = av$ does not change the element since $((\det f)(av))^{-1} = a^{-1}((\det f)(v))^{-1}$. Suppose a different generator $g : A \cong B$ were chosen. Then $f = ug$ for a unit $u \in A$. Then $(ug)^{\otimes m} = \overline{u}^m g^{\otimes m}$ (where $\overline{u}$ denotes the class of u mod $\mathfrak{m}$) and $((\det ug)(v))^{-1} = (\det(u))^{-1}((\det g)(v))^{-1}$. Lemma 4.1.1 implies $\overline{u}^m = \det(u)$, hence a different choice of generator for B does not change the element.

If $B \not\subset \mathfrak{m}$, then the term $(B/B^2 \otimes_A A/\mathfrak{m})^{\otimes m}$ is trivial. But then B has a canonical generator as A-module, namely the preimage of 1 under $A \to A/J$ (equivalently $A \to A/\mathfrak{m}$), hence $\det_k(B \otimes_A A/J) \cong \det_k(A/J)$, canonically. $\square$

4.2 Algebra of Hilbert-Chow for codimension 1 subschemes

Let $A \to B$ be a finite locally free ring map, and let $M^\bullet$ be a bounded complex of finite locally free B-modules. The goal of this section is to find a formula relating $\det_A M^\bullet$ and $\det_A \det_B M^\bullet$, and to show the formula descends to the derived category.

Lemma 4.2.1. *Let $A \to B$ be a finite locally free ring homomorphism such that B has rank n as A-module. Let $M_1, \dots, M_r$ be locally free B-modules of rank 1. Then there exists a canonical isomorphism:*

$$(\wedge_A^n B)^{\otimes r-1} \otimes_A \wedge_A^n (M_1 \otimes_B \cdots \otimes_B M_r) = \wedge_A^n M_1 \otimes_A \cdots \otimes_A \wedge_A^n M_r.$$

Proof. Localizing on B we may assume the M_i are free. Localizing on A we may assume B is A-free. Now choose bases $M_i = B \cdot e_i$ and $B = \oplus_j A \cdot f_j$. Then we have:

$\wedge_A^n B = A \cdot f_1 \wedge \cdots \wedge f_n$,

$M_1 \otimes_B \cdots \otimes_B M_r = B \cdot e_1 \otimes \cdots \otimes e_r$, and

$\wedge_A^n (M_1 \otimes_B \cdots \otimes_B M_r) = A \cdot f_1(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge f_n(e_1 \otimes \cdots \otimes e_r)$.

Likewise:

$M_i = \oplus_j A \cdot f_j e_i$ and $\wedge_A^n M_i = A \cdot f_1 e_i \wedge \cdots \wedge f_n e_i$.

We claim the isomorphism (of free A-modules of rank 1)

$$\begin{gathered}\varphi : (\wedge_A^n B)^{\otimes r-1} \otimes_A \wedge_A^n (M_1 \otimes_B \cdots \otimes_B M_r) \to \wedge_A^n M_1 \otimes_A \cdots \otimes_A \wedge_A^n M_r \\ (f_1 \wedge \cdots \wedge f_n)^{\otimes r-1} \otimes f_1(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge f_n(e_1 \otimes \cdots \otimes e_r) \mapsto \\ (f_1 e_1 \wedge \cdots \wedge f_n e_1) \otimes \cdots \otimes (f_1 e_r \wedge \cdots \wedge f_n e_r)\end{gathered}$$

is canonical, i.e. independent of the choice of bases. So suppose different bases ${e_i}'$ and ${f_j}'$ were chosen. Then for some $\alpha_i \in B, \beta_{jb} \in A$, we have ${e_i}' = \alpha_i e_i$ and ${f_j}' = \sum_b \beta_{jb} f_b$; and $({f_1}' \wedge \cdots \wedge {f_n}') = \det(\beta)(f_1 \wedge \cdots \wedge f_n)$. Let $\Theta = \prod_{i=1}^r \alpha_i \in B$, and for any $s \in B$, let $\det_A(s) \in A$ denote the determinant of multiplication by $s : B \to B$. Then:

${f_1}'({e_1}' \otimes \cdots \otimes {e_r}') \wedge \cdots \wedge {f_n}'({e_1}' \otimes \cdots \otimes {e_r}')$

$= {f_1}'(\alpha_1 e_1 \otimes \cdots \otimes \alpha_r e_r) \wedge \cdots \wedge {f_n}'(\alpha_1 e_1 \otimes \cdots \otimes \alpha_r e_r)$

$= \Theta {f_1}'(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge \Theta {f_n}'(e_1 \otimes \cdots \otimes e_r)$

$= \det_A(\Theta) {f_1}'(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge {f_n}'(e_1 \otimes \cdots \otimes e_r)$

$= \det_A(\Theta)(\sum_b \beta_{1b} f_b)(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge (\sum_b \beta_{nb} f_b)(e_1 \otimes \cdots \otimes e_r)$

$= \det_A(\Theta) \det(\beta) f_1(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge f_n(e_1 \otimes \cdots \otimes e_r).$

Therefore:

$({f_1}' \wedge \cdots \wedge {f_n}')^{\otimes r-1} \otimes {f_1}'({e_1}' \otimes \cdots \otimes {e_r}') \wedge \cdots \wedge {f_n}'({e_1}' \otimes \cdots \otimes {e_r}')$

$= \det(\beta)^{r-1} \det_A(\Theta) \det(\beta)(f_1 \wedge \cdots \wedge f_n)^{\otimes r-1} \otimes f_1(e_1 \otimes \cdots \otimes e_r) \wedge \cdots \wedge f_n(e_1 \otimes \cdots \otimes e_r).$

On the other hand:

$({f_1}'{e_1}' \wedge \cdots \wedge {f_n}'{e_1}') \otimes \cdots \otimes ({f_1}'{e_r}' \wedge \cdots \wedge {f_n}'{e_r}')$

$= (\alpha_1 {f_1}' e_1 \wedge \cdots \wedge \alpha_1 {f_n}' e_1) \otimes \cdots \otimes (\alpha_r {f_1}' e_r \wedge \cdots \wedge \alpha_r {f_n}' e_r)$

$= \det_A(\alpha_1)({f_1}' e_1 \wedge \cdots \wedge {f_n}' e_1) \otimes \cdots \otimes \det_A(\alpha_r)({f_1}' e_r \wedge \cdots \wedge {f_n}' e_r)$

$= \prod_i \det_A(\alpha_i)(\sum_b \beta_{1b} f_b e_1 \wedge \cdots \wedge \sum_b \beta_{nb} f_b e_1) \otimes \cdots \otimes (\sum_b \beta_{1b} f_b e_r \wedge \cdots \wedge \sum_b \beta_{nb} f_b e_r)$

$= \det_A(\Theta) \det(\beta)^r (f_1 e_1 \wedge \cdots \wedge f_n e_1) \otimes \cdots \otimes (f_1 e_r \wedge \cdots \wedge f_n e_r).$

So if φ' is the map defined by the same formula in terms of the ${e_i}'$ and ${f_j}'$, $\varphi = \varphi'$. $\square$

Lemma 4.2.2. *Let $A \to B$ be a finite locally free ring homomorphism such that B has rank n as A-module. Let M be a locally free B-module of rank k. Then there exists a canonical isomorphism:*

$$\wedge_A^{nk} M = (\wedge_A^n B)^{\otimes k-1} \otimes_A \wedge_A^n (\wedge_B^k M).$$

It also holds that:

$$\wedge_A^{nk} Hom_A(M, A) = (\wedge_A^n B)^{\otimes -k-1} \otimes_A \wedge_A^n (\wedge_B^k Hom_B(M, B)).$$

Proof. After localizing so that M is B-free and B is A-free, choose bases $M = \oplus_i B \cdot m_i$ and $B = \oplus_j A \cdot f_j$. Then we have:

$\wedge_A^n B = A \cdot f_1 \wedge \cdots \wedge f_n,$

$\wedge_B^k M = B \cdot m_1 \wedge \cdots \wedge m_k$,

$\wedge_A^n(\wedge_B^k M) = A \cdot f_1(m_1 \wedge \cdots \wedge m_k) \wedge \cdots \wedge f_n(m_1 \wedge \cdots \wedge m_k)$, and

$\wedge_A^{nk} M = A \cdot f_1 m_1 \wedge \cdots \wedge f_n m_k$.

We claim the isomorphism

$$\varphi : \wedge_A^{nk} M \to (\wedge_A^n B)^{\otimes k-1} \otimes_A \wedge_A^n(\wedge_B^k M)$$

$$f_1 m_1 \wedge \cdots \wedge f_n m_k \mapsto (f_1 \wedge \cdots \wedge f_n)^{k-1} \otimes f_1(m_1 \wedge \cdots \wedge m_k) \wedge \cdots \wedge f_n(m_1 \wedge \cdots \wedge m_k)$$

is canonical, i.e. independent of the choice of bases. So suppose different bases $m_i{}'$ and $f_j{}'$ were chosen. Then for some $\alpha_{ia} \in B, \beta_{jb} \in A$, we have $m_i{}' = \sum_a \alpha_{ia} m_a$ and $f_j{}' = \sum_b \beta_{jb} f_b$. Then:

$f_1{}' m_1{}' \wedge \cdots \wedge f_n{}' m_k{}'$

$= (f_1{}' \sum_a \alpha_{1a} m_a \wedge \cdots \wedge f_1{}' \sum_a \alpha_{ka} m_a) \wedge \cdots \wedge (f_n{}' \sum_a \alpha_{1a} m_a \wedge \cdots \wedge f_n{}' \sum_a \alpha_{ka} m_a)$

$= (\det(\alpha) f_1{}' m_1 \wedge \cdots \wedge f_1{}' m_k) \wedge \cdots \wedge (\det(\alpha) f_n{}' m_1 \wedge \cdots \wedge f_n{}' m_k)$

$= \det_A(\det(\alpha))(f_1{}' m_1 \wedge \cdots \wedge f_1{}' m_k \wedge \cdots \wedge f_n{}' m_1 \wedge \cdots \wedge f_n{}' m_k)$

$= \det_A(\det(\alpha))(\sum_b \beta_{1b} f_b m_1 \wedge \cdots \wedge \sum_b \beta_{1b} f_b m_k \wedge \cdots \wedge \sum_b \beta_{nb} f_b m_1 \wedge \cdots \wedge \sum_b \beta_{nb} f_b m_k)$

$= \det_A(\det(\alpha))(\det(\beta))^k f_1 m_1 \wedge \cdots \wedge f_n m_k$.

On the other hand:

$(f_1{}' \wedge \cdots \wedge f_n{}')^{k-1} \otimes f_1{}'(m_1{}' \wedge \cdots \wedge m_k{}') \wedge \cdots \wedge f_n{}'(m_1{}' \wedge \cdots \wedge m_k{}')$

$= (f_1{}' \wedge \cdots \wedge f_n{}')^{k-1} \otimes f_1{}'(\sum_a \alpha_{1a} m_a \wedge \cdots \wedge \sum_a \alpha_{ka} m_a) \wedge \cdots \wedge f_n{}'(\sum_a \alpha_{1a} m_a \wedge \cdots \wedge \sum_a \alpha_{ka} m_a)$

$= (\det(\beta))^{k-1} (f_1 \wedge \cdots \wedge f_n)^{k-1} \otimes \det(\alpha) f_1{}'(m_1 \wedge \cdots \wedge m_k) \wedge \cdots \wedge \det(\alpha) f_n{}'(m_1 \wedge \cdots \wedge m_k)$

$= \det_A(\det(\alpha))(\det(\beta))^{k-1} (f_1 \wedge \cdots \wedge f_n)^{k-1} \otimes f_1{}'(m_1 \wedge \cdots \wedge m_k) \wedge \cdots \wedge f_n{}'(m_1 \wedge \cdots \wedge m_k)$

$= \det_A(\det(\alpha))(\det(\beta))^{k-1} (f_1 \wedge \cdots \wedge f_n)^{k-1} \otimes \sum_b \beta_{1b} f_b(m_1 \wedge \cdots \wedge m_k) \wedge \cdots \wedge \sum_b \beta_{nb} f_b(m_1 \wedge \cdots \wedge m_k)$

$= \det_A(\det(\alpha))(\det(\beta))^k (f_1 \wedge \cdots \wedge f_n)^{k-1} \otimes f_1(m_1 \wedge \cdots \wedge m_k) \wedge \cdots \wedge f_n(m_1 \wedge \cdots \wedge m_k)$.

So if φ' is the map defined by the same formula in terms of the $m_i{}'$ and $f_j{}'$, $\varphi = \varphi'$. This proves the first statement.

To prove the second statement, let $M_1 = \wedge_B^k M$ and $M_2 = \mathrm{Hom}_B(\wedge_B^k M, B)$ in Lemma 4.2.1. Then $M_1 \otimes_B M_2 = B$, so:

$(\wedge_A^n B)^{\otimes 2} = \wedge_A^n(\wedge_B^k M) \otimes_A \wedge_A^n(\mathrm{Hom}_B(\wedge_B^k M, B))$.

Since $\mathrm{Hom}_B(\cdot, B)$ and $\wedge_B^k$ are compatible, we find:

$\wedge_A^n(\wedge_B^k \mathrm{Hom}_B(M, B)) = \mathrm{Hom}_A(\wedge_A^n(\wedge_B^k M), A) \otimes_A (\wedge_A^n B)^{\otimes 2}$.

By the first statement of Lemma 4.2.2 just proven, we can replace $\wedge_A^n(\wedge_B^k M)$ on the RHS to obtain:

$\mathrm{RHS} = \mathrm{Hom}_A(\wedge_A^{nk} M \otimes_A (\wedge_A^n B)^{\otimes 1-k}, A) \otimes_A (\wedge_A^n B)^{\otimes 2}$

$= \mathrm{Hom}_A(\wedge_A^{nk} M, A) \otimes_A (\wedge_A^n B)^{\otimes k+1}$, since $\mathrm{Hom}_A(\cdot, A)$ and $\otimes_A$ are compatible.

Rearranging, we have:

$\wedge_A^{nk}(\mathrm{Hom}(M, A)) = \mathrm{Hom}_A(\wedge_A^{nk} M, A) = (\wedge_A^n B)^{\otimes -k-1} \otimes_A \wedge_A^n(\wedge_B^k \mathrm{Hom}_B(M, B))$

as desired. □

Proposition 4.2.3. *Let $A \to B$ be a finite locally free ring homomorphism such that B has rank n as A-module. Let $M^\bullet$ be a bounded complex of finite locally free B-modules with $rk_B(M^i) = m_i$. Then there is a canonical isomorphism*

$$\alpha_A^M : \det_A M^\bullet \cong (\det_A B)^{rk(M^\bullet)-1} \otimes_A \det_A \det_B M^\bullet$$

with the following properties.

(1) The α are compatible with base change: if $A \to A'$ is any ring map, then $\alpha_A \otimes_A A' = \alpha_{A'}$.

(2) The α are compatible with short exact sequences of complexes of locally free sheaves of finite rank: if $0 \to M_1^\bullet \to M_2^\bullet \to M_3^\bullet \to 0$ is such, then the dia-

gram:

$$\begin{tikzcd} \det_A M_1 \otimes \det_A M_3 \arrow[r, "\alpha^{M_1}\otimes\alpha^{M_3}"] \arrow[d] & (\det_A B)^{rk(M_1)+rk(M_3)-2} \otimes \det_A \det_B M_1 \otimes \det_A \det_B M_3 \arrow[d] \\ \det_A M_2 \arrow[r, "\alpha^{M_2}"] & (\det_A B)^{rk(M_2)-1} \otimes \det_A \det_B M_2 \end{tikzcd}$$

in which the vertical arrows are induced by the exact sequence, commutes.

(3) *The α are compatible with the canonical trivialization of an acyclic complex of locally free sheaves of finite rank: if $H^\bullet$ is such an acyclic complex, then the diagram:*

$$\begin{tikzcd} \det_A H^\bullet \arrow[r, "\alpha"] \arrow[d] & (\det_A B)^{-1} \otimes \det_A \det_B H^\bullet \arrow[d] \\ A \arrow[r, "can"] & (\det_A B)^{-1} \otimes \det_A B \end{tikzcd}$$

in which the vertical arrows are induced by the quasi-isomorphism $H^\bullet \to 0^\bullet$, commutes.

(4) *The α are compatible with quasi-isomorphisms of bounded complexes of locally free sheaves of finite rank: if $q : M^\bullet \to N^\bullet$ is such, then the diagram*

$$\begin{tikzcd} \det_A M^\bullet \arrow[r, "\alpha^M"] \arrow[d, "\det_A(q)"] & (\det_A B)^{rk(M)-1} \otimes \det_A \det_B M^\bullet \arrow[d, "1\otimes\det_A\det_B(q)"] \\ \det_A N^\bullet \arrow[r, "\alpha^N"] & (\det_A B)^{rk(N)-1} \otimes \det_A \det_B N^\bullet \end{tikzcd}$$

commutes.

Proof. By Lemma 4.2.2, $(\det_A(\det_B M^i))^{-1} = (\det_A B)^{\otimes -2} \otimes_A \det_A((\det_B M^i)^{-1})$

$\det_A(\det_B M^\bullet) := \det_A(\otimes_{B,i}(\det_B M^i)^{(-1)^i})$ by definition

$= \otimes_{A,i} \det_A((\det_B M^i)^{(-1)^i}) \otimes_A (\det_A B)^{1-\sum_i 1}$ by Lemma 4.2.1

$= \otimes_{A,i}(\det_A \det_B M^i)^{(-1)^i} \otimes_A (\det_A B)^{(\sum_{i \text{odd}} 2)+1-\sum_i 1}$ by above (second statement of Lemma 4.2.2)

$= \otimes_{A,i}((\det_A B)^{\otimes 1-m_i} \det_A M^i)^{(-1)^i} \otimes_A (\det_A B)^{(\sum_{i \text{odd}} 2)+1-\sum_i 1}$ by Lemma 4.2.2

Recall that by definition $\det_A M^\bullet = \otimes_{A,i}(\det_A M^i)^{(-1)^i}$. Therefore:

$$(\det_A B)^{\mathrm{rk}(M^\bullet)-1} \otimes_A \det_A \det_B M^\bullet = \det_A M^\bullet \otimes_A (\det_A B)^{\sum_i (-1)^i + \sum_{i\mathrm{odd}} 2 - \sum_i 1}.$$

So it suffices to check $\sum_i (-1)^i + \sum_{i\mathrm{odd}} 2 - \sum_i 1 = 0$, which holds one i at a time.

Compatibility with base change. Suppose we used the A-basis $\{b_j\}$ for B. Given $A \to A'$, the identification $\otimes_A A'$ is the formula with basis $\{b_j \otimes 1\}$ for $B' := B \otimes_A A'$. Over A' we may choose a different basis and the map $\alpha_{A'}$ is the same.

Compatibility with short exact sequences. By definition [23, p.31], the isomorphism determined by a short exact sequence of complexes is the tensor product of the isomorphisms in each degree, with sign. Therefore it suffices to show the formula for α is compatible with short exact sequences of locally free sheaves of finite rank. Let $0 \to M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \to 0$ be such a sequence, and (localizing if necessary) let $\{m_{1,i}\}_{i\in I}, \{\psi(m_{2,j})\}_{j\in J}$ be B-bases for M_1, M_3 respectively. Also let $\{b_k\}_{k\in K}$ be an A-basis for B. Then the isomorphism $\det_B M_1 \otimes \det_B M_3 \cong \det_B M_2$ is described by:

$$\bigwedge_{i\in I} m_{1,i} \otimes \bigwedge_{j\in J} \psi(m_{2,j}) \mapsto \bigwedge_{i\in I} \varphi(m_{1,i}) \bigwedge_{j\in J} m_{2,j}$$

Taking $\det_A$ of this identification introduces the extra factor of $(\det_A B)^{-1}$ in the upper right corner, and shows the route passing through this corner (with respect to the chosen bases) results in $(\bigwedge_{k\in K} b_k)^{rk(M_2)-1} \otimes \bigwedge_{k\in K} b_k(\bigwedge_{i\in I} \varphi(m_{1,i}) \bigwedge_{j\in J} m_{2,j})$.

On the other hand, the isomorphism $\det_A M_1 \otimes \det_A M_3 \cong \det_A M_2$ is described by:

$$\bigwedge_{k\in K, i\in I} b_k m_{1,i} \otimes \bigwedge_{k\in K, j\in J} b_k \psi(m_{2,j}) \mapsto \bigwedge_{k\in K, i\in I} b_k \varphi(m_{1,i}) \bigwedge_{k\in K, j\in J} b_k m_{2,j}$$

Applying α^{M_2}, we get the same result as above.

Compatibility with acyclic complexes. We proceed by induction on the length of the acyclic complex. The length 2 case amounts to showing the α isomorphism is compatible with isomorphisms of locally free sheaves of finite rank, which is a special case of compatibility with short exact sequences of such. Let $H^\bullet$ be an acyclic complex of length $n+1$ supported in the interval $[i, i+n]$. Following the method and notation of [23, p.34], we use the short exact sequence of complexes $0 \to H_{\mathrm{I}}^{\bullet} \to H^{\bullet} \to H_{\mathrm{II}}^{\bullet} \to 0$ with $H_{\mathrm{I}}^{\bullet}$ acyclic of length 2 and $H_{\mathrm{II}}^{\bullet}$ acyclic of length n. Set $RHS(M) = (\det_A B)^{rk(M)-1} \otimes \det_A \det_B M$. The sequence of acyclic complexes determines a cube:

$$\begin{tikzcd}
\det_A H_{\mathrm{I}} \otimes \det_A H_{\mathrm{II}} \arrow[rr, "\alpha^{H_{\mathrm{I}}} \otimes \alpha^{H_{\mathrm{II}}}"] \arrow[dr] \arrow[dd, "\det(0)\otimes\det(0)"] & & RHS(H_{\mathrm{I}}) \otimes RHS(H_{\mathrm{II}}) \arrow[dr] \arrow[dd, "(1\otimes\wedge_A\wedge_B(0))^{\otimes 2}"] & \\
& \det_A H \arrow[rr, "\alpha^H"] \arrow[dd, "\det(0)"] & & RHS(H) \arrow[dd, "1\otimes\wedge_A\wedge_B(0)"] \\
A \otimes A \arrow[rr] \arrow[dr] & & (\wedge_A B)^{-2} \otimes (\wedge_A B)^2 \arrow[dr] & \\
& A \arrow[rr] & & (\wedge_A B)^{-1} \otimes \wedge_A B
\end{tikzcd}$$

The left and right faces commute by the inductive definition of the trivialization of the determinant of an acyclic complex [23, p.34]. The back face commutes by the induction hypothesis. The top face commutes because the α isomorphisms are compatible with short exact sequences of complexes of finite rank locally frees. The commutativity of the bottom face is trivial as the unlabeled arrows are the canonical ones. Therefore the front face commutes as well, which is what we aimed to show.

Compatibility with quasi-isomorphisms. To define the determinant of a quasi-isomorphism of bounded complexes of locally free sheaves of finite rank, we use the short exact sequences (again following the notation of [23, p.29]):

$$0 \to M^{\bullet} \xrightarrow{(100)} Z_q^{\bullet} \to \mathrm{Cok}(100) \to 0$$

$$0 \to N^{\bullet} \xrightarrow{(010)} Z_q^{\bullet} \to \mathrm{Cok}(010) \to 0$$

Then $\det_A(q) : \det_A M^{\bullet} \to \det_A N^{\bullet}$ is defined to be the composition:

$$\det_A M^{\bullet} \leftarrow \det_A M^{\bullet} \otimes \det_A \mathrm{Cok}(100) \to \det_A Z_q^{\bullet} \leftarrow \det_A N^{\bullet} \otimes \det_A \mathrm{Cok}(010) \to \det_A N^{\bullet}.$$

Now we claim the following diagram commutes:

$$\begin{tikzcd}
\det_A M^\bullet \arrow[d, "\alpha^M"] & \det_A M^\bullet \otimes \det_A \mathrm{Cok}(100) \arrow[l] \arrow[r] \arrow[d, "\alpha^M \otimes \alpha^{\mathrm{Cok}(100)}"] & \det_A Z_q^\bullet \arrow[d, "\alpha^{Z_q}"] \\
RHS(M^\bullet) & RHS(M^\bullet) \otimes RHS(\mathrm{Cok}(100)) \arrow[l] \arrow[r] & RHS(Z_q^\bullet)
\end{tikzcd}$$

The left square commutes since the α isomorphisms are compatible with acyclic complexes. The right square commutes by compatibility with short exact sequences. Attaching to the rightmost arrow the analogous diagram with $N^\bullet$ and (010), we obtain a diagram whose top row is $\det_A(q) : \det_A M^\bullet \to \det_A N^\bullet$ and which expresses the compatibility of the α isomorphisms with quasi-isomorphisms. □

Proposition 4.2.4. *Keep the notation from Proposition* 4.2.3*. The α isomorphisms are compatible with distinguished triangles in $Parf^0$. If in addition A, B are reduced, then the α isomorphisms are compatible with arbitrary distinguished triangles.*

Proof. Consider the commutative diagram resulting from a given triangle $\mathcal{E}_1^\bullet \to \mathcal{E}_2^\bullet \to \mathcal{E}_3^\bullet \to^{+1}$:

$$\begin{tikzcd}
\det(\mathcal{E}_1^\bullet) \otimes \det(\mathcal{E}_3^\bullet) \arrow[r, "\text{triangle}"] \arrow[d] & \det(\mathcal{E}_2^\bullet) \arrow[d] \\
\otimes_n (\det H^n(\mathcal{E}_1^\bullet))^{(-1)^n} \otimes \otimes_n (\det H^n(\mathcal{E}_3^\bullet))^{(-1)^n} \arrow[r] & \otimes_n (\det H^n(\mathcal{E}_2^\bullet))^{(-1)^n}
\end{tikzcd}$$

From the second and fourth properties of α in Proposition 4.2.3, we deduce compatibility with arbitrary short exact sequences of perfect complexes. Since the good truncation of a complex involves short exact sequences of perfect complexes, the α isomorphisms are compatible with passing to the cohomology sheaves of a complex (the vertical arrows). Since the α isomorphisms are compatible with acyclic complexes, they are compatible with the bottom arrow. Therefore they are compatible with the top row, i.e. the original triangle, as well. □

Remark 4.2.5. In the next section we will see the role of the α isomorphisms of Proposition 4.2.3 in defining the descent datum in the $\mathscr{H}^{\leq \dim(X)-1} \to \mathscr{C}_{\dim(X)-1}$

direction. The α isomorphisms will also play an important role in the discussion of 1-cycles on a threefold.

4.3 The Hilbert-Chow morphism for codimension 1 subschemes

Notation. Let X be a smooth projective scheme over a field k. We keep the notation from Section 4.1.

Let $i_V : V \hookrightarrow X \times \mathscr{H}^P(X)$ denote the universal closed subscheme with Hilbert polynomial P, and $i_Z : Z \hookrightarrow X \times \mathscr{H}^m(X)$ the universal closed subscheme with (constant) Hilbert polynomial m. By abuse of notation let V, Z also refer to the closed subschemes of $X \times \mathscr{H}^m(X) \times \mathscr{H}^P(X)$ corresponding to the projection morphisms from $\mathscr{H}^m(X) \times \mathscr{H}^P(X)$; likewise the inclusions. Let $p = pr \circ i_Z : Z \to \mathscr{H}^m(X)$ denote the universal finite flat morphism of degree m (or its product with $\mathscr{H}^P(X)$ or $\mathscr{H}^{P_1}(X) \times_{\mathrm{CDiv}(X)} \mathscr{H}^{P_2}(X)$).

Then by Construction 2.4.1 there is a well-defined invertible sheaf on $\mathscr{H}^m(X) \times \mathscr{H}^P(X)$:

$$\mathcal{L} = \det \mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z) \otimes^{\mathbf{L}} i_{V*}(\mathcal{O}_V)).$$

There is a morphism $FC : \mathscr{H}(X) \to \mathrm{CDiv}(X)$ which maps a proper closed subscheme to its divisorial part. The fundamental cycle of a subscheme supported in codimension 1 is its divisorial part; a subscheme supported in codimension ≥ 2 maps to the trivial Cartier divisor [10, 3. 2]. The morphism FC gives rise to the fiber square:

$$\begin{tikzcd}
\mathscr{H}(X) \times_{\mathrm{CDiv}(X)} \mathscr{H}(X) \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathscr{H}(X) \arrow[d, "FC"] \\
\mathscr{H}(X) \arrow[r, "FC"] & \mathrm{CDiv}(X)
\end{tikzcd}$$

Remark 4.3.1. We really have a fiber square for every pair of Hilbert polynomials (P_1, P_2), and the fiber product is nonempty if and only if $\mathscr{H}^{P_1}(X)$ and $\mathscr{H}^{P_2}(X)$ map to the same component of CDiv(X); this means the degree $\dim(X)-1$ terms of P_1 and P_2 have the same coefficient. So throughout we will think of a pair of Hilbert schemes mapping to the same space of Cartier divisors.

Taking the product with $\mathscr{H}^m(X)$ yields pullback morphisms (let $p_i = \mathrm{Id}\times p_i$):

$$\begin{array}{ccc} \mathscr{H}^m(X)\times\mathscr{H}^{P_1}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_2}(X) & \xrightarrow{p_2} & \mathscr{H}^m(X)\times\mathscr{H}^{P_2}(X) \\ \downarrow{\scriptstyle p_1} & & \\ \mathscr{H}^m(X)\times\mathscr{H}^{P_1}(X) & & \end{array}$$

Then for $i=1,2$ we can define $\mathcal{L}_i$ on $\mathscr{H}^m(X)\times\mathscr{H}^{P_i}(X)$ by replacing V in the above formula with V_i. Additionally we may form the triple fiber product $\mathscr{H}^{P_1}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_2}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_3}(X)$, which admits canonical maps q_{12}, q_{13}, q_{23} to the appropriate $\mathscr{H}^{P_i}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_j}(X)$. The purpose of this section is to construct the isomorphisms in the following theorem.

Theorem 4.3.2. *Use the notation from above. There exists an isomorphism* $\phi : {p_1}^*\mathcal{L}\cong {p_2}^*\mathcal{L}$ *of invertible sheaves on* $\mathscr{H}^m(X)\times\mathscr{H}^{P_1}(X)\times_{CDiv(X)}\mathscr{H}^{P_2}(X)$*. The isomorphism* ϕ *satisfies the cocycle condition:* $q_{12}^*(\phi)\circ q_{23}^*(\phi) = q_{13}^*(\phi)$ *on* $\mathscr{H}^m(X)\times\mathscr{H}^{P_1}(X)\times_{CDiv(X)}\mathscr{H}^{P_2}(X)\times_{CDiv(X)}\mathscr{H}^{P_3}(X)$*.*

Proof. First we interpret $FC(V_1)=FC(V_2)$ as an isomorphism of line bundles. Then we manipulate the formula for $\mathcal{L}$ so this isomorphism determines ϕ as in the theorem.

Interpretation of $FC(V_1)=FC(V_2)$. We follow [10, p.69], whose Inv is our det. For $i=1,2$, let V_i denote the closed subscheme of $X\times\mathscr{H}^{P_1}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_2}(X)$ corresponding to the projection morphism $p_i:\mathscr{H}^{P_1}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_2}(X)\to\mathscr{H}^{P_i}(X)$. Each $\mathcal{O}_{V_i}$ is a torsion sheaf of finite Tor-dimension on $X\times\mathscr{H}^{P_1}(X)\times_{\mathrm{CDiv}(X)}\mathscr{H}^{P_2}(X)$ since X is smooth. Therefore $\mathcal{O}_{V_i}$ defines a canonical global section of its determinant:

$$s_i:\mathcal{O}_{X\times\mathscr{H}\times_{\mathrm{CDiv}}\mathscr{H}}\to\det(\mathcal{O}_{V_i}).$$

Now $FC(V_1) = FC(V_2)$ means V_1 and V_2 determine the same Cartier divisor on $X \times \mathscr{H}^{P_1}(X) \times_{\mathrm{CDiv}(X)} \mathscr{H}^{P_2}(X)$: in addition to the fact that $\det(\mathcal{O}_{V_1}) \cong \det(\mathcal{O}_{V_2})$ as abstract invertible sheaves, there is a unique isomorphism φ_{12} making the diagram commute:

$$\begin{tikzcd}
\mathcal{O}_{X\times H\times_{CD} H} \arrow[r, "s_1"] \arrow[d, "="] & \det(\mathcal{O}_{V_1}) \arrow[d, "\varphi_{12}"] \\
\mathcal{O}_{X\times H\times_{CD} H} \arrow[r, "s_2"] & \det(\mathcal{O}_{V_2})
\end{tikzcd}$$

By uniqueness, we get $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$.

To define ϕ, we will construct, for every affine scheme S and every morphism $f : S \to \mathscr{H}^m(X) \times \mathscr{H}^{P_1}(X) \times_{\mathrm{CDiv}(X)} \mathscr{H}^{P_2}(X)$, an isomorphism $\phi_f : {p_1}^*\mathcal{L}|_S \cong {p_2}^*\mathcal{L}|_S$, compatible with arbitrary base change (among affine schemes).
So supposing such an f is given, set $f_i = p_i \circ f : S = \mathrm{Spec}\ A \to \mathscr{H}^m(X) \times \mathscr{H}^{P_i}(X)$, and let Z_S, V_{iS} denote the corresponding subschemes of $X \times S$. Set $\Gamma(Z_S, \mathcal{O}_{Z_S}) = A_Z$. First we define

$$\vartheta_1 : f^*{p_1}^*\det \mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z)\otimes^{\mathbf{L}} i_{V*}(\mathcal{O}_V)) \cong (\wedge_A A_Z)^{-1} \otimes \wedge_A \wedge_{A_Z} \mathbf{L}{i_{Z_S}}^* i_{V_{1S}*}(\mathcal{O}_{V_{1S}})$$

to be the following composition:

$$\begin{tikzcd}
f^*{p_1}^*\det \mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z)\otimes^{\mathbf{L}} i_{V*}(\mathcal{O}_V)) \arrow[d, "\mathrm{can}"] & (\wedge_A A_Z)^{-1} \otimes \wedge_A \wedge_{A_Z} \mathbf{L}{i_{Z_S}}^* i_{V_{1S}*}(\mathcal{O}_{V_{1S}}) \\
{f_1}^*\det \mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z)\otimes^{\mathbf{L}} i_{V*}(\mathcal{O}_V)) \arrow[d, "\eta_{f_1}"] & \det p_{S*}(\mathbf{L}{i_{Z_S}}^*(i_{V_{1S}*}(\mathcal{O}_{V_{1S}}))) \arrow[u, "\alpha_S"'] \\
\det \mathbf{L}{f_1}^*\mathbf{R}pr_{23*}(i_{Z*}(\mathcal{O}_Z)\otimes^{\mathbf{L}} i_{V*}(\mathcal{O}_V)) \arrow[d, "\text{base change}"] & \det \mathbf{R}p_{S*}(\mathbf{L}{i_{Z_S}}^*(i_{V_{1S}*}(\mathcal{O}_{V_{1S}}))) \arrow[u, "2.2.7"'] \\
\det \mathbf{R}pr_{S*}\mathbf{L}{f_1}'^*(i_{Z*}(\mathcal{O}_Z)\otimes^{\mathbf{L}} i_{V*}(\mathcal{O}_V)) \arrow[r, "{Z,V\,\mathrm{flat}}"] & \det \mathbf{R}pr_{S*}(i_{Z_S*}(\mathcal{O}_{Z_S})\otimes^{\mathbf{L}} i_{V_{1S}*}(\mathcal{O}_{V_{1S}})) \arrow[u, "2.2.5"']
\end{tikzcd}$$

Now we define $\phi_f := (\vartheta_2)^{-1} \circ (1 \otimes \wedge_A(\varphi_{12}|_{Z_S})) \circ \vartheta_1$.

Compatibility with base change and cocycle condition. Suppose also given a morphism $g : T = \mathrm{Spec}\ B \to S = \mathrm{Spec}\ A$. That $g^*\phi_f = \phi_{fg}$ follows from the commutativity of a gigantic diagram we suppress. The basic properties needed are the following: the isomorphisms used to define the ϑ_i are compatible with base change; the φ_{ij}, being canonical, are compatible with base change; and the base change isomorphisms are compatible with compositions of morphisms (Axiom IV. of Definition-Theorem 2.1.2).

The cocycle condition follows easily from the equalities $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ and $q_{ij}^*(\phi_f) = (\vartheta_j)^{-1} \circ (1 \otimes \wedge_A(\varphi_{ij}|_{Z_S})) \circ \vartheta_i$. $\square$

Remark 4.3.3. Previously we defined $\mathcal{L}$ directly on $\mathscr{H}^m(X) \times \mathrm{CDiv}(X)$, so there is no question about the effectiveness of the descent datum we have just constructed. Now we reformulate the results of this chapter in a manner suggestive of the contents of the next chapter. Omitting the X from the notation, we set:

$C := \mathscr{C}_{0,m} \times \mathscr{C}_{\dim(X)-1}$;

$V_i := \mathscr{H}^m \times_{\mathscr{C}_{0,m}} \ldots \times_{\mathscr{C}_{0,m}} \mathscr{H}^m \quad (i+1 \text{ factors})$;

$W_i := ((\mathscr{H}^{\leq \dim(X)-1})^{sn} \times_{\mathscr{C}_{\dim(X)-1}} \ldots \times_{\mathscr{C}_{\dim(X)-1}} (\mathscr{H}^{\leq \dim(X)-1})^{sn})^{sn} \;\; (i+1 \text{ factors})$;

and

$Y_i := V_i \times W_i$ for $i \geq 0$.

The Hilbert-Chow morphism $Y_0 \to C$ gives rise to a proper hypercovering $Y_\bullet$ augmented over C, with ith term Y_i. Since seminormalization is a functor, the morphisms in $Y_\bullet$ are simply the seminormalizations of the canonical morphisms. We have the incidence bundle $\mathcal{L} \in \mathrm{Pic}(Y_0)$. In this chapter we have shown:

(1) $\mathcal{L}$ extends to a line bundle on $Y_\bullet$, and

(2) $\mathcal{L}$ descends to C via the morphism $Y_\bullet \to C$.

Chapter 5

Pairs of 1-cycles in a threefold

5.1 One variable descent isomorphism

Notation and conventions. Let P be a smooth projective threefold over a perfect field k. For a k-scheme T, set $P_T := P \times_{\mathrm{Spec}\, k} T$, and let $\pi : P_T \to T$ denote the projection. As usual, if $T = \mathrm{Spec}\, R$, we will write P_R for P_T. For $\mathcal{F}, \mathcal{G} \in \mathrm{Parf}(P_T)$, put:

$$f_T(\mathcal{F}, \mathcal{G}) := \det{}_T \mathbf{R}\pi_*(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{P_T}} \mathcal{G}).$$

Since π is a proper flat morphism, this is well-defined.

If $\mathcal{F} \in \mathrm{Coh}(P_T)$ is T-flat, then $[\mathcal{F}]$, the fundamental cycle of $\mathcal{F}$, is a well-defined family of cycles on P over T [24, I.3.15].

Now assume in addition T is the spectrum of a DVR R with maximal ideal $\mathfrak{m}$. We will use $k_0 := R/\mathfrak{m}$, $K := \mathrm{Frac} R$, and $P_0 := P_{k_0}$; for $M \in R-\mathrm{mod}$, we set $M_0 := M \otimes_R k_0$ and $M_K := M \otimes_R K$. If, for the T-flat $\mathcal{F} \in \mathrm{Coh}(P_T)$, we write $[\mathcal{F}] = \sum_i n_i [C_i]$, then the C_i are R-flat subvarieties (flatness by [18, III.9.7]) of maximal relative dimension (equal to the relative dimension of $\mathrm{Supp}(\mathcal{F})$), and we may define

$$f_R([\mathcal{F}],\mathcal{G}) := \otimes_i(\det{}_R\mathbf{R}\pi_*(\mathcal{O}_{C_i}\otimes^{\mathbf{L}}\mathcal{G}))^{\otimes n_i}.$$

If the sheaf $\mathcal{F}$ is not flat, we have a canonical exact sequence $0 \to \mathrm{Tor}(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}^{fl} \to 0$, where $\mathrm{Tor}(\mathcal{F})$ denotes the torsion subsheaf of $\mathcal{F}$. In this case we define $[\mathcal{F}] := [\mathcal{F}^{fl}]$, the cycle of the flat limit of $\mathcal{F} \otimes_R K$.

For $\mathcal{F} \in \mathrm{Parf}(P_T)$, we set $\mathrm{Supp}(\mathcal{F}) := \cup_q \mathrm{Supp}(\mathcal{H}^q(\mathcal{F}))$. We set $[\mathcal{F}] := \sum(-1)^q[\mathcal{H}^q(\mathcal{F})]$, and therefore $f_R([\mathcal{F}],\mathcal{G}) = \otimes_q(f_R([\mathcal{H}^q(\mathcal{F})],\mathcal{G}))^{(-1)^q}$.

For $r \geq 0$, we define $\mathrm{Coh}_{d\leq r}(P_R)$ to be the subcategory of coherent sheaves $\mathcal{F}$ such that $\dim(\mathrm{Supp}(\mathcal{F})) \leq r$ (we mean relative dimension as Definition 2.2.8). Let $D_{d\leq r}(P_R) \subset \mathrm{Parf}(P_R)$ denote the subcategory of perfect complexes $\mathcal{F}$ satisfying $\mathrm{Supp}(\mathcal{F})$ has relative dimension $\leq r$, i.e. $\mathcal{H}^q(\mathcal{F}) \in \mathrm{Coh}_{d\leq r}(P_R)$ for all q. For a DVR R, let $D_{d\leq 0;1}(P_R) \subset D_{d\leq 1}(P_R)$ denote the subcategory of complexes such that

- $\dim(\mathrm{Supp}(\mathcal{F} \otimes_R K)) = 0$, and
- $\dim(\mathrm{Supp}(\mathcal{H}^q(\mathcal{F}) \otimes k_0)) \leq 1$ for all q.

Remark 2.2.9 implies that if $\mathcal{F} \in D_{d\leq r}(P_T)$, then for any base change $T' \to T$, the derived pullback $\mathcal{F}\otimes^{\mathbf{L}}_T T'$ lies in $D_{d\leq r}(P_{T'})$.

Situation 5.1.1. The following data are fixed throughout this section.

As above P is a smooth projective threefold over the (fixed) perfect field k.

The complex $\mathcal{G}$ will be assumed to satisfy/come equipped with:

- $\mathcal{G} \in \mathrm{Parf}(P_T)$ for every base T (so "the" complex $\mathcal{G}$ is a really a compatible system of complexes; in the application there will be a universal $\mathcal{G}$);
- $\mathrm{rk}(\mathcal{G}) = 0$; and
- there are given isomorphisms $\gamma_T : \det_{\mathcal{O}_{P_T}}(\mathcal{G}) \cong \mathcal{O}_{P_T}$, for every T which is the spectrum of a DVR or a field, compatible with quasi-isomorphisms (as we think of $\mathcal{G}$ as an object in the derived category) and base change (among DVRs and fields).

On the variable objects in this section, we impose the following hypotheses.

Hypotheses 5.1.2. All DVRs and fields contain k.

If R is a DVR, we assume that in the diagram:

$$\begin{tikzcd} R \arrow[r] & k_0 \\ k \arrow[u] & \end{tikzcd}$$

the map $R \to k_0$ has a k-linear section. This holds, for example, if k is perfect and R is complete [25, 28.J Thm. 60; Cor. 2].

We assume the complex $\mathcal{F}$ belongs to $D_{d\leq 1}(P_R)$.

Goal. We denote $f_T(\mathcal{F}) := f_T(\mathcal{F}, \mathcal{G})$. The purpose of this section is to construct a coherent system of isomorphisms $\varphi_T^{\mathcal{F}} : f_T(\mathcal{F}) \to f_T([\mathcal{F}])$, for T the spectrum of a DVR or field. To define the isomorphisms $\varphi^{\mathcal{F}}$ will involve some choices; that the resulting map is independent of these choices is the main content of this section. Unless otherwise stated, the system of isomorphisms constructed depends on γ; in the application there will be a natural candidate for γ.

The key property of these isomorphisms is captured in the following definition.

Definition 5.1.3. Let $D' \subset D_{d\leq 1}(P_R)$ be a subcategory. We say a collection of isomorphisms $\varphi_R^{\mathcal{F}} : f(\mathcal{F}) \to f([\mathcal{F}])$, one for each $\mathcal{F} \in D'$, is *compatible with triangles in* D' if whenever $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to^{+1}$ is a triangle in $\mathrm{Parf}(P_R)$ such that all $\mathcal{F}_i \in D'$, the diagram:

$$\begin{tikzcd} f_R(\mathcal{F}_1) \otimes f_R(\mathcal{F}_3) \arrow[r] \arrow[d, "\varphi_R^{\mathcal{F}_1} \otimes \varphi_R^{\mathcal{F}_3}"] & f_R(\mathcal{F}_2) \arrow[d, "\varphi_R^{\mathcal{F}_2}"] \\ f_R([\mathcal{F}_1]) \otimes f_R([\mathcal{F}_3]) \arrow[r, "+"] & f_R([\mathcal{F}_2]) \end{tikzcd}$$

in which the triangle induces the top row, and addition of cycles induces the bottom, is commutative. Note that when $[\mathcal{F}] = \emptyset$, $f_R([\mathcal{F}]) = f_R(0) = R$, canonically.

This is equivalent to saying the trivialization $f(\mathcal{F}_1)\otimes(f(\mathcal{F}_2))^{-1}\otimes f(\mathcal{F}_3) \xrightarrow{\varphi^{\mathcal{F}_1}\otimes(\varphi^{\mathcal{F}_2})^{-1}\otimes\varphi^{\mathcal{F}_3}} f([\mathcal{F}_1]) \otimes (f([\mathcal{F}_2]))^{-1} \otimes f([\mathcal{F}_3]) \xrightarrow{\text{can}} \mathcal{O}$ is equal to the trivialization induced by the triangle.

Remark 5.1.4. When X is reduced, the isomorphism induced by a distinguished triangle is defined by replacing the distinguished triangle with a true triangle [23, Prop. 7]. If the $\mathcal{F}_i$ comprising the triangle belong to $\mathrm{Parf}^0(X)$, we can equally use the acyclicity of the long exact cohomology sequence, thought of as a complex [23, Cor. 2].

Remark 5.1.5. The fact that $[-]$ is additive on triangles follows from two observations: localization is exact, and length is additive.

Definition 5.1.6. Let $T = \operatorname{Spec} R$ for R a DVR or a field, and let $\mathcal{F}$ be a coherent sheaf on P_R such that $\operatorname{Supp}(\mathcal{F})$ has relative dimension ≤ 1. Then $\mathcal{F}$ has a filtration $0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = \mathcal{F}$ such that each $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_{C_i}(n_i)$ for some subvariety $C_i \subset P_R$, some $n_i \in \mathbb{Z}$ [18, I.7.4]. We will call such a filtration a *cycle filtration* of $\mathcal{F}$. A cycle filtration of a perfect complex $\mathcal{F}$ means a finite collection of triangles each of whose "terminal" pieces is isomorphic to some $\mathcal{O}_C(n)$ as above; equivalently, the collection expresses $\mathcal{F} = \sum_i \mathcal{O}_{C_i}(n_i)$ in $K_0(P_R)$. One can obtain a cycle filtration of a complex by taking a cycle filtration of each of its cohomology sheaves.

The condition in Hypotheses 5.1.2 on the dimension of the support of $\mathcal{F}$ implies every C_i appearing in a cycle filtration has one of the following forms:

- (V) subvariety of dimension ≤ 1 in the closed fiber,
- (Z) flat family of zero-cycles over R, or
- (C) flat family of curves over R.

Construction 5.1.7. A cycle filtration of $\mathcal{F} \in \mathrm{Parf}(P_R)$ induces a "filtration" of $\mathbf{R}\pi_*(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G})$ whose graded pieces are $\mathbf{R}\pi_*((\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes^{\mathbf{L}} \mathcal{G})$, and therefore an isomorphism $f_R(\mathcal{F}) \to \otimes_i f_R(\mathcal{F}_i/\mathcal{F}_{i-1})$, as follows:

The filtration can be thought of as a collection of triangles $\mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1} \to^{+1}$, and the operations $-\otimes^{\mathbf{L}}\mathcal{G}$ and $\mathbf{R}\pi_*(-)$ preserve triangles. Then for every i we obtain an isomorphism $f_R(\mathcal{F}_i) \xrightarrow{\sim} f_R(\mathcal{F}_{i-1}) \otimes f_R(\mathcal{F}_i/\mathcal{F}_{i-1})$, and therefore an identification $f_R(\mathcal{F}) \xrightarrow{\sim} \otimes_i f_R(\mathcal{F}_i/\mathcal{F}_{i-1})$ as asserted.

Now we prove a lemma which reduces "compatibility with triangles" to "additivity on short exact sequences of coherent sheaves."

Lemma 5.1.8. *With notation as above, let $C \subset Coh(P_R) \cap D_{d\leq 1}(P_R)$ be a subcategory with kernels and cokernels, and let $D(C) \subset D_{d\leq 1}(P_R)$ denote the subcategory of complexes such that $\mathcal{H}^q(\mathcal{F}) \in C$ for all q. Suppose that we have an additive system of isomorphisms $\{\varphi^{\mathcal{F}}\}_{\mathcal{F}\in C}$, i.e. for every $\mathcal{F} \in C$, we have an isomorphism $\varphi^{\mathcal{F}} : f(\mathcal{F}) \to f([\mathcal{F}])$, and for every short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in C, the following diagram commutes:*

$$\begin{tikzcd}
f(\mathcal{F}_1) \otimes f(\mathcal{F}_3) \arrow[r] \arrow[d, "\varphi^{\mathcal{F}_1} \otimes \varphi^{\mathcal{F}_3}"] & f(\mathcal{F}_2) \arrow[d, "\varphi^{\mathcal{F}_2}"] \\
f([\mathcal{F}_1]) \otimes f([\mathcal{F}_3]) \arrow[r, "+"] & f([\mathcal{F}_2])
\end{tikzcd}$$

Then these data determine a unique collection of isomorphisms $\{\varphi^{\mathcal{F}}\}_{\mathcal{F}\in D(C)}$ with the following properties:

- *the $\varphi^{\mathcal{F}}$ agree on C, and*
- *the $\varphi^{\mathcal{F}}$ are compatible with triangles in $D(C)$.*

In fact the $\varphi^{\mathcal{F}}$ are induced by the canonical filtration of $\mathcal{F}$.

Proof. Uniqueness is clear, since for any $\mathcal{F} \in D(C)$, we can find a finite collection of triangles expressing $\mathcal{F} = \sum (-1)^q \mathcal{H}^q(\mathcal{F})$ in K-theory. Then the $\varphi^{\mathcal{H}^q(\mathcal{F})}$ determine $\varphi^{\mathcal{F}}$ by compatibility with triangles.

To show existence, we define $\varphi^{\mathcal{F}} : f(\mathcal{F}) \cong \otimes_q (f(\mathcal{H}^q(\mathcal{F})))^{(-1)^q} \cong \otimes_q (f([\mathcal{H}^q(\mathcal{F})]))^{(-1)^q} =: f([\mathcal{F}])$ via first the canonical filtration, then the $\{\varphi^{\mathcal{F}}\}_{\mathcal{F}\in C}$. We verify this is compatible with triangles by induction on the length of the amplitude interval of $\mathcal{F}_3$.

Suppose the amplitude interval of $\mathcal{F}_3$ is $[a, b]$. Then using the canonical truncation triangle $\tau^{\leq b-1}\mathcal{F}_3 \to \mathcal{F}_3 \to \mathcal{H}^b(\mathcal{F}_3)[-b]$ and compatibility with triangles of triangles, we reduce to the case where the amplitude interval has length 0, i.e $\mathcal{F}_3 \in C$ is a sheaf. This completes the induction step.

So we must prove compatibility for triangles of the shape $\mathcal{F}_1 \to \mathcal{F}_2 \to A[a] \to^{+1}$ with $A \in C$. Using the canonical truncations we reduce to the case where $\mathcal{F}_1$ and $\mathcal{F}_2$ are 0 except possibly in degrees a and $a+1$. Then we use the diagram

$$\begin{tikzcd}
\tau^{\leq a}\mathcal{F}_1 \arrow[r] \arrow[d] & \tau^{\leq a}\mathcal{F}_2 \arrow[r] \arrow[d] & I[a] \arrow[d] \\
\mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & A[a] \arrow[d] \\
\tau^{>a}\mathcal{F}_1 \arrow[r] & \tau^{>a}\mathcal{F}_2 \arrow[r] & A/I[a]
\end{tikzcd}$$

Here $I \subset A$ is the image of $\mathcal{F}_2^a$. The first and third rows, and the third column, consist of objects of C. The first and second columns are the canonical truncations. Hence the $\varphi^{\mathcal{F}}$ are compatible with the second row as well. $\square$

Only the subvarieties of type (C) contribute to $[\mathcal{F}]$; first we discuss the trivializations of $f(\mathcal{F}_i/\mathcal{F}_{i-1})$ for the C_i of types (V) and (Z). In fact these trivializations will be needed to construct $f(\mathcal{F}_i/\mathcal{F}_{i-1}) \xrightarrow{\sim} f(\mathcal{O}_{C_i})$ for the C_i of type (C).

Unless otherwise stated, the symbols $R, \mathcal{F}, \mathcal{G}, \gamma$ are as in Situation 5.1.1 *and Hypotheses* 5.1.2.

Let R be a DVR and let $\pi \in R$ be a uniformizer. By $D_{vert}(P_R) \subset D_{d\leq 1}(P_R)$ we denote the subcategory of perfect complexes $\mathcal{F}$ such that $\mathrm{Supp}(\mathcal{F}) \subset P_0$ is a subset of dimension ≤ 1 in the closed fiber. Then $\mathcal{F} \in D_{vert}(P_R)$ implies all $\mathcal{H}^q(\mathcal{F})$ are annihilated by some π^i. Let $A_i \subset \mathrm{Coh}(P_R)$ denote the subcategory of coherent sheaves $\mathcal{F}$ with $\dim(\mathrm{Supp}(\mathcal{F})) \leq 1$ and $\pi^i\mathcal{F} = 0$. Let $D_{vert,i}(P_R) \subset D_{vert}(P_R)$ denote the subcategory of complexes such that $\mathcal{H}^q(\mathcal{F}) \in A_i$ for all q.

Proposition 5.1.9. *Let R be a DVR and let $\pi \in R$ be a uniformizer. Then there is a collection of trivializations $\{\varphi^{\mathcal{F}} : f_R(\mathcal{F}) \to R\}_{\mathcal{F} \in D_{vert}(P_R)}$ compatible with triangles in $D_{vert}(P_R)$.*

The $\varphi^{\mathcal{F}}$ are characterized by the properties:

- *they are compatible with triangles in $D_{vert}(P_R)$, and*
- *they extend the normalization trivialization $\nu : f_K(\mathcal{F} \otimes_R K) = f_K(0) \cong K$ over Spec R.*

In fact the second property alone characterizes the $\varphi^{\mathcal{F}}$.

The construction of the $\varphi^{\mathcal{F}}$, for $\mathcal{F} \in \cup_i A_i \subset \mathrm{Coh}(P_R)$, will proceed in three steps. First we will construct the trivialization in the case $\pi\mathcal{F} = 0$, and show compatibility with short exact sequences in A_1. Then we will show that $\varphi^{\mathcal{F}}$ for all $\mathcal{F} \in A_{n-1}$ (compatible with short exact sequences in A_{n-1}) induces $\varphi^{\mathcal{F}}$ for all $\mathcal{F} \in A_n$. Finally we will show that compatibility with short exact sequences in A_{n-1} implies compatibility with short exact sequences in A_n.

Lemma 5.1.10. *Let R be a DVR and let $\pi \in R$ be a uniformizer. Then there is a collection of trivializations $\{\varphi^{\mathcal{F}} : f_R(\mathcal{F}) \to R\}_{\mathcal{F} \in D_{vert,1}(P_R)}$ compatible with triangles in $D_{vert,1}(P_R)$.*

The $\varphi^{\mathcal{F}}$ extend the normalization trivialization $\nu : f_K(\mathcal{F} \otimes_R K) = f_K(0) \cong K$ over Spec R.

Proof. Since A_1 has kernels and cokernels, by Lemma 5.1.8 we reduce to the case $\mathcal{F} \in \mathrm{Coh}(P_R)$ and $\pi\mathcal{F} = 0$, i.e. $\mathcal{F} \in A_1$.

Construction. We may regard $\mathcal{F}$ as a coherent sheaf on P_0. Since P_0 is a nonsingular scheme, by [11, B.8.3], $\mathcal{F}$ has a finite locally free $\mathcal{O}_{P_0}$ resolution $\mathcal{E}^\bullet$. Then the total complex of the double complex $[\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m} \to \mathcal{E}^\bullet \otimes_{k_0} R]$ is a locally free $\mathcal{O}_{P_R}$ resolution of $\mathcal{F}$. (Here we use the assumption that $R \to k_0$ has a k-linear section.) Since the total complex is the cone of the morphism $\mathcal{E}^\bullet \otimes \mathfrak{m} \to \mathcal{E}^\bullet \otimes R$, there is a distinguished triangle in $\mathrm{Parf}(P_R)$:

$$\mathcal{E}^\bullet \otimes \mathfrak{m} \to \mathcal{E}^\bullet \otimes R \to \mathcal{F}.$$

Applying Construction 5.1.7 to this triangle induces an isomorphism:

$$f(\mathcal{F}) \to (f(\mathcal{E}^\bullet \otimes \mathfrak{m}))^{-1} \otimes f(\mathcal{E}^\bullet \otimes R).$$

To obtain the isomorphism $f(\mathcal{F}) \to R$, we seek an isomorphism $f(\mathcal{E}^\bullet \otimes \mathfrak{m}) \cong f(\mathcal{E}^\bullet \otimes R)$ compatible with quasi-isomorphisms $\mathcal{E}^\bullet \to \mathcal{E}'^\bullet$.

Now $\mathfrak{m} \cong R$ as R-modules, and an isomorphism corresponds to a choice of uniformizing parameter. Given two isomorphisms $g_1, g_2 : \mathfrak{m} \to R$, there is a unique element $a \in R^\times$ making the following diagram of complexes of $\mathcal{O}_{P_R}$-modules commute:

$$\begin{tikzcd} & \mathcal{E}^\bullet \otimes R \arrow[dd, "a"] \\ \mathcal{E}^\bullet \otimes \mathfrak{m} \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & \\ & \mathcal{E}^\bullet \otimes R \end{tikzcd}$$

Therefore we have:

$$\begin{tikzcd} & f(\mathcal{E}^\bullet \otimes R) \arrow[dd, "f(a)"] \\ f(\mathcal{E}^\bullet \otimes \mathfrak{m}) \arrow[ur, "f(g_1)"] \arrow[dr, "f(g_2)"'] & \\ & f(\mathcal{E}^\bullet \otimes R) \end{tikzcd}$$

The isomorphism $f(a)$ is multiplication by a to the exponent

$$\sum_i (-1)^i R^i \pi_*((\mathcal{E}^\bullet \otimes R) \otimes^{\mathbf{L}} \mathcal{G}) = \sum_i (-1)^i H^i((\mathcal{E}^\bullet \otimes R) \otimes^{\mathbf{L}} \mathcal{G}) = \chi((\mathcal{E}^\bullet \otimes R) \otimes^{\mathbf{L}} \mathcal{G}).$$

The last quantity is zero by Lemma 2.2.11, so $f(g_1) = f(g_2)$. Since a quasi-isomorphism $q : \mathcal{E}^\bullet \to \mathcal{E}'^\bullet$ is compatible with multiplication by scalars, the identification is compatible with quasi-isomorphisms in the sense that the following diagram commutes:

$$\begin{tikzcd}
f(\mathcal{E}^\bullet \otimes \mathfrak{m}) \arrow[r, "f(g_1)"] \arrow[d, "f(q)"] & f(\mathcal{E}^\bullet \otimes R) \arrow[d, "f(q)"] \\
f(\mathcal{E}'^\bullet \otimes \mathfrak{m}) \arrow[r, "f(g_1)"] & f(\mathcal{E}'^\bullet \otimes R)
\end{tikzcd}$$

Compatibility. For any short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of coherent sheaves on P_0 satisfying the hypotheses of Lemma 5.1.10, we claim the following diagram commutes:

$$\begin{tikzcd}
f(\mathcal{F}_1) \otimes f(\mathcal{F}_3) \arrow[r] \arrow[d, "\varphi^{\mathcal{F}_1} \otimes \varphi^{\mathcal{F}_3}"] & f(\mathcal{F}_2) \arrow[d, "\varphi^{\mathcal{F}_2}"] \\
R \otimes R \arrow[r, "\text{mult}"] & R
\end{tikzcd}$$

To show this, it suffices to show commutativity at the generic point:

$$\begin{tikzcd}
f(\mathcal{F}_1)_K \otimes f(\mathcal{F}_3)_K \arrow[r] \arrow[d, "\varphi^{\mathcal{F}_1}_K \otimes \varphi^{\mathcal{F}_3}_K"] & f(\mathcal{F}_2)_K \arrow[d, "\varphi^{\mathcal{F}_2}_K"] \\
K \otimes K \arrow[r, "\text{mult}"] & K
\end{tikzcd}$$

Since the base change isomorphisms η are compatible with triangles, we have a commutative diagram whose top row is $\varphi^{\mathcal{F}} \otimes K$ (for $\mathcal{F} = \mathcal{F}_i, i = 1, 2, 3$):

$$\begin{tikzcd}
f(\mathcal{F})_K \arrow[r] \arrow[d, "\eta"] & f^{-1}(\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m})_K \otimes f(\mathcal{E}^\bullet \otimes_{k_0} R)_K \arrow[r, "f(\text{choice } \pi)\text{,mult}"] \arrow[d, "\eta\otimes\eta"] & R \otimes_R K \arrow[d, "\eta"] \\
f(0) \arrow[r] & f^{-1}(\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m} \otimes_R K) \otimes f(\mathcal{E}^\bullet \otimes_{k_0} R \otimes_R K) \arrow[r, "\psi"] & K
\end{tikzcd}$$

Here ψ results from the canonical identification $K \cong \mathfrak{m} \otimes_R K \cong R \otimes_R K$. To show the right square commutes, we must show ψ is equal to the isomorphism resulting from the (non-canonical) identification $\mathfrak{m} \otimes_R K \xrightarrow{\sim} R \otimes_R K$ induced by the choice of π in the top row. On $\mathcal{E}^\bullet \otimes_{k_0} K$ these differ by an element of K (namely, the choice of generator), which does not influence the resulting map on $f(\mathcal{E}^\bullet \otimes_{k_0} K)$ by Lemma 2.2.11.

Since ψ is induced by the canonical identifications $K \cong \mathfrak{m} \otimes_R K \cong R \otimes_R K$, the triangle giving rise to the bottom row is $\mathcal{E}^\bullet \otimes K \xrightarrow{1} \mathcal{E}^\bullet \otimes K \to 0$. By Axiom II.A. of Definition-Theorem 2.1.2, the bottom row is the normalization isomorphism $\nu : f(0) \cong K$ from Axiom I. of Definition-Theorem 2.1.2. Therefore $\eta \circ (\varphi^{\mathcal{F}} \otimes_R K) = \nu \circ \eta$. Since the ν's are additive on short exact sequences, the desired compatibility follows and the lemma is proved. □

Now we show that a compatible system of $\varphi^{\mathcal{F}}$ for all $\mathcal{F} \in A_{n-1}$ induces $\varphi^{\mathcal{F}}$ for all $\mathcal{F} \in A_n$.

Lemma 5.1.11. *Let A_i be as above. Let $T_{n-1} = \{\varphi^{\mathcal{A}} : f_R(\mathcal{A}) \to R\}_{\mathcal{A} \in A_{n-1}}$ be a collection of trivializations compatible with short exact sequences in A_{n-1}. Then T_{n-1} determines a unique collection $T_n = \{\varphi^{\mathcal{A}} : f_R(\mathcal{A}) \to R\}_{\mathcal{A} \in A_n}$ of trivializations compatible with short exact sequences in A_n and agreeing with T_{n-1} on A_{n-1}.*

Proof. Let $\mathcal{F} \in A_n \setminus A_{n-1}$. For $r \in R$, let $\mathrm{Ann}^{\mathcal{F}}(r)$ denote the subsheaf of $\mathcal{F}$ annihilated by r. In the exact sequence $0 \to \mathrm{Ann}^{\mathcal{F}}(\pi^{n-1}) \to \mathcal{F} \to \overline{\mathcal{F}} \to 0$, we have $\pi\overline{\mathcal{F}} = 0$, so this sequence determines $\varphi^{\mathcal{F}}$ by forcing the following diagram to commute.

$$\begin{tikzcd}
f(\mathrm{Ann}^{\mathcal{F}}(\pi^{n-1})) \otimes f(\overline{\mathcal{F}}) \arrow[r] \arrow[d, "\varphi\otimes\varphi"] & f(\mathcal{F}) \arrow[d, "\varphi^{\mathcal{F}}"] \\
R \otimes_R R \arrow[r, "\mathrm{mult}"] & R
\end{tikzcd}$$

Uniqueness. There may be many short exact sequences $0 \to \mathcal{A} \to \mathcal{F} \to \mathcal{B} \to 0$ with $\pi^{n-1}\mathcal{A} = \pi^{n-1}\mathcal{B} = 0$, but the $\varphi^{\mathcal{F}}$ thus determined is the same. Given two such, we have the following exact square (all rows and columns are exact sequences).

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r] \arrow[d] & \mathcal{A}' \arrow[r] \arrow[d] & \mathcal{C}_2 \arrow[d] \\
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{C}_3 \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{C}_4
\end{tikzcd} \tag{5.1.1}$$

For all objects $X \neq \mathcal{F}$ in (5.1.1), we have $\varphi^X : f(X) \to R$ compatible with the four triangles not involving $\mathcal{F}$ (the outer rows and columns). By Axiom II.C. of Definition-Theorem 2.1.2, we have the following commutative cube, which expresses $\varphi^{\mathcal{F}}_{\mathcal{A}\to\mathcal{F}\to\mathcal{B}} = \varphi^{\mathcal{F}}_{\mathcal{A}'\to\mathcal{F}\to\mathcal{B}'}$.

$$\begin{tikzcd}
\otimes_i f(\mathcal{C}_i) \arrow[rr] \arrow[dr] \arrow[dd, "\otimes\varphi^{\mathcal{C}_i}"'] & & f(\mathcal{A}') \otimes f(\mathcal{B}') \arrow[dr] \arrow[dd, "\varphi^{\mathcal{A}'}\otimes\varphi^{\mathcal{B}'}"] & \\
& f(\mathcal{A}) \otimes f(\mathcal{B}) \arrow[rr] \arrow[dd, "\varphi^{\mathcal{A}}\otimes\varphi^{\mathcal{B}}"] & & f(\mathcal{F}) \arrow[dd, "\varphi^{\mathcal{F}}"] \\
R^{\otimes 4} \arrow[rr, "{12,34}"] \arrow[dr, "{13,24}"'] & & R^{\otimes 2} \arrow[dr] & \\
& R^{\otimes 2} \arrow[rr] & & R
\end{tikzcd}$$

Compatibility. Finally we show the $\varphi^{\mathcal{F}}$ just constructed are compatible with short exact sequences in A_n. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is such a sequence, we use the following exact square.

$$\begin{tikzcd}
\mathrm{Ann}^{\mathcal{F}_1}(\pi^{n-1}) \arrow[r] \arrow[d] & \mathrm{Ann}^{\mathcal{F}_2}(\pi^{n-1}) \arrow[r] \arrow[d] & {\mathcal{F}_3}' \arrow[d] \\
\mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{F}_3 \arrow[d] \\
\overline{\mathcal{F}_1} \arrow[r] & \overline{\mathcal{F}_2} \arrow[r] & {\mathcal{F}_3}''
\end{tikzcd}$$

Note that if $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$, then $\mathrm{Ann}^{\mathcal{F}_1}(r) = \mathrm{Ann}^{\mathcal{F}_2}(r) \cap \mathcal{F}_1$. Since $\pi\overline{\mathcal{F}_2} = 0$, we have $\pi{\mathcal{F}_3}'' = 0$; likewise we have $\pi^{n-1}{\mathcal{F}_3}' = 0$. We may apply the induction hypothesis to the top and bottom rows in this square, yielding the following cube (denote $A\mathcal{F} = \mathrm{Ann}^{\mathcal{F}}(\pi^{n-1})$).

$$\begin{tikzcd}
f(A\mathcal{F}_1) \otimes f({\mathcal{F}_3}') \otimes f(\overline{\mathcal{F}_1}) \otimes f({\mathcal{F}_3}'') \arrow[rr] \arrow[dr] \arrow[dd, "\varphi^{\otimes 4}"'] & & f(A\mathcal{F}_2) \otimes f(\overline{\mathcal{F}_2}) \arrow[dr] \arrow[dd, "\varphi\otimes\varphi"] & \\
& f(\mathcal{F}_1) \otimes f(\mathcal{F}_3) \arrow[rr] \arrow[dd, "\varphi^{\mathcal{F}_1}\otimes\varphi^{\mathcal{F}_3}"] & & f(\mathcal{F}_2) \arrow[dd, "\varphi^{\mathcal{F}_2}"] \\
R^{\otimes 4} \arrow[rr, "{12,34}"] \arrow[dr, "{13,24}"'] & & R^{\otimes 2} \arrow[dr] & \\
& R^{\otimes 2} \arrow[rr] & & R
\end{tikzcd}$$

The top face commutes by Axiom II.C. of Definition-Theorem 2.1.2; the back face commutes by the induction hypothesis; the left and right faces commute by definition; the bottom face, whose arrows are the (possibly transposed) multiplication maps, clearly commutes. Therefore the front face commutes too. □

Proof of Proposition 5.1.9. The category $\cup_i A_i$ has kernels and cokernels, so by Lemma 5.1.8 we reduce to short exact sequences in $\cup_i A_i \subset \mathrm{Coh}(P_R)$. To see the $\varphi^{\mathcal{F}}$ are compatible with short exact sequences, apply Lemma 5.1.11 to the construction in Lemma 5.1.10.

To finish the proof of Proposition 5.1.9, note that any map of free R-modules is determined by its restriction to the generic point, so it suffices to show the $\varphi^{\mathcal{F}}$ extend the normalization trivialization. We proceed by induction on the minimal n such that $\pi^n\mathcal{F} = 0$. The case $n = 1$ we observed in the construction. We use the induction hypothesis on the first and third terms in the sequence $0 \to \mathrm{Ann}^{\mathcal{F}}(\pi^{n-1}) \to \mathcal{F} \to \overline{\mathcal{F}} \to 0$. Since the base change and normalization isomorphisms are compatible with triangles, the conclusion follows. □

Now we deal with factors of type (Z) where the base of the family is a field L (not necessarily algebraically closed) containing the original field k.

Proposition 5.1.12. *Let $L \supset k$ be a field. Then there is a collection of trivializations $\{\varphi^{\mathcal{F}} : f_L(\mathcal{F}) \to L\}_{\mathcal{F}\in D_{d\leq 0}(P_L)}$ compatible with triangles in $D_{d\leq 0}(P_L)$. The $\varphi^{\mathcal{F}}$ are characterized by the properties:*

- *they are compatible with triangles in $D_{d\leq 0}(P_L)$; and*
- *on $\mathcal{F} \in Coh(P_L)$ of type (Z), they agree with the trivialization (assume $\mathcal{F} \cong \mathcal{O}_Z$):*

$$\varphi_L^Z : f_L(\mathcal{F}) \cong det_L \pi_*(\mathcal{O}_Z \otimes^{\boldsymbol{L}} \mathcal{G}) \overset{\alpha_L}{\cong} (det_L \mathcal{O}_Z)^{-1} \otimes_L det_L(det_Z(\mathcal{G})) \xrightarrow{1\otimes \det_L(\gamma|_Z)} L$$

with α_L as in Proposition 4.2.3.

Remark 5.1.13. Since α and γ are compatible with quasi-isomorphisms and base change, the trivialization of $\mathcal{F} \in \mathrm{Coh}(P_L)$ of type (Z) may depend on the isomorphism $\mathcal{F} \cong \mathcal{O}_Z$, but nothing else. Part of the assertion is that any identification $\mathcal{F} \cong \mathcal{O}_Z$ results in the same trivialization.

The category $\mathrm{Coh}_{d\leq 0}(P_L)$ has kernels and cokernels, so by Lemma 5.1.8 we reduce to short exact sequences in $\mathrm{Coh}_{d\leq 0}(P_L)$. Since $\mathrm{Supp}(\mathcal{F})$ is a finite set, it is contained in some affine in P_L, and the problem is essentially algebraic. For this reason we formulate the necessary lemmas algebraically, and use the notation $f(M) := f(\widetilde{M})$.

Now we show that for $\mathcal{F} \in \mathrm{Coh}(P_L)$ of type (Z), the identification $\mathcal{F} \cong \mathcal{O}_Z$ does not influence the resulting map on determinants.

Lemma 5.1.14. *Let $(A, \mathfrak{m})$ be an Artinian local L-algebra which is a quotient of $\Gamma(U, \mathcal{O}_{P_L})$ for some affine open $U \subset P_L$. Let M be an A-module such that $M \cong A/\mathfrak{m}$, and let β, β' be isomorphisms $M \to A/\mathfrak{m}$. Then $f_L(\beta) = f_L(\beta') : f_L(M) \to f_L(A/\mathfrak{m})$.*

Proof. This is a special case of Corollary 2.2.12. □

Corollary 5.1.15. *Let $(A, \mathfrak{m})$ be as in Lemma 5.1.14. Let M be a finite type A-module, and let $\{M_i\}$ be a filtration of M such that, for all i, $M_i/M_{i-1} \cong A/\mathfrak{m}$. Let $\{\beta_i\}, \{\beta'_i\}$ be two families of isomorphisms: $\beta_i, \beta'_i : M_i/M_{i-1} \to A/\mathfrak{m}$. Then $\otimes_i f(\beta_i) = \otimes_i f(\beta'_i) : f(M) \cong \otimes_i f(M_i/M_{i-1}) \cong f(A/\mathfrak{m})^{\ell_A(M)}$.*

Proof. Tensor the result of Lemma 5.1.14 over all i. □

Remark 5.1.16. The maximal ideal $\mathfrak{m}_p = \mathfrak{m} \subset A$ is also the unique prime ideal in A, so such a filtration exists.

We know that for a fixed filtration, the trivializations of the graded pieces do not intervene; now we compare two filtrations.

Lemma 5.1.17. *Let $(A, \mathfrak{m})$ be as in Lemma 5.1.14. Let M be a finite type A-module, set $\ell := \ell_A(M)$, and let*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{\ell-1} \subset M_\ell = M \qquad \textit{and}$$

$$0 = N_0 \subset N_1 \subset \ldots \subset N_{\ell-1} \subset N_\ell = M$$

be filtrations of M as in Corollary 5.1.15*. Then the isomorphisms:*

$$f(M) \cong \otimes_i f(M_i/M_{i-1}) \to f(A/\mathfrak{m})^{\otimes \ell} \qquad \text{and}$$

$$f(M) \cong \otimes_j f(N_j/N_{j-1}) \to f(A/\mathfrak{m})^{\otimes \ell}$$

are the same.

Remark 5.1.18. The maps $\otimes_i f(M_i/M_{i-1}) \to f(A/\mathfrak{m})^{\otimes \ell}$ and $\otimes_j f(N_j/N_{j-1}) \to f(A/\mathfrak{m})^{\otimes \ell}$ are unambiguous by Lemma 5.1.14.

Proof. We use the compatibility between a pair of filtrations [23, p.22], which is a consequence of the compatibility of det with triangles of triangles. Set $K_{i,j} = M_i \cap N_j/(M_{i-1} \cap N_j) + (M_i \cap N_{j-1})$. Then we have the following commutative diagram of isomorphisms.

$$\begin{tikzcd} f(M) \arrow[r] \arrow[d] & \otimes_i f(M_i/M_{i-1}) \arrow[d] \\ \otimes_j f(N_j/N_{j-1}) \arrow[r] & \otimes_{i,j} f(K_{i,j}) \end{tikzcd}$$

Fix i for a moment. The $\{K_{i,j}\}_j$ are filtration quotients for the filtration $\{M_i \cap N_j/M_{i-1} \cap N_j\}_j$ of M_i/M_{i-1}. Since $M_i/M_{i-1} \cong A/\mathfrak{m}$ is a simple A-module, all of the $K_{i,j}$ are 0 except one, say K_{i,j_i}, and the filtration gives an identification $M_i/M_{i-1} \xrightarrow{\approx} K_{i,j_i}$. Composing with β_i gives a trivialization we will call ${\beta_i}^M : K_{i,j_i} \xrightarrow{\approx} A/\mathfrak{m}$. Tensoring over all i we obtain the a commutative diagram:

$$\begin{tikzcd} f(M) \arrow[r, "\cong"] \arrow[d, "\cong"] & \otimes_i f(M_i/M_{i-1}) \arrow[d, "\otimes_i f(\beta_i)"] \\ \otimes_i f(K_{i,j_i}) \arrow[r, "\otimes_i f({\beta_i}^M)"] & f(A/\mathfrak{m})^{\otimes \ell} \end{tikzcd} \tag{5.1.2}$$

where the bottom map is unambiguous by Lemma 5.1.14.

The same argument applied to the $\beta_j : N_j/N_{j-1} \cong A/\mathfrak{m}$ produces trivializations ${\beta_j}^N : K_{i_j,j} \xrightarrow{\approx} A/\mathfrak{m}$ and a commutative diagram similar to (5.1.2) whose bottom arrow is $\otimes_j f({\beta_j}^N)$. Using the identifications $\otimes_{i,j} f(K_{i,j}) = \otimes_i f(K_{i,j_i}) = \otimes_j f(K_{i_j,j})$, we obtain isomorphisms $\otimes_i f({\beta_i}^M)$ and $\otimes_j f({\beta_j}^N) : \otimes_{i,j} f(K_{i,j}) \rightarrow f(A/\mathfrak{m})^{\otimes \ell}$. Now by Corollary 5.1.15 applied to $\{K_{i,j}\}$ and the trivializations $\{{\beta_i}^M\}, \{{\beta_j}^N\}$, we conclude $\otimes_i f({\beta_i}^M) = \otimes_j f({\beta_j}^N)$. Then we assemble the two commutative diagrams of type (5.1.2) into one whose outer square is the desired one:

$$\begin{tikzcd} f(M) \arrow[r, "\cong"] \arrow[d, "\cong"] & \otimes_i f(M_i/M_{i-1}) \arrow[d, "\otimes_i f(\beta_i)"] \\ \otimes_j f(N_j/N_{j-1}) \arrow[r, "\otimes_j f(\beta_j)"] & f(A/\mathfrak{m})^{\otimes \ell} \end{tikzcd}$$

□

Proof of Proposition 5.1.12. Now we can define the $\varphi^{\mathcal{F}}$ whose existence is asserted in Proposition 5.1.12. Any $\mathcal{F} \in \mathrm{Coh}_{d \leq 0}(P_L)$ can be written as $\mathcal{F} = \oplus_i \mathcal{F}_i$ with each $\mathcal{F}_i = \widetilde{M}$ for M a finite type module over an Artinian local ring $(A_i, \mathfrak{m}_i)$, and each $\mathrm{Supp}(\mathcal{F}_i) = \{p_i\}$, p_i a closed point. We obtain $f_L(\mathcal{F}) \cong \otimes_i f_L(\mathcal{F}_i)$, canonically, so we will trivialize each $f_L(\mathcal{F}_i)$ and multiply to obtain $\varphi_L^{\mathcal{F}} : f_L(\mathcal{F}) \cong L$.

To fix notation, say $p_i \in P_L(L_i)$, where $L \to L_i = A_i/\mathfrak{m}_i$ is a finite field extension. Also set $\ell_i := \ell_{\mathcal{O}_{p_i}}(\mathcal{F}_i)$. By taking a cycle filtration of $\mathcal{F}_i$ and trivializing each graded piece, then using the identification α (Proposition 4.2.3), and finally restricting γ to p_i, we define $\varphi_L^{\mathcal{F}_i}$ using Construction 5.1.7:

$$f_L(\mathcal{F}_i) \cong (f_L(\mathcal{O}_{p_i}))^{\ell_i} \overset{\alpha^{\otimes \ell_i}}{\cong} ((\det_L \mathcal{O}_{p_i})^{-1} \otimes \det_L \det_{p_i} \mathcal{G})^{\ell_i} \xrightarrow{(1 \otimes \det(\gamma|_{p_i}))^{\ell_i}} L^{\otimes \ell_i} \xrightarrow{\text{mult}} L.$$

Compatibility. Thinking of a short exact sequence as a filtration possibly needing refinement to become a cycle filtration, the additivity of φ on short exact sequences in $\mathrm{Coh}_{d \leq 0}(P_L)$ is a consequence of the independence of the φ of the cycle filtration. More explicitly, let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \xrightarrow{h} \mathcal{F}_3 \to 0$ be an exact sequence in $\mathrm{Coh}_{d \leq 0}(P_L)$. Then cycle filtrations $\{\mathcal{F}_{1i}\}, \{\mathcal{F}_{3j}\}$ on $\mathcal{F}_1, \mathcal{F}_3$ (of lengths a, b respectively) induce a cycle filtration of $\mathcal{F}_2$:

$$0 = \mathcal{F}_{10} \subset \mathcal{F}_{11} \subset \ldots \subset \mathcal{F}_{1a-1} \subset \mathcal{F}_{1a} = h^{-1}(0) \subset h^{-1}(\mathcal{F}_{31}) \subset \ldots \subset h^{-1}(\mathcal{F}_{3b}) = \mathcal{F}_2.$$

The map $\varphi^{\mathcal{F}_2}$ is independent of the cycle filtration, so we are free to choose one compatible with cycle filtrations of $\mathcal{F}_1$ and $\mathcal{F}_3$.

Our $\varphi^{\mathcal{F}}$ are compatible with triangles and, for $\mathcal{F} \in \mathrm{Coh}(P_L)$ of type (Z), are defined to be the trivialization φ_L^Z in Proposition 5.1.12. These properties clearly characterize the isomorphisms. □

Now we deal with factors of type (Z) over a DVR; we will use the case of (Z) over a field. Since we encounter factors of type (V) in constructing the trivialization, and because we will need the more general statement later, we formulate the statement for $\mathcal{F} \in D_{d\leq 0;1}(P_R)$.

Proposition 5.1.19. *Let R be a DVR. Then there is a collection of trivializations $\{\varphi_R^{\mathcal{F}} : f_R(\mathcal{F}) \to R\}_{\mathcal{F}\in D_{d\leq 0;1}(P_R)}$ compatible with triangles in $D_{d\leq 0;1}(P_R)$.*

The $\varphi_R^{\mathcal{F}}$ are characterized by the properties:

- *they are compatible with triangles in $D_{d\leq 0;1}(P_R)$, and*
- *they are compatible with restriction to the generic fiber in the sense that $\varphi_R^{\mathcal{F}} \otimes_R K$ is equal to the trivialization constructed in Proposition* 5.1.12.

In fact the second property alone characterizes the $\varphi_R^{\mathcal{F}}$.

Proof. The category $\mathrm{Coh}(P_R) \cap D_{d\leq 0;1}(P_R)$ has kernels and cokernels, so by Lemma 5.1.8, we are reduced to $\mathcal{F} \in \mathrm{Coh}(P_R)$ satisfying the support condition, and we must show compatibility with short exact sequences of such.

Construction. Any $\mathcal{F} \in \mathrm{Coh}(P_R)$ satisfying the hypotheses of Proposition 5.1.19 has a cycle filtration such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_{V_i}(n_i)$ for V_i of types (V) and (Z). For factors of type (V), we constructed $f(\mathcal{O}_{V_i}(n_i)) \cong R$ in Proposition 5.1.9; we call this trivialization φ_R^V.

Consider now a factor of type (Z): suppose $\mathcal{F} \cong \mathcal{O}_Z(n)$ for a subvariety $i : Z \hookrightarrow P_R$ which is finite and flat over Spec R. First note we may assume $n = 0$ by Lemma 2.2.13 and Corollary 2.2.12: the choice of identification $\mathcal{O}_Z \cong \mathcal{O}_Z(n)$ does not influence the resulting map on determinants.

Denote by $p = \pi \circ i : Z \to \operatorname{Spec} R$. We define $\varphi_R^{\mathcal{F}}$ using the identification α (Proposition 4.2.3) and $\gamma|_Z$, as we did in Proposition 5.1.12. (We suppress the identification $f_R(\mathcal{F}) \cong \det_R \mathbf{R}\pi_*(\mathcal{O}_Z \otimes^{\mathbf{L}} \mathcal{G})$.)

$$\varphi_R^Z : \det_R \mathbf{R}\pi_*(\mathcal{O}_Z \otimes^{\mathbf{L}} \mathcal{G}) \cong \det_R p_*(\mathbf{L}i^*\mathcal{G}) \overset{\alpha_R}{\cong} (\det_R \mathcal{O}_Z)^{-1} \otimes \det_R(\det_Z(\mathcal{G})) \xrightarrow{1 \otimes \det_R(\gamma|_Z)} R$$

Construction 5.1.7 then gives a trivialization:

$$\varphi_R^{\mathcal{F}} : f_R(\mathcal{F}) \xrightarrow{\text{cycle filt}} \otimes_i f_R(\mathcal{F}_i/\mathcal{F}_{i-1}) \xrightarrow{\varphi_R^V, \varphi_R^Z} R^{\otimes} \xrightarrow{\text{mult}} R.$$

It remains to be checked: the trivialization is independent of the choice of the filtration; and that the collection of trivializations is compatible with short exact sequences and with restriction to the generic fiber.

Independence of cycle filtration. It suffices to show different filtrations induce the same map on the generic fiber. So let two cycle filtrations be given, and denote $\mathcal{F}_i := \operatorname{gr}^i(\mathcal{F}), \mathcal{G}_j := \operatorname{gr}^j(\mathcal{F})$. Since we have shown $\varphi_R^V \otimes_R K : f_R(\mathcal{O}_V \otimes_R K) \to K$ is the normalization isomorphism from Axiom I. of Definition-Theorem 2.1.2, it suffices to show the following diagram commutes.

$$\begin{tikzcd} f(\mathcal{F}) \otimes_R K \arrow[r] \arrow[d] & \otimes_{i \in \mathrm{I}^{fl}} f(\mathcal{F}_i) \otimes_R K \arrow[d] \\ \otimes_{j \in \mathrm{J}^{fl}} f(\mathcal{G}_j) \otimes_R K \arrow[r] & K \end{tikzcd} \tag{5.1.3}$$

where $\mathrm{I}^{fl} = \{i | \mathcal{F}_i \text{ is finite flat over} R\}$ and likewise J^{fl}. To see the two compositions are equal, we will show they both agree with the trivialization constructed over a field in Proposition 5.1.12.

We have a diagram:

$$\begin{tikzcd}
f(\mathcal{F}) \otimes K \arrow[r, "\mathrm{filt}\otimes K"] \arrow[d, "\eta"] & \otimes_i f(\mathcal{F}_i) \otimes K \arrow[r, "\otimes(\alpha_R,\gamma_R)_K"] \arrow[d, "\eta"] & \otimes_i (R \otimes K) \arrow[r, "\mathrm{mult}"] \arrow[d, "\eta"] & R \otimes K \arrow[d, "\eta"] \\
f(\mathcal{F} \otimes K) \arrow[r] & \otimes_i f(\mathcal{F}_i \otimes K) \arrow[r, "\otimes(\alpha_K,\gamma_K)"] & \otimes_i K \arrow[r, "\mathrm{mult}"] & K
\end{tikzcd}$$

which commutes. The filtration isomorphisms consist of triangle isomorphisms, and the triangle isomorphisms are compatible with base change [23, Thm. 2]. The identification α is compatible with base change (Proposition 4.2.3), as is γ. This shows the diagram commutes.

In fact the bottom row is the trivialization $\varphi_K^{\mathcal{F}\otimes_R K}$ constructed in Proposition 5.1.12: $R \to K$ is a localization, so $\Gamma(Z, \mathcal{O}_Z) \otimes_R K$ is a domain, so (cycle filtration) $\otimes_R K$ is a cycle filtration of $\mathcal{F} \otimes_R K$. The construction in Proposition 5.1.12 uses α_K and γ_K, as we have here. Therefore each route in (5.1.3) is equal to $\varphi_K^{\mathcal{F}\otimes_R K} \circ \eta$. We have also shown the $\varphi^{\mathcal{F}}$ (for $\mathcal{F}$ as in Proposition 5.1.19) are compatible with restriction to the generic fiber.

Compatibility. To show compatibility with triangles, by the above remark we can check on the generic fiber, which was done in Proposition 5.1.12. Alternatively, as before we may view a short exact sequence as a filtration requiring refinement, and then independence of cycle filtration (which was shown by looking on the generic fiber!) gives the additivity. As before, the restriction to the generic point determines the isomorphism. $\square$

In the construction of the $\varphi^{\mathcal{F}}$ for $\dim(\mathrm{Supp}(\mathcal{F})) = 1$, we need a trivialization of $f(\mathcal{Q})$, $\dim(\mathrm{Supp}(\mathcal{Q})) = 0$, for *varying* $\mathcal{Q}$. Since the $\mathcal{Q}$ will be parameterized by a seminormal base, it suffices to show the construction over a DVR is compatible with base change to the generic and closed points. The generic point is covered by Proposition 5.1.19.

First we study algebraically the base change to the closed fiber of the factors of type (Z).

Situation 5.1.20. Let R be a DVR or a field. Let S be a domain which is a quotient of $\Gamma(U, \mathcal{O}_{P_R})$ for some affine open $U \subset P_R$, and suppose that the natural map $R \to S$ is finite and flat.

Let K' be a field and $R \to K'$ a ring map. Then $S' := S \otimes_R K'$ is an Artinian K'-algebra which splits canonically as $S' = \oplus A_r$, with $(A_r, \mathfrak{m}_r)$ an Artinian local K'-algebra with residue field κ_r. Each A_r has a filtration (necessarily of length $\ell(A_r)$) by A_r-submodules with graded pieces $\mathrm{gr}^k(A_r) \cong \kappa_r$. Let $\psi_{kr} : \mathrm{gr}^k(A_r) \cong \kappa_r$ be a collection of isomorphisms.

Lemma 5.1.21. *Let $R, S, K', (A_r, \mathfrak{m}_r), \psi_{kr}$ be as in Situation* 5.1.20*. Let $\mathcal{G}$ be as in Situation* 5.1.1*. Set $N = S \otimes^{\boldsymbol{L}}_{\mathcal{O}_{P_R}} \mathcal{G}$ and $N' = N \otimes_R K' = N \otimes_S S'$. Let $\gamma : \det_S N \to S$ denote also the restriction of γ to Spec S. Then:*

(1) For each r, the composition

$$\det_{K'}(A_r \otimes N') \cong \otimes_k \det_{K'}(gr^k(A_r) \otimes N') \xrightarrow{\text{via } \psi_{kr}} \det_{K'}(\kappa_r \otimes N')^{\otimes \ell(A_r)}$$

is independent of the choice of filtration and isomorphisms ψ_{kr}.

(2) The following diagram commutes.

$$
\begin{tikzcd}
(\det_R N) \otimes_R K' \arrow[r, "(1\otimes\det_R(\gamma))_{K'}\circ(\alpha_R\otimes K')"] \arrow[d, "\eta"] & ((\det_R S)^{-1} \otimes \det_R S)_{K'} \arrow[d, "\eta\otimes\eta"] \\
\det_{K'}(N') \arrow[r, "(1\otimes\det_{K'}(\gamma_{S'}))\circ\alpha_{K'}"] \arrow[d, "="] & (\det_{K'} S')^{-1} \otimes_{K'} \det_{K'} S' \arrow[ddd] \\
\otimes_r \det_{K'}(A_r \otimes N') \arrow[d, "\cong"] & \\
\otimes_{k,r} \det_{K'}(gr^k(A_r) \otimes N') \arrow[d] & \\
\otimes_r \det_{K'}(\kappa_r \otimes N')^{\ell(A_r)} \arrow[r, "\otimes_r(1\otimes\det_{K'}(\gamma\otimes\kappa)\circ\alpha_r)^{\ell(A_r)}"] & \otimes_r((\det_{K'} \kappa_r)^{-1} \otimes \det_{K'} \kappa_r)^{\ell(A_r)}
\end{tikzcd}
$$

Here α is as in Proposition 4.2.3.

Proof. The first item is a combination of Lemmas 5.1.14 and 5.1.17.

To see the second, first observe the top rectangle commutes since α and γ are compatible with base change. For a single r, the bottom row is the following composition ($\wedge = \det$) to the tensor power of $\ell(A_r)$:

$$\wedge_{K'}(\kappa_r \otimes N') \xrightarrow{\alpha_r} (\wedge_{K'}\kappa_r)^{-1} \otimes \wedge_{K'} \wedge_{\kappa_r} (\kappa_r \otimes N') \xrightarrow{1\otimes\wedge_{K'}(\gamma\otimes\kappa)} (\wedge_{K'}\kappa_r)^{-1} \otimes (\wedge_{K'}\kappa_r).$$

Recall (Proposition 4.2.3) the α isomorphisms are defined (for a single locally free S-module N) by choosing bases $S = \oplus_{i\in I} R \cdot s_i$ and $N = \oplus_{j\in J} S \cdot n_j$ (I and J are finite sets consisting of a and b elements respectively), then using the natural map

$$s_1n_1 \wedge \cdots \wedge s_an_b \mapsto (s_1 \wedge \cdots \wedge s_a)^{\otimes(b-1)} s_1(n_1 \wedge \cdots \wedge n_b) \wedge \cdots \wedge s_a(n_1 \wedge \cdots \wedge n_b).$$

The choice $S = \oplus_{i\in I} R \cdot s_i$ determines $S' = \oplus_{i\in I} K' \cdot s_i$. Since we have $S' = \oplus A_r$, the choice of s_i induces also $A_r = \oplus_{i\in I_r} K' \cdot s_i$, where $r \neq r' \Rightarrow I_r \cap I_{r'} = \emptyset$ and $\cup_r I_r = I$. We have also $\mathrm{gr}^k(A_r) = \oplus_{i\in I_{rk}} K' \cdot s_i$, where $k \neq k' \Rightarrow I_{rk} \cap I_{rk'} = \emptyset$ and $\cup_k I_{rk} = I_r$.

In particular we have $\kappa_r = \mathrm{gr}^{\ell(A_r)}(A_r) = \oplus_{i\in I_{r\ell}} K' \cdot s_i$. Such a decomposition is used to define the α_r.

Then the identification (along the left column) $\det_{K'}(N') \cong \otimes_r \det_{K'}(\kappa_r \otimes N')^{\ell(A_r)}$ is given by the formula:

$$\bigwedge_{i\in I, j\in J} s_i n_j \mapsto \bigotimes_{k,r}\Big(\bigwedge_{i\in I_{rk}, j\in J} s_i n_j\Big).$$

A single α_r is described by:

$$\bigwedge_{i\in I_{rk}, j\in J} s_i n_j \mapsto \Big(\bigwedge_{i\in I_{rk}} s_i\Big)^{\otimes(b-1)}\Big(\bigwedge_{i\in I_{rk}} s_i\big(\bigwedge_{j\in J} n_j\big)\Big)$$

which is manifestly compatible with the $\alpha_{K'}$ recalled above. □

We cannot yet deal with base change to the closed fiber in the case $\dim(\mathrm{Supp}(\mathcal{F}\otimes^{\mathbf{L}} k_0)) = 1$, so the hypotheses in the following lemma are more restrictive than in Proposition 5.1.19.

Lemma 5.1.22. *Let R be a DVR. Let $\mathcal{F} \in Coh_{d\leq 0}(P_R)$. Then the trivialization $\varphi^{\mathcal{F}}$ constructed in Proposition* 5.1.19 *is compatible with restriction to the closed fiber in the sense that the following diagram commutes.*

$$\begin{tikzcd}
f_R(\mathcal{F}) \otimes_R k_0 \arrow[r, "\varphi_R^{\mathcal{F}} \otimes_R k_0"] \arrow[d, "\eta"] & R \otimes_R k_0 \arrow[d, "\eta"] \\
f_{k_0}(\mathcal{F} \otimes^{\mathbf{L}} k_0) \arrow[r, "\varphi_{k_0}^{\mathcal{F}\otimes^{\mathbf{L}} k_0}"] & k_0
\end{tikzcd}$$

The bottom arrow is the trivialization constructed in Proposition 5.1.12*.*

Remark 5.1.23. For $\mathcal{F}$ as in Lemma 5.1.22, we have $\dim(\mathrm{Supp}(\mathcal{F}\otimes^{\mathbf{L}} k_0)) \leq 0$ by Remark 2.2.9, so the construction in Proposition 5.1.12 defines $\varphi_{k_0}^{\mathcal{F}\otimes^{\mathbf{L}} k_0}$.

Proof. Take a cycle filtration of $\mathcal{F}$. The base change isomorphisms are compatible with triangles, thus filtrations; and the $\varphi^{\mathcal{F}}$ constructed in Proposition 5.1.12 are compatible with triangles. Therefore we are reduced to showing the diagram commutes when $\mathcal{F}$ is of type (V) (and zero-dimensional, so the structure sheaf of a point in the closed fiber) or of type (Z).

Type (V). Let $\mathcal{E}^\bullet$ be a locally free $\mathcal{O}_{P_0}$-resolution of $\mathcal{F}$. We observe:

(1) $\mathrm{Tot}[\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m} \to \mathcal{E}^\bullet \otimes_{k_0} R]$ is adapted to $-\otimes^{\mathbf{L}} k_0$; and

(2) $\mathrm{Tot}[\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m} \to \mathcal{E}^\bullet \otimes_{k_0} R] \otimes^{\mathbf{L}} k_0 = \mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m}/\mathfrak{m}^2[1] \oplus \mathcal{E}^\bullet$.

Now we claim the following diagram, in which the top path is $\varphi_R^{\mathcal{F}} \otimes_R k_0$ and the leftmost path is $\varphi_{k_0}^{\mathcal{F}\otimes^{\mathbf{L}} k_0} \circ \eta$, commutes.

$$\begin{tikzcd}
f(\mathcal{F})_0 \arrow[r] \arrow[d, "\eta"] & f^{-1}(\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m})_0 \otimes f(\mathcal{E}^\bullet \otimes_{k_0} R)_0 \arrow[r, "(\mathfrak{m}\cong R)\otimes 1"] \arrow[d, "\eta\otimes\eta"] & f^{-1}(\mathcal{E}^\bullet \otimes_{k_0} R)_0 \otimes f(\mathcal{E}^\bullet \otimes_{k_0} R)_0 \arrow[d] \\
f(\mathcal{F}\otimes^{\mathbf{L}} k_0) \arrow[r] \arrow[dr, "\text{take } \mathcal{H}^q"'] & f^{-1}(\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m}/\mathfrak{m}^2) \otimes f(\mathcal{E}^\bullet) \arrow[r, "(\mathfrak{m}/\mathfrak{m}^2\cong k_0)\otimes 1"] \arrow[d, "\text{take } \mathcal{H}^q"] & f^{-1}(\mathcal{E}^\bullet) \otimes f(\mathcal{E}^\bullet) \arrow[d, "\text{take } \mathcal{H}^q"] \\
& f^{-1}(\mathcal{F} \otimes_{k_0} \mathfrak{m}/\mathfrak{m}^2) \otimes f(\mathcal{F}) \arrow[r, "(\mathfrak{m}/\mathfrak{m}^2\cong k_0)\otimes 1"] \arrow[d, "(\varphi_{k_0}^{\mathcal{F}\otimes\mathfrak{m}/\mathfrak{m}^2})^{-1} \otimes \varphi_{k_0}^{\mathcal{F}}"] & f^{-1}(\mathcal{F}) \otimes f(\mathcal{F}) \arrow[d, "(\varphi^{\mathcal{F}})^{-1} \otimes \varphi^{\mathcal{F}}"] \\
& f^{-1}([\mathcal{F} \otimes_{k_0} \mathfrak{m}/\mathfrak{m}^2]) \otimes f([\mathcal{F}]) \arrow[r, "="] & f^{-1}([\mathcal{F}]) \otimes f([\mathcal{F}])
\end{tikzcd}$$

All squares are trivial except the bottom right. Any $g : \mathfrak{m}/\mathfrak{m}^2 \cong k_0$ induces an isomorphism $\mathcal{F} \otimes \mathfrak{m}/\mathfrak{m}^2 \cong \mathcal{F}$. Since the φ (only on $\mathrm{Coh}_{d\le 0}(P_0)$) are compatible with triangles, we have a commutative diagram:

$$\begin{tikzcd}
f(\mathcal{F} \otimes \mathfrak{m}/\mathfrak{m}^2) \arrow[r, "f(g)"] \arrow[d, "\varphi^{\mathcal{F}\otimes\mathfrak{m}/\mathfrak{m}^2}"] & f(\mathcal{F}) \arrow[d, "\varphi^{\mathcal{F}}"] \\
f([\mathcal{F} \otimes \mathfrak{m}/\mathfrak{m}^2]) \arrow[r, "="] & f([\mathcal{F}])
\end{tikzcd}$$

which (after dualizing) shows the bottom right square is commutative.

Type (Z). This part of the argument is valid for any base change $R \to K'$ where K' is a field.

Let $Z = \mathrm{Spec}\, S \subset P_R$ be an R-flat closed subvariety of relative dimension zero. Then the following diagram commutes.

$$\begin{tikzcd}
f(\mathcal{O}_Z) \otimes_R K' \arrow[r, "\varphi_R^Z \otimes K'"] \arrow[d, "\eta"] & R \otimes_R K' \arrow[d, "\eta"] \\
f(\mathcal{O}_Z \otimes_R K') \arrow[r, "\varphi_{K'}^{Z\otimes_R K'}"] & K'
\end{tikzcd}$$

The commutativity is the content of Lemma 5.1.21. □

In the application $\dim(\mathrm{Supp}(\mathcal{F})), \dim(\mathrm{Supp}(\mathcal{G})) \leq 1$, so we expect $\mathrm{Supp}(\mathcal{F})$ and $\mathrm{Supp}(\mathcal{G})$ to be disjoint. If in addition $\mathcal{G}$ is exact on $\mathrm{Supp}(\mathcal{F})$, $f(\mathcal{F})$ has another trivialization induced by the quasi-isomorphism $\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G} \xrightarrow{\sim} 0$. Eventually we will establish the $\varphi^{\mathcal{F}}$ are compatible with this quasi-isomorphism; for now we establish it among coherent sheaves with zero-dimensional support.

Proposition 5.1.24. *Let R be a DVR or a field. Let $i : Z \subset P_R$ be a closed subvariety which is finite and flat over Spec R, and let $p = \pi \circ i : Z \to$ Spec R denote the structure map. In addition to Situation* 5.1.1*, suppose $\mathcal{G}$ is exact on Z. (For example, $\mathcal{G} \xrightarrow{\sim} \mathcal{O}_D$, where $D \subset P_R$ is a closed subscheme of relative dimension ≤ 1 such that $Z \cap D = \emptyset$.)*

Then the trivialization $\varphi : f_R(\mathcal{O}_Z) \to R$ constructed in Proposition 5.1.19*, with the canonical γ of Lemma* 2.1.4*, is equal to the one induced by the quasi-isomorphism $\mathcal{O}_Z \otimes^{\boldsymbol{L}} \mathcal{G} \xrightarrow{\sim} 0$.*

Proof. The composition:

$$i^*(\det_{P_R}(\mathcal{G})) \xrightarrow{\eta} \det_Z(\mathbf{L}i^*\mathcal{G}) \xrightarrow{\det(0)} \det(0) = \mathcal{O}_Z$$

is equal to the restriction of γ to Z.

This gives the commutativity of the bottom triangle in the following diagram.

$$\begin{tikzcd}
\det \mathbf{R}\pi_*(\mathcal{O}_Z \otimes^{\mathbf{L}} \mathcal{G}) \arrow[r, "\det \mathbf{R}\pi_*(0)"] \arrow[d, "="] & \det \mathbf{R}\pi_*(0) = R \arrow[d, "="] \\
\det p_*(\mathbf{L}i^*\mathcal{G}) \arrow[r, "\det p_*(0)"] \arrow[d, "\alpha_R"] & \det p_*(0) = R \arrow[d, "\alpha_R"] \\
\det_R(\mathcal{O}_Z)^{-1} \otimes \det_R(\det_Z(\mathbf{L}i^*\mathcal{G})) \arrow[r, "1 \otimes \det_R(\det_Z(0))"] \arrow[dr, "1 \otimes \det_R(\gamma|_Z)"'] & \det_R(\mathcal{O}_Z)^{-1} \otimes \det_R(\mathcal{O}_Z) \arrow[d, "="] \\
 & \det_R(\mathcal{O}_Z)^{-1} \otimes \det_R(\mathcal{O}_Z)
\end{tikzcd}$$

The claim follows. Note that the middle square uses the compatibility of α with quasi-isomorphisms of complexes of locally free sheaves of finite rank (Proposition 4.2.3). $\square$

Before we consider the case (C), we need a preliminary lemma.

Lemma 5.1.25. *Let $K \supset k$ be an algebraically closed field, and let $\mathcal{F}_1, \mathcal{F}_2$ be torsion free sheaves of rank 1 on a proper variety X/K.*

(1) The functor $F : K - algebras \to Sets$

$$F(A) = Hom_{\mathcal{O}_X}(\mathcal{F}_1 \otimes_K A, \mathcal{F}_2 \otimes_K A)$$

is isomorphic to the functor $Hom_{Sch/K}(-, \mathbb{A}^n_K)$, where $n = \dim_K Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)$.

Let $M = Hom(\mathcal{F}_1, \mathcal{F}_2) \setminus 0 \cong \mathbb{A}^n_K \setminus 0$ denote the K-scheme parameterizing nonzero morphisms between $\mathcal{F}_1$ and $\mathcal{F}_2$.

(2) If $\dim(X) = 1$ and $X \subset P$, the universal cokernel $\mathcal{Q}^U$ is finite and flat over M, and the trivializations $\{\varphi^{\mathcal{Q}_\alpha}\}_{\alpha \in M}$ constructed in Proposition 5.1.19 glue to a trivialization:

$$\varphi^U : det_M \boldsymbol{R}\pi_{2*}(\mathcal{Q}^U \otimes^{\boldsymbol{L}} \mathcal{G}) \cong \mathcal{O}_M.$$

Proof. For the first item, use the identities:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1 \otimes_K A, \mathcal{F}_2 \otimes_K A)\\
&= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2 \otimes_K A)\\
&= (\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)) \otimes_K A\\
&\cong \mathrm{Hom}_{K-vs}(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)^*, A) \text{ since } V^{**} \cong V\\
&= \mathrm{Hom}_{K-alg}(\mathrm{Sym}^*_K(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)^*), A)\\
&= \mathrm{Hom}_{Sch/K}(\mathrm{Spec}\ A, \mathrm{Spec}\ \mathrm{Sym}^*_K(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)^*)).
\end{aligned}$$

We proceed to the second item. On $X \times_K M$, we have the universal nonzero homomorphism ${\pi_1}^*(\mathcal{F}_1) \to {\pi_1}^*(\mathcal{F}_2)$. Any nonzero map between torsion free rank 1 sheaves on a proper variety over an algebraically closed field must be injective with finite cokernel. By [25, 20.E] the universal cokernel is a coherent sheaf $\mathcal{Q}^U$ on $X \times_K M \subset P \times_K M$ which is finite and flat over M. We have the line bundle $\det_M \mathbf{R}\pi_{2*}(\mathcal{Q}^U \otimes^{\mathbf{L}} \mathcal{G})$ on M

whose fiber over $\alpha \in M$ is precisely $f(\mathcal{Q}_\alpha)$. The trivializations constructed in Proposition 5.1.19 are compatible with base change to the generic and closed (Lemma 5.1.22) fibers, so they glue as claimed by the seminormality of $\mathbb{A}^n_K \setminus 0$ and Theorem 3.1.5. □

Now we begin to tackle the factors of type (C): C_i is a flat family of curves.

Proposition 5.1.26. *Let $L \supset k$ be an algebraically closed field. Then there is a collection of isomorphisms $\{\varphi_L^{\mathcal{F}} : f_L(\mathcal{F}) \to f_L([\mathcal{F}])\}_{\mathcal{F} \in D_{d\leq 1}(P_L)}$ compatible with triangles in $D_{d\leq 1}(P_L)$. The $\varphi_L^{\mathcal{F}}$ are characterized by the following properties:*

- *they are compatible with triangles in $D_{d\leq 1}(P_L)$;*
- *on $\mathcal{F} \in Coh(P_L)$ of type (Z), they agree with the trivialization constructed in Proposition 5.1.12; and*
- *if $C \subset P_L$ is a reduced and irreducible curve, then $\varphi_L^{\mathcal{O}_C} : f(\mathcal{O}_C) \to f([\mathcal{O}_C])$ is the identity.*

Proof. The category $\mathrm{Coh}_{d\leq 1}(P_L)$ has kernels and cokernels, so by Lemma 5.1.8, we are reduced to short exact sequences in $\mathrm{Coh}_{d\leq 1}(P_L)$.

Construction. Any $\mathcal{F} \in \mathrm{Coh}(P_L)$ satisfying the hypotheses of Proposition 5.1.26 has a cycle filtration such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_{V_i}(n_i)$ for V_i of types (Z) and (C). We have trivialized the factors of type (Z). Now we construct $S^{\mathcal{F}} : f(\mathcal{F}) \cong f(\mathcal{O}_C)$ for coherent sheaves $\mathcal{F}$ such that $\mathrm{Supp}(\mathcal{F})$ is a reduced and irreducible curve C, and $\mathcal{F}|_C$ is torsion free of rank 1.

By Serre's Theorem [18, II.5.17], there is a very ample divisor $H \subset P_L$ such that $\mathcal{F}(H)$ (regarded as a sheaf on the subvariety on which it is scheme-theoretically supported) has a global section. This gives rise to an exact sequence:

$$0 \to \mathcal{O}_C(-H) \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{Q}_{\alpha,\beta} \to 0$$

where $\mathcal{Q}$ satisfies the hypotheses of Proposition 5.1.12. Using the trivialization there constructed and the triangle determined by (α, β), we obtain an isomorphism:

$$f_L(\mathcal{O}_C(-H)) \otimes_L L \xrightarrow{1\otimes(\varphi^{\mathcal{Q}})^{-1}} f_L(\mathcal{O}_C(-H)) \otimes_L f_L(\mathcal{Q}_{\alpha,\beta}) \xrightarrow{f(\alpha,\beta)} f_L(\mathcal{F}).$$

First we observe this isomorphism does not depend on β: the cokernel is unique up to isomorphism, and the isomorphism is a (very simple) triangle involving coherent sheaves satisfying the hypotheses of Proposition 5.1.12. Therefore we denote $f(\alpha) = f(\alpha,\beta) \circ 1 \otimes (\varphi^{\mathcal{Q}})^{-1} : f(\mathcal{O}_C(-H)) \to f(\mathcal{F})$.

Independence of section. We study the effect of changing α. First we show that for $\lambda \in L^\times$, we have $f(\lambda\alpha) = f(\alpha)$. We have an isomorphism of triangles:

$$\begin{tikzcd}
\mathcal{O}_C(-H) \arrow[r, "\lambda\alpha"] \arrow[d, "\lambda"] & \mathcal{F} \arrow[r] \arrow[d, "1"] & \mathcal{Q}_{\lambda\alpha} \arrow[d] \\
\mathcal{O}_C(-H) \arrow[r, "\alpha"] & \mathcal{F} \arrow[r] & \mathcal{Q}_{\alpha}
\end{tikzcd}$$

which induces the following commutative diagram (we suppress the $f(\mathcal{Q}_\alpha)$).

$$\begin{tikzcd}
f(\mathcal{O}_C(-H)) \arrow[r, "f(\lambda\alpha)"] \arrow[d, "f(\lambda)"] & f(\mathcal{F}) \arrow[d, "1"] \\
f(\mathcal{O}_C(-H)) \arrow[r, "f(\alpha)"] & f(\mathcal{F})
\end{tikzcd}$$

Since $f(\lambda) = 1$ by Lemma 2.2.11, we have $f(\lambda\alpha) = f(\alpha)$.

Now we claim the map

$$\theta : \mathbb{A}^n_L \setminus 0 \cong \mathrm{Hom}_{\mathcal{O}_{P_L}}(\mathcal{O}_C(-H), \mathcal{F}) \setminus 0 \to \mathrm{Isom}_L(f(\mathcal{O}_C(-H)), f(\mathcal{F})) \cong \mathbb{A}^1_L \setminus 0$$

$$\theta(\alpha) = f(\alpha)$$

is a morphism of schemes. If so, then since θ is invariant under scaling by $L^\times$, it descends to a map $\mathbb{P}^{n-1}_L \to \mathbb{A}^1_L$, which must be constant; so in fact $f(\alpha) : f(\mathcal{O}_C(-H)) \to f(\mathcal{F})$ is independent of α.

To prove the claim, notice the left hand side is endowed with a scheme structure by the first part of Lemma 5.1.25. Now $\mathbf{R}\pi_*(\alpha \otimes^{\mathbf{L}} 1) : \mathbf{R}\pi_*(\mathcal{O}_C(-H) \otimes^{\mathbf{L}} \mathcal{G}) \to \mathbf{R}\pi_*(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G})$ is even L-linear. Since the determinant is given by a polynomial (say after choosing an L-basis of the vector space $\mathrm{Hom}(\mathcal{O}_C(-H), \mathcal{F})$), the map $f(\alpha, \beta)$ varies algebraically with α. By the second part of Lemma 5.1.25, $(\varphi^{\mathcal{Q}_\alpha})$ also varies algebraically with α. Therefore $f(\alpha) = f(\alpha, \beta) \circ 1 \otimes (\varphi^{\mathcal{Q}_\alpha})^{-1}$ varies algebraically with α, and θ is a morphism of schemes.

We have also an exact sequence

$$0 \to \mathcal{O}_C(-H) \xrightarrow{i} \mathcal{O}_C \to \mathcal{Q}_i \to 0$$

where $\dim(\mathrm{Supp}(\mathcal{Q}_i)) = 0$, so the above argument produces $f(i) : f(\mathcal{O}_C(-H)) \to f(\mathcal{O}_C)$, independent of i. Then we define $S_H^{\mathcal{F}} = f(i) \circ (f(\alpha))^{-1} : f(\mathcal{F}) \to f(\mathcal{O}_C)$. In fact the choice of H is irrelevant.

Independence of very ample divisor. Suppose we used a different pair of exact sequences:

$$\mathcal{O}_C(-H') \xrightarrow{\alpha'} \mathcal{F} \to \mathcal{Q}_{\alpha'} \qquad \mathcal{O}_C(-H') \xrightarrow{i'} \mathcal{O}_C \to \mathcal{Q}_{i'}.$$

We claim $S_H^{\mathcal{F}} = S_{H'}^{\mathcal{F}}$. We have the following commutative squares of injective maps.

$$\begin{tikzcd} \mathcal{O}_C(-H-H') \arrow[r, "i'(-H)"] \arrow[d, "i(-H')"] & \mathcal{O}_C(-H) \arrow[d, "\alpha"] \\ \mathcal{O}_C(-H') \arrow[r, "\alpha'"] & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_C(-H-H') \arrow[r, "i'(-H)"] \arrow[d, "i(-H')"] & \mathcal{O}_C(-H) \arrow[d, "i"] \\ \mathcal{O}_C(-H') \arrow[r, "i'"] & \mathcal{O}_C \end{tikzcd}$$

To prove the claim, it suffices to show $S_H = S_{H+H'}$. Since every S is independent of the sections used to define it, it suffices to show

$$f(i) \circ (f(\alpha))^{-1} = f(i \circ i'(-H)) \circ (f(\alpha \circ i'(-H)))^{-1}. \tag{5.1.4}$$

To show (5.1.4) it suffices to check

$$f(\alpha \circ i'(-H)) = f(\alpha) \circ f(i'(-H)) \tag{5.1.5}$$

and likewise for $i \circ i'(-H)$. To see (5.1.5) we use the following exact square.

$$\begin{tikzcd} \mathcal{O}_C(-H-H') \arrow[r, "\alpha\circ i'(-H)"] \arrow[d, "i'(-H)"] & \mathcal{F} \arrow[r] \arrow[d, "="] & \mathcal{Q}_{\alpha\circ i'(-H))} \arrow[d] \\ \mathcal{O}_C(-H) \arrow[r, "\alpha"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{Q}_\alpha \arrow[d] \\ \mathcal{Q}_{i'}(-H) \arrow[r] & 0 \arrow[r] & \mathcal{Q}'' \end{tikzcd}$$

Note the rightmost column is in general a distinguished triangle, not a short exact sequence. All of the $\mathcal{Q}$ terms satisfy the hypotheses of Proposition 5.1.12, so the trivializations are compatible with the triangles in this exact square. Therefore we obtain a commutative square:

$$\begin{tikzcd} f(\mathcal{O}_C(-H-H')) \arrow[r, "f(\alpha\circ i'(-H))"] \arrow[d, "f(i'(-H))"] & f(\mathcal{F}) \arrow[d, "1"] \\ f(\mathcal{O}_C(-H)) \arrow[r, "f(\alpha)"] & f(\mathcal{F}) \end{tikzcd}$$

which proves (5.1.5). The proof that $f(i \circ i'(-H)) = f(i) \circ f(i'(-H))$ is identical. The claim follows.

Independence of cycle filtration. We have constructed $S^{\mathcal{F}} : f(\mathcal{F}) \cong f(\mathcal{O}_C)$ for coherent sheaves $\mathcal{F}$ such that $\mathrm{Supp}(\mathcal{F})$ is a reduced and irreducible curve C, and $\mathcal{F}|_C$ is torsion free of rank 1. In particular we have constructed $f(\mathcal{F}) \cong f([\mathcal{F}])$ for factors of type (C). Now using the trivialization of the (Z) factors constructed in Proposition 5.1.12, we define the isomorphism $\varphi_L^{\mathcal{F}}$, relative to a cycle filtration, using Construction 5.1.7:

$$f_L(\mathcal{F}) \xrightarrow{\text{cycle filt}} \otimes_i f_L(\mathcal{F}_i/\mathcal{F}_{i-1}) \xrightarrow{5.1.12, S^{\mathcal{F}}} L^{\otimes} \otimes f_L([\mathcal{F}]) \xrightarrow{\text{mult}} f_L([\mathcal{F}]).$$

In fact this map is independent of the filtration: let $\mathcal{F} \in \mathrm{Coh}_{d\le 1}(P_L)$, and let $\{\mathcal{F}_i\}, \{\mathcal{G}_j\}$ be two cycle filtrations of $\mathcal{F}$. Then we claim the maps

$$f(\mathcal{F}) \to \otimes_i f(\mathcal{F}_i/\mathcal{F}_{i-1}) \xrightarrow{S\otimes\varphi^Z} f([\mathcal{F}]) \text{ and } f(\mathcal{F}) \to \otimes_j f(\mathcal{G}_j/\mathcal{G}_{j-1}) \xrightarrow{S\otimes\varphi^Z} f([\mathcal{F}])$$

are equal. For fixed i, let $\mathcal{K}_{ij}$ denote the quotients of the filtration $\{\mathcal{F}_i \cap \mathcal{G}_j / \mathcal{F}_{i-1} \cap \mathcal{G}_j\}$ of $\mathcal{F}_i/\mathcal{F}_{i-1}$. Then we have the following commutative diagram.

$$\begin{tikzcd}
f(\mathcal{F}) \arrow[r] \arrow[d] & \otimes_i f(\mathcal{F}_i/\mathcal{F}_{i-1}) \arrow[d] \\
\otimes_j f(\mathcal{G}_j/\mathcal{G}_{j-1}) \arrow[r] & \otimes_{i,j} f(\mathcal{K}_{ij})
\end{tikzcd}$$

The idea is to show $\otimes_i f(\mathcal{F}_i/\mathcal{F}_{i-1}) \xrightarrow{S\otimes\varphi^Z} f([\mathcal{F}])$ and $\otimes_j f(\mathcal{G}_j/\mathcal{G}_{j-1}) \xrightarrow{S\otimes\varphi^Z} f([\mathcal{F}])$ induce the same map $\otimes_{i,j} f(\mathcal{K}_{ij}) \to f([\mathcal{F}])$.

The quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ of type (Z) are (isomorphic to) structure sheaves of L-points (as $L = \overline{L}$), so, as in Lemma 5.1.17, the subsheaves $\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j$ are equal to $\mathcal{F}_i/\mathcal{F}_{i-1}$ for a while, then become 0. The trivialization $\varphi^{\mathcal{K}_{ij}} : f(\mathcal{K}_{ij}) \to L$ constructed in Proposition 5.1.12 is compatible with the trivialization $\varphi^{\mathcal{F}_i/\mathcal{F}_{i-1}}$.

For a factor $\mathcal{F}_i/\mathcal{F}_{i-1}$ of type (C), there exists j_i such that:

$\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j$ is torsion free of rank 1 on C for $j \geq j_i$; and

$\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j = 0$ for $j < j_i$.

If necessary, twist $\mathcal{F}$ itself by some very ample $H \subset P_R$ so that all sheaves of the shape $\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j$ have global sections. We will suppress the H because if the maps to $f(\mathcal{O}_C(-H))$ are the same, by composing with $f(\mathcal{O}_C(-H)) \to f(\mathcal{O}_C)$, we see the maps to $f(\mathcal{O}_C)$ are the same.

By choosing a section of $\mathcal{F}_i \cap \mathcal{G}_{j_i}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j_i}$, for all $j > j_i$, we have a triangle of triangles:

$$\begin{tikzcd}
\mathcal{O}_C \arrow[r, "="] \arrow[d] & \mathcal{O}_C \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{F}_i \cap \mathcal{G}_{j-1}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j-1} \arrow[r] \arrow[d] & \mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j \arrow[r] \arrow[d] & \mathcal{K}_{ij} \arrow[d] \\
\mathcal{Q}_1 \arrow[r] & \mathcal{Q}_2 \arrow[r] & \mathcal{K}_{ij}
\end{tikzcd} \tag{5.1.6}$$

which induces the following commutative diagram.

$$\begin{tikzcd}
f(\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j) \arrow[r] \arrow[d] & f(\mathcal{O}_C) \otimes f(\mathcal{Q}_2) \arrow[r, "1\otimes\varphi^{\mathcal{Q}_2}"] \arrow[d] & f(\mathcal{O}_C) \arrow[d, "1"] \\
f(\mathcal{F}_i \cap \mathcal{G}_{j-1}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j-1}) \otimes f(\mathcal{K}_{ij}) \arrow[r] & f(\mathcal{O}_C) \otimes f(\mathcal{Q}_1) \otimes f(\mathcal{K}_{ij}) \arrow[r, "1\otimes\varphi^{\mathcal{Q}_1}\otimes\varphi^{\mathcal{K}_{ij}}"] & f(\mathcal{O}_C)
\end{tikzcd}$$

The top row is the map $S : f(\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j) \cong f([\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j])$ constructed above, and the bottom row is the map $f(\mathcal{F}_i \cap \mathcal{G}_{j-1}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j-1}) \cong f([\mathcal{F}_i \cap \mathcal{G}_{j-1}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j-1}])$ tensored with the trivialization $\varphi^{\mathcal{K}_{ij}} : f(\mathcal{K}_{ij}) \to L$ constructed in Proposition 5.1.12. On $\mathcal{F}_i \cap \mathcal{G}_{j_i}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j_i}$ itself, $S^{\mathcal{F}_i/\mathcal{F}_{i-1}}$ induces $S^{\mathcal{F}_i \cap \mathcal{G}_{j_i}/\mathcal{F}_{i-1} \cap \mathcal{G}_{j_i}}$.

Therefore $S \otimes \varphi^Z : \otimes_i f(\mathcal{F}_i/\mathcal{F}_{i-1}) \to f([\mathcal{F}])$ induces $S \otimes \varphi^Z : \otimes_{i,j} f(\mathcal{K}_{ij}) \to f([\mathcal{F}])$. Of course $S \otimes \varphi^Z : \otimes_j f(\mathcal{G}_j/\mathcal{G}_{j-1}) \to f([\mathcal{F}])$ does as well, and the claim follows.

Compatibility. As in Proposition 5.1.12, compatibility with triangles follows from independence of cycle filtration: given a short exact sequence in $\mathrm{Coh}_{d\leq 1}(P_L)$, we use a cycle filtration of the middle term which induces cycle filtrations of the outer terms.

Characterizing properties. Our construction clearly has the stated properties, so it suffices to show the properties determine the isomorphisms. For any very ample $H \subset P_L$, the sequence

$$0 \to \mathcal{O}_C(-H) \to \mathcal{O}_C \to \mathcal{Q} \to 0$$

determines $\varphi^{\mathcal{O}_C(-H)} : f(\mathcal{O}_C(-H)) \to f([\mathcal{O}_C(-H)]) = f(\mathcal{O}_C)$, since we have additivity on the sequence and we have specified the isomorphisms on the second and third terms of the sequence. Then $\varphi^{\mathcal{F}}$ is determined for any $\mathcal{F}$ such that $\mathrm{Supp}(\mathcal{F})$ is a reduced and irreducible curve C and $\mathcal{F}|_C$ is torsion free of rank 1, because we can always find a very ample H such that there is a nonzero map $\mathcal{O}_C(-H) \to \mathcal{F}$ whose cokernel is (necessarily) supported in dimension zero. Then the normalization on $\mathcal{F} \in \mathrm{Coh}(P_L)$ of type (Z) and compatibility with filtrations determines $\varphi^{\mathcal{F}}$ for any $\mathcal{F} \in \mathrm{Coh}_{d\leq 1}(P_L)$. □

Corollary 5.1.27. *Let $L \supset k$ be an algebraically closed field. Let $\mathcal{F}_1, \mathcal{F}_2 \in Coh_{d\leq 1}(P_L)$. Let $\alpha, \alpha' : \mathcal{F}_1 \to \mathcal{F}_2$ be injective morphisms, and suppose that* $\dim(Supp(Cok(\alpha))) =$

$\dim(Supp(Cok(\alpha'))) = 0$. *Then* $f(\alpha) = f(\alpha') : f(\mathcal{F}_1) \xrightarrow{\sim} f(\mathcal{F}_2)$.

Proof. Both $f(\alpha)$ and $f(\alpha')$ are equal to $(\varphi^{\mathcal{F}_2})^{-1} \circ \varphi^{\mathcal{F}_1}$. □

We will need the result Proposition 5.1.26 when the base is a DVR or an arbitrary field. A minor difference between the DVR case and the case of an algebraically closed field is that filtrations quotients may be vertical. Since all cokernels still lie in $D_{d\leq 0;1}(P_R)$, this is not a problem. More serious is the failure of the second part of Lemma 5.1.25 to hold, even over a field K: $C_{\overline{K}}$ need not be a variety even if C_K is, and then the locus of injective maps inside $\mathrm{Hom}_{C_{\overline{K}}}(\mathcal{F}_1, \mathcal{F}_2)$ is mysterious, even in the case $\mathcal{F}_1$ and $\mathcal{F}_2$ are torsion free of rank 1. It seems difficult to argue that $\mathrm{Hom}_{C_{\overline{K}}}(\mathcal{F}_1, \mathcal{F}_2)/\overline{K}^{\times}$ is a variety without nonconstant functions.
To get around this, we extract the necessary result (independence of choice of section) by deducing it from the case $L = \overline{K}$.

Lemma 5.1.28. *Let R be a DVR or a field. Let C_R be a reduced and irreducible relative curve, and let $\mathcal{F}_1, \mathcal{F}_2$ be torsion free sheaves of rank* 1 *on C_R. Let $\alpha, \alpha' : \mathcal{F}_1 \to \mathcal{F}_2$ be injective morphisms; then both $Cok(\alpha)$ and $Cok(\alpha')$ satisfy the hypotheses of Proposition* 5.1.19*. Then $f(\alpha) = f(\alpha')$, that is:*

$$1 \otimes (\varphi^{Cok(\alpha)})^{-1} \circ f(\alpha, \beta) = 1 \otimes (\varphi^{Cok(\alpha')})^{-1} \circ f(\alpha', \beta') : f(\mathcal{F}_1) \xrightarrow{\sim} f(\mathcal{F}_2).$$

Here the β's denote the canonical maps to the cokernels.

Proof. It suffices to show $f(\alpha) \otimes_R \overline{K} = f(\alpha') \otimes_R \overline{K}$, where $K = \mathrm{Frac}\, R$ and $\overline{K}$ denotes its algebraic closure. Since the base change isomorphisms are compatible with triangles, we have $f(\alpha, \beta) \otimes_R \overline{K} = f(\alpha \otimes_R \overline{K}, \beta \otimes_R \overline{K})$.
Likewise the $\varphi^{\mathcal{Q}}$'s are compatible with the base change $R \to \overline{K}$: $\varphi^{\mathrm{Cok}(\alpha)} \otimes_R \overline{K} = \varphi^{\mathrm{Cok}(\alpha \otimes_R \overline{K})}$. The compatibility with $R \to K$ follows from Proposition 5.1.19, and with $K \to \overline{K}$ from the argument in "Type (Z)" in Lemma 5.1.22.
Now $R \to \overline{K}$ is flat, so $\alpha \otimes_R \overline{K}$ and $\alpha' \otimes_R \overline{K}$ are injective. Corollary 5.1.27 implies $f(\alpha \otimes_R \overline{K}) = f(\alpha' \otimes_R \overline{K})$, so we are done.
When R is a field, only the base change $K \to \overline{K}$ is interesting. □

Theorem 5.1.29. *Let R be a DVR or a field. Then there is a collection of isomorphisms $\{\varphi_R^{\mathcal{F}} : f_R(\mathcal{F}) \to f_R([\mathcal{F}])\}_{\mathcal{F} \in D_{d\leq 1}(P_R)}$ compatible with triangles in $D_{d\leq 1}(P_R)$. The $\varphi_R^{\mathcal{F}}$ are characterized by the following properties:*

- *they are compatible with triangles in $D_{d\leq 1}(P_R)$;*
- *on $\mathcal{F} \in D_{d\leq 0;1}(P_R)$, they agree with the trivialization constructed in Proposition* 5.1.19; *and*
- *if $C \subset P_R$ is a reduced and irreducible R-flat curve, then $\varphi_R^{\mathcal{O}_C} : f(\mathcal{O}_C) \to f([\mathcal{O}_C])$ is the identity.*

Proof. The construction is the same as in Proposition 5.1.26: take a cycle filtration of $\mathcal{F}$, use the trivializations constructed in Proposition 5.1.19 on the terms of type (Z) and (V), twist and take sections on the terms of type (C), and multiply these together using Construction 5.1.7. By Lemma 5.1.28 we obtain independence of the choice of section. The previous argument for independence of the choice of very ample divisor is valid over a DVR as well. The argument for independence of cycle filtration requires a mild strengthening because filtrations over a DVR may be more interesting. So let $\mathcal{F} \in \mathrm{Coh}(P_R)$, and suppose two cycle filtrations $\{\mathcal{F}_i\}, \{\mathcal{G}_j\}$ are given. Let $\mathcal{K}_{ij}$ denote the subquotients of the filtration $\{\mathcal{F}_i \cap \mathcal{G}_j / \mathcal{F}_{i-1} \cap \mathcal{G}_j\}$ of $\mathcal{F}_i/\mathcal{F}_{i-1}$.

For a factor $\mathcal{F}_i/\mathcal{F}_{i-1}$ of type (V) or (Z), all of the subsheaves $\mathcal{F}_i \cap \mathcal{G}_j/\mathcal{F}_{i-1} \cap \mathcal{G}_j$ are of the same type, so the trivializations $\varphi^{\mathcal{K}_{ij}}$ are compatible with $\varphi^{\mathcal{F}_i/\mathcal{F}_{i-1}}$ by Proposition 5.1.19.

For a factor $\mathcal{F}_i/\mathcal{F}_{i-1}$ of type (C), we construct a triangle of triangles as in (5.1.6) where $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{K}_{ij}$ all satisfy the hypotheses of Proposition 5.1.19.

As both $\otimes_i f(\mathcal{F}_i/\mathcal{F}_{i-1}) \to f([\mathcal{F}])$ and $\otimes_j f(\mathcal{G}_j/\mathcal{G}_{j-1}) \to f([\mathcal{F}])$ induce the "expected" map $S \otimes \varphi^V \otimes \varphi^Z : \otimes_{i,j} f(\mathcal{K}_{ij}) \to f([\mathcal{F}])$, we conclude $f(\mathcal{F}) \to f([\mathcal{F}])$ is independent of the cycle filtration.

Our construction clearly has the stated properties. Since any $\mathcal{F} \in D_{d\leq 1}(P_R)$ has a filtration whose factors lie in $D_{d\leq 0;1}(P_R)$ or are reduced and irreducible relative curves, the stated properties determine the isomorphisms on the subcategory $D_{d\leq 1}(P_R)$. $\square$

Summary. For $\mathcal{F} \in D_{d\leq 1}(P_R)$, we constructed $\varphi_R^{\mathcal{F}} : f_R(\mathcal{F}) \cong f_R([\mathcal{F}])$ as follows:

$$f(\mathcal{F}) \xrightarrow{\text{cycle filtration}} \otimes_i f(\mathcal{F}_i) \xrightarrow{\varphi^V \otimes \varphi^Z \otimes S} f([\mathcal{F}])$$

where each $\mathcal{F}_i$ is a subquotient of the cycle filtration (so of type (V), (Z), or (C)).

5.2 Compatibilities among the φ.

Eventually we will show the φ are compatible with base change from Spec R to the closed and generic fibers; now we prove the φ are compatible with field extensions. We renew the conditions in Hypotheses 5.1.2 and the fixed data in Situation 5.1.1. Further decorations of the φ symbol will be explained as we encounter them.

Field extensions

First we analyze the behavior under a field extension K'/K of the trivialization $\varphi^{\mathcal{F}}$ for $\mathcal{F} \in \mathrm{Coh}(P_K)$ of type (Z). Then we do the same for the isomorphism $\varphi^{\mathcal{F}}$ for $\mathcal{F} \in \mathrm{Coh}(P_K)$ of type (C). Finally we combine these results to show the φ are compatible with field extensions.

Remark 5.2.1. For $r = 0, 1$, if $\mathcal{F} \in D_{d\leq r}(P_K)$, then $\mathcal{F}_{K'} \in D_{d\leq r}(P_{K'})$ by Remark 2.2.9. We use this to know certain $\varphi_{K'}$ are defined.

Lemma 5.2.2. *Let K'/K be a field extension. Let $Z = Spec\ L \subset P_K$ be a zero-dimensional closed subvariety. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{O}_Z) \otimes K' \arrow[r, "\varphi_K^Z \otimes K'"] \arrow[d, "\eta"] & K \otimes K' \arrow[d, "\eta"] \\ f(\mathcal{O}_Z \otimes K') \arrow[r, "\varphi_{K'}^{Z \otimes K'}"] & K' \end{tikzcd}$$

Proof. This follows from Lemma 5.1.21 with $R = K$. $\square$

Corollary 5.2.3. *Let K'/K be a field extension. Then the trivializations $\{\varphi_K^{\mathcal{F}} : f_K(\mathcal{F}) \to K\}_{\mathcal{F}\in D_{d\leq 0}(P_K)}$ are compatible with base change to K'. This means for any $\mathcal{F} \in D_{d\leq 0}(P_K)$, the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}) \otimes K' \arrow[r, "\varphi_K^{\mathcal{F}}\otimes K'"] \arrow[d, "\eta"] & K \otimes K' \arrow[d, "\eta"] \\
f(\mathcal{F} \otimes K') \arrow[r, "\varphi_{K'}^{\mathcal{F}\otimes K'}"] & K'
\end{tikzcd}.$$

Proof. Both $\{\varphi_K^{\mathcal{F}} \otimes K'\}$ and $\{\eta^{-1} \circ \varphi_{K'}^{\mathcal{F}\otimes K'} \circ \eta\}$ are compatible with triangles, and by Lemma 5.2.2 they agree on the pullbacks of coherent sheaves of type (Z). These properties characterize the trivializations on the image of $D_{d\leq 0}(P_K)$ in $D_{d\leq 0}(P_{K'})$ via pullback. $\square$

Lemma 5.2.4. *Let K'/K be a field extension. Let $\mathcal{F} \in Coh(P_K)$ be such that $Supp(\mathcal{F})$ is a reduced and irreducible curve C, and $\mathcal{F}|_C$ is torsion free of rank 1. Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}) \otimes_K K' \arrow[r, "\varphi_K^{\mathcal{F}}\otimes K'"] \arrow[d, "\eta"] & f(\mathcal{O}_C) \otimes_K K' \arrow[d, "\eta"] \\
f(\mathcal{F} \otimes_K K') \arrow[dr, "\varphi_{K'}^{\mathcal{F}\otimes K'}"'] & f(\mathcal{O}_{C_{K'}}) \arrow[d, "\varphi_{K'}^{\mathcal{O}_{C_{K'}}}"] \\
 & f([\mathcal{O}_{C_{K'}}])
\end{tikzcd}$$

Remark 5.2.5. The subscheme $C_{K'} \subset P_{K'}$ need not be reduced nor irreducible. The isomorphism $\varphi_{K'}^{\mathcal{O}_{C_{K'}}}$ is constructed in Theorem 5.1.29.

Proof. Suppose $\varphi_K^{\mathcal{F}} : f_K(\mathcal{F}) \to f_K([\mathcal{F}]) = f_K(\mathcal{O}_C)$ is realized by a short exact sequence $0 \to \mathcal{O}_C \to \mathcal{F} \to \mathcal{Q} \to 0$, the general case being obtained by splicing two together. Then the sequence $0 \to \mathcal{O}_{C_{K'}} \to \mathcal{F}_{K'} \to \mathcal{Q}_{K'} \to 0$ is exact. We claim the following diagram commutes.

$$\begin{tikzcd}
f(\mathcal{F})_{K'} \arrow[r] \arrow[d, "\eta"] & (f(\mathcal{O}_C) \otimes f(\mathcal{Q}))_{K'} \arrow[r, "(1\otimes\varphi^{\mathcal{Q}})_{K'}"] \arrow[d, "\eta"] & f(\mathcal{O}_C)_{K'} \arrow[d, "\eta"] \\
f(\mathcal{F} \otimes K') \arrow[r] \arrow[d, "\varphi^{\mathcal{F}\otimes K'}"] & f(\mathcal{O}_{C_{K'}}) \otimes f(\mathcal{Q}_{K'}) \arrow[r, "1\otimes\varphi^{\mathcal{Q}_{K'}}"] \arrow[d, "\varphi^{\mathcal{O}_{C_{K'}}}\otimes\varphi^{\mathcal{Q}_{K'}}"] & f(\mathcal{O}_{C_{K'}}) \arrow[d, "\varphi^{\mathcal{O}_{C_{K'}}}"] \\
f([\mathcal{F} \otimes K']) \arrow[r, "+"] & f([\mathcal{O}_{C_{K'}}]) \otimes K' \arrow[r] & f([\mathcal{O}_{C_{K'}}])
\end{tikzcd}$$

The bottom left square commutes because the φ are compatible with triangles. The top right square commutes since the φ on $D_{d\leq 0}(P_K)$ are compatible with field extensions by Corollary 5.2.3. The commutativity of the bottom right square is trivial. The commutativity of the outer square is what we aimed to prove. □

Proposition 5.2.6. *Let K'/K be a field extension. For any $\mathcal{F} \in D_{d\leq 1}(P_K)$, the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}) \otimes_K K' \arrow[r, "\varphi_K^{\mathcal{F}}\otimes K'"] \arrow[d, "\eta"] & f([\mathcal{F}]) \otimes_K K' \arrow[d, "\eta"] \\
f(\mathcal{F} \otimes_K K') \arrow[dr, "\varphi_{K'}^{\mathcal{F}\otimes_K K'}"'] & f([\mathcal{F}] \otimes_K K') \arrow[d, "\varphi_{K'}^{[\mathcal{F}]\otimes_K K'}"] \\
& f([\mathcal{F} \otimes_K K'])
\end{tikzcd}$$

Remark 5.2.7. Suppose $[\mathcal{F}] = \sum n_i C_i$ and $[C_{iK'}] = \sum m_{ij} C'_j$. Then $[\mathcal{F}_{K'}] = \sum n_i m_{ij} C'_j$. By definition, $\varphi_{K'}^{[\mathcal{F}]\otimes_K K'} = \otimes_i (\varphi_{K'}^{\mathcal{O}_{C_{iK'}}})^{\otimes n_i}$.

Proof. Since the base change and φ isomorphisms are compatible with triangles, it suffices to show the diagram commutes when $\mathcal{F} \in \mathrm{Coh}(P_K)$ is of type (Z) or (C). These were done in Lemmas 5.2.2 and 5.2.4 respectively. □

Corollary 5.2.8. *Let K'/K be a field extension. For any $\mathcal{F}_1, \mathcal{F}_2 \in D_{d\leq 1}(P_K)$ such that $[\mathcal{F}_1] = [\mathcal{F}_2]$, the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}_1) \otimes_K K' \arrow[r, "((\varphi_K^{\mathcal{F}_2})^{-1}\circ\varphi_K^{\mathcal{F}_1})\otimes_K K'"] \arrow[d, "\eta"] & f(\mathcal{F}_2) \otimes_K K' \arrow[d, "\eta"] \\
f(\mathcal{F}_1 \otimes_K K') \arrow[r, "(\varphi_{K'}^{\mathcal{F}_2\otimes K'})^{-1}\circ\varphi_{K'}^{\mathcal{F}_1\otimes K'}"] & f(\mathcal{F}_2 \otimes_K K')
\end{tikzcd}$$

Having constructed the isomorphism over fields and DVRs, we proceed to show these constructions are compatible with base change from Spec R to the closed and generic fibers.

Base change to the generic fiber

Proposition 5.2.9. *Let R be a DVR and K its fraction field. For any $\mathcal{F} \in D_{d\leq 1}(P_R)$, the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}) \otimes_R K \arrow[r, "\varphi_R^{\mathcal{F}} \otimes_R K"] \arrow[d, "\eta"] & f([\mathcal{F}]) \otimes_R K \arrow[d, "\eta"] \\ f(\mathcal{F} \otimes_R K) \arrow[r, "\varphi_K^{\mathcal{F} \otimes_R K}"] & f([\mathcal{F}] \otimes_R K) = f([\mathcal{F} \otimes_R K]) \end{tikzcd}$$

Remark 5.2.10. Because $R \to K$ is a localization, varieties over R restrict to varieties over K. Hence the diagram here is simpler than in Proposition 5.2.6.

Proof. Since the base change isomorphisms are compatible with triangles, it suffices to show the diagram commutes when $\mathcal{F} \in \mathrm{Coh}(P_R)$ is of type (V), (Z), or (C). On those of types (V) and (Z), the trivializations φ_R are compatible with restriction to the generic fiber by Proposition 5.1.19.

For those of type (C), we argue as in Lemma 5.2.4. Suppose an exact sequence $0 \to \mathcal{O}_C \to \mathcal{F} \to \mathcal{Q} \to 0$ on P_R is used to construct $\varphi_R^{\mathcal{F}} : f_R(\mathcal{F}) \to f_R([\mathcal{F}]) = f_R(\mathcal{O}_C)$. This pulls back to an exact sequence on P_K, so is compatible with the construction of $\varphi_K^{\mathcal{F} \otimes_R K}$. Since the trivialization of $f(\mathcal{Q})$ is compatible with restriction to the generic fiber by Proposition 5.1.19, we obtain a commutative diagram as in the proof of Lemma 5.2.4. $\square$

Corollary 5.2.11. *Keep the notation from Proposition* 5.2.9. *For any $\mathcal{F}_1, \mathcal{F}_2 \in D_{d\leq 1}(P_R)$ such that $[\mathcal{F}_1] = [\mathcal{F}_2]$, the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}_1) \otimes_R K \arrow[r, "((\varphi_R^{\mathcal{F}_2})^{-1} \circ \varphi_R^{\mathcal{F}_1}) \otimes_R K"] \arrow[d, "\eta"] & f(\mathcal{F}_2) \otimes_R K \arrow[d, "\eta"] \\ f(\mathcal{F}_1 \otimes_R K) \arrow[r, "(\varphi_K^{\mathcal{F}_2 \otimes K})^{-1} \circ \varphi_K^{\mathcal{F}_1 \otimes K}"] & f(\mathcal{F}_2 \otimes_R K) \end{tikzcd}$$

From compatibility with field extensions and with base change to the generic fiber, compatibility with base change between DVRs follows formally.

Corollary 5.2.12. *Let R and R' be DVRs with fraction fields K and K', and let $R \to R'$ be a ring map. For any $\mathcal{F}_1, \mathcal{F}_2 \in D_{d\leq 1}(P_R)$ such that $[\mathcal{F}_1] = [\mathcal{F}_2]$, the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}_1) \otimes_R R' \arrow[r, "((\varphi_R^{\mathcal{F}_2})^{-1} \circ \varphi_R^{\mathcal{F}_1}) \otimes_R R'"] \arrow[d, "\eta"] & f(\mathcal{F}_2) \otimes_R R' \arrow[d, "\eta"] \\ f(\mathcal{F}_1 \otimes_R R') \arrow[r, "(\varphi_{R'}^{\mathcal{F}_2 \otimes R'})^{-1} \circ \varphi_{R'}^{\mathcal{F}_1 \otimes R'}"] & f(\mathcal{F}_2 \otimes_R R') \end{tikzcd}$$

Proof. It suffices to show the diagram commutes after further base change to K'. Since the intermediate step in the base change $R \to K'$ does not affect the end result, we can think of the top row as $((\varphi_R^{\mathcal{F}_2})^{-1} \circ \varphi_R^{\mathcal{F}_1}) \otimes_R K \otimes_K K'$. Then by compatibility with field extensions (Corollary 5.2.8) and with base change to the generic fiber (Corollary 5.2.11), the top row is equal to $(\varphi_{K'}^{\mathcal{F}_2 \otimes K'})^{-1} \circ \varphi_{K'}^{\mathcal{F}_1 \otimes K'}$. This is equal to the bottom row by Corollary 5.2.11 again. □

Base change to the closed fiber

As in the case of a field extension, we analyze the restriction of the trivializations $\varphi_R^{\mathcal{F}}$ on $\mathcal{F} \in \mathrm{Coh}(P_R)$ of types (V) and (Z), then we study the restriction of the isomorphisms $\varphi_R^{\mathcal{F}}$ on those of type (C). Finally we combine these to show the φ are compatible with restriction to the closed fiber.

Proposition 5.2.13. *Let R be a DVR and k_0 its residue field. Let $\mathcal{F} \in D_{vert}(P_R)$. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}) \otimes k_0 \arrow[r, "\varphi_R^{\mathcal{F}} \otimes k_0"] \arrow[d, "\eta"] & R \otimes k_0 \arrow[d, "\eta"] \\ f(\mathcal{F} \otimes^{\mathbf{L}} k_0) \arrow[r, "\varphi_{k_0}^{\mathcal{F} \otimes^{\mathbf{L}} k_0}"] & k_0 \end{tikzcd}$$

Remark 5.2.14. If $\mathcal{F} \in D_{vert}(P_R)$, then $[\mathcal{F} \otimes^{\mathbf{L}} k_0] = \emptyset$. It is possible there exist q such that $[\mathcal{H}^q(\mathcal{F} \otimes^{\mathbf{L}} k_0)] \neq \emptyset$; in this case $[\mathcal{F} \otimes^{\mathbf{L}} k_0] = \emptyset$ means the nontrivial terms cancel.

Over any base, whenever $[\mathcal{F}] = \emptyset$, by $\varphi^{\mathcal{F}}$ is meant

$$f(\mathcal{F}) \to \otimes_q (f(\mathcal{H}^q(\mathcal{F})))^{(-1)^q} \xrightarrow{\otimes \varphi^{\mathcal{H}^q(\mathcal{F})}} \otimes_q (f([\mathcal{H}^q(\mathcal{F})]))^{(-1)^q} \xrightarrow{\text{pair canceling terms}} \mathcal{O}.$$

We may use any collection of triangles which decomposes $\mathcal{F}$ into coherent sheaves of type (Z), (V), and (C).

Proof. Since the η and φ are compatible with triangles, we may suppose $\mathcal{F} \in \mathrm{Coh}(P_R)$ is scheme-theoretically supported on P_0 and $\dim(\mathrm{Supp}(\mathcal{F})) \leq 1$.
Let $\mathcal{E}^\bullet$ be a locally free $\mathcal{O}_{P_0}$-resolution of $\mathcal{F}$. As in the proof of Lemma 5.1.22, we have:

(1) $\mathrm{Tot}[\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m} \to \mathcal{E}^\bullet \otimes_{k_0} R]$ is adapted to $- \otimes^{\mathbf{L}} k_0$; and

(2) $\mathrm{Tot}[\mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m} \to \mathcal{E}^\bullet \otimes_{k_0} R] \otimes^{\mathbf{L}} k_0 = \mathcal{E}^\bullet \otimes_{k_0} \mathfrak{m}/\mathfrak{m}^2[1] \oplus \mathcal{E}^\bullet$.

Then we obtain a large diagram as in the proof of Lemma 5.1.22, with the bottom right square possibly involving complexes supported on curves. Since the φ_{k_0} are compatible with triangles in $D_{d\leq 1}(P_0)$, we have a commutative diagram:

$$\begin{tikzcd} f(\mathcal{F} \otimes \mathfrak{m}/\mathfrak{m}^2) \arrow[r, "f(g)"] \arrow[d, "\varphi^{\mathcal{F}\otimes \mathfrak{m}/\mathfrak{m}^2}"] & f(\mathcal{F}) \arrow[d, "\varphi^{\mathcal{F}}"] \\ f([\mathcal{F} \otimes \mathfrak{m}/\mathfrak{m}^2]) \arrow[r, "="] & f([\mathcal{F}]) \end{tikzcd}$$

which (after dualizing) shows the bottom right square is commutative. □

Proposition 5.2.15. *Let R be a DVR. Let $Z \subset P_R$ be a subvariety which is finite and flat over Spec R. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{O}_Z) \otimes k_0 \arrow[r, "\varphi_R^Z \otimes k_0"] \arrow[d, "\eta"] & R \otimes k_0 \arrow[d, "\eta"] \\ f(\mathcal{O}_Z \otimes k_0) \arrow[r, "\varphi_{k_0}^{Z\otimes k_0}"] & k_0 \end{tikzcd}$$

Proof. Because Z is R-flat, $-\otimes^{\mathbf{L}} k_0$ may be replaced with $-\otimes k_0$ via the base change isomorphisms [20, 8.3.2.2] $\mathbf{L}i^*\mathbf{R}\pi_*\mathcal{O}_Z \cong \mathbf{R}\pi'_* i'^*\mathcal{O}_Z = \mathbf{R}\pi'_*(\mathcal{O}_Z \otimes k_0)$ for the following diagram.

$$\begin{tikzcd} P_0 \arrow[r, "i'"] \arrow[d, "\pi'"] & P_R \arrow[d, "\pi"] \\ \operatorname{Spec} k_0 \arrow[r, "i"] & \operatorname{Spec} R \end{tikzcd}$$

The commutativity of the diagram is an entirely algebraic claim and follows from Lemma 5.1.21 with $K' = k_0$. ∎

Corollary 5.2.16. *Let R be a DVR. For any $\mathcal{F} \in Coh(P_R)$ such that $[\mathcal{F}] = \emptyset$, i.e. one with a cycle filtration all of whose graded pieces are of types (V) and (Z), the trivialization $\varphi_R^{\mathcal{F}}$ is compatible with restriction to the closed fiber: the diagram*

$$\begin{tikzcd} f(\mathcal{F}) \otimes k_0 \arrow[r, "\varphi_R^{\mathcal{F}} \otimes k_0"] \arrow[d, "\eta"] & R \otimes k_0 \arrow[d, "\eta"] \\ f(\mathcal{F} \otimes^{\boldsymbol{L}} k_0) \arrow[r, "\varphi_{k_0}^{\mathcal{F} \otimes^{\boldsymbol{L}} k_0}"] & k_0 \end{tikzcd}$$

commutes.

Proposition 5.2.17. *Let R be a DVR. Let $\mathcal{F} \in Coh(P_R)$ be an R-flat sheaf such that $Supp(\mathcal{F})$ is a reduced and irreducible relative curve C, and $\mathcal{F}|_C$ is torsion free of rank* 1*. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}) \otimes k_0 \arrow[rr, "\varphi_R^{\mathcal{F}} \otimes k_0"] \arrow[d, "\eta"] & & f(\mathcal{O}_C) \otimes k_0 \arrow[d, "\eta"] \\ f(\mathcal{F} \otimes k_0) \arrow[drr, "\varphi_{k_0}^{\mathcal{F} \otimes k_0}"] & & f(\mathcal{O}_{C_0}) \arrow[d, "\varphi_{k_0}^{\mathcal{O}_{C_0}}"] \\ & & f([\mathcal{F} \otimes k_0] = f([\mathcal{O}_{C_0}]) \end{tikzcd}$$

Remark 5.2.18. The fact that taking the fundamental cycle of a flat sheaf is compatible with specialization, here $[\mathcal{O}_{C_0}] = [\mathcal{F} \otimes k_0]$, follows from [24, I.3.2.2].

Proof. Suppose $\varphi_R^{\mathcal{F}}$ is constructed using the exact sequence $0 \to \mathcal{O}_C \xrightarrow{\alpha} \mathcal{F} \to \mathcal{Q} \to 0$, the general case being obtained by splicing two together. Then we have the following commutative diagram.

$$\begin{tikzcd}
f(\mathcal{F})_0 \arrow[r, "f(\alpha)^{-1}\otimes k_0"] \arrow[d, "\eta"] & f(\mathcal{O}_C)_0 \otimes f(\mathcal{Q})_0 \arrow[r, "\varphi_R^{\mathcal{Q}}\otimes k_0"] \arrow[d, "\eta\otimes\eta"] & f(\mathcal{O}_C)_0 \arrow[d, "\eta"] \\
f(\mathcal{F}\otimes k_0) \arrow[r, "f(\alpha\otimes^{\mathbf{L}}k_0)^{-1}"] & f(\mathcal{O}_{C_0}) \otimes f(\mathcal{Q}\otimes^{\mathbf{L}}k_0) \arrow[r, "1\otimes\varphi_{k_0}^{\mathcal{Q}\otimes^{\mathbf{L}}k_0}"] & f(\mathcal{O}_{C_0})
\end{tikzcd}$$

The right hand square commutes by Corollary 5.2.16. Since $\mathcal{O}_{C_0} \xrightarrow{\alpha\otimes^{\mathbf{L}}k_0} \mathcal{F}\otimes k_0 \to \mathcal{Q}\otimes^{\mathbf{L}}k_0 \to^{+1}$ is a triangle, we have a commutative diagram:

$$\begin{tikzcd}
f(\mathcal{O}_{C_0}) \otimes f(\mathcal{Q}\otimes^{\mathbf{L}}k_0) \arrow[r, "f(\alpha\otimes^{\mathbf{L}}k_0)"] \arrow[d, "\varphi^{\mathcal{O}_{C_0}}\otimes\varphi^{\mathcal{Q}\otimes^{\mathbf{L}}k_0}"] & f(\mathcal{F}\otimes k_0) \arrow[d, "\varphi^{\mathcal{F}\otimes k_0}"] \\
f([\mathcal{O}_{C_0}]) \otimes k_0 \arrow[r, "\mathrm{mult}"] & f([\mathcal{F}\otimes k_0])
\end{tikzcd}$$

which, pasted on the bottom of the previous diagram, gives the desired diagram. ☐

Proposition 5.2.19. *Let R be a DVR and k_0 its residue field. For any $\mathcal{F} \in D_{d\le 1}(P_R)$, the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}) \otimes_R k_0 \arrow[r, "\varphi_R^{\mathcal{F}}\otimes_R k_0"] \arrow[d, "\eta"] & f([\mathcal{F}]) \otimes_R k_0 \arrow[d, "\eta"] \\
f(\mathcal{F}\otimes_R^{\boldsymbol{L}}k_0) \arrow[dr, "\varphi_{k_0}^{\mathcal{F}\otimes_R^{\boldsymbol{L}}k_0}"'] & f([\mathcal{F}] \otimes_R k_0) \arrow[d, "\varphi_{k_0}^{[\mathcal{F}]\otimes_R k_0}"] \\
 & f([\mathcal{F}\otimes_R^{\boldsymbol{L}}k_0])
\end{tikzcd}$$

Proof. Since the base change and φ isomorphisms are compatible with triangles, by taking a cycle filtration of $\mathcal{F}$ we reduce to showing commutativity in the case $\mathcal{F} \in \mathrm{Coh}(P_R)$ is type (V), (Z), or (C). These were done in Propositions 5.2.13, 5.2.15, and 5.2.17, respectively. ☐

Corollary 5.2.20. *Keep the notation from Proposition* 5.2.19*. For any $\mathcal{F}_1, \mathcal{F}_2 \in D_{d\le 1}(P_R)$ such that $[\mathcal{F}_1] = [\mathcal{F}_2]$, the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}_1) \otimes_R k_0 \arrow[r, "((\varphi_R^{\mathcal{F}_2})^{-1}\circ\varphi_R^{\mathcal{F}_1})\otimes_R k_0"] \arrow[d, "\eta"] & f(\mathcal{F}_2) \otimes_R k_0 \arrow[d, "\eta"] \\
f(\mathcal{F}_1\otimes_R^{\boldsymbol{L}}k_0) \arrow[r, "(\varphi_{k_0}^{\mathcal{F}_2\otimes^{\boldsymbol{L}}k_0})^{-1}\circ\varphi_{k_0}^{\mathcal{F}_1\otimes^{\boldsymbol{L}}k_0}"] & f(\mathcal{F}_2\otimes_R^{\boldsymbol{L}}k_0)
\end{tikzcd}$$

Theorem 5.2.21. *Let B be a seminormal k-scheme. Let $\mathcal{F}_1, \mathcal{F}_2 \in D_{d\leq 1}(P_B)$ be such that $[\mathcal{F}_1] = [\mathcal{F}_2]$. (For example, $\mathcal{F}_i = \mathcal{O}_{C_i}$, where $C_1, C_2 \subset P_B$ are B-flat closed subschemes of relative dimension ≤ 1 such that $[C_1] = [C_2]$.)*
Let $\mathcal{G}$ be as in Situation 5.1.1. *(For example, $\mathcal{G}$ is a locally free resolution of a B-flat subscheme $D \subset P_B$ of relative dimension ≤ 1.)*
Then there is an isomorphism $\phi_{\mathcal{G}}^{\mathcal{F}_1,\mathcal{F}_2} : f_B(\mathcal{F}_1) \cong f_B(\mathcal{F}_2)$ of line bundles on B.
The ϕ's satisfy the cocycle condition: if $\mathcal{F}_3 \in D_{d\leq 1}(P_B)$ satisfies $[\mathcal{F}_3] = [\mathcal{F}_i]$ for $i = 1, 2$, then $\phi_{\mathcal{G}}^{\mathcal{F}_1,\mathcal{F}_3} = \phi_{\mathcal{G}}^{\mathcal{F}_2,\mathcal{F}_3} \circ \phi_{\mathcal{G}}^{\mathcal{F}_1,\mathcal{F}_2}$.
The isomorphism $\phi := \phi_{\mathcal{G}}^{\mathcal{F}_1,\mathcal{F}_2}$ is characterized by the property that for any field K and any morphism Spec $K \to B$,

$$\phi|_{Spec\ K} = \left(\varphi_K^{\mathcal{F}_2 \otimes^{\mathbf{L}} K}\right)^{-1} \circ \varphi_K^{\mathcal{F}_1 \otimes^{\mathbf{L}} K}.$$

Proof. By Corollary 3.1.11, an isomorphism of line bundles $L \cong M$ on a seminormal scheme B is equivalent to isomorphisms $f^*L \cong f^*M$ for every $f : \operatorname{Spec} R \to B$, R a complete DVR or a field, compatible with restriction to the generic and closed points. We obtain such isomorphisms using the construction in Theorem 5.1.29:

$$\left(\varphi_R^{\mathcal{F}_2}\right)^{-1} \circ \varphi_R^{\mathcal{F}_1} : f_R(\mathcal{F}_1) \to f_R([\mathcal{F}_1]) = f_R([\mathcal{F}_2]) \to f_R(\mathcal{F}_2).$$

These isomorphisms are compatible with restriction to the generic point by Corollary 5.2.11, and to the closed point by Corollary 5.2.20.
The cocycle condition is clear: $\left(\varphi_R^{\mathcal{F}_3}\right)^{-1} \circ \varphi_R^{\mathcal{F}_1} = \left(\varphi_R^{\mathcal{F}_3}\right)^{-1} \circ \varphi_R^{\mathcal{F}_2} \circ \left(\varphi_R^{\mathcal{F}_2}\right)^{-1} \circ \varphi_R^{\mathcal{F}_1}$.
Any isomorphism of line bundles is determined by its fibers. □

Further compatibility with disjoint cycles. In Proposition 5.1.24 we showed (over a field L) that the trivialization $\varphi_L^Z : f_L(\mathcal{O}_Z) \to L$ is the one induced by the quasi-isomorphism $\mathcal{O}_Z \otimes^{\mathbf{L}} \mathcal{G} \xrightarrow{\sim} 0$ if $\mathcal{G}$ is exact on Z. Now we prove an amplification of this result.

Proposition 5.2.22. *Let $K \supset k$ be a field. Let $\mathcal{F}, \mathcal{G} \in D_{d\leq 1}(P_K)$ satisfy Supp$(\mathcal{F}) \cap$ Supp$(\mathcal{G}) = \emptyset$. Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}) = \det \boldsymbol{R}\pi_*(\mathcal{F}\otimes^{\mathbf{L}}\mathcal{G}) \arrow[rr, "\det \boldsymbol{R}\pi_*(0)"] \arrow[dr, "\varphi^{\mathcal{F}}"'] & & \det(0) = K \\
& f([\mathcal{F}]) \arrow[ur, "\otimes \det \boldsymbol{R}\pi_*(0)"'] &
\end{tikzcd}$$

Proof. By Proposition 5.2.6, we may suppose $K = \overline{K}$. We can find a cycle filtration of $\mathcal{F}$ using triangles all of whose objects have support disjoint from the support of $\mathcal{G}$. Then the quasi-isomorphism $\mathcal{F}\otimes^{\mathbf{L}}\mathcal{G} \xrightarrow{\sim} 0$ is compatible with triangles: suppose the triangle $A \to \mathcal{F} \to B \to^{+1}$ is used. Then the diagram:

$$\begin{tikzcd}
A\otimes^{\mathbf{L}}\mathcal{G} \arrow[r] \arrow[d, "\wr"] & \mathcal{F}\otimes^{\mathbf{L}}\mathcal{G} \arrow[r] \arrow[d, "\wr"] & B\otimes^{\mathbf{L}}\mathcal{G} \arrow[d, "\wr"] \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

induces the following commutative diagram.

$$\begin{tikzcd}
f(A) \otimes f(B) \arrow[r] \arrow[d] & f(\mathcal{F}) \arrow[d] \\
f(0) \otimes f(0) \arrow[r] & f(0)
\end{tikzcd} \tag{5.2.1}$$

Since the φ are also compatible with triangles, we reduce the proposition to the case $\mathcal{F} \in \mathrm{Coh}(P_K)$ and $\mathcal{F}$ is of type (Z) or (C). Factors of type (Z) were handled in Proposition 5.1.24.

There is nothing to prove in case $\mathcal{F} = \mathcal{O}_C$. By using $\mathcal{O}_C(-H) \to \mathcal{O}_C \to \mathcal{Q}_1$ and a diagram similar to (5.2.1), we deduce the proposition for any $\mathcal{O}_C(-H)$. For a general $\mathcal{F}$ of type (C), use $\mathcal{O}_C(-H) \to \mathcal{F} \to \mathcal{Q}_2$. □

Corollary 5.2.23. *With notation and hypotheses as in Theorem 5.2.21, suppose also Supp$(\mathcal{F}_1) \cap$ Supp$(\mathcal{G})$ = Supp$(\mathcal{F}_2) \cap$ Supp$(\mathcal{G}) = \emptyset$. Then the following diagram commutes.*

$$\begin{tikzcd}
f_B(\mathcal{F}_1) \arrow[r, "\det \boldsymbol{R}\pi_*(0)"] \arrow[d, "\phi"] & \mathcal{O}_B \arrow[d, "="] \\
f_B(\mathcal{F}_2) \arrow[r, "\det \boldsymbol{R}\pi_*(0)"] & \mathcal{O}_B
\end{tikzcd}$$

5.3 The two variable problem

Goal. In constructing the descent datum, we encounter the following problem: given $(C,D),(C',D') \in \mathscr{H} \times \mathscr{H}$ with $[C]=[C']$ and $[D]=[D']$, identify the fibers $f(C,D) \cong f(C',D')$. Both fibers map to $f([C],[D])$, but in (possibly) two ways: we can use the operations $C \rightsquigarrow [C]$, then $D \rightsquigarrow [D]$; or vice versa. The goal of this section is to show these define the same isomorphism.

In this section only the threefold P/k is fixed, and we allow $\mathcal{G}$ to vary. In addition to Hypotheses 5.1.2, in this section we impose the following conditions on the variable objects.

Hypotheses 5.3.1. The complexes $\mathcal{F}$ and $\mathcal{G}$ belong to $D_{d\leq 1}(P_R)$.

Both $\mathcal{F}$ and $\mathcal{G}$ are equipped with the canonical trivializations (see Lemma 2.1.4).

$\gamma_{\mathcal{F}} : \det_{\mathcal{O}_{P_T}}(\mathcal{F}) \cong \mathcal{O}_{P_T}$

$\gamma_{\mathcal{G}} : \det_{\mathcal{O}_{P_T}}(\mathcal{G}) \cong \mathcal{O}_{P_T}$

Notation. Since we will operate on both variables, we further decorate the notation from the previous section. Set $\varphi_R^{\mathcal{F}} = {\varphi_{\mathcal{G},\gamma}^{\mathcal{F}\rightsquigarrow[\mathcal{F}]}}_R : f_R(\mathcal{F},\mathcal{G}) \to f_R([\mathcal{F}],\mathcal{G})$.

By operating on the second variable and using the canonical trivializations above, we define ${\varphi^{\mathcal{F},\gamma_{\mathcal{F}}}_{\mathcal{G}\rightsquigarrow[\mathcal{G}],\gamma_{\mathcal{G}}}}_R : f_R(\mathcal{F},\mathcal{G}) \to f_R(\mathcal{F},[\mathcal{G}])$.

If $[\mathcal{G}] = \sum_i n_i C_i$, we define $\varphi^{\mathcal{F}\rightsquigarrow[\mathcal{F}],\gamma_{\mathcal{F}}}_{[\mathcal{G}],\gamma_{[\mathcal{G}]}} = \otimes_i (\varphi^{\mathcal{F}\rightsquigarrow[\mathcal{F}],\gamma_{\mathcal{F}}}_{\mathcal{O}_{C_i},\gamma_{\mathcal{O}_{C_i}}})^{\otimes n_i}$.

Lemma 5.3.2. *Let $K \supset k$ be a field. In addition to Hypotheses 5.3.1, suppose $\mathcal{F} \in Coh(P_K)$ is of type (Z), i.e. $\mathcal{F} \cong \mathcal{O}_p$ for some $p \in P(L)$, with L/K finite. Let $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to^{+1}$ be a triangle in $D_{d\leq 1}(P_K)$. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{O}_p,\mathcal{G}_1) \otimes f(\mathcal{O}_p,\mathcal{G}_3) \arrow[r] \arrow[d, "\varphi^{\mathcal{O}_p\rightsquigarrow\emptyset}_{\mathcal{G}_1} \otimes \varphi^{\mathcal{O}_p\rightsquigarrow\emptyset}_{\mathcal{G}_3}"] & f(\mathcal{O}_p,\mathcal{G}_2) \arrow[d, "\varphi^{\mathcal{O}_p\rightsquigarrow\emptyset}_{\mathcal{G}_2}"] \\ K \otimes K \arrow[r, "mult"] & K \end{tikzcd} \tag{5.3.1}$$

Proof. Let $\mathcal{E}_i$ be a bounded resolution of $\mathcal{G}_i$ by finite locally free $\mathcal{O}_{P_K}$-modules. Let $i_p : p \hookrightarrow P_K$ denote the inclusion and $\pi_p : p \to \operatorname{Spec} K$ the structure morphism. Then the following diagram commutes ($\wedge := \det_K$).

$$\begin{tikzcd}
f(\mathcal{O}_p, \mathcal{G}_1) \otimes f(\mathcal{O}_p, \mathcal{G}_3) \arrow[r] \arrow[d] & f(\mathcal{O}_p, \mathcal{G}_2) \arrow[d] \\
\wedge \pi_* \mathbf{L} i_p{}^* \mathcal{E}_1 \otimes \wedge \pi_* \mathbf{L} i_p{}^* \mathcal{E}_3 \arrow[r] \arrow[d, "\alpha\otimes\alpha"] & \wedge \pi_* \mathbf{L} i_p{}^* \mathcal{E}_2 \arrow[d, "\alpha"] \\
(\wedge \mathcal{O}_p)^{-1} \otimes \wedge \det_p \mathcal{E}_1 \otimes (\wedge \mathcal{O}_p)^{-1} \otimes \wedge \det_p \mathcal{E}_3 \arrow[r] \arrow[d, "1\otimes\wedge(\gamma_{\mathcal{G}_1})\otimes 1\otimes\wedge(\gamma_{\mathcal{G}_3})"] & (\wedge \mathcal{O}_p)^{-1} \otimes \wedge \det_p \mathcal{E}_2 \arrow[d, "1\otimes\wedge(\gamma_{\mathcal{G}_2})"] \\
K \otimes K \arrow[r] & K
\end{tikzcd} \tag{5.3.2}$$

The middle square commutes because the α isomorphisms are compatible with triangles (Proposition 4.2.4). To see the bottom square commutes, restrict the commutative diagram from the third statement of Lemma 2.1.4 to p, apply $\det_K$, and use the identity 4.2.1 for 1-dimensional $\mathcal{O}_p$-vector spaces A, B: $\det_K(A \otimes_{\mathcal{O}_p} B) = (\det_K \mathcal{O}_p)^{-1} \otimes_K \det_K A \otimes_K \det_K B$.

We obtain the following diagram which, when tensored with $(\wedge \mathcal{O}_p)^{-1}$, proves the commutativity of the bottom square in (5.3.2). (Again set $\wedge := \det_K$.)

$$\begin{tikzcd}
(\wedge \mathcal{O}_p)^{-1} \otimes \wedge(\wedge_p \mathcal{E}_1) \otimes \wedge(\wedge_p \mathcal{E}_3) \arrow[r, "="] \arrow[d, "1\otimes\wedge(\gamma_1|_p)\otimes\wedge(\gamma_3|_p)"] & \wedge(\wedge_p \mathcal{E}_1 \otimes_p \wedge_p \mathcal{E}_3) \arrow[r] \arrow[d, "\wedge(\gamma_1|_p\otimes\gamma_3|_p)"] & \wedge(\wedge_p \mathcal{E}_2) \arrow[d, "\wedge(\gamma_2|_p)"] \\
(\wedge \mathcal{O}_p)^{-1} \otimes \wedge \mathcal{O}_p \otimes \wedge \mathcal{O}_p \arrow[r, "="] & \wedge(\mathcal{O}_p \otimes \mathcal{O}_p) \arrow[r] & \wedge \mathcal{O}_p
\end{tikzcd}$$

$\square$

Now we deduce the result of Lemma 5.3.2 for any $\mathcal{F} \in \mathrm{Coh}_{d\leq 0}(P_K)$.

Corollary 5.3.3. *Let $K \supset k$ be a field. In addition to Hypotheses* 5.3.1, *suppose $\mathcal{F} \in Coh_{d\leq 0}(P_K)$. Let $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to^{+1}$ be a distinguished triangle in $D_{d\leq 1}(P_K)$. Then we have a commutative diagram akin to* (5.3.1) *except we replace $\mathcal{O}_p$ with $\mathcal{F}$.*

Proof. We may operate on $\mathcal{F}$ one summand at a time, so it suffices to prove Corollary 5.3.3 in the case $\mathrm{Supp}(\mathcal{F}) = \{p\}$ for $p \in P(L)$, L/K a finite field extension.

We induct on $\ell_{\mathcal{O}_p}(\mathcal{F})$. The case $\ell = 1$ is Lemma 5.3.2. In general we can find an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ with $\ell(\mathcal{F}'), \ell(\mathcal{F}'') < \ell(\mathcal{F})$. This gives rise to the following exact square.

$$\begin{tikzcd}
\mathcal{F}' \otimes^{\mathbf{L}} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{F}' \otimes^{\mathbf{L}} \mathcal{G}_2 \arrow[r] \arrow[d] & \mathcal{F}' \otimes^{\mathbf{L}} \mathcal{G}_3 \arrow[d] \\
\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}_2 \arrow[r] \arrow[d] & \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}_3 \arrow[d] \\
\mathcal{F}'' \otimes^{\mathbf{L}} \mathcal{G}_1 \arrow[r] & \mathcal{F}'' \otimes^{\mathbf{L}} \mathcal{G}_2 \arrow[r] & \mathcal{F}'' \otimes^{\mathbf{L}} \mathcal{G}_3
\end{tikzcd} \tag{5.3.3}$$

The trivializations are compatible with the columns by the 1-variable case (Proposition 5.1.12), and with the first and third rows by the induction hypothesis. Therefore they are compatible with the second row as well. □

Now we show the φ^C are compatible with triangles in the other variable.

Lemma 5.3.4. *Let $K \supset k$ be a field. In addition to Hypotheses* 5.3.1*, suppose $\mathcal{F} \in Coh(P_K)$ is of type (C), i.e. $Supp(\mathcal{F})$ is a reduced and irreducible curve C, and $\mathcal{F}|_C$ is torsion free of rank* 1*. Let $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to^{+1}$ be a distinguished triangle in $D_{d\leq 1}(P_K)$. Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}, \mathcal{G}_1) \otimes f(\mathcal{F}, \mathcal{G}_3) \arrow[r] \arrow[d, "\varphi_{\mathcal{G}_1}^{\mathcal{F}\rightsquigarrow\mathcal{O}_C} \otimes \varphi_{\mathcal{G}_3}^{\mathcal{F}\rightsquigarrow\mathcal{O}_C}"] & f(\mathcal{F}, \mathcal{G}_2) \arrow[d, "\varphi_{\mathcal{G}_2}^{\mathcal{F}\rightsquigarrow\mathcal{O}_C}"] \\
f(\mathcal{O}_C, \mathcal{G}_1) \otimes f(\mathcal{O}_C, \mathcal{G}_3) \arrow[r] & f(\mathcal{O}_C, \mathcal{G}_2)
\end{tikzcd}$$

Here both the top and bottom rows are induced by triangles.

Proof. Without loss of generality, suppose $\varphi^{\mathcal{F}} : f(\mathcal{F}) \cong f(C)$ is realized by $0 \to \mathcal{O}_C \to \mathcal{F} \to \mathcal{Q} \to 0$. Form the exact square analogous to (5.3.3) by replacing $\mathcal{F}'$ with $\mathcal{O}_C$ and $\mathcal{F}''$ with $\mathcal{Q}$. This gives the commutativity of the top square in the following diagram.

$$\begin{tikzcd}
& f(\mathcal{F},\mathcal{G}_1)\otimes f(\mathcal{F},\mathcal{G}_3) \arrow[rr] \arrow[d] & & f(\mathcal{F},\mathcal{G}_2) \arrow[d] \\
& f(\mathcal{O}_C,\mathcal{G}_1)\otimes f(\mathcal{Q},\mathcal{G}_1)\otimes f(\mathcal{O}_C,\mathcal{G}_3)\otimes f(\mathcal{Q},\mathcal{G}_3) \arrow[rr] \arrow[d, "1\otimes\varphi_{\mathcal{G}_1}^{\mathcal{Q}\rightsquigarrow\emptyset}\otimes 1\otimes\varphi_{\mathcal{G}_3}^{\mathcal{Q}\rightsquigarrow\emptyset}"] & & f(\mathcal{O}_C,\mathcal{G}_2)\otimes f(\mathcal{Q},\mathcal{G}_2) \arrow[d, "1\otimes\varphi_{\mathcal{G}_2}^{\mathcal{Q}\rightsquigarrow\emptyset}"] \\
& f(\mathcal{O}_C,\mathcal{G}_1)\otimes f(\mathcal{O}_C,\mathcal{G}_3) \arrow[rr] & & f(\mathcal{O}_C,\mathcal{G}_2)
\end{tikzcd}$$

The bottom square commutes by Corollary 5.3.3. The columns are $\varphi_{\mathcal{G}_i}^{\mathcal{F}\rightsquigarrow\mathcal{O}_C}$. □

Lemma 5.3.5. *Let $K \supset k$ be a field. Let p, q be closed points of P_K. Then $\varphi_{\mathcal{O}_q}^{\mathcal{O}_p\rightsquigarrow\emptyset} = \varphi_{\mathcal{O}_q\rightsquigarrow\emptyset}^{\mathcal{O}_p} : f(\mathcal{O}_p,\mathcal{O}_q) \to K$.*

Proof. Let $i_p : p \hookrightarrow P_K$ denote the inclusion and $\pi_p : p \to \operatorname{Spec} K$ the structure morphism. Let $\mathcal{E}_p^\bullet$ denote a locally free $\mathcal{O}_{P_K}$-resolution of $\mathcal{O}_p$, and let $\gamma_p : \det_{P_K}(\mathcal{E}_p^\bullet) \to \mathcal{O}_{P_K}$ denote the canonical trivialization.
We claim the following diagram commutes.

$$\begin{tikzcd}
f(\mathcal{O}_p,\mathcal{O}_q) \arrow[r] \arrow[d] & \det\pi_{q*}\mathbf{L}i_q^*(\mathcal{E}_p^\bullet) \arrow[r, "\alpha"] & (\det_K \mathcal{O}_q)^{-1}\otimes\det_K(\det_q \mathcal{E}_p^\bullet) \arrow[dd, "1\otimes\det_K(\gamma_p|_q)"] \\
\det\pi_{p*}\mathbf{L}i_p^*(\mathcal{E}_q^\bullet) \arrow[d, "\alpha"] & & \\
(\det_K \mathcal{O}_p)^{-1}\otimes\det_K(\det_p \mathcal{E}_q^\bullet) \arrow[rr, "1\otimes\det_K(\gamma_q|_p)"] & & K
\end{tikzcd}$$

In case $p \neq q$, both compositions are equal to $f(\mathcal{O}_p,\mathcal{O}_q) \xrightarrow{f(\mathcal{O}_p\otimes^{\mathbf{L}}\mathcal{O}_q \xrightarrow{\sim} 0)} f(0) = K$ by Proposition 5.1.24.
In case $p = q$, the compositions differ by $\prod_i \det H^i(\sigma)$, where $\sigma : \mathcal{E}^\bullet\otimes^{\mathbf{L}}\mathcal{E}^\bullet \to \mathcal{E}^\bullet\otimes^{\mathbf{L}}\mathcal{E}^\bullet$ denotes the 'switch' map, defined (following [7, 1.3]) with sign $(-1)^{ab}$ on $\mathcal{E}_a^\bullet \otimes \mathcal{E}_b^\bullet$ (a summand of the degree $a+b$ term of $\mathcal{E}^\bullet\otimes^{\mathbf{L}}\mathcal{E}^\bullet$). Because p is a local complete intersection in P_K, and because p is affine (!), we can compute explicitly using the Koszul resolution. So let $U = \operatorname{Spec} A \ni p$ be an open affine in P_K such that we have

$(f_1, f_2, f_3) = \mathfrak{m}_p$ for $f_1, f_2, f_3 \in A$. Let $M \cong A^3$ be the A-module on generators f_i. Then we can take $\mathcal{E}^\bullet$ to be the complex of A-modules:

$$\wedge^3 M \xrightarrow{d_3} \wedge^2 M \xrightarrow{d_2} M \xrightarrow{d_1} A$$

with differentials:

$d_3(a \wedge b \wedge c) = a(b \wedge c) - b(a \wedge c) + c(a \wedge b)$,

$d_2(a \wedge b) = a(b) - b(a)$, and

$d_1(a) = a$, i.e. $d_1(v_1, v_2, v_3) = v_1 f_1 + v_2 f_2 + v_3 f_3$.

Then $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{E}^\bullet$ is (the single complex associated to) the double complex which appears below with its maps to $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{O}_p$ and $\mathcal{O}_p \otimes^{\mathbf{L}} \mathcal{E}^\bullet$; the sign convention on the double complex again follows [7].

$$\begin{tikzcd}
A/\mathfrak{m} \otimes \wedge^3 M \arrow[r, "0"] & A/\mathfrak{m} \otimes \wedge^2 M \arrow[r, "0"] & A/\mathfrak{m} \otimes M \arrow[r, "0"] & A/\mathfrak{m} \otimes A & \\
A \otimes \wedge^3 M \arrow[r, "1\otimes d_3"] & A \otimes \wedge^2 M \arrow[r, "1\otimes d_2"] & A \otimes M \arrow[r, "1\otimes d_1"] \arrow[u] & A \otimes A & A \otimes A/\mathfrak{m} \\
 & M \otimes \wedge^2 M \arrow[r, "-1\otimes d_2"] \arrow[u, "d_1\otimes 1"] & M \otimes M \arrow[r, "-1\otimes d_1"] \arrow[u, "d_1\otimes 1"] & M \otimes A \arrow[u, "d_1\otimes 1"] \arrow[r] & M \otimes A/\mathfrak{m} \arrow[u, "0"] \\
 & & \wedge^2 M \otimes M \arrow[r, "1\otimes d_1"] \arrow[u, "d_2\otimes 1"] & \wedge^2 M \otimes A \arrow[u, "d_2\otimes 1"] & \wedge^2 M \otimes A/\mathfrak{m} \arrow[u, "0"] \\
 & & & \wedge^3 M \otimes A \arrow[u, "d_3\otimes 1"] & \wedge^3 M \otimes A/\mathfrak{m} \arrow[u, "0"]
\end{tikzcd}$$

We calculate the sign of σ on each H^i separately, moving from the right hand column to the top row.

$H^i(\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{O}_p) \leftrightarrow$ class in $\oplus_{r=0}^{r=i} \wedge^{i-r} M \otimes \wedge^r M \leftrightarrow H^i(\mathcal{O}_p \otimes^{\mathbf{L}} \mathcal{E}^\bullet)$

$\alpha \otimes 1 \leftrightarrow [\alpha \otimes 1] = [1 \otimes \alpha] \leftrightarrow 1 \otimes \alpha \Rightarrow H^0(\sigma) = 1$

$a \otimes 1 \leftrightarrow [a \otimes 1, -1 \otimes a] \leftrightarrow -1 \otimes a \Rightarrow H^1(\sigma) = -1$

$a \wedge b \otimes 1 \leftrightarrow [(a \wedge b) \otimes 1, b \otimes a - a \otimes b, 1 \otimes (a \wedge b)] \leftrightarrow 1 \otimes a \wedge b \Rightarrow H^2(\sigma) = 1$

$a \wedge b \wedge c \otimes 1 \leftrightarrow [(a \wedge b \wedge c) \otimes 1, -(b \wedge c) \otimes a + (a \wedge c) \otimes b - (a \wedge b) \otimes c, c \otimes (a \wedge b) + a \otimes (b \wedge c) - b \otimes (a \wedge c), -1 \otimes (a \wedge b \wedge c)] \leftrightarrow -1 \otimes a \wedge b \wedge c \Rightarrow H^3(\sigma) = -1$

Therefore $\prod_i \det H^i(\sigma) = 1$ and the original diagram commutes. □

Remark 5.3.6. Thinking of the possible failure of the diagram to commute as a function $R : P(K) \times P(K) \to \{-1, 1\}$, since we know $R|_{P(K)\times P(K)\setminus\Delta} = 1$, it seems quite likely that $R \equiv 1$.

Corollary 5.3.7. *Let $K \supset k$ be a field. Let $\mathcal{F}, \mathcal{G} \in Coh_{d\leq 0}(P_K)$. Then $\varphi_{\mathcal{G}}^{\mathcal{F}\rightsquigarrow\emptyset} = \varphi^{\mathcal{F}}_{\mathcal{G}\rightsquigarrow\emptyset} : f(\mathcal{F}, \mathcal{G}) \to K$.*

Lemma 5.3.8. *Let $K \supset k$ be a field. Let $p \in P_K$ be a closed point, and suppose $\mathcal{G} \in Coh(P_K)$ is of type (C), i.e. Supp($\mathcal{G}$) is a reduced and irreducible curve C, and $\mathcal{G}|_C$ is torsion free of rank 1. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{O}_p, \mathcal{G}) \arrow[r, "\varphi^{\mathcal{O}_p}_{\mathcal{G}\rightsquigarrow\mathcal{O}_C}"] \arrow[d, "\varphi_{\mathcal{G}}^{\mathcal{O}_p\rightsquigarrow\emptyset}"] & f(\mathcal{O}_p, \mathcal{O}_C) \arrow[d, "\varphi_{\mathcal{O}_C}^{\mathcal{O}_p\rightsquigarrow\emptyset}"] \\ K \arrow[r, "="] & K \end{tikzcd}$$

Proof. Without loss of generality suppose $\varphi^{\mathcal{G}} : f(\mathcal{G}) \cong f(\mathcal{O}_C)$ is constructed via a short exact sequence $0 \to \mathcal{O}_C \to \mathcal{G} \to \mathcal{Q} \to 0$. We claim the following diagram commutes.

$$\begin{tikzcd} f(\mathcal{O}_p, \mathcal{G}) \arrow[r] \arrow[d, "\varphi_{\mathcal{G}}^{\mathcal{O}_p\rightsquigarrow\emptyset}"] & f(\mathcal{O}_p, \mathcal{O}_C) \otimes f(\mathcal{O}_p, \mathcal{Q}) \arrow[r, "1\otimes\varphi^{\mathcal{O}_p}_{\mathcal{Q}\rightsquigarrow\emptyset}"] \arrow[d, "\varphi_{\mathcal{O}_C}^{\mathcal{O}_p\rightsquigarrow\emptyset}\otimes\varphi_{\mathcal{Q}}^{\mathcal{O}_p\rightsquigarrow\emptyset}"] & f(\mathcal{O}_p, \mathcal{O}_C) \arrow[d, "\varphi_{\mathcal{O}_C}^{\mathcal{O}_p\rightsquigarrow\emptyset}"] \\ K \arrow[r, "\mathrm{mult}^{-1}"] & K \otimes K \arrow[r, "\mathrm{mult}"] & K \end{tikzcd}$$

The left square commutes by Lemma 5.3.2 and the right by 5.3.7. □

Corollary 5.3.9. *Let $K \supset k$ be a field. Let $\mathcal{F} \in Coh_{d\leq 0}(P_K)$, and let $\mathcal{G} \in Coh(P_K)$ be of type (C), i.e. Supp($\mathcal{G}$) is a reduced and irreducible curve C, and $\mathcal{G}|_C$ is torsion free of rank 1. Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}, \mathcal{G}) \arrow[r, "\varphi^{\mathcal{F}}_{\mathcal{G}\rightsquigarrow\mathcal{O}_C}"] \arrow[d, "\varphi_{\mathcal{G}}^{\mathcal{F}\rightsquigarrow\emptyset}"] & f(\mathcal{F}, \mathcal{O}_C) \arrow[d, "\varphi_{\mathcal{O}_C}^{\mathcal{F}\rightsquigarrow\emptyset}"] \\ K \arrow[r, "="] & K \end{tikzcd}$$

Proof. We may take a cycle filtration of $\mathcal{F}$ by Lemma 5.3.4 (with the roles of $\mathcal{F}, \mathcal{G}$ reversed), then apply Lemma 5.3.8 to each of the factors. $\square$

Lemma 5.3.10. *Let $K \supset k$ be a field. Let $\mathcal{F}, \mathcal{G} \in Coh_{d\leq 1}(P_K)$ be such that $Supp(\mathcal{F}) = C_1$ a reduced and irreducible curve, and $\mathcal{F}|_{C_1}$ is torsion free of rank 1; similarly for $\mathcal{G}$ with a reduced and irreducible curve C_2. Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F},\mathcal{G}) \arrow[r, "\varphi^{\mathcal{F}}_{\mathcal{G}\rightsquigarrow\mathcal{O}_{C_2}}"] \arrow[d, "\varphi_{\mathcal{G}}^{\mathcal{F}\rightsquigarrow\mathcal{O}_{C_1}}"'] & f(\mathcal{F},\mathcal{O}_{C_2}) \arrow[d, "\varphi_{\mathcal{O}_{C_2}}^{\mathcal{F}\rightsquigarrow\mathcal{O}_{C_1}}"] \\
f(\mathcal{O}_{C_1},\mathcal{G}) \arrow[r, "\varphi^{\mathcal{O}_{C_1}}_{\mathcal{G}\rightsquigarrow\mathcal{O}_{C_2}}"] & f(\mathcal{O}_{C_1},\mathcal{O}_{C_2})
\end{tikzcd}$$

Proof. Without loss of generality suppose $\varphi^{\mathcal{F}} : f(\mathcal{F}) \cong f(\mathcal{O}_{C_1})$ is constructed via a short exact sequence $0 \to \mathcal{O}_{C_1} \to \mathcal{F} \to \mathcal{Q}_1 \to 0$, and $\varphi^{\mathcal{G}}$ via $0 \to \mathcal{O}_{C_2} \to \mathcal{G} \to \mathcal{Q}_2 \to 0$. We claim the diagram:

$$\begin{tikzcd}
f(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d, "\varphi_{\mathcal{G}}^{\mathcal{F}\rightsquigarrow\mathcal{O}_{C_1}}"'] & f(\mathcal{F},\mathcal{O}_{C_2}) \otimes f(\mathcal{F},\mathcal{Q}_2) \arrow[r, "1\otimes\varphi^{\mathcal{F}}_{\mathcal{Q}_2\rightsquigarrow\emptyset}"] \arrow[d, "\varphi_{\mathcal{O}_{C_2}}^{\mathcal{F}\rightsquigarrow\mathcal{O}_{C_1}}\otimes\varphi_{\mathcal{Q}_2}^{\mathcal{F}\rightsquigarrow\mathcal{O}_{C_1}}"'] & f(\mathcal{F},\mathcal{O}_{C_2}) \arrow[d, "\varphi_{\mathcal{O}_{C_2}}^{\mathcal{F}\rightsquigarrow\mathcal{O}_{C_1}}"] \\
f(\mathcal{O}_{C_1},\mathcal{G}) \arrow[r] & f(\mathcal{O}_{C_1},\mathcal{O}_{C_2}) \otimes f(\mathcal{O}_{C_1},\mathcal{Q}_2) \arrow[r, "1\otimes\varphi^{\mathcal{O}_{C_1}}_{\mathcal{Q}_2\rightsquigarrow\emptyset}"] & f(\mathcal{O}_{C_1},\mathcal{O}_{C_2})
\end{tikzcd}$$

commutes. The left square commutes by Lemma 5.3.4 and the right by Corollary 5.3.9. $\square$

Proposition 5.3.11. *Let $K \supset k$ be a field. Let $\mathcal{F}, \mathcal{G} \in D_{d\leq 1}(P_K)$ be as in Hypotheses* 5.3.1*. Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F},\mathcal{G}) \arrow[r, "\varphi^{\mathcal{F}}_{\mathcal{G}\rightsquigarrow[\mathcal{G}]}"] \arrow[d, "\varphi_{\mathcal{G}}^{\mathcal{F}\rightsquigarrow[\mathcal{F}]}"'] & f(\mathcal{F},[\mathcal{G}]) \arrow[d, "\varphi_{[\mathcal{G}]}^{\mathcal{F}\rightsquigarrow[\mathcal{F}]}"] \\
f([\mathcal{F}],\mathcal{G}) \arrow[r, "\varphi^{[\mathcal{F}]}_{\mathcal{G}\rightsquigarrow[\mathcal{G}]}"] & f([\mathcal{F}],[\mathcal{G}])
\end{tikzcd}$$

Proof. Choose a cycle filtration of $\mathcal{F}$ (resp. $\mathcal{G}$) whose graded pieces are denoted by

$\mathcal{F}_i$ (resp. $\mathcal{G}_j$). Then we need to show the following diagram commutes.

$$\begin{tikzcd}
f(\mathcal{F},\mathcal{G}) \arrow[r, "\text{filt } \mathcal{G}"] \arrow[d, "\text{filt } \mathcal{F}"] & \otimes_j f(\mathcal{F},\mathcal{G}_j) \arrow[r, "\otimes_j \varphi^{\mathcal{F}}_{\mathcal{G}_j \rightsquigarrow [\mathcal{G}_j]}"] \arrow[d, "\otimes\text{filt } \mathcal{F}"] & f(\mathcal{F},[\mathcal{G}]) \arrow[d, "\otimes\text{filt } \mathcal{F}"] \\
\otimes_i f(\mathcal{F}_i,\mathcal{G}) \arrow[r, "\otimes\text{filt } \mathcal{G}"] \arrow[d, "\otimes_i \varphi_{\mathcal{G}}^{\mathcal{F}_i \rightsquigarrow [\mathcal{F}_i]}"] & \otimes_{i,j} f(\mathcal{F}_i,\mathcal{G}_j) \arrow[r, "\otimes_{i,j} \varphi^{\mathcal{F}_i}_{\mathcal{G}_j \rightsquigarrow [\mathcal{G}_j]}"] \arrow[d, "\otimes_{i,j} \varphi_{\mathcal{G}_j}^{\mathcal{F}_i \rightsquigarrow [\mathcal{F}_i]}"] & \otimes_i f(\mathcal{F}_i,[\mathcal{G}]) \arrow[d, "\otimes_i \varphi_{[\mathcal{G}]}^{\mathcal{F}_i \rightsquigarrow [\mathcal{F}_i]}"] \\
f([\mathcal{F}],\mathcal{G}) \arrow[r, "\otimes\text{filt } \mathcal{G}"] & \otimes_j f([\mathcal{F}],\mathcal{G}_j) \arrow[r, "\otimes_j \varphi^{[\mathcal{F}]}_{\mathcal{G}_j \rightsquigarrow [\mathcal{G}_j]}"] & f([\mathcal{F}],[\mathcal{G}])
\end{tikzcd} \tag{5.3.4}$$

The commutativity of the top left square is trivial. The top right and bottom left squares in (5.3.4) assert that φ (on sheaves of types (Z) and (C)) are compatible with triangles in the other variable. We proved this for type (Z) in Lemma 5.3.2 and for type (C) in Lemma 5.3.4.

There are three cases of the bottom right square: (point, point), (point, curve), and (curve, curve). The commutativity in these cases was verified in Lemmas 5.3.5, 5.3.8, and 5.3.10, respectively. $\square$

Theorem 5.3.12. *Let B be a seminormal k-scheme.*

Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1, \mathcal{G}_2 \in D_{d\leq 1}(P_B)$ be such that $[\mathcal{F}_1] = [\mathcal{F}_2]$ and $[\mathcal{G}_1] = [\mathcal{G}_2]$. Then there is an isomorphism $\phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_1,\mathcal{G}_2} : f_B(\mathcal{F}_1,\mathcal{G}_1) \cong f_B(\mathcal{F}_2,\mathcal{G}_2)$ of line bundles on B.

The ϕ's satisfy the cocycle condition: if $\mathcal{F}_3, \mathcal{G}_3 \in D_{d\leq 1}(P_B)$ satisfy $[\mathcal{F}_3] = [\mathcal{F}_i]$ and $[\mathcal{G}_3] = [\mathcal{G}_i]$ for $i = 1, 2$, then $\phi^{\mathcal{F}_1,\mathcal{F}_3}_{\mathcal{G}_1,\mathcal{G}_3} = \phi^{\mathcal{F}_2,\mathcal{F}_3}_{\mathcal{G}_2,\mathcal{G}_3} \circ \phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_1,\mathcal{G}_2} : f_B(\mathcal{F}_1,\mathcal{G}_1) \cong f_B(\mathcal{F}_3,\mathcal{G}_3)$.

Proof. By Theorem 5.2.21 we have two candidates: $\phi^{\mathcal{F}_2}_{\mathcal{G}_1,\mathcal{G}_2} \circ \phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_1}$ and $\phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_2} \circ \phi^{\mathcal{F}_1}_{\mathcal{G}_1,\mathcal{G}_2}$, corresponding to the two outermost ways of traversing the following square.

$$\begin{tikzcd}
f(\mathcal{F}_1,\mathcal{G}_1) \arrow[r] \arrow[d] & f(\mathcal{F}_1,[\mathcal{G}_1]) \arrow[d] & f(\mathcal{F}_1,\mathcal{G}_1) \arrow[l] \arrow[d] \\
f([\mathcal{F}_1],\mathcal{G}_1) \arrow[r] & f([\mathcal{F}_1],[\mathcal{G}_1]) & f([\mathcal{F}_1],\mathcal{G}_2) \arrow[l] \\
f(\mathcal{F}_2,\mathcal{G}_1) \arrow[r] \arrow[u] & f(\mathcal{F}_2,[\mathcal{G}_1]) \arrow[u] & f(\mathcal{F}_2,\mathcal{G}_2) \arrow[l] \arrow[u]
\end{tikzcd}$$

It suffices to check the commutativity at every field-valued point of B, and at these points it suffices to check the commutativity of the smaller squares. These commute by Proposition 5.3.11. Therefore we may define $\phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_1,\mathcal{G}_2} := \phi^{\mathcal{F}_2}_{\mathcal{G}_1,\mathcal{G}_2} \circ \phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_1} = \phi^{\mathcal{F}_1,\mathcal{F}_2}_{\mathcal{G}_2} \circ \phi^{\mathcal{F}_1}_{\mathcal{G}_1,\mathcal{G}_2}$. The preceding equality, together with the cocycle condition with one factor fixed (Theorem 5.2.21), implies the cocycle condition:

$$\phi^{23}_{23}\phi^{12}_{12} = (\phi^{23}_{3}\phi^{2}_{23})(\phi^{2}_{12}\phi^{12}_{1}) = \phi^{23}_{3}\phi^{2}_{13}\phi^{12}_{1} = \phi^{23}_{3}(\phi^{12}_{3}\phi^{1}_{13}) = \phi^{13}_{3}\phi^{1}_{13} = \phi^{13}_{13}.$$

□

Theorem 5.3.13. *Use the notation from Construction* 2.4.1 *(with q_1, q_2 of degree one), Theorem* 5.2.21*, and Theorem* 5.3.12*. There exists an isomorphism $\phi : {p_1}^*\mathcal{L} \cong {p_2}^*\mathcal{L}$ on $(\mathscr{H}_1 \times \mathscr{H}_2 \times_{\mathscr{C}_1\times\mathscr{C}_2} \mathscr{H}_1 \times \mathscr{H}_2)^{sn}$ satsifying the cocycle condition: $q_{12}^*(\phi)\circ q_{23}^*(\phi) = q_{13}^*(\phi)$ on $(\mathscr{H}_1 \times \mathscr{H}_2 \times_{\mathscr{C}_1\times\mathscr{C}_2} \mathscr{H}_1 \times \mathscr{H}_2 \times_{\mathscr{C}_1\times\mathscr{C}_2} \mathscr{H}_1 \times \mathscr{H}_2)^{sn}$.*

Proof. This is simply Theorem 5.3.12 with $B = (\mathscr{H}_1 \times \mathscr{H}_2 \times_{\mathscr{C}_1\times\mathscr{C}_2} \mathscr{H}_1 \times \mathscr{H}_2)^{sn}$ and $\mathcal{F}_i, \mathcal{G}_i$ the structure sheaves of the appropriate universal flat families. □

5.4 Descent property

In this section we show the descent datum just defined is effective. We freely use notation from the previous three sections.

Notation. Let P/k be the fixed smooth projective threefold from before, with k now supposed algebraically closed. Let $\mathscr{C}_1$ (resp. $\mathscr{C}_2$) denote its Chow variety of curves of degree d_1 (resp. d_2). Let q be a linear numerical polynomial with leading coefficient d_1, and let $(\mathscr{H}_1^q)'$ denote the closed subscheme of the Hilbert scheme consisting of subschemes $T \subset P$ such that all irreducible components of T are one-dimensional. Let $\mathscr{H}_1 = \coprod_q (\mathscr{H}_1^q)'^{sn}$ denote the disjoint union of the seminormalizations of the $(\mathscr{H}_1^q)'$. We have the universal family $\mathscr{U}_1 \hookrightarrow P \times \mathscr{H}_1$ and a morphism $\pi_1 : \mathscr{H}_1 \to \mathscr{C}_1$. (Similarly when 1 is replaced with 2.) We have the line bundle $\mathcal{L} := \det \mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathbf{L}} \mathcal{O}_{\mathscr{U}_2})$ on

$\mathscr{H}_1 \times \mathscr{H}_2$ and the isomorphism $\phi : p_1{}^*\mathcal{L} \cong p_2{}^*\mathcal{L}$ on $(\mathscr{H}_1 \times \mathscr{H}_2 \times_{\mathscr{C}_1 \times \mathscr{C}_2} \mathscr{H}_1 \times \mathscr{H}_2)^{sn}$. All γ's are understood to be the canonical ones as in Lemma 2.1.4, and all fields are extensions of k.

Strategy. First we use the moving lemma (in the form of Lemma 2.2.14) to construct trivializations locally on the Chow varieties. As a result, we conclude $(\mathcal{L}, \phi)$ descends to both $\mathscr{C}_1 \times \mathscr{H}_2$ and $\mathscr{H}_1 \times \mathscr{C}_2$. Next we apply Proposition 3.2.11 to show the descended bundles lift to simplicial line bundles. Finally we show these bundles descend to $\mathscr{C}_1 \times \mathscr{C}_2$.

Construction 5.4.1. Let $K \supset k$ be a field and let $Y_K \subset P_K := P \times_k K$ be a one-dimensional subscheme corresponding to $y \in \mathscr{H}_2(K)$. Let C be a 1-cycle on P corresponding to $c \in \mathscr{C}_1(k)$.

Now apply Lemma 2.2.14 with $X = P_K$: choose a collection of short exact sequences $\{\alpha\}$ in $\mathrm{Coh}(P_K)$ expressing $[\mathcal{O}_{Y_K}] = \sum_i n_i[T_i]$ with $T_i \in \mathrm{Coh}(P_K)$ as in Lemma 2.2.14, i.e. $\mathrm{Supp}(T_i) \cap \mathrm{Supp}(C_K) = \emptyset$ for all i. Let $a_{c,y}$ be the unique automorphism of $\mathcal{O}_{P_K}$ making the following diagram commute:

$$\begin{tikzcd}
\det_{P_K} \mathcal{O}_{Y_K} \arrow[r, "\otimes \det(\alpha)"] \arrow[d, "\gamma_{\mathcal{O}_{Y_K}}"] & \otimes_i (\det_{P_K} T_i)^{\otimes n_i} \arrow[d, "\otimes_i (\gamma_{T_i})^{\otimes n_i}"] \\
\mathcal{O}_{P_K} \arrow[r, "a_{c,y}"] & \mathcal{O}_{P_K}
\end{tikzcd}$$

Since P is projective, we have $a_{c,y} \in K$.

Define $U \subset \mathscr{C}_1$ to consist of those cycles C' satisfying $\mathrm{Supp}(C'_K) \cap \mathrm{Supp}(T_i) = \emptyset$ for all i. Thus U is an open set containing c. We have $V := \pi^{-1}(U) = \coprod_q V_q \subset \mathscr{H}_1$ as q ranges over certain numerical polynomials.

Define $T_{c,y} : \mathcal{L}|_{V_q \times y} \cong \mathcal{O}_{V_q \times y}$, on each V_q separately, as follows:

$$\det \mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_{V_q}} \otimes^{\mathbf{L}} \mathcal{O}_{Y_K}) \xrightarrow{\alpha} \otimes_i (\det \mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_{V_q}} \otimes^{\mathbf{L}} T_i))^{\otimes n_i} \xrightarrow{\beta} \mathcal{O}_{V_q \times y} \xrightarrow{\cdot a_{c,y}{}^{-q(0)}} \mathcal{O}_{V_q \times y}.$$

Here α means the map induced by the collection $\{\alpha\}$, and β is induced by the natural maps

$$\mathbf{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_{V_q}} \otimes^{\mathbf{L}} T_i) \xrightarrow{\sim} 0$$

which are quasi-isomorphisms because a locally free resolution of T_i is acyclic on $\mathscr{U}_{V_q}$.

Now we study how the construction interacts with the isomorphism ϕ.

Lemma 5.4.2. *Let $\mathcal{F} \in Coh(P)$ satisfy $\mathcal{F} \cong \mathcal{O}_p$ for $p \in P(k)$, and let $\psi : \mathcal{F} \cong \mathcal{O}_p$ be a chosen identification. Let Y_K, T_i be obtained by Construction* 5.4.1. *Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}_K, \mathcal{O}_Y) \arrow[r, "\alpha"] \arrow[d, "f(\psi_K)"'] & \otimes f(\mathcal{F}_K, T_i)^{\otimes n_i} \arrow[d, "\otimes f(\psi_K)"'] \arrow[dr, "\beta"] & \\
f(\mathcal{O}_{p_K}, \mathcal{O}_Y) \arrow[r, "\alpha"] \arrow[d, "="'] & \otimes f(\mathcal{O}_{p_K}, T_i)^{\otimes n_i} \arrow[d, "="'] & K \\
\det_{p_K} \mathcal{O}_{Y_K} \arrow[r] \arrow[d, "\gamma_{\mathcal{O}_{Y_K}}|_{p_K}"'] & \otimes(\det_{p_K} T_i)^{\otimes n_i} \arrow[d, "\otimes(\gamma_{T_i}|_{p_K})^{\otimes n_i}"'] & \\
K \arrow[r, "\cdot a_{c,y}"'] & K \arrow[uur, "="'] &
\end{tikzcd}$$

Proof. The commutativity of the squares is trivial. The triangle commutes by Proposition 5.1.24. □

Corollary 5.4.3. *Keep the notation from Construction* 5.4.1 *and Lemma* 5.4.2. *Let $p_j \in P(k)$ be closed points, and let $a_j \in \mathbb{Z}$. Then the following diagram commutes.*

$$\begin{tikzcd}
\otimes_j f(\mathcal{O}_{p_j K}, \mathcal{O}_{Y_K})^{\otimes a_j} \arrow[r, "\alpha"] \arrow[d, "\otimes\varphi_{\mathcal{O}_{Y_K}}^{\mathcal{O}_{p_j}\rightsquigarrow\emptyset}"] & \otimes_{i,j}(f(\mathcal{O}_{p_j K}, T_i)^{\otimes a_j})^{\otimes n_i} \arrow[d, "\beta"] \\
K \arrow[r, "{a_{c,y}}^{\sum a_j}"] & K
\end{tikzcd}$$

Proposition 5.4.4. *Keep the notation from Construction* 5.4.1. *Let $\mathcal{F} \in Coh_{d\leq 1}(P)$, and suppose chosen short exact sequences of coherent sheaves on P expressing $\mathcal{F} = [\mathcal{F}] + \sum a_j p_j$. Set $A = \sum a_j$. Then the following diagram commutes.*

$$\begin{tikzcd}
f(\mathcal{F}_K, \mathcal{O}_{Y_K}) \arrow[r, "\beta\alpha"] \arrow[d, "\varphi_{\mathcal{O}_Y}^{\mathcal{F}\rightsquigarrow[\mathcal{F}]}"'] & K \arrow[d, "{a_{c,y}}^{-A}"] \\
f([\mathcal{F}_K], \mathcal{O}_{Y_K}) \arrow[r, "\beta\alpha"] & K
\end{tikzcd}$$

Remark 5.4.5. Since k is algebraically closed, a cycle filtration of $\mathcal{F}$ induces one of $\mathcal{F}_K$, and the notation $[\mathcal{F}]_K = [\mathcal{F}_K]$ is unambiguous. Henceforth we omit the subscript K.

Proof. The isomorphism $\varphi_{\mathcal{O}_Y}^{\mathcal{F}\rightsquigarrow[\mathcal{F}]}$ admits the following description: choose short exact sequences of coherent sheaves on P expressing $\mathcal{F} = [\mathcal{F}] + \sum a_j p_j$, for example via a cycle filtration, and trivialize the factors of the form $f(\mathcal{O}_{p_j}, \mathcal{O}_Y)$. Using many triangles of triangles, we obtain a commutative diagram:

$$\begin{tikzcd} f(\mathcal{F},\mathcal{O}_Y) \arrow[r, "\alpha"] \arrow[d] & \otimes_i(f(\mathcal{F},T_i))^{\otimes n_i} \arrow[r, "\beta"] \arrow[d] & K \arrow[d, "="] \\ f([\mathcal{F}],\mathcal{O}_Y)\otimes\otimes_j f(\mathcal{O}_{p_j},\mathcal{O}_Y)^{\otimes a_j} \arrow[r, "\alpha"] & \otimes_i(f([\mathcal{F}],T_i)\otimes\otimes_j f(\mathcal{O}_{p_j},T_i)^{\otimes a_j})^{\otimes n_i} \arrow[r, "\beta"] & K \end{tikzcd}$$

Combining this with the diagram in Corollary 5.4.3 gives the result. $\square$

Proposition 5.4.6. *Keep the notation from Construction* 5.4.1 *and Proposition* 5.4.4. *Suppose* $\mathcal{F}_1, \mathcal{F}_2 \in Coh_{d\leq 1}(P)$ *satisfy* $[\mathcal{F}_1] = [\mathcal{F}_2]$. *Suppose chosen short exact sequences expressing* $\mathcal{F}_1 = [\mathcal{F}_1] + \sum a_j p_j$ *and* $\mathcal{F}_2 = [\mathcal{F}_2] + \sum b_k q_k$. *Set* $A = \sum a_j$ *and* $B = \sum b_k$. *Then the following diagram commutes.*

$$\begin{tikzcd} f(\mathcal{F}_1,\mathcal{O}_Y) \arrow[r, "\beta\alpha"] \arrow[d, "\varphi_{\mathcal{O}_Y}^{\mathcal{F}_1\rightsquigarrow[\mathcal{F}_1]}"'] & K \arrow[r, "{a_{c,y}}^{-\chi(\mathcal{F}_1)}"] \arrow[d, "{a_{c,y}}^{-A}"] & K \arrow[dd, "="] \\ f([\mathcal{F}_1],\mathcal{O}_Y) \arrow[r, "\beta\alpha"] & K & \\ f(\mathcal{F}_2,\mathcal{O}_Y) \arrow[r, "\beta\alpha"] \arrow[u, "\varphi_{\mathcal{O}_Y}^{\mathcal{F}_2\rightsquigarrow[\mathcal{F}_2]}"] & K \arrow[r, "{a_{c,y}}^{-\chi(\mathcal{F}_2)}"'] \arrow[u, "{a_{c,y}}^{-B}"'] & K \end{tikzcd}$$

Proof. The squares on the left commute by Corollary 5.4.3, and the right side is clearly commutative. Note the left column is $\phi : {p_1}^*\mathcal{L} \cong {p_2}^*\mathcal{L}$ on the fiber at $(\mathcal{F}_1, \mathcal{O}_Y)\times_{[\mathcal{F}_1],\mathcal{O}_Y} (\mathcal{F}_2, \mathcal{O}_Y) \in (\mathscr{H}_1 \times \mathscr{H}_2 \times_{\mathscr{C}_1\times\mathscr{C}_2} \mathscr{H}_1 \times \mathscr{H}_2)^{sn}$, and the top and bottom rows are $T_{c,y}$. $\square$

Theorem 5.4.7. *For any* $(c, y) \in \mathscr{C}_1(k) \times \mathscr{H}_2(K)$, *there exist an open* $U \subset \mathscr{C}_1$ *containing* c, *and a trivialization (on* $\mathscr{H}_1 \times y$*)* $T_{c,y} : \mathcal{L}|_{\pi_1{}^{-1}(U)} \cong \mathcal{O}_{\pi_1{}^{-1}(U)}$ *such that* *the following diagram commutes.*

$$\begin{tikzcd}
{p_1}^*(\mathcal{L}|_{V_q\times y}) \arrow[r, "{p_1}^*T_{c,y}"] \arrow[d, "\phi"] & \mathcal{O}_{(p_1)^{-1}(V_q\times y)} \arrow[d, "="] \\
{p_2}^*(\mathcal{L}|_{V_q\times y}) \arrow[r, "{p_2}^*T_{c,y}"] & \mathcal{O}_{(p_2)^{-1}(V_q\times y)}
\end{tikzcd}$$

Proof. Construction 5.4.1 gives an open neighborhood U and a trivialization $T_{c,y}$. (Note that if $\mathrm{Supp}(c) \cap \mathrm{Supp}(y) = \emptyset$, no "moving" is necessary.)

To see $T_{c,y}$ is compatible with the descent datum, it is enough the check the commutativity of the diagram pointwise. So we must show: if $\mathcal{F}_1, \mathcal{F}_2$ are the structure sheaves of subschemes with Hilbert polynomials q_1, q_2 such that $[\mathcal{F}_1] = [\mathcal{F}_2] = c$, then the following diagram commutes.

$$\begin{tikzcd}
f(\mathcal{F}_1, \mathcal{O}_Y) \arrow[r, "T_{c,y}"] \arrow[d, "\varphi"'] \arrow[dd, bend right=60, "\phi"'] & K \arrow[dd, "="] \\
f([\mathcal{F}_1], [\mathcal{O}_Y]) & \\
f(\mathcal{F}_2, \mathcal{O}_Y) \arrow[r, "T_{c,y}"] \arrow[u, "\varphi"] & K
\end{tikzcd}$$

Since the maps $f([\mathcal{F}_1], \mathcal{O}_Y) \to f([\mathcal{F}_1], [\mathcal{O}_Y])$ and $f([\mathcal{F}_2], \mathcal{O}_Y) \to f([\mathcal{F}_2], [\mathcal{O}_Y])$ are equal, we may replace $f([\mathcal{F}_1], [\mathcal{O}_Y])$ with $f([\mathcal{F}_1], \mathcal{O}_Y)$ in this diagram. Then the commutativity follows from Proposition 5.4.6. $\square$

Theorem 5.4.8. *Keep the notation from the beginning of this section. The line bundle $\mathcal{L}$ on $\mathscr{H}_1 \times \mathscr{H}_2$ descends to $\mathscr{C}_1 \times \mathscr{H}_2$ and to $\mathscr{H}_1 \times \mathscr{C}_2$.*

Proof. We form a proper hypercovering $Y^1_\bullet$ augmented over the seminormal scheme $\mathscr{C}_1$ as follows

$Y^1_0 := \mathscr{H}_1$

$Y^1_1 := (\mathscr{H}_1 \times_{\mathscr{C}_1} \mathscr{H}_1)^{sn}$

$Y^1_n := (\mathscr{H}_1 \times_{\mathscr{C}_1} \ldots \times_{\mathscr{C}_1} \mathscr{H}_1)^{sn}$ $\quad$ ($n+1$ factors)

with the obvious maps. To obtain the maps we use that seminormalization is a functor. By restricting the descent datum from Theorem 5.3.13, we see the incidence bundle $\mathcal{L}$ on $Y_0^1 \times \mathscr{H}_2$ has the descent datum ϕ on $Y_1^1 \times \mathscr{H}_2$ satisfying the cocycle condition on $Y_2^1 \times \mathscr{H}_2$. We think of these data as a morphism $\mathscr{H}_2 \to \underline{\mathrm{Pic}}_{Y_\bullet^1}$. The descent criterion in Corollary 3.2.4 and the descent property in Theorem 5.4.7 say that for every $h \in \mathscr{H}_2$, the morphism lifts to $\underline{\mathrm{Pic}}_{\mathscr{C}_1}$. By Theorem 3.2.10, the pointwise lifts glue to a morphism $\mathscr{H}_2 \to \underline{\mathrm{Pic}}_{\mathscr{C}_1}$. To see we obtain a line bundle this way, consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(\mathscr{H}_2) \arrow[r] \arrow[d, "="] & \mathrm{Pic}(\mathscr{C}_1 \times \mathscr{H}_2) \arrow[r] \arrow[d] & \underline{\mathrm{Pic}}_{\mathscr{C}_1}(\mathscr{H}_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}(\mathscr{H}_2) \arrow[r] & \mathrm{Pic}(Y_\bullet^1 \times \mathscr{H}_2) \arrow[r] & \underline{\mathrm{Pic}}_{Y_\bullet^1}(\mathscr{H}_2) &
\end{tikzcd}$$

By considering this diagram this diagram one connected component of $\mathscr{C}_1$ at a time, we argue as in Theorem 4.1.8 that the morphism $\mathscr{H}_2 \to \underline{\mathrm{Pic}}_{\mathscr{C}_1}$ is induced by a line bundle on $\mathscr{C}_1 \times \mathscr{H}_2$. Since the top row is exact and the rightmost column is injective, there is a unique such line bundle. $\square$

Remark 5.4.9. Strictly speaking, we should choose a finite collection F_1 of numerical polynomials q such that $\coprod_{F_1} (\mathscr{H}_1^q)'^{sn} \to \mathscr{C}_1$ is surjective, and similarly for $\mathscr{C}_2$; form the proper hypercovering along $\coprod_{F_1} (\mathscr{H}_1^q)'^{sn} \to \mathscr{C}_1$; and replace $\mathscr{H}_2$ with $\coprod_{F_2} (\mathscr{H}_2^q)'^{sn}$. Since the result holds for any pair of finite collections, and the resulting line bundle (in this case, on a particular $\mathscr{C}_1 \times (\mathscr{H}_2^q)'^{sn}$) does not depend on the collections chosen, we make the statement and give the proof without reference to any discrete invariants.

Remark 5.4.10. We would be done if we could show the following. For $i = 1, 2$, let $f_i : Y_i \to X_i$ be a (proper, surjective) morphism (between seminormal k-schemes, k algebraically closed). Suppose $\mathcal{L} \in \mathrm{Pic}(Y_1 \times Y_2)$ descends to $X_1 \times Y_2$ and $Y_1 \times X_2$. Then $\mathcal{L}$ descends to $X_1 \times X_2$. Unfortunately we are not able to do this.

Corollary 5.4.11. *Keep the notation from the beginning of this section and Theorem* 5.4.8*. The descended line bundles on $\mathscr{C}_1 \times \mathscr{H}_2$ and $\mathscr{H}_1 \times \mathscr{C}_2$ extend to line bundles on $\mathscr{C}_1 \times Y_\bullet^2$ and $Y_\bullet^1 \times \mathscr{C}_2$.*

Proof. This is a special case of Proposition 3.2.11. $\square$

Theorem 5.4.12. *Keep the notation from the beginning of this section and Theorem* 5.4.8*. The line bundle $\mathcal{L}$ descends to $\mathscr{C}_1 \times \mathscr{C}_2$.*

Proof. Thus far we have constructed the commutative diagram on the right hand side of the following diagram.

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \mathscr{H}_2 \arrow[r] \arrow[d] & \underline{\operatorname{Pic}}_{\mathscr{C}_1} \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \mathscr{C}_2 \arrow[r] & \underline{\operatorname{Pic}}_{Y^1_\bullet} \end{tikzcd}$$

Note the arrow $\mathscr{C}_2 \to \underline{\operatorname{Pic}}_{Y^1_\bullet}$ is a consequence of Corollary 5.4.11. Now let $\operatorname{Spec} K \to \mathscr{C}_2$ be a morphism, with $K \supset k$ a field. Since $\mathscr{H}_2 \to \mathscr{C}_2$ is surjective, we can lift it to a field-valued point $\operatorname{Spec} K' \to \mathscr{H}_2$. We would like to find a lift $\operatorname{Spec} K \to \underline{\operatorname{Pic}}_{\mathscr{C}_1}$. The bundle corresponding to $\operatorname{Spec} K' \to \underline{\operatorname{Pic}}_{\mathscr{C}_1}$ inherits a descent datum on $\mathscr{C}_1 \times_k (K' \times_K K')$ by restricting the line bundle on $\mathscr{C}_1 \times Y^2_\bullet$. Since $\operatorname{Spec} K' \to \operatorname{Spec} K$ is faithfully flat, this descent datum is effective. Therefore we have a lift $\operatorname{Spec} K \to \underline{\operatorname{Pic}}_{\mathscr{C}_1}$.

By Theorem 3.2.10, the lifts glue to a morphism $\mathscr{C}_2 \to \underline{\operatorname{Pic}}_{\mathscr{C}_1}$. To see this morphism determines a unique line bundle on $\mathscr{C}_1 \times \mathscr{C}_2$, consider the diagram with exact rows and injective outer (hence middle) columns:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Pic}(\mathscr{C}_2) \arrow[r] \arrow[d, "="] & \operatorname{Pic}(\mathscr{C}_1 \times \mathscr{C}_2) \arrow[r] \arrow[d] & \underline{\operatorname{Pic}}_{\mathscr{C}_1}(\mathscr{C}_2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Pic}(\mathscr{C}_2) \arrow[r] & \operatorname{Pic}(Y^1_\bullet \times \mathscr{C}_2) \arrow[r] & \underline{\operatorname{Pic}}_{Y^1_\bullet}(\mathscr{C}_2) & \end{tikzcd}$$

□

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