Splice diagrams. Singularity Links and Universal Abelian Covers

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Abstract

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To a rational homology sphere graph manifold one can associate a weighted tree invariant called splice diagram. In this thesis we prove a sufficient numerical condition on the splice diagram for a graph manifold to be a singularity link. We also show that if two manifolds have the same splice diagram, then their universal abelian covers are homeomorphic. To prove the last theorem we have to generalize our notions to orbifolds.

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Chapter 1

Introduction

For a prime 3-manifold M one has several decomposition theorems, like the geometric decomposition which cuts M along embedded tori and Klein bottles into geometric pieces, or the JSJ decomposition which cuts M along embedded tori into simple and Seifert fibered pieces. A graph manifold is a manifold that does not have any hyperbolic pieces in its geometric decomposition, or equivalently only has Seifert fibered pieces in its JSJ decomposition. To a graph manifold one can associate several graph invariants, and in this thesis we are going to describe the properties of such an invariant, the splice diagram.

Splice diagrams were originally introduced in [EN85] and [Sie80], but only for manifolds that are integer homology spheres. Splice diagrams were then generalized to rational homology spheres in [NW02], and used extensively in [NW05a] and [NW05b]. Our splice diagrams differ from the ones in [EN85] in that we do not allow negative weights on edges, and from the ones in [NW02], [NW05a] and [NW05b] in that we have decorations on the nodes; however it is shown in [NW05a] that in the case of singularity links their splice diagrams are the same as ours.

It has long been known that the link of a isolated complex surface singularity is a graph manifold having a plumbing diagram with only orientable base surfaces and having a negative definite intersection form. Grauert showed in [Gra62] that the converse is also true. In [Neu97] Neumann described conditions on the decomposition graph, which are equivalent to the manifold being a singularity link. It is shown in Appendix 1 of [NW05a] that a rational homology sphere singularity link has a splice diagram without any negative decorations at nodes and has all edge determinants positive. In this thesis we prove the other direction to get the following theorem.

Theorem 1. *Let M be a rational homology sphere graph manifold with splice diagram Γ. Then M is a singularity link if and only if Γ has no negative decorations at nodes and all edge determinants are positive.*

Another interesting subject in the study of 3 manifolds is the theory of abelian covers. Since our manifolds are rational homology spheres their universal abelian covers are finite covers. We show that the splice diagram of a manifold determines its universal abelian cover.

Theorem 2. *Let M and M' be rational homology sphere graph manifolds with the same splice diagram Γ. Then M and M' have isomorphic universal abelian covers.*

For singularity links the result of theorem 2 was obtained in [NW05a]. These will be referred to as Main Theorem 1 and 2 throughout this thesis, or as first and second Main Theorem.

The first 2 chapters will provide a general background on 3-manifolds and links of normal surface singularities, and will not include any proofs. In the last 3 chapters complete proofs are presented for most results. Where references are given in lieu of proofs, it is because presenting the proof would be beyond the scope of this thesis.

In chapter 2 we describe several notions and results we need from 3-manifold theory. Section 2.1 contains the description of the prime and JSJ decompositions (of which we will primarily use variations of the latter). In section 2.2 we define Seifert fibered manifolds, which are going to be the basic pieces of the decompositions we are interested in. We will give two different definitions. The first as being "almost" circle fibrations over surfaces, and the second as being actual circle fibrations, but in the category of orbifolds instead of in the category of manifolds. The first definition is good for getting invariants, while the second is useful since we are able to generalize it to what we will call *graph orbifolds*, which are needed for the proof of Main Theorem 2. The invariants obtained from Seifert fibered manifolds will then be used to study universal and universal abelian covers of Seifert fibered manifolds in the last part of this section.

We are going to study graph manifolds more closely in Section 2.3. First we introduce the decomposition graph of a graph manifold, which is an invariant obtained from the JSJ decomposition. Then we describe the plumbing construction of a graph manifold, which yields a 4-manifold having the given 3-manifold as a boundary. We then discuss the non-uniqueness of the plumbing constructions, and deduce an invariant of the graph manifold. The last part of the chapter is devoted to the intersection matrix, which is very important in the study of graph manifolds, and its use to see when graph manifolds are rational homology spheres.

Chapter 3 is about normal surface singularities. First we discuss the topology of normal surface singularities by studying the 3-manifold called the *link* of the singularity. It turns out that the link determines the topology. To determine which 3-manifolds are links of normal surface singularities, we study the resolution of the singularity, and arrive at the dual resolution graph, which turns out to be a plumbing graph of the link. We then describe Grauert's result about which 3-manifolds are singularity links, and Neumann's version of this using the decomposition graph.

In Section 3.2 we describe a particularly nice type of singularity known as the Brieskorn complete intersection, which turns out to have Seifert fibered links. We give a description of their Seifert invariants from their equations, and use this to classify universal abelian covers of rational homology sphere Seifert fibered manifolds.

In Chapter 4 we define our version of splice diagrams, and prove several facts we need about them, included how one gets a splice diagram from a plumbing diagram.

Chapter 5 contains the proof of the two Main Theorems, which are in 5.2 and 5.4 respectively. Section 5.1 has some important lemmas, which among other things show that the splice diagram together with the order of the first homology group determine the decomposition graph. In Section 5.3 we define graph orbifolds and examine some of their properties, since they will be needed to prove Main Theorem 2.

The final chapter is about some corollaries of the second Main Theorem and its proof. We determine from the splice diagram when the universal abelian cover is an integer homology sphere, and find a necessary condition on the splice diagram for the universal abelian cover to be a rational homology sphere. In the case of singularity links, the condition also turns out to be sufficient.

Chapter 2

3-Manifolds

2.1 Decomposition theorems

In understanding and classifying closed oriented 3-manifolds, we are going to do as one often does in science: divide the manifold into simpler, more understandable pieces. But before we start applying this procedure to 3-manifolds, let us look at how it is done with 2-manifolds. The procedure one uses to decompose 2-manifolds into simpler pieces is called *connected sum* and is defined as follows in all dimensions: let M_1 and M_2 be closed manifolds of dimension n, let $\widetilde{M_i}$ be M_i with a ball removed. Note that $\partial\widetilde{M_i} = S^{n-1}$, and let M_1 connect sum M_2, denoted $M_1 \# M_2$, be defined as $\widetilde{M_1} \bigcup_\varphi -\widetilde{M_2}$ where $\varphi\colon S^{n-1} \to S^{n-1}$ is a homeomorphism. This gives a natural way to compose manifolds, and just as multiplication of natural numbers lets us define prime numbers we can define prime manifolds. A manifold $M \neq S^n$ is *prime* if $M = M_1 \# M_2$ implies that M_1 or M_2 is homeomorphic to S^n. Note that S^n is a unit for the connected sum, since S^n with a ball removed is just a ball.

In the case of 2 dimensional manifolds there are only two different prime manifolds, namely S^2 and the torus T^2 (if we include unorientable manifolds then $\mathbb{R}P^2$ is also prime). We have a unique prime decomposition, every orientable surface is homeomorphic to $\#_g T^2$ (where we take the empty sum to be S^2) where g is the genus of the surface. This actually gives a semi group isomorphism between oriented surfaces modulo homeomorphisms with connected sum and $\mathbb{N}$ with plus. For unoriented surfaces there is no unique factorization, since $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ is homeomorphic to $T^2 \# \mathbb{R}P^2$.

A manifold is *irreducible* if every embedded sphere bounds a disk. It is clear that irreducible implies prime, since if a manifold is not prime, the sphere along which the connected sum is taken does not bound a disk. On the other hand the torus is prime but not irreducible, since T^2 has lots of circles, which are not null-homologous. So a prime 2-manifold is either irreducible or T^2. In three dimensions we see the same behaviour.

Proposition 2.1. *If M, a closed oriented 3 dimensional manifold, is prime, then either M is irreducible or M is homeomorphic to $S^2 \times S^1$.*

We have the following theorem which is analogous to the prime decomposition for 2-manifolds.

Theorem 2.2 (Kneser-Milnor)**.** *Let M be a closed orientable 3-manifold, then M is homeomorphic to $\#_{i=1}^{n} P_i$, where P_i are prime, and the decomposition is unique up to permutation of the P_i's.*

The prime decomposition of 2-manifolds is fairly simple, since the pieces of the decomposition all tori. This is not the case for 3-manifolds. There are infinitely many prime 3-manifolds, and prime 3-manifolds may be very complicated. In fact there is currently no complete classification of prime 3-manifolds.

In the following any closed 3-manifold will be prime. To understand prime 3-manifolds and try to classify them, we are going to cut them along tori. But cutting a manifold along a torus has the following two complications.

The first is that gluing two manifolds along torus boundaries, is not a unique operation. The gluing depends on the isotopy class of the diffeomorphism of the torus we use in the gluing. Keeping track of the gluing data, i.e. the isotopy class of the gluing map, is very important and in section 2.3 we give several ways to do it for the type of manifolds we are particularly interested in. In the prime decomposition where we only considered closed manifolds, because after we cut the manifold into two pieces, we glued a ball into the boundaries of the pieces, which is a unique operation. We can not do this in the case of torus boundary since there is not a unique way to glue in a solid torus. Hence the pieces of the decomposition have to be thought of as manifolds with boundary.

The second complication is that we need to consider which embedded tori we are going to cut along. We do not want to cut along too many tori since it would leave too much information in the gluing maps, and not in the pieces of the decomposition. Saying this in another way, well understood manifolds should not be cut. Consider the following, the class of lens spaces are a nice class of manifold, but every lens space L has an embedded tori T^2, such that if we cut L along T^2 we get two solid tori. If we cut along tori of this type, then every lens space would decompose into two solid tori, so the only way to distinguish two lens space would be by the gluing map. We are going to restrict the types of tori we cut along to avoid things like this.

An embedded surface $F \subset M$ is called *boundary parallel* if it is isotopic to a boundary component of M. Cutting along a boundary parallel torus leaves a component homeomorphic to $T^2 \times [0,1]$, and a component which is homeomorphic to the piece we had before we cut. Hence we have not gained any new information.Next we define a restriction on the tori to avoid the example above with separating lens spaces into two solid tori.

Definition 2.3. Let $F \subset M$ be an embedded surface in an oriented 3-manifold. We call F *incompressible* if either (i) F is S^2, does not bound a ball and is not boundary parallel: or (ii) $\pi_1(F) \to \pi_1(M)$ is an injection.

We call a surface *compressible*, if it is not incompressible. For tori we get the following result:

Proposition 2.4. *Let $T^2 \subset M$ be an embedded torus in an oriented irreducible* *manifold. T^2 is compressible if and only if either T^2 bounds a solid torus or $T^2 \subset D^3 \subset M$ and T^2 bounds a knot complement, where D^3 is a 3-ball.*

So cutting along incompressible tori, ensures that we do not cut lens spaces into solid tori. We say that two submanifolds are *parallel* if they are isotopic. We have the Kneser-Haken Finiteness Theorem:

Theorem 2.5. *Let M be a compact irreducible 3-manifold, then there exists a bound on the number of pairwise non-parallel incompressible surfaces in M.*

Now to separate the incompressible surfaces we have:

Lemma 2.6. *Let $S_1, \dots, S_n \subset M$ incompressible surfaces, such that every pair S_i, S_j can be isotoped to be disjoint. Then one can make them simultaneously disjoint, and $S_1 \bigcup \cdots \bigcup S_n \subset M$ is determined up to isotopy.*

We call an incompressible torus *canonical* if it can be isotoped to be disjoint from any other incompressible torus. By the Kneser-Haken Finiteness Theorem there are only finitely many non-parallel canonical tori. These are the tori we are going to cut along. A manifold is called *atoroidal* or *simple* if every incompressible tori is boundary parallel.

Theorem 2.7 (JSJ Decomposition)**.** *Let M be a irreducible 3-manifold and $\{T_1, \dots, T_n\}$ be a maximal set of non-parallel canonical tori. Then $T = \bigcup_{i=1}^n T_i$ cuts M into simple or Seifert fibered pieces. Furthermore T is uniquely minimal up to isotopy among all collections of incompressible tori which cut M into simple or Seifert fibered pieces.*

We will describe Seifert fibered manifolds in greater detail in the next section. It should be said that a piece could be both Seifert fibered and simple, e.g. S^3 or lens spaces.

We are in this thesis primarily going to be interested in *graph manifolds*, which are defined to be manifolds, where all the pieces of their JSJ decomposition are Seifert fibered.

There is another decomposition of 3-manifolds, namely the geometric decomposition, which decomposes a 3-manifold into geometric pieces. This differs from the JSJ-decomposition in several ways. Firstly, in the geometric decomposition one cuts along both tori and Klein bottles; where the geometric decomposition would cut along a Klein bottles, the JSJ-decomposition cut along the torus boundary of its orientation I-bundle instead. Also there are several manifolds which are geometric but would be cut in the JSJ decomposition, e.g. manifolds with the Sol geometry. Also $\mathbb{R}P^3 \# \mathbb{R}P^3$ is geometric and would thus not be cut in the geometric decomposition even though it is not prime.

2.2 Seifert fibered manifolds

2.2.1 Definitions and invariants

The following discussion of Seifert fibered manifolds is based on [Neu99] and [Neu07]. Let us start looking at the Seifert fibered manifolds which come up as one of the types of pieces in the JSJ decomposition. They are defined as follows

Definition 2.8. A *generalized Seifert fibration* on a 3-manifold M consist of a 2-manifold S (possible with boundary) and a map $\pi\colon M \to S$ such that all the fibers are circles, and there is a smallest finite (possible empty) set of fibers $F_1, \dots, F_n \subset M - \partial M$, such that if $H = \bigcup_{i=1}^n F_i$, $\pi|_{(M-H)}\colon (M-H) \to (S - \pi(H))$ is an fibration. One calls the F_i *singular fibers*. A 3-manifold M is Seifert fibered if there exist a Seifert fibration on M.

To a Seifert fibration one associates several natural invariants. The simplest is the genus of the base surface. If S in unorientable then by the genus of S we mean minus the rank of the first singular homology group. Even if S is not orientable, we always assume from now on that M is orientable. For the next invariants let us look more closely at the fibers. Let F be a fiber (possibly singular). Then we can find a regular neighbourhood N of F such that $N = \pi^{-1}(D)$ where D is a disk around $\pi(F)$. N is homeomorphic to a solid torus, and thus $T = \partial N$ is a torus. We can choose N such that no additional singular fibers are included. We get a natural fibration $\pi|_T \colon T \to S^1$, by restricting π. Choose a longitude $l \subset T$ and a meridian $m \subset T$, i.e. simple closed curves such that their homology classes generate $H_1(T)$, and such that the class of l generate $H_1(N)$, while m is contractible in N. Then the class of a fiber f of the fibration $\pi|_T \colon T \to S^1$ is equal to $\alpha l + qm$ for some coprime α, q. One can see $N - F$ as made of toral shells the following way. Consider $D - \pi(F) = \bigcup_{r\in(p,1]} S^1_r$ where S^1_r is a circle of radius r, then $N - F = \bigcup_{r\in(o.1]} \pi^{-1}(S^1_r)$, and $T = \pi^{-1}(S^1_r)$ is a torus. More over all the T's are fibered the same way, since $\pi|_{(N-F)} \colon (N - F) \to (D - \pi(F))$ is a fibration. Seeing $N - F$ as made up of toral shells one can see that nearby fibers converge to αF, or saying it in an other way, nearby fibers cover F α times. One therefore calls α the *degree* of the fiber F. If $\alpha = 1$ then F is a regular fiber, i.e not a singular fiber. That $\alpha \geq 0$ follows by choosing the right orientation on f, and if $\alpha = 0$ then M is in fact a connected sum of lens spaces, and hence not prime. We will prove the last fact in a more general setting as part of the proof of the second main theorem see 5.14.

Choose a section s of the fibration of T. Then we have the following homological relation $m = \alpha s + \beta f$ where $\beta q \equiv 1 \pmod{\alpha}$, we call the pair (α, β) for the *Seifert pair* of the the fiber F. Now β depends on the choice of section s. Two different sections s and s' differs in homology by a multiple of f, and $s + f$ is also a section. Hence changing sections changes β by adding multiples of α. There is a unique β satisfying $0 \leq \beta < \alpha$; we will say that the Seifert pair is *normalized* if this is satisfied. In defining α, β there is a choice of orientation of T. Now the choice of orientation will not change α but changing the orientation of T will change q to $-q$, and therefore β to $\alpha - \beta$. We will use the conventions of [Neu97] to choose our orientations (which is the opposite of the more standard ones used in [JN83], [Neu83b], [Orl72]) and [Sco83], because Seifert pairs that will be obtained from plumbing diagrams will have a nicer presentation, as we will see.

Choose a collection of fibers $F_1, \ldots, F_n$ which include all the singular fibers. Let $F_i \subset N_i$ be neighbourhoods of the fibers as above, let $\mathring{N}_i$ be the interior of N_i and let $M_0 = M - \bigcup_{i=1}^n \mathring{N}_i$. Now M_0 is a genuine fibration over $S_0 = S - \pi(\bigcup \mathring{N}_i)$. Let s be a section of the fibration. s restricts to a section on each of the boundary components of M_0, so one get a section s_i of the fibration of ∂N_i, and hence as above a Seifert pair (α_i, β_i) for each of the fibers F_i. Now changing the section s will change all the s_i by $n_i f_i$, for some $n_i \in \mathbb{Z}$ subject to the restraint $\sum n_i = 0$. Thus changing the section keeps fixed the congruence class $\beta_i \pmod{\alpha_i}$ and $e = \sum \frac{\beta_i}{\alpha_i}$. We call the sum e the *rational euler number* of the Seifert fibration. Note that if we had chosen the other convention for Seifert pairs then one defines $e = -\sum \frac{\beta_i}{\alpha_i}$ in order to get the same number.

If $\alpha_i = 1$ then F_i is not a singular fiber, as mentioned before, so the fibration on M_0 can be extended to a fibration on $M_0 \bigcup N_i$. If furthermore $\beta_i = 0$ then s can be extended to a section on all of $M_0 \bigcup N_i$, hence we can discard pair of Seifert invariants of the form $(1, 0)$. Discarding such pairs does not change e. Assuming that M is prime, the only way we can change the collection of Seifert pairs is by adding $n_i\alpha_i$ to β_i such that $\sum_i n_i = 0$ and discarding pairs of the form $(1, 0)$. Since e is preserved under these operations, e is an invariant of the Seifert fibration.

Choose a collection of fibres $F_1, \ldots, F_n$ which includes all the singular fibers and at least one non singular fiber. Then choose any section of $\pi \colon M_0 \to S_0$ one gets a n-tuple of Seifert pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)$. By changing the Seifert invariant as described above and reordering the fibers, the Seifert invariants can be brought to the following form: $(1, b), (\alpha'_1, \beta'_1), \ldots, (\alpha'_{n'}, \beta'_{n'})$ where $\alpha'_{i+1} \geq \alpha'_i$ for all i and $\alpha'_1 > 1$, and if $\alpha'_i = \alpha'_{i+1}$ then $\beta'_i \leq \beta'_{i+1}$. We say collection of Seifert pairs in this form are in *normal form*. Viewed in this way, the set of α'_i, β'_i gives a section of $\partial(M - \bigcup_{i=1}^{n'} N_i)$ and b is the obstruction to extending this section to all of $M - \bigcup_{i=1}^{n'} N_i$.

Given a manifold M and a Seifert fibration of M, we associate a set of Seifert invariants given by $(g, r, (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ where g is the genus of the base surface S, r is the number of boundary components of M, and $(1, b), (\alpha_1, \beta_1) \ldots, (\alpha_n, \beta_n)$ are the collection of Seifert pairs of the fibers in normal form. This invariant uniquely describes the given Seifert fibration, and it also determines the topology of M except in the few cases where M has several Seifert fibrations.

Theorem 2.9. *Let $(g, r, (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ be a set of integers where $r \geq 0$, the n-tuple of α_i, β_i are ordered by the lexicographic ordering, $\alpha_1 > 1$, $\alpha_i > \beta_i \geq 0$ and $\gcd(\alpha_i, \beta_i) = 1$. Then there exists a 3-manifold $M(g, r, (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ with that set of integers as its Seifert invariants in normal form. Furthermore this is the unique Seifert fibration of M except in the following cases:*

$M(0, 1, (\alpha, \beta))$*- There are infinitely many Seifert fibrations of the solid torus.*

$M(0, 1, (2, 1), (2, 1)) = M(-1, 1, (1, 0))$.

$M(0, 0, (\alpha_1, \beta_1), (\alpha_2, \beta_2))$*- There are many fibrations of S^3, $S^1 \times S^2$ and the lens spaces.*

$M(0, 0, (1, b), (2, 1), (2, 1), (\alpha, \beta)) = M(-1, 0, (\alpha(b - 1) - \beta, \alpha))$*- (Note the second expression might not be in normal form, so we mean the Seifert invariant obtained by putting it in normal form, which depends on α and β.)*

$M(0, 0, (1, -2), (2, 1), (2, 1), (2, 1), (2, 1)) = M(-2, 0, (1, 0))$.

Example 2.10. $M(0, 0, (1, -2), (2, 1), (3, 2), (5, 4))$ is the Poincare homology sphere, and the rational euler number of the Poincare homology sphere is $-\frac{1}{30}$.

Given the Seifert invariants of a manifold M one can get the usual algebraic topological invariants in the following way.

Theorem 2.11. *Let $M = M(g, 0, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ if $g \geq 0$ then*

$$\pi_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n, h \mid [h, a_i] = [h, b_i] = [h, q_i] = 1, \\ q_j^{\alpha_j} h^{\beta_J} = 1, \Big(\prod_{j+1}^{n} q_j\Big)\Big(\prod_{i=1}^{g} [a_i, b_i]\Big) = 1\rangle \tag{2.1}$$

if $g < 0$ then

$$\pi_1(M) = \langle a_1, \ldots, a_g, q_1, \ldots, q_n, h \mid a_i^{-1} h a_i = h^{-1},\ [h, q_i] = 1,\ q_j^{\alpha_j} h^{\beta_J} = 1, \\ \Big(\prod_{j+1}^{n} q_j\Big)\Big(\prod_{i=1}^{|g|} a_i^2\Big) = 1\rangle. \tag{2.2}$$

For proofs see [JN83] or [Orl72]. In [Orl72] he also proves under a condition on the Seifert invariants he call "large", that $\pi_1(M)$ determines the Seifert invariants. This is clearly not always the case since we had the examples before of manifolds with more than one Seifert fibration.

Corollary 2.12. *Let $M = M(g, 0, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ where $g \geq 0$ then*

$$H_1(M) = \mathbb{Z}^{2g} \oplus \langle Q_1, \ldots, Q_n, H \mid \alpha_j Q_j + \beta_j H = 0, \sum_{i=1}^{n} Q_i = 0\rangle. \tag{2.3}$$

In particular if $e(M) \neq 0$ then $H_1(M) = \mathbb{Z}^{2g} \oplus T$ where T is a finite abelian group of order $|T| = \alpha_1 \alpha_2 \ldots \alpha_n |e|$ and if $e(M) = 0$ then $H_1(M) = \mathbb{Z}^{2g+1} \oplus T$ where T is a finite group.

It is clear from the corollary that M is a rational homology sphere if and only if $e(M) \neq 0$ and $g = 0$; Moreover M is an integer homology sphere if and only if $g = 0$ and $|e(M)| = \alpha_1 \alpha_2 \ldots \alpha_n$. In particular this proves that the Poincare homology sphere is an integer homology sphere. From this it is easy to see that the α_i's have to be pairwise coprime for M to be a integer homology sphere. We will see later how one constructs from any collection of pairwise coprime numbers an integer homology sphere which is a Seifert fibered manifold, and that such a manifold is unique.

2.2.2 Alternative definition

Instead of defining Seifert fibrations as "almost" circle fibrations, we will extend the category we are working in to one where Seifert fibrations are actual S^1 fibrations. To do this we need the following definition.

Definition 2.13. An n-dimensional *orbifold atlas* on a paracompact Hausdorff space M consist of an open cover $\{U_i\}_{i\in I}$ closed under finite intersections, and for each $i \in I$ a homeomorphism $\varphi_i \colon \widetilde{U}_i/G_i \to U_i$ where $\widetilde{U}_i$ is a open subset of $\mathbb{R}^n$ and G_i is a finite group acting smoothly on $\widetilde{U}_i$. Furthermore it satisfies the compatibility condition that whenever $U_j \subseteq U_i$ there is an inclusion $f_{ji} \colon G_j \to G_i$ and an embedding $\varphi_{ji} \colon \widetilde{U}_j \to \widetilde{U}_i$, equivariant with respect to f_{ji} such that the following diagram commutes

$$\begin{tikzcd}
\widetilde{U}_j \arrow[r, "\widetilde{\varphi}_{ji}"] \arrow[d] & \widetilde{U}_i \arrow[d] \\
\widetilde{U}_j/G_j \arrow[r, "\varphi_{ji}"] \arrow[dd, "\varphi_i"] & \widetilde{U}_i/f_{ji}G_j \arrow[d] \\
 & \widetilde{U}_i/G_i \arrow[d, "\varphi_i"] \\
U_j \arrow[r, hook] & U_i.
\end{tikzcd} \tag{2.4}$$

Each of the $(U_i, \widetilde{U}_i, G_i)$ is called an orbifold chart.

An orbifold atlas is of course not unique but, as with manifolds, we define a *n-dimensional orbifold* to be a paracompact Hausdorff space equipped with a maximal orbifold atlas. Clearly all smooth manifolds are orbifolds, using smooth charts with trivial G_i action. More generally, if all the G_i act freely then M is a manifold. Also note that if M is an n-dimensional orbifold then there is an open dense subset $U \subseteq M$, such that for each $x \in U$, x has a neighbourhood diffeomorphic to $\mathbb{R}^n$.

Let $f \colon M \to N$ be a continuous map between two orbifolds. We say f is a *orbifold map* if for every $x \in M$ there exist orbifold charts $(U_x, \widetilde{U}_x, G_x)$, $(U_{f(x)}, \widetilde{U}_{f(x)}, G_{f(x)})$

with $x \in U_x \subseteq M$ and $f(x) \in U_{f(x)} \subseteq M$ such that $f(U_x) \subseteq U_{f(x)}$, a homeomorphism $\psi\colon G_x \to G_{f(x)}$, and a smooth map $\widetilde{f}\colon \widetilde{U}_x \to \widetilde{U}_{f(x)}$ equivariant with respect to the actions of G_x and $G_{f(x)}$ (i.e $\psi(g)\widetilde{f}(y) = \widetilde{f}(gy)$ for $g \in G_x$ and $y \in \widetilde{U}_x$)) such that the following diagram commutes

$$\begin{tikzcd}
\widetilde{U}_x \arrow[rr, "\widetilde{f}|_{U_x}"] \arrow[d] & & \widetilde{U}_{f(x)} \arrow[d] \\
\widetilde{U}_x/G_x \arrow[d, "\varphi_x"] & & \widetilde{U}_{f(x)}/G_{f(x)} \arrow[d, "\varphi_{f(x)}"] \\
U_x \arrow[rr, "f|_{U_x}"] & & U_{f(x)}.
\end{tikzcd} \tag{2.5}$$

If M and N are smooth manifolds, then all orbifold maps are smooth maps. A isomorphism of orbifolds is a homeomorphism $f\colon M \to N$ such that f and f^{-1} are orbifold maps.

An *orbifold fibration with fiber* F is defined to be a orbifold map $f\colon M \to N$ such that each point $x \in N$ has a neighbourhood U_x homeomorphic to $\widetilde{U}_x/G_x$, F is a manifold with a G_x-action for each x, and such that the orbifold chart for $f^{-1}(U_x)$ is $(f^{-1}(U_x), \widetilde{U}_x \times F, G_x)$ with G_x acting diagonally on $\widetilde{U}_x \times F$ with quotient $f^{-1}(U_x)$. We also require that the overlap between charts in M respects the fibered structure. Notice that if the diagonal action of G_x on $\widetilde{U}_x \times F$ is free for all G_x then M is a manifold, even if N is not. Now we can make another definition of Seifert fibered manifolds.

Definition 2.14. A Seifert fibered manifold is an orbifold fibration $f\colon M \to S$ with fibers S^1, where S is a 2 dimensional orbifold, and M is a manifold with the given orbifold charts.

This definition is equivalent to our previous definition. Since M is a manifold, the only singular points of S, i.e. points where where G_x does not act freely on U_x, are cone points i.e. points where $G_x \cong \mathbb{Z}/p\mathbb{Z}$ acts as a subgroup of $SO(2)$. There are only finitely many cone points, each corresponds to a singular fiber of M, and the first of the Seifert pair α, β associated to the singular fiber is p, i.e. $\alpha = p$.

	$\chi^{orb} > 0$	$\chi^{orb} = 0$	$\chi^{orb} < 0$
$e = 0$	$\mathbb{S}^2 \times \mathbb{E}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{E}$
$e \neq 0$	$\mathbb{S}^3$	$\mathbb{N}il$	$\mathbb{P}SL$

Table 2.1: Geometries of Seifert fibered manifolds

The definition of Seifert fibered manifolds as orbifold fibrations gives rise to two invariants of the Seifert fibration. The first is the orbifold euler characteristic of S given by

$$\chi^{orb}(S) = \chi(S) - \sum_i (1 - \tfrac{1}{p_i}), \tag{2.6}$$

where $\chi(S)$ is the topological euler characteristic of S and the sum is taken over all points where $G_i \cong \mathbb{Z}/p_i\mathbb{Z}$ for $p_i \neq 1$. The other invariant is the rational euler number we defined above, but in this definition it is actual the euler number of the orbifold circle fibration, hence an obstruction to the existence of a section. So the Poincare homology sphere has $\chi^{orb} = \frac{1}{30}$, and earlier we saw that its rational euler number is $\frac{-1}{30}$. These two invariants are important since they determine the geometric structure on Seifert fibered manifolds according to Table 2.1.

By this we see that the geometry of the Poincare homology sphere is spherical, which of course is obvious from most of its constructions.

One can easily define orbifolds with boundaries by using a half space instead of $\mathbb{R}^n$ in the definition. Then $\chi^{orb}(S)$ is defined the same way, but we need more information before we can define e. Let M be a Seifert fibered manifold with boundary, choose a simple closed curve in each of the boundary pieces transverse to the fibrations. We call such a choice a *system of meridians*. Given a system of meridians one can form a closed Seifert fibered manifold $\overline{M}$ by gluing a solid torus in each of the boundary components, identifying a meridian of the solid torus with the simple closed curve of the boundary piece. Then one defines the rational euler number of M with a system of meridians to be $e(\overline{M})$. We will later see how the JSJ decomposition gives a system of meridians.

2.2.3 Covers of Seifert fibered manifolds

We have the following useful theorem about maps between Seifert fibered spaces

Theorem 2.15. *Let $g\colon M_1 \to M_2$ be a map between Seifert fibered manifolds such that if $\pi_1\colon M_1 \to S_1$ and $\pi_2\colon M_2 \to S_2$ are the Seifert fibrations, and there is a map $g'\colon S_1 \to S_2$ such that the following diagram commutes*

$$\begin{tikzcd} M_1 \arrow[r, "g"] \arrow[d, "\pi_1"] & M_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "g'"] & S_2. \end{tikzcd} \tag{2.7}$$

Then $e(M_2) = \frac{b}{f}e(M_1)$ where $b = \deg(g')$ and f is the degree of g restricted to a general fiber, i.e. $\deg(g) = bf$.

For a proof of this see [JN83].

This theorem is particularly useful when looking at (finite) coverings of Seifert fibered manifolds. Universal covers are completely determined by χ^{orb} and e, deduced from the geometries of the Seifert fibered spaces (see Table 2.1). If $\chi^{orb} \leq 0$ the the universal cover is $\mathbb{R}^3$, if $\chi^{orb} > 0$ and $e = 0$ the universal cover is $S^2 \times \mathbb{R}$. Only in the case $\chi^{orb} < 0$ and $e \neq 0$ is the universal cover a finite cover, and hence S^3. Playing with the condition $\chi^{orb} < 0$ and $e \neq 0$ one can determine that the only Seifert fibered manifolds with cover S^3 are either lens space or spaces with genus 0 having 3 singular fibers of degrees $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$.

We are in general in this thesis interested in universal abelian covers, i.e. the cover that has $H_1(M)$ as covering transformation group. The universal abelian cover of a manifold M is finite if and only if $H_1(M)$ is finite. We saw in Theorem 2.12 that a Seifert fibered manifold has finite $H_1(M)$ if and only if the genus is 0 and $e \neq 0$. While only a few Seifert fibered manifolds have finite universal covers there are many that have finite universal abelian covers. Given any finite collection of pairs of coprime integers greater than 0 there are infinitely many rational homology spheres with that set as its Seifert pairs. Now notice that any finite cover of a Seifert fibered manifold is Seifert fibered, so by Theorem 2.15, one can determine e of the cover just by knowing the degree of the cover, and either the degree of the covering restricted to the fibers or the degree of the covering restricted to the base. Notice that the covering restricted to the base need not be a covering map, but can be branched along the singular points; if we see this as a covering in the category of orbifolds, then map restricted to the base is an actual orbifold cover. We will later show a way to construct the universal abelian cover of a Seifert fibered manifold, knowing the Seifert invariants.

2.3 Graph Manifolds

In this thesis most of the manifolds we are going to consider are graph manifolds. We have defined these before, but we repeat the definition for convenience.

Definition 2.16. A *graph manifold* is a 3-manifold which only has Seifert fibered pieces in its JSJ-decomposition.

We have an alternative characterisation of graph manifolds given by Waldhausen see [Wal67]:

Theorem 2.17 (Waldhausen)**.** *A manifold M is a graph manifold if and only if M can be cut along embedded tori into pieces of the form $\Sigma \times S^1$ where Σ is a surface.*

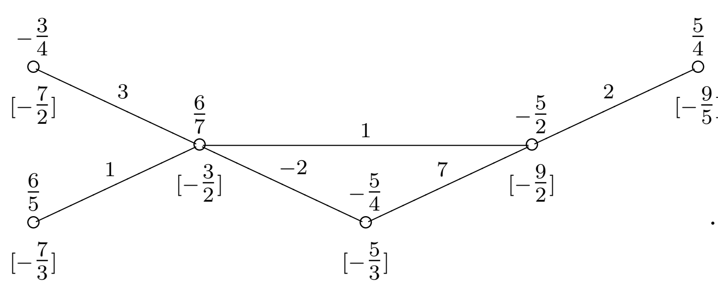
2.3.1 The decomposition graph

We want to construct invariants of graph manifolds. As is fitting for the name "graph manifolds", most of the invariants are going to be (weighted) graphs. The first invariant we will construct is the *decomposition graph*. It has a vertex for each Seifert fibered piece in the JSJ decomposition of M, and an edge corresponding to each torus we have cut along in the JSJ decomposition, attached to the vertices where each side of the torus was glued (this could mean that our graph has loops).

One adds the following decorations to the graph $\Gamma(M)$. Each of the pieces is a Seifert fibered manifold with boundary (at least if there are any edges in the diagram), so each of the boundary components has a natural fibration. If two vertices v_i and v_j (corresponding to Seifert fibered pieces M_i and M_j) are connected by an edge, then M_i and M_j are glued along a torus T. Take f_i and f_j to be fibers of the boundary pieces of M_i and M_j which are glued along T. Then we can view f_i and f_j as simple closed curves in T, and hence they have a intersection number $p_{ij} = f_i \cdot f_j$. To make sure p_{ij} is well-defined we always orient T as the boundary of the piece M_i. Then $f_i \cdot f_j = f_j \cdot f_i$, since we change the orientation on T when we change the order of factors. Notice that $p_{ij} \neq 0$, because if $p_{ij} = 0$ then fibers from each of the side would not intersect, hence are homologous and the fibration can be extended over T, so $M_i \bigcup_T M_j$ is Seifert fibered and T is not a torus we would cut along in the JSJ decomposition.

To each vertex v_i we associate two numbers. The first is χ^{orb} of the piece M_i, which we donote by $[\chi^o rb]$. The other is the rational euler number e of M_i with the following system of meridians: for each boundary component of M_i choose a fiber f_j of the boundary of the pieces M_j glued to M_i along the torus corresponding to the edge, and let the simple closed curve be the image of f_j in M_i.

Example 2.18.



Obvious the decomposition graph for a Seifert fibered manifold is just one vertex, with χ^{orb} and e as added decorations.

Given a decomposition graph one can form the *decomposition matrix*, which is a symmetric $n \times n$ matrix if the decomposition graph has n vertices. The entries are as follows

$$a_{ii} = e(M_i) + 2 \sum_{k \in \left\{ \substack{\text{edges from} \\ v_i \text{ to } v_i} \right\}} \frac{1}{|p_k|} \tag{2.8}$$

$$a_{ij} = \sum_{k \in \left\{ \substack{\text{edges from} \\ v_i \text{ to } v_j} \right\}} \frac{1}{|p_k|} \tag{2.9}$$

We will later use the decomposition matrix to determine which graph manifolds are singularity links, which are going to be important for the proof of the first Main Theorem. There are other uses of the decomposition matrix, such as determining when a graph manifold fibers over the circle, and when a graph manifold has an embedded surface transverse to a Seifert fiber at one of the Seifert fibered pieces. This is shown in part 2 and 3 of Theorem D in [Neu97], which is also is the article where Neumann introduce the decomposition graph and decomposition matrix.

2.3.2 The plumbing graph

We now describe another "invariant" of graph manifold, the *plumbing graph* or *plumbing diagram*. This graph serves two purposes simultaneously, it describes a certain way to decompose M into pieces along tori (generally cutting along more tori than in the JSJ decomposition), and it gives a construction of a 4-manifold X whose boundary is M. Since this effectively construct M, the plumbing diagram is a complete "invariant", so two graph manifolds M_1 and M_2 with the same plumbing diagram are diffeomorphic. However there may be many different cuttings, or different 4-manifolds with the same boundary, so that the plumbing graph depends on the chosen construction. Nonetheless there is a calculus of plumbing graphs which enable us to eventually define a particular plumbing graph called the normal form, which is an invariant of M.

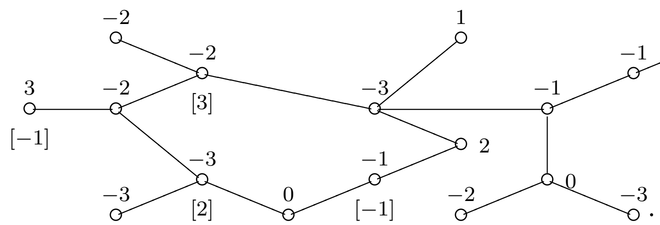
The decomposition of M we are going to get is not quite the decomposition given by Waldhausen's theorem 2.17, but will correspond to this immediate corollary of it.

Corollary 2.19. *A manifold M is a graph manifold if and only if it can be cut along tori into pieces of the form S^1-bundle over a surface Σ.*

Moreover, any piece which is a non trivial S^1-bundle over Σ can further be cut into pieces of the form of Theorem 2.17: cut Σ along a circle into two surfaces with boundaries; any S^1-bundle over a surface with boundary is trivial. However, considering more general S^1-bundles will give us a more interesting data. The corresponding plumbing graph will effectively have vertices corresponding to these pieces, with decorations which identify the S^1-bundle.

To find a 4-manifold whose boundary is a S^1-bundle, one should look for a disk bundle. This inspires the approach below. Start with a collection of disk bundles $X_i \to \Sigma_i$ where the Σ_i's are surfaces of genus g_i, remember that $-g_i < 0$ means that Σ_i is unorientable and rank $H_1(\Sigma_i) = g_i$. One can then glue X_i to X_j in the following way. Take disc $D_i \subseteq \Sigma_i$ and $D_j \subseteq \Sigma_j$, and restricting the fibrations to D_i and D_j gives subsets $D_i \times D_i' \subseteq X_i$ and $D_j \times D_j' \subseteq X_j$ where D_i', D_j' are disks, such that $X_i - D_i \times D_i'$ fibers over $\Sigma_i - D_i$ and $X_j - D_j \times D_j'$ fibers over $\Sigma_j - D_j$. Now glue X_i to X_j by identifying $D_i \times D_i'$ with $D_j \times D_j'$ with the map that reverses the factors, i.e. identify all the disk with the standard disk and let the map be $(x, y) \to (y, x)$. So a collection $\{X_i\}$'s and data describing gluings of various pieces gives rise to a 4-manifold which we will called *plumbed* according to this data. A convenient way to record the data is in a graph Γ, which we will call a *plumbing graph* or *plumbing diagram*. It has a vertex corresponding to each X_i, and an edge between vertices v_i and v_j for each time we glued X_i to X_j. We decorate each vertex with g_i and the euler number e_i of $X_j \to \Sigma_i$ which also corresponds to the self-intersection number of the zero section of the bundle $X_i \to \Sigma_i$. When drawing plumbing graphs, by convention we generally omit the genus label when $g_i = 0$.

Example 2.20.



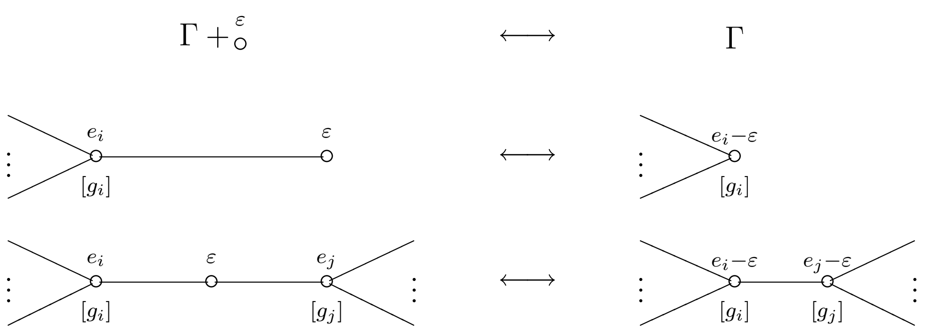
Consider the boundary of a plumbed 4-manifold. It is not smooth, so we smooth it; the result is unique up to diffeomorphism. The boundary M_i of each X_i before the plumbing is an S^1-bundle over Σ_i, and moreover all S^1 bundles over Σ_i can be obtained this way, e_i is the euler number of that bundle. Removing $D_i \times D_i'$ from X_i corresponds to removing a solid torus $D_i \times S^1$ from M_i, and $M_i - D_i \times S^1$ fibers over $\Sigma_i - D_i$ with torus boundary. When we then glue X_i to X_j, we identify the boundaries of $M_i - D_i \times S^1$ with the boundary $M_j - D_j \times S^1$ by exchanging the fibers and sections. This then shows that the boundary of the plumbed manifold can be cut along a collection of tori, one for each edge in Γ, into pieces which are S^1 bundles over surfaces.

This shows that all boundaries of plumbed 4-manifolds are graph manifolds by Corollary 2.19. There is still one obstruction to all graph manifolds being of this form. Decompose an arbitrary graph manifold into pieces of the form of S^1 bundle over surfaces, make a graph Γ with a vertex for each piece and a edge for each torus in the decomposition; if there is a cycle in Γ then we may not be able to orient the pieces compatibly for this to be a plumbing as above. To extend the plumbing graph construction to all graph manifolds, we associate a map $\tau\colon H_1(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$ such that τ of a cycle is 0 if we can orient the pieces compatibly, 1 otherwise. Given a plumbing graph with a τ as above, we first make a graph manifold M' as before according to Γ, and then for each cycle where $\tau = 1$, we take any one of the tori corresponding to an edge in the cycle, cut the manifold M' along it, and reglue by the map $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Doing this appropriately will return our original graph manifold M.

One can also extend the plumbing graph to manifolds which are not prime, but which only have graph manifolds in its prime decomposition. The plumbing graph is then going to be the disjoint union of the plumbing graphs of the the pieces of the prime decomposition. This means when we have a plumbing graph with two disjoint pieces it represents the connected sum of the graph manifolds associated to the pieces.

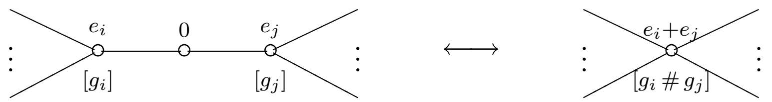
2.3.3 Plumbing calculus and normal form

We saw in last section how to associate to a graph manifold a plumbing graph. This is unfortunately highly non-unique. Given a graph manifold M there are in general infinitely many different plumbing graphs for M. This is because there are several ways to change X without changing its boundary, the most important of which are blowing up and blowing down. There are two types of blow ups, $+1$ and -1. A -1 blow up corresponds to taking a connect sum with $\overline{\mathbb{C}P^2}$ while a $+1$ corresponds to taking a connect sum with $\mathbb{C}P^2$. The -1 also has another description in the algebraic setting which make it very usefull in the next chapter. Blow downs are the inverses of these operations. These operations have one of the following effects on the plumbing graph for X, depending where the points one blows up are. In the following $\varepsilon = \pm 1$, $+$ denotes disjoint union of graphs and the left hand side is the blow up of the right hand side



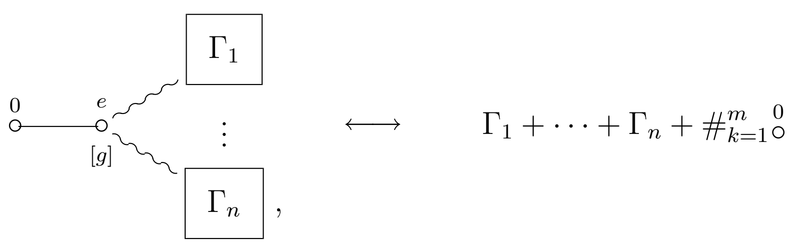
There are also blow ups which interact with loops and 2 cycles in a graph, but since the plumbing diagrams we generally deal with do not have these we will not describe this, as it involves a subtlety with the map τ.

Another valid transformation of plumbing graphs is:



where $g_i \# g_j = g_i + g_j$ if they both have the same sign, and if one of them (say g_i) is negative the $g_i \# g_j = g_i - 2g_j$. This corresponds to the extension of the fibration of each of the pieces over the torus corresponding to the edge, to get a fibration on the union of the pieces. (Again, there is a sublety with the map τ, which we can ignore since we are only interested in trees.)

If a vertex v_i with $e_i = 0$ and $g_i = 0$ has valence one then the associated manifold is not prime, being a connected sum of the manifolds one get from plumbing on the connected pieces of Γ minus the vertex v_i and minus the v_j vertex adjacent to v_i and a number of $S^1 \times S^2$'s.



Where indicates that there may be several edges starting at v_i and ending at a vertex of Γ_l.

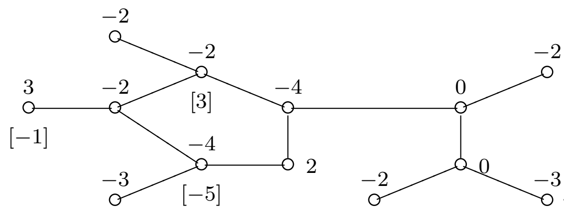
To see the reminder of the plumbing graph moves, and for a proof that this "plumbing calculus" works, see [Neu81]. One can also define a plumbing calculus for graph manifolds with boundaries, see [Neu81].

Using all these moves, one can get the plumbing diagram to a normal form. By a *string* we mean a chain of valence two vertices with $g_i = 0$.

Theorem 2.21. *Every graph manifold has a unique normal form plumbing diagram, such that the number of vertices with negative genus is minimal and on all strings the e_i's are ≤ -2.*

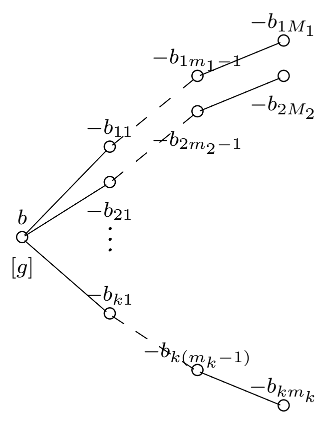
This is not quite the same normal form as in [Neu81] but using the plumbing calculus mentioned above we can get from Neumann's normal form to the one we use see [Neu89b].

Example 2.22. The plumbing diagram in Example 2.20 was not in normal form, here is its normal form



We can now describe the plumbing diagram for any Seifert fibered manifold.

Theorem 2.23. *A Seifert fibered manifold* $M = M(g, 0, (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ *has the following normal form plumbing graph*

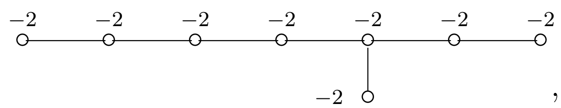


where

$$\frac{\alpha_i}{\beta_i} = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \ldots}}. \tag{2.10}$$

Notice that since $\alpha_i > \beta_i$ and $\gcd(\alpha_i.\beta_i) = 1$, $\frac{\alpha_i}{\beta_i}$ has a unique continued fraction expression with the entries greater than or equal to 2.

Example 2.24. The following weighted graph is a plumbing graph for the Poincare homology sphere.



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2.3.4 Intersection form

The plumbed manifold X deformation retracts down to the union of the Σ_i all intersecting transversely, hence the Σ_i's generate $H_2(X, \mathbb{Z})$. The plumbing graph encodes how the Σ_i's intersect each other, so from the plumbing graph Γ one can make the *intersection matrix* $I(\Gamma)$, which is a representation of the intersection form on $H_2(X)$ in the generating set given by the Σ_i's. $I(\Gamma)$ is an $n \times n$ matrix (a_{ij}), where n is the number of vertices of Γ, with $a_{ii} = e_i$ and a_{ij} is equal to the number of edges between v_i and v_j in Γ for $i \neq j$. In the next chapter we will see some important application of the intersection matrix. In [Neu97] Neumann shows how to get the decomposition matrix from the intersection matrix.

We are in general going to be most interested in rational homology sphere graph manifolds, so let us see how to deduce from the plumbing diagram that a manifold is a rational homology sphere. The first thing to notice is that there is no cycle. Otherwise M is the union of two connected pieces M_1 and M_2, such that $M_1 \bigcap M_2$ has at least two connected components, and Meyer-Vietoris then implies that the image of $H_1(M)$ in $H_0(M_1 \bigcap M_2)$ has a $\mathbb{Z}$ summand, hence that $H_1(M)$ has a $\mathbb{Z}$ summand. The second thing to notice is that if any of the pieces is plumbed over a surface with $g_i \neq 0$ then one gets first homology classes of infinite order; this follows by doing induction of the number of pieces of the plumbing (using Meyer-Vietoris on the gluing), and in the case of one node from Leray-Hirsch. Finally if there are no cycles and $g_i = 0$ for all i, then if we look at $I(\Gamma)$ as a map from $\mathbb{Z}^n$ to $\mathbb{Z}^n$, we get $H_1(M, \mathbb{Z}) \cong \operatorname{Coker} I(\Gamma)$. For $H_1(M)$ to be finite $\det(I(\Gamma)) \neq 0$. Thus we conclude the following.

Proposition 2.25. *A graph manifold is a rational homology sphere if and only if its normal form plumbing graph is a tree, it is plumbed over surfaces of genus zero and* $\det(I(\Gamma)) \neq 0$.

Chapter 3

Normal Surface Singularities

3.1 The topology of normal surface singularities

3.1.1 The link of a singularity

Let X be a normal complex analytic surface. Then every singular point $p \in X$ is isolated, and there exists a neighbourhood U of P such that p is the only singular point in U, and U embeds into $\mathbb{C}^N$ for some N. We choose the embedding so that p is sent to the origin of $\mathbb{C}^N$. Let $M_\varepsilon = U \bigcap S_\varepsilon^{2N-1}$ where S_ε^{2N-1} is the sphere of radius ε in $\mathbb{C}^N$.

Theorem 3.1. *For sufficiently small ε, M_ε is a smooth real manifold, and the diffeomorphism type of M_ε does not depend on ε or the embedding. In this case we call M_ε the* link *of the singular point. Let B_ε be the ball of radius ε. Then for sufficiently small ε $(U \bigcap B_\varepsilon, M_\varepsilon)$ is homeomorphic to (CM, M) where M is the link of the singular point p and CM is the cone over M.*

For a proof, see [Mil68]. We will be interested in the local topology at a singular point, so when we talk about a singularity X, we mean a neighbourhood of a singular point, such that X is homeomorphic to the cone over the link.

The first result regarding the topology of a complex singularity is the following theorem due to Mumford.

Theorem 3.2. *Let $p \in X$. If the link of p is diffeomorphic to S^3, then the point is smooth.*

The reason we look at normal singularities instead of more general singularities is that the normalization of an irreducible singular point does not change its local topology.

3.1.2 Resolution of singularities

We want to see which 3-manifolds arise as links of singularities, and to do this we need the following construction.

Definition 3.3. A resolution of a singularity $p \in X$ is a smooth complex variety $\widetilde{X}$ and a surjective and proper morphism of analytic varieties $\pi \colon \widetilde{X} \to X$ such that, $\pi|_{(\widetilde{X}-\pi^{-1}(p))} \colon (\widetilde{X} - \pi^{-1}(p)) \to X - p$ is an isomorphism.

Theorem 3.4. *Resolutions of singularities always exist and the map π is a composition of blow ups.*

A proof of this in all dimensions is due to Hironaka, but for surfaces this was proven much earlier.

Given a resolution $\pi \colon \widetilde{X} \to X$ of a singular point p in a surface, one defines the *exceptional divisor* $E = \pi^{-1}(p)$. Then $E = \sum_i E_i$ where E_i are irreducible curves of $\widetilde{X}$. One calls a resolution *good* if E is a normal crossing divisor, in other words that all the E_i's are smooth curves which intersect transversely, at most two at a time. It is always possible to get a good resolution, just by continuing to blowing up the points of $\widetilde{X}$, where the E_i are singular, intersect non transversely, or more than two intersect at the point. Henceforth we assume that all resolutions are good; this is necessary for the next definition, and therefore for the rest of the thesis.

Given a resolution one can encode the information given by the exceptional divisor in a graph called the *dual resolution graph.* Take a vertex for each E_i, connect the vertices v_i and v_j by $E_i \cdot E_j$ edges. Analytic curves never intersect negatively so this makes sense. One decorates a vertex with self-intersection number E_i^2 and the genus $g(E_i)$.

3.1.3 Links are plumbed manifolds

The dual resolution graph looks superficially very much like a plumbing diagram, so to examine this connection, look closer at what the resolution does to the topology of X.

Since the link $M \subset X$ is in the complement of the singular point, $\pi^{-1}(M)$ is diffeomorphic to M, where $\pi\colon \widetilde{X} \to X$ is a resolution. We are only looking at the neighbourhood of the singular point p, so we can assume that X is homeomorphic to the cone over M, so that $\pi^{-1}(M) = \partial\overline{\widetilde{X}}$. $\widetilde{X}$ deformation retracts onto E. Hence if the resolution is good, $\widetilde{X}$ is is union of disk bundles over each of the E_i, and the Tubular neighbourhood theorem says they look like a neighbourhood of the zerosections of the bundles, and thus is a plumbed manifold whit euler numbers E_i^2.

We may conclude that singularity links are plumbed 3-manifolds, and the dual resolution graph is simultaneously a plumbing graph for M. One says that a resolution is *minimal* if there is no curve E_i of E with self intersection number $E_i^2 = -1$, since if there were such E_i, it could then be blown down by Castelnuovo's criterion. It is know that there exist a unique minimal good resolution, by blowing down E_i with $E_i^2 = -1$ such that we still have a good resolution after blowing down. A minimal good resolution need not be minimal, but can have E_i's with $E_i^2 = -1$, such E_i have no self intersection (i.e. are embedded), has at least 3 intersection with other E_j's or 3 intersections with one single other E_j. The dual resolution graph for the minimal good resolution is the normal form plumbing graph for the link, as defined in the last chapter.

Not every plumbed manifold is the link of a isolated normal surface singularity. One easy observation is that each component of the exceptional divisor is an analytic curve, hence is orientable. So a singularity link is plumbed over orientable surfaces. A complete classification of which plumbed manifolds are singularity links is given by the Grauert's Theorem.

Theorem 3.5. *Let M be a closed graph manifold. Then M is the link of a normal complex surface singularity if and only if M is plumbed over orientable surfaces, τ is the constant map 0 and the intersection matrix I is negative definite.*

As a historical note, Grauert only proved that if a plumbed manifold satisfies the criterion then it is an singularity link; the other direction was proved earlier by Du Val, although even this direction is sometimes referred to as Mumford's Theorem due to a very nice proof he gave.

Now in Theorem 4.1 in [Neu97] Neumann showed how the intersection matrix is related to the decomposition matrix, and hence gave the following classification of singularity links using the decomposition matrix. Remember that *good* means that all the Seifert fibered pieces have orientable base.

Definition 3.6. A graph manifold is *Very good* if the pieces of the decomposition graph can be compatibly oriented such that the p_i's are all positive. This corresponds to $\tau = 0$.

Theorem 3.7. *Let M be a closed graph manifold. If M does not fiber over the circle with torus fibers, then M is a singularity link if and only if it is good and very good, and the decomposition matrix is negative definite. If M is a T^2 bundle over S^1, it is a singularity link if and only if the monodromy is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$ with $b > 0$, or has trace > 2.*

3.2 Brieskorn complete intersections

In this section we are going to look closer at a particular type of singularity that has a particularly nice topology. It also turns up that the links of these singularities play an important role in the theory of abelian covers of Seifert fibered spaces.

Definition 3.8. The *Brieskorn complete intersection* $V_A(\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i \in \mathbb{N}$ and $\alpha_i > 1$ is defined as the vanishing locus of the $n-2$ equations

$$a_{1i}X_1^{\alpha_1} + a_{2i}X_2^{\alpha_2} + \cdots + a_{ni}X_n^{\alpha_n} = 0 \qquad \text{for } i = 1, 2, \dots, n-2 \tag{3.1}$$

such that all maximal minors of the $n \times (n-2)$ matrix A with entries a_{ji} are non-zero.

Proposition 3.9. *Brieskorn complete intersections define normal surfaces with a unique singular point* $(0, 0, \dots, 0)$*. Furthermore, the topological type of* V_A *does not depend on the choice of* A *satisfying the criterion.*

Since the choice of A does not change the topological type we define $\Sigma(\alpha_1, \alpha_2, \dots, \alpha_n)$ to be the link of $V_A(\alpha_1, \dots, \alpha_n)$ for any choice of A. It turns out $\Sigma(\alpha_1, \dots, \alpha_n)$ is Seifert fibered, and its Seifert invariant can be calculated using the following theorem.

Theorem 3.10. *Let* $\Sigma(\alpha_1, \dots, \alpha_n)$ *be the link of a Brieskorn complete intersection. Then it is homeomorphic to* $M(g, 0, (1, b), \underbrace{(p_1, q_1), \dots, (p_1, q_1)}_{t_1}, \underbrace{(p_2, q_2) \dots, (p_2, q_2)}_{t_2}, \dots, \underbrace{(p_n, q_n), \dots, (p_n, q_n)}_{t_n})$*, where*

$$p_i = \frac{\mathrm{lcm}_j(\alpha_j)}{\mathrm{lcm}_{j \neq i}(\alpha_j)} \tag{3.2}$$

$$t_i = \frac{\prod_{j \neq i} \alpha_j}{\mathrm{lcm}_{j \neq i}(\alpha_j)} \tag{3.3}$$

$$g = \tfrac{1}{2}\left(2 + \frac{(n-2)\prod_i \alpha_i}{\mathrm{lcm}_i(\alpha_i)} - \sum_{i=1}^{n} t_i\right). \tag{3.4}$$

One deduces the values of q_i *from the equations* $\frac{\mathrm{lcm}(\alpha_j)}{\alpha_i} q_i \equiv -1 \pmod{p_i}$*, and* b *is then given by*

$$b = -\frac{\prod_i \alpha_i + \mathrm{lcm}_i(\alpha_i) \sum_i q_i \prod_{j \neq i} \alpha_j}{(\mathrm{lcm}_i(\alpha_i))^2} \tag{3.5}$$

The rational euler number is given

$$e = -\frac{\prod_i \alpha_i}{(\operatorname{lcm}_i \alpha_i)^2}. \tag{3.6}$$

For a proof of this see [JN83], but be aware the reference uses the opposite orientation convention when defining Seifert invariants.

Example 3.11. Consider the hypersurface given by $X^2 + Y^3 + Z^5 = 0$. It is a Brieskorn complete intersection since $A = (\,1\;1\;1\,)$. By the last theorem $\Sigma(2,3,5)$ is homeomorphic to $M(0,0,(1,-2),(2,1),(3,2),(5,4))$, which we saw in last chapter is the Poincare homology sphere.

Notice that if $\alpha_1,\ldots,\alpha_n$ are pairwise coprime then one finds that $p_i = \alpha_i$, $t_i = 1$, $g = 0$ and $e = -\frac{1}{\alpha_1\ldots\alpha_n}$. Remember from the discussion following 2.12 that if $g = 0$ and $e \neq 0$ then the order of the first homology group is $\frac{\alpha_1\ldots\alpha_n}{|e|}$. Hence if $\alpha_1,\ldots,\alpha_n$ are pairwise, coprime $\Sigma(\alpha_1,\ldots,\alpha_n)$ is a integer homology sphere. In fact, this manifold is the unique integer homology sphere with Seifert invariants $p_i = a_i$ for given set α_i and $e < 0$. The manifold with the opposite orientation is clearly also an integer homology sphere, but changing orientation change the rational euler number to $-e$. This follows from the classification of universal abelian covers of rational homology sphere Seifert fibered manifolds, given in the next theorem.

Theorem 3.12. *Let M be a rational homology sphere Seifert fibered manifold, with Seifert invariants $(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)$. Then the universal abelian cover of M is homeomorphic to $\Sigma(\alpha_1,\alpha_2,\ldots,\alpha_n)$.*

If $e > 0$ one compose the cover of the manifold with the other orientation with a orientation reversing map. For proofs of see [Neu83a] and [Neu83b].

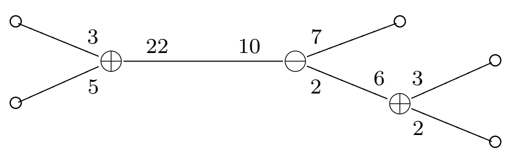
We are going to generalise this theorem to all graph manifolds in the second main Theorem. In the next chapter, we define define what should replace the Seifert invariants in the case of an arbitrary rational homology sphere graph manifold.

Chapter 4

Splice Diagrams

4.1 Defining splice diagrams

A *splice diagram* is a tree which has vertices of valence one, which we call *leaves*, and vertices of valence greater than or equal to 3, which we call nodes. The end of an edge adjacent to a node of the splice diagram is decorated with a non-negative integer, and each node is decorated with either a plus or a minus sign; in general one only writes the minus signs. Here is an example:



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We will in general not distinguish between a leaf and the edge leading to that leaf, except when it would be confusing not to.

Given a rational homology sphere graph manifold M, one can construct the splice diagram $\Gamma(M)$ as follows. Take a node for each Seifert fibered piece. Attach an edge between two nodes if the Seifert fibered pieces are glued along a torus, and attach a leaf to a node for each singular fiber of the Seifert fibered piece. This makes sense since the JSJ decomposition of a rational homology sphere is a tree according to 2.25, i.e. the decomposition graph is a tree, and the nodes of the splice diagram correspond to the vertices of the decomposition graph.

If v is a node of $\Gamma(M)$ then M_v is the corresponding Seifert fibered piece of the JSJ decomposition. Let v be a node of $\Gamma(M)$ and let e be a edge at v, we define the manifold M_{ve} as follows. Cut M along the torus corresponding to e, and let M' be the connected component after cutting not containing M_v. Then $M_{ve} = M' \bigcup_T (S^1 \times D^2)$, by gluing a meridian of the solid torus to a fiber of M_v.

The decoration d_{ve} on edge e at node v is the order of the first homology of M_{ve}. We assign 0 if the homology is infinite.

For the decorations on nodes, we need the following definition.

Definition 4.1. Let $L_0, L_1 \subset M$ be two disjoint knots in a rational homology sphere. Let $C_1 \subset M$ be a submanifold, such that $\partial C_1 = d_1 L_1$, for some integer d_1. Then the linking number of L_0 with L_1 is defined to be $\operatorname{lk}(L_0, L_1) = \frac{1}{d_1} L_0 \bullet C_1$, where $\bullet$ denotes the intersection product in M.

Such a C_1 always exists since M is a rational homology sphere. To see that $\operatorname{lk}(L_0, L_1)$ is well defined, we need to show that if $C_1' \subset M$ is a submanifold such that $\partial C_1' = d_1' L_1$ then $\frac{1}{d_0} L_0 \bullet C_1 = \frac{1}{d_0'} L_0 \bullet C_1'$. Since $\partial C_1' = d_1' L_1$ we have that $\partial(d_1 C_1') = d_1 d_1' L_1$, in the same way we have that $\partial(d_1' C_1) = d_1 d_1' L_1$. We can then form a closed submanifold $N = d_1 C_1' \bigcup_{d_1 d_1' L_1} -d_1' C_1$. Since M is a rational homology sphere, the homology class of N is 0, so $L_0 \bullet N = 0$. But then $0 = L_0 \bullet N = L_0 \bullet (d_1 C_1' \bigcup_{d_1 d_1' L_0} -d_1' C_1) = L_0 \bullet d_1 C_1' - L_0 \bullet d_1' C_1$. Since the intersection product is bilinear we get the result we desire, after dividing by $d_1 d_1'$.

To show that $\operatorname{lk}(L_0, L_1) = \operatorname{lk}(L_1, L_0)$, we will define another notion of linking number equivalent to our first definition, and this alternative definition is symmetric with respect to L_0, L_1

Definition 4.2. Let $L_0, L_1 \subset M$ be knots in a rational homology sphere, let X be a compact 4-manifold such that $M = \partial X$. Let $A_0, A_1 \subset X$ be submanifolds such that $\partial A_i = d_i L_i$ for some integers d_0, d_1, and such that either A_1 or A_0 has zero intersection with any 2-cycle in X. Then let $\widetilde{\mathrm{lk}}(L_0, L_1) = \frac{1}{d_0 d_1} A_0 \cdot A_1$, where $\cdot$ denotes the intersection product in X.

A_i exists since M is a rational homology sphere, and we can the choose $A_i \subset M$. That one of the A_i's can be chosen so that it does not intersect any closed cycles of X follows because a collar neighbourhood of M has zero second homology, since $H_2((0,1] \times M) \cong H_2(M) = \{0\}$, and hence we can just choose $A_i \subset (0,1] \times M$. Since the definition is symmetric we henceforth assume that A_0 has zero intersection with all 2-cycles. To show $\widetilde{\mathrm{lk}}(L_0, L_1)$ is well defined we start by showing that if $A_1' \subset X$ is such that $\partial A_1' = d_1' L_1$ then $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0 d_1'} A_0 \cdot A_1'$. We form $N = (d_1' A_1 \bigcup_{d_1 d_1' L_1} -d_1 A_1')$, then $A_0 \cdot N = 0$ since N is a closed 2-cycle, and it follows that $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0 d_1'} A_0 \cdot A_1'$ as above. Now assume $A_0' \subset X$ is such that $\partial A_0' = d_0' L_0$ and A_0' has zero intersection with all 2-cycles in X. To show that $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0' d_1} A_0' \cdot A_1$ we can choose a $A_1 \subset X$ such that A_1 has zero intersection with any 2-cycles, since changing A_1 does not change $\widetilde{\mathrm{lk}}(L_0, L_1)$ as just shown. Then form $N' = (d_0' A_0 \bigcup_{d_0 d_0' L_0} -d_0 A_0')$. Now N is a 2-cycle and by our choice of A_1, $N \cdot A_1 = 0$ it follows that $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0' d_1} A_0' \cdot A_1$ and therefore that $\widetilde{\mathrm{lk}}(L_0, L_1)$ is well-defined.

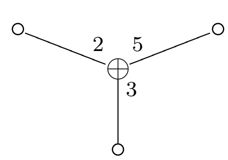
Proposition 4.3. $\mathrm{lk}(L_0, L_1) = \widetilde{\mathrm{lk}}(L_0, L_1)$

Proof. We choose A_0 such that we have that $A_0 \cap (0,1] \times M = (0,1] \times d_0 L_0$, and we choose A_1 such that $A_1 \subset M$. Then we get that $\widetilde{\mathrm{lk}}(L_0, L_1) = \frac{1}{d_0 d_1}(0,1] \times d_0 L_0 \cdot A_1 = \frac{1}{d_0 d_1}(\{1\} \times d_0 L_0) \cdot A_1 = \frac{d_0}{d_0 d_1} L_0 \bullet A_1 = \mathrm{lk}(L_0, L_1)$ □

Hence $\mathrm{lk}(L_0, L_1)$ is symmetric.

We decorate a node of the splice diagram we a sign ε_v corresponding to the sign of the linking number of two non singular fibers in the Seifert fibration.

Example 4.4. The splice diagram for the Poincare homology sphere is the following



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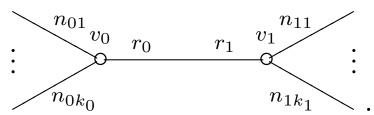
4.2 Edge determinants and orientations

A general edge between two nodes looks like



To such an edge we assign a number called the *edge determinant*.

Definition 4.5. The edge determinant D associated to a edge between nodes v_0 and v_1 is defined by the equation

$$D = r_0 r_1 - \varepsilon_0 \varepsilon_1 N_0 N_1, \tag{4.1}$$

where N_i is the product of all the edge weights adjacent to v_i, (except r_i). I.e. $N_i = \prod_{j=1}^{k_i} n_{ij}$, and ε_i is the sign on the node v_i, if we interpret a plus sign as 1 and a minus sign as -1.

An edge in our splice diagram between nodes v_0 and v_1 corresponds to a torus T^2 along which pieces are glued. In that torus we get several natural knots from the Seifert fibered structure on each side: namely, a fiber F_i and a section S_i from the fibration of M_{v_i}. We are going to be interested in the fiber intersection of F_0 with F_1 in the torus T^2. We use the following convention: when we write $F_0 \cdot F_1$ we mean the intersection product in T^2, where T^2 is oriented as the boundary of M_{v_0}; when we write $F_1 \cdot F_0$ we mean the intersection product in T^2 oriented as the boundary of M_{v_1}. In this way $F_0 \cdot F_1 = F_1 \cdot F_0$, since we change the orientation on T^2 when we interchange F_0 and F_1.

Because a splice diagram is a tree, it is possible to orient the F_i's and S_i's such that the intersection number of the fibers F_i and F_j from the Seifert fibered pieces on each side of a separating torus is always positive, and so that $F_i \cdot S_i = 1$. It should be mentioned that one can change S_i by adding a multiple of F_i, but for choosing a orientation of the F_i's and the S_i's in the fibration, it does not matter.

One does this by choosing an end node v, i.e. a node only connected to one other node v', and choose a orientation on M_v. This then gives an orientation on the fiber in the boundary of M_v and we the choose the orientation of the sections to satisfy the above. Choose the orientation on the $M_{v'}$, such that the fiber intersection number is positive, and choose the orientation of the sections to satisfy the above. Then continue to do the same in the other boundary pieces of $M_{v'}$, and so on. The process will terminate since $\Gamma(M)$ is a tree.

We will always assume our fibers and sections are oriented this way.

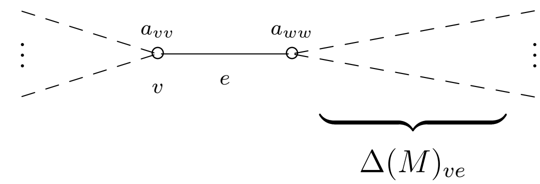
4.3 Constructing splice diagrams from plumbing graphs

We can deduce the splice diagram for a manifold directly from a plumbing diagram for the manifold. Suppose M has a plumbing diagram $\Delta(M)$ which we assume is in normal form, see Theorem 2.21. In particular, all base surfaces are orientable and the decorations on strings are less than or equal to -2.

We then get a splice diagram $\Gamma(M)$ by taking one node for each *node* in $\Delta(M)$, i.e. vertex with more than 3 edges, or with genus $\neq 0$. Note that for rational homology spheres, every vertex of the plumbing diagram has genus $= 0$ by Proposition 2.25. Connect two nodes in $\Gamma(M)$ if there is a string between the corresponding nodes in $\Delta(M)$, and add a leaf at a node in $\Gamma(M)$ for each string starting at that node in $\Delta(M)$ and not ending at any node. We also get from Proposition 2.25 that $\Delta(M)$ is a tree, so this makes sense.

We denote the intersection matrix by $A(\Delta(M))$ and define $\det(\Delta(M))$ to be equal to $\det(-A(\Delta(M)))$.

Lemma 4.6. *Let v be a node in $\Gamma(M)$, and e be a edge on that node. We get the weight d_{ve} on that edge by $d_{ve} = |\det(\Delta(M)_{ve})|$, where $\Delta(M)_{ve}$ is is the connected component of $\Delta(M) - e$ which does not contain v.*

$\Delta(M) =$ 

Proof. First we notice that the manifold given by the plumbing diagram $\Delta(M)_{ve}$, is the manifold corresponding to gluing a solid torus in the boundary of the manifold obtained by cutting M along the torus corresponding to e, and taking the connected component not containing M_v. The solid torus is glued as we described in the definition of M_{ve}, and hence $\Delta(M)_{ve}$ is a plumbing diagram of M_{ve}. The result now follows from the fact that the absolute value of the determinant of the intersection matrix of a rational homology sphere graph manifold is the order of the first homology group, and the determinant is 0 if the first homology group is infinite. □

Note that we used that $\Delta(M)_{ve}$ is plumbing diagram for M_{ve}, a fact we will use several times later. We will in general denote $\Delta(M_{ve})$ by Δ_{ve}.

Lemma 4.7. *Let v be a node in $\Gamma(M)$. Then the sign ε at v is $\varepsilon = -\operatorname{sign}(a_{vv})$, where a_{vv} is the entry of $A(\Delta(M))^{-1}$ corresponding to the node v.*

Proof. To prove this we calculate $\widetilde{\mathrm{lk}}(L_v, L_w)$ where L_v is a nonsingular fiber at the v'th node and L_w is a non singular fiber at the w'th node. Let X be the plumbed 4-manifold given by $\Delta(M)$. Then each vertex of $\Delta(M)$ corresponds to a circle bundle over a 2-manifold in the plumbing. The i'th node gives us a 2-cycle E_i in X, and the collection of all the E_i's generate $H_2(X)$. The intersection matrix $A(\Delta(M))$ is the matrix representation for the intersection form on $H_2(X)$ with respect to this generating set. So to construct A_0 with zero intersection with all 2-cycles, we just need that $A_0 \cdot E_i = 0$ for all i. Let D_v and D_w be transverse disks to E_v and E_w respectively, with boundaries L_v and L_w. If $v = w$, choose D_v and D_w to be disjoint. Set $A_0 = \det(\Delta(M))D_v - \sum_i \det(\Delta(M))(a_{vi})E_i$, where a_{ij} is the ij'th entry of $A(\Delta(M))^{-1}$, and choose $A_1 = D_w$. Then $A_0 \cdot E_i = 0$ for all i and $\widetilde{\mathrm{lk}}(L_v, L_w) =$

$\frac{1}{\det(\Delta(M))} A_0 \cdot D_w = -\frac{1}{\det(\Delta(M))} \det(\Delta(M))(a_{vw}) E_w \cdot D_w = -a_{vw}$, since $E_i \cdot D_w = 0$ if $i \neq w$ and $E_w \cdot D_w = 1$. □

The proof here is the same as that given for proposition 9.1 in [NW08].

Corollary 4.8. *No edge weight on any leaf is* 0.

Proof. We assume that our plumbing diagram is in normal form such that that all weights on strings are ≤ -2. Then the weight on an edge to a leaf is the determinant of a matrix of the form

$$\begin{pmatrix} b_{11} & -1 & 0 & \dots & 0 \\ -1 & b_{22} & -1 & \dots & 0 \\ 0 & -1 & b_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{nn} \end{pmatrix}$$

where $b_{ii} \leq 2$. Determinants of such matrices are never 0. □

4.4 Unnormalized and maximal splice diagrams

We now introduce two more diagrams associated to a given plumbing diagram of a rational homology sphere, which we will call the *unnormalized splice diagram* $\widetilde{\Gamma}(M)$ and the *maximal splice diagram* $\widetilde{\Delta}(M)$.

Definition 4.9. The *unnormalized splice diagram* $\widetilde{\Gamma}(M)$ is a tree, with the same graph structure as the splice diagram $\Gamma(M)$, but it has no signs at nodes, and the weights at edges are defined to be $\tilde{d}_{ve} = \det(\Delta(M_{ve}))$.

The unnormalized splice diagram forgets the sign at each vertex, but remember the sign of $\det(\Delta(M_{ve}))$.

Definition 4.10. The *maximal splice diagram* $\widetilde{\Delta}(M)$ of a manifold M with plumbing diagram $\Delta(M)$ has the same underlying graph as the graph of the plumbing diagram

$\Delta(M)$. On edges one adds decorations as in the construction of a unnormalized splice diagram from the plumbing diagram $\Delta(M)$.

Where the maximal splice diagram has a string of vertices, the unnormalized splice diagram has a single edge.

We will in the rest of this section see how the splice diagram can be obtained from the unnormalized splice diagram, and how the unnormalized splice diagram can be obtained from the maximal splice diagram if they both arise from the same plumbing diagram. We will also see that the maximal splice diagram is equivalent to the plumbing diagram.

To get the unnormalized splice diagram from the maximal splice diagram, one removes the vertices of valence two and removes the decoration on edges next to vertices of valence one.

We have following theorem from appendix 1 of [NW05a] concerning maximal splice diagrams.

Theorem 4.11. *For any pair of vertices v_i and v_j in a maximal splice diagram $\widetilde{\Delta}(M)$, let $l_{v_iv_j}$ be the product of all the weights adjacent to, but not on, the shortest path from v_i to v_j in $\widetilde{\Delta}(M)$. Then the matrix $L = (l_{v_iv_j})$ satisfies $\frac{1}{\det(\Delta(M))}L = -A(\Delta(M))^{-1}$.*

Proof. We need to order the vertices of $\Delta(M)$ to give a matrix representation of the intersection form $A(\Delta)$. We construct the ordering the following way. As the first vertex, take the vertex in Δ corresponding to v_i in $\widetilde{\Delta}$. Let the second vertex be the one adjacent to the first in Δ, along the shortest path in Δ between the first vertex and the vertex in Δ corresponding to v_j in $\widetilde{\Delta}$. Continue choosing vertices this way until we reach the vertex corresponding to v_j. Let k be the number in the ordering of the vertex corresponding to v_j. Removing the chain of vertices from 1 to k gives us s disjoint plumbing diagrams $\Delta_1, \ldots, \Delta_s$. We order the vertices of $\Delta(M)$ minus the string from 1 to k such that if u, v are vertices in Δ_l ordered $n_u < n_v$, then there does not exist a vertex w not in Δ_s, such that $n_u < n_w < n_v$. In this ordering we get that

$$-A(\Delta) = \left(\begin{array}{cccc|ccc} b_1 & -1 & 0 & & & & \\ -1 & b_2 & -1 & & & C & \\ & & \ddots & -1 & & & \\ & & -1 & b_k & & & \\ \hline & & & & -A(\Delta_1) & 0 & \\ & C^t & & & 0 & \ddots & 0 \\ & & & & & 0 & -A(\Delta_s) \end{array}\right).$$

The matrix C consists of entries of 0 except for a single -1 above each of the $-A(\Delta_l)$'s. Furthermore we could have chosen the ordering such that the -1 corresponding to the l'th such entry occurs above the $-A(\Delta_l)$ in the n_l'th row, and that $n_1 \leq n_2 \leq \cdots \leq n_s$. The $(1, k)$ minor of $-A(\Delta)$ is then given by

$$\left(\begin{array}{cccc|ccc} -1 & b_2 & & & & & \\ & \ddots & & & & C' & \\ & & -1 & b_{k-1} & & & \\ & & & -1 & & & \\ \hline & & & & -A(\Delta_1) & 0 & \\ & C_k^t & & & 0 & \ddots & 0 \\ & & & & & 0 & -A(\Delta_k) \end{array}\right).$$

We can add multiples of the $k-1$ column to clear the $(k-1)$'th row of C', then continue by using the $(k-2)$'nd column to clear the $(k-2)$'nd row, and so on. Thus one can clear C'. Since we chose $n_1 \leq n_2 \leq \ldots n_s$ we get that after clearing C', the

$(1, k)$'th minor is the following

$$\left(\begin{array}{cccc|ccc}
-1 & b_2 & & & & & \\
 & \ddots & & & & 0 & \\
 & & -1 & b_{k-1} & & & \\
 & & & -1 & & & \\ \hline
 & & & & -A(\Delta_1) & * & * \\
 & C_k^t & & & 0 & \ddots & * \\
 & & & & & 0 & -A(\Delta_s)
\end{array}\right).$$

It is now easy to see that the determinant of the (i, k)'th minor is given by the expression $(-1)^{1-k}\prod_{l=1}^{s}\det(\Delta_s)$. To get the $(1, k)$'th entry of the adjoint matrix of $-A(\Delta)$ we have to multiply this number by $(-1)^{1+k}$. But notice that $\prod_{l=1}^{k}\det(\Delta_l)$ is equal to the product of all the weights adjacent to but not on the path from v_i to v_j in the maximal splice diagram $\widetilde{\Delta}(M)$. Hence the adjoint matrix of $-A(\Delta)$ is L, and the theorem follows. $\square$

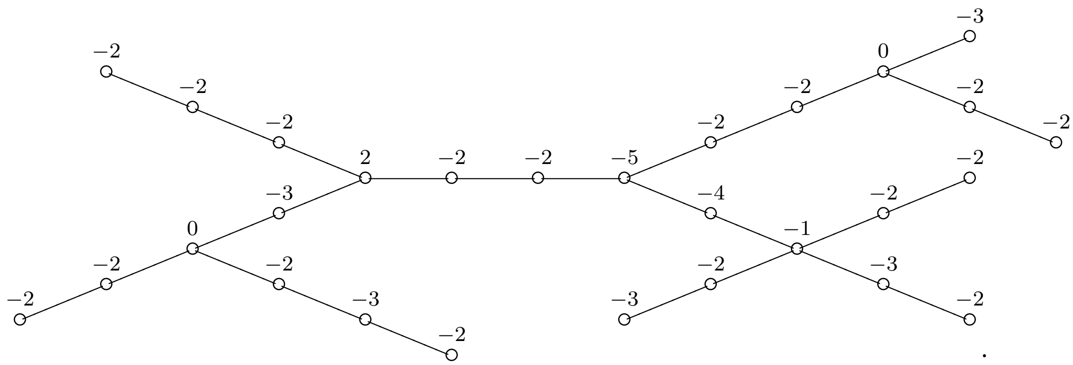
Corollary 4.12. *Let $\widetilde{\Delta}(M)$ be the maximal splice diagram of M, and $\widetilde{\Gamma}(M)$ be a unnormalized splice diagram of M, both obtained from the same plumbing diagram. Let v_i and v_j be two vertices of $\widetilde{\Gamma}(M)$. Define $\widetilde{l}_{ij}$ to be the product of all edge weights adjacent to but not on the shortest path from v_i to v_j. Then $\widetilde{l}_{IJ} = l_{ij}$, where l_{ij} is as defined in Theorem 4.11.*

Proof. The products are the same, since the edge weights at vertices of valence different from two in $\widetilde{\Delta}(M)$ are the same as the corresponding edge weights in $\widetilde{G}(M)$, and no edge weight on vertices of valence two are going to be adjacent to the shortest path between two vertices that do not have valence two. $\square$

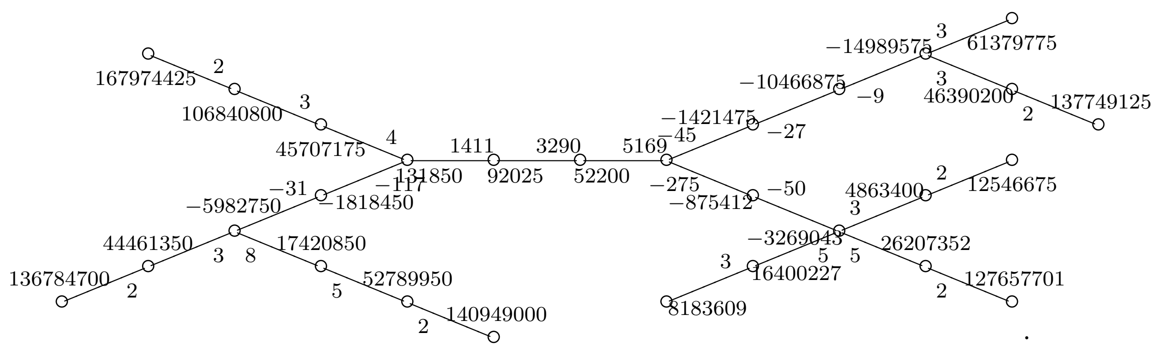
Lemma 4.13. *Let $\widetilde{\Gamma}(M)$ be an unnormalized splice diagram of the rational homology sphere graph manifold M. Then $\widetilde{\Gamma}(M)$ has the same underlying graph as $\Gamma(M)$. For each node v, $d_{ve} = |\tilde{d}_{ve}|$ and $\varepsilon_v = \operatorname{sign}(\Delta(M))\prod_e \operatorname{sign}(\tilde{d}_{ve})$, where the product is taken over all edges at v.*

Proof. That the graph has the same form and $d_{ve} = |\tilde{d}_{ve}|$ is clear from the constructions. The last statement follows from Theorem 4.11, Lemma 4.7, and Corollary 4.12. $\square$

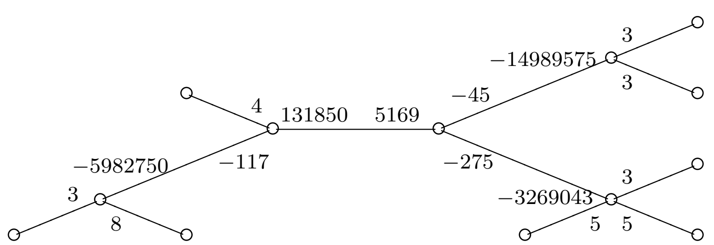
Example 4.14. Let M be a graph manifold with the following plumbing diagram



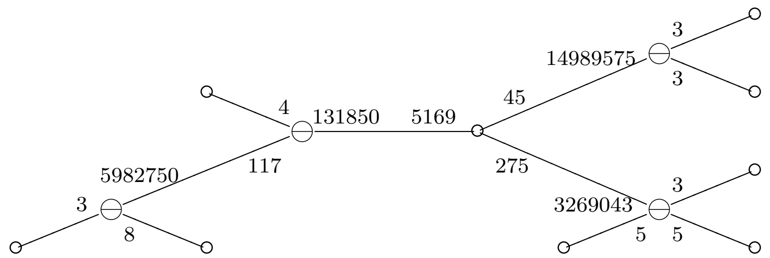
$\det(I) = 229108050$ and the maximal splice diagram of M is



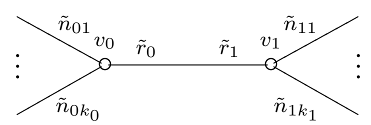
One then easily see that the unnormalized splice diagram is



Finally we obtain the splice diagram which is



If we have an edge between two nodes in an unnormalized splice diagram that look like:



Then we define the unnormalized edge determinant $\widetilde{D}$ associated to an edge to be

$$\widetilde{D} = \tilde{r}_0\tilde{r}_1 - \widetilde{N}_0\widetilde{N}_1 \tag{4.2}$$

where $\widetilde{N}_i = \prod_{j=1}^{k_i} \tilde{n}_{ij}$. It is clear that $D = \operatorname{sign}(\tilde{r}_0)\operatorname{sign}(\tilde{r}_1)\widetilde{D}$.

4.5 The ideal generator condition

The following section is basically the same as appendix 2 in [NW05a], the only difference is the definition of splice diagram used, and that we do not demand our manifolds to be negative definite plumbings. Let Γ be a splice diagram, not necessarily coming from a manifold. If v, w are two nodes of Γ, recall that l_{vw} is the product of all the edge weights adjacent to, but not on, the path from v to w. Define l'_{vw} the same way, except that the product does not include the weights at v and w. We assume that empty products are 1. Let v be an node of Γ and e an edge at v, and Γ_{ve} the connected component of $\Gamma - e$ not containing v. One defines the *ideal generator* $\overline{d}_{ve}$ associated to v and e to be the positive generator of the following ideal in $\mathbb{Z}$

$$\langle l'_{vw} | w \text{ is a leaf of } \Gamma_{ve}\rangle. \tag{4.3}$$

Definition 4.15. A splice diagram Γ is said to *satisfy the ideal condition*, if for every node v and every edge e at v, the edge weight d_{ve} is divisible by the ideal generator $\overline{d}_{ve}$.

Proposition 4.16. *If M is a rational homology sphere graph manifold, then $\Gamma(M)$ satisfies the ideal condition.*

To prove this, we will first find a nice topological description of the ideal generator. It is actually this topological description of the ideal generator which will make the ideal generator important to the proof of the second Main Theorem.

Theorem 4.17. *Let M be a rational homology sphere graph manifold, v be a node of $\Gamma(M)$, and e an edge at v. Then $\overline{d}_{ve} = |H_1(M/M'_v;\mathbb{Z})|$, where M'_v is the connected component containing M_v of M cut along the torus corresponding to e.*

First we prove that this implies the ideal condition. Notice that M/M'_v is the same as $M'_{v'}/\partial M'_{v'}$ where $M'_{v'}$ is the other connected component of M cut along the torus corresponding to e. We defined the edge weight d_{ve} to be $|H_1(M_{ve})|$ where M_{ve} is $M'_{v'}$ with a solid torus glued in the boundary. Let $K \subset M_{ve}$ be the knot corresponding to the core of the solid torus we glued into $M'_{v'}$ to make M_{ve}. Then $M_{ve}/K = M'_{v'}/\partial M'_{v'}$, and hence the ideal condition follows from the theorem and the fact that $H_1(M'_{v'}) \to H_1(M'_{v'}/K)$ is surjective by the long exact sequence in homology.

We are going to prove Theorem 4.17 by induction. We start the proof by a computational lemma.

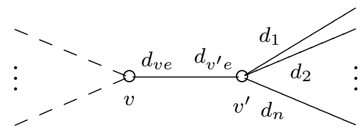
Lemma 4.18. *Let v' be the node connected to v by e, denote the edges at v' not equal to e by e_i for $i = 1, \dots, n$. Suppose that the ideal generator $\overline{d}_i$ is known at the edge e_i at v'. Then*

$$\overline{d}_{ve} = \gcd_{i=1}^{n} \left(\overline{d}_i \prod_{j \neq i}^{n} d_j \right), \tag{4.4}$$

where $d_j = d_{v'e_j}$.

The splice diagram looks like

$\Gamma =$



Proof. $\overline{d_i}$ is the generator of the ideal $\langle l'_{v'w} | w$ is a leaf of Γ in $\Gamma_{v'e_i}\rangle$, and hence the ideal $\langle l'_{vw} | w$ is a leaf of Γ in $\Gamma_{ve}\rangle$ is generated by the elements $\overline{d}_i \prod_{j\neq i}^{n} d_j$, and the statement follows. $\square$

Proof of Theorem 4.17. Assume our splice diagram looks like the one above. Let $M'_i = M_{v'e_i}$, then $d_i = |H_1(M'_i)|$. Let $K_i \subset M'_i$ be the core of the solid torus glued into the connected piece M_i to get M'_i. Let $\overline{d}_i = |H_1(M'_i/K_i)|$. We want to show that the $\overline{d}_i$'s satisfy the inductive formula of Lemma 4.5, and therefore are the ideal generators.

If we cut M along the tori corresponding to the edges e and e_i for $i = 1, 2 \ldots, n$, we then get that the piece containing the node v' is a S^1 bundle over S, homeomorphic to S^2 with $n+1$ punctures. Let f be a fiber of this fibration, and let $q_e, q_1, \ldots, q_n$ be the restriction of a section of the bundle to the boundary pieces corresponding to edges $e, e_i, \ldots, e_n$ respectively. Remember that M'_i is gotten from M_i by gluing in a solid torus, identifying the meridian with f. This implies

$$H_1(M'_i) = H_1(M_i)/\langle [f]\rangle. \tag{4.5}$$

so the order of $H_1(M_i)/\langle [f]\rangle$ is d_i. This gives us that

$$H_1(M'_{v'})/\langle [f]\rangle = \bigoplus_{i=1}^{n} H_1(M_i)/\langle [f]\rangle \tag{4.6}$$

has order $d_1 d_2 \ldots d_n$.

By definition $\overline{d}_i = |H_1(M_i)|/\langle [f], [q_i]\rangle$, so the order of $[q_i] \in H_1(M_i)/\langle [f]\rangle$ is $d_i/\overline{d}_i$. Hence the element $[q_1]+[q_2]+\cdots+[q_n] \in H_1(M_{v'})/\langle [f]\rangle$ has order $\mathrm{lcm}(d_1/\overline{d}_1, \ldots, d_n/\overline{d}_n)$. Now

$$H_1(M/M_v) = H_1(M_{v'})/\langle [f], [q_e]\rangle = H_1(M_{v'})/\langle [f], [q_1] + \cdots + [q_n]\rangle \tag{4.7}$$

since $q_e \cup q_1 \cup \cdots \cup q_n$ is a boundary. Hence $|H_1(M/M_v)| = |H_1(M_{v'})/\langle f\rangle|/\operatorname{lcm}_i(d_i/\overline{d}_i)$. This is the same as

$$\frac{d_1 d_2 \dots d_n}{\operatorname{lcm}(d_1/\overline{d}_i, \dots, d_n/\overline{d}_n)} = \gcd_{i=1}^{n}\left(\overline{d}_i \prod_{j\neq i}^{n} d_j\right) \tag{4.8}$$

which proves the theorem. □

Definition 4.19. We say that an edge weight r of a splice diagram *sees* a vertex v (or edge e) of the splice diagram if, when we delete the node which r is adjacent to, the vertex v (or the edge e) and the edge which r is on are in the same connected component.

Remark 4.20. Given a vertex v on any edge e between nodes, one of the edge weights at e sees v and the other does not see v. Let us introduce the following notation. Let v be a vertex of Γ and let v' be a node of Γ, where $v \neq v'$. Then let $r_{v'}(v)$ be the unique edge weight at an edge adjacent to v' which sees v. Likewise let $d_{v'}(v)$ be the unique ideal generator associated to v', which sees v.

Definition 4.21. We say that an edge weight r_v sees an edge weight $r_{v'}$ if r_v sees v' and $r_{v'} \neq r_{v'}(v)$. Likewise for ideal generators.

Proposition 4.22. *Let v be a node of a splice diagram Γ of a manifold M. Let r_v be a edge weight adjacent to v and let d_v be the corresponding ideal generator. Then r_v and d_v are divisible by every ideal generator they see. Moreover if r_v and d_v see a node v' and n, n' are edge weights at v' and $n, n' \neq r_{v'}(v)$, then* $\gcd(n, n') \mid r_v, d_v$.

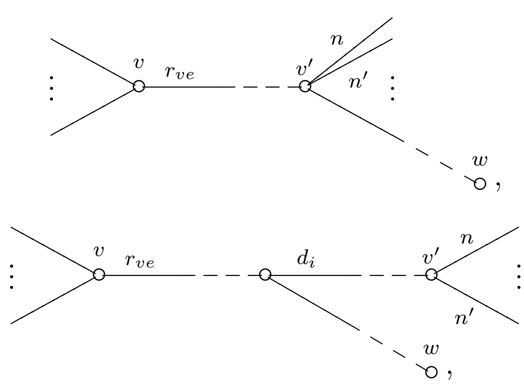
Proof. We first observe that it is enough to show the proposition only for d_v, since $d_v \mid r_v$ by 4.16. Let e be the edge d_v is on.

We will show this by induction on the number of edges between v and v'. If e is adjacent to v', then d_v is the generator of an ideal I, which can be generated by elements, each of which is divisible by the product of all but one of the edge weights at v' not on e. But this implies that each of the elements in the generating set is divisible by either n or n', and hence each the elements is divisible by $\gcd(n,n')$, and therefore d_v is divisible by $\gcd(n,n')$.

Assume by induction that if there are k edges between v'' and v' then $d_{v''}(v')$ are divisible by $\gcd(n,n')$. Assume that there are $k+1$ edges between v and v'. Let d_i for $i \in 1,\dots,k$ be $d_{\widetilde{v}}(v')$ on the vertex $\widetilde{v}$ on i'th edge between v and v', then by induction $\gcd(n,n') \mid d_i$ for all i. Remember that d_v is the generator of the ideal

$$\langle l'_{vw} | w \text{ is a leaf of } \Gamma_{ve} \rangle. \tag{4.9}$$

where l'_{vw} is the product of the edge weights adjacent to but not on the path from v to w. Now there are two types of leaves w: the leaves w where the path between w and v goes through v', and the one where the path does not go through v'. In the first case, $n \mid l'_{v'w}$ or $n' \mid l'_{vw}$ or both, so in this case $\gcd(n,n') \mid l'_{vw}$. In the second case one of the d_i's will divide l'_{vw}. This implies that $\gcd(n,n') \mid l'_{vw}$ for all w, and hence $\gcd(n,n')$ divides the generator of the ideal d_v. The following illustrates how Γ looks in the first and the second case of the induction. In the first case, the path to w can also pass through the edges with n or n'.



The statement about d_v being divisible by ideal generators it sees follows from a similar argument as above. □

Chapter 5

The main results

5.1 Determining the decomposition graph from the splice diagram

In this section we will show that the splice diagram almost determines the decomposition graph (associated to the JSJ decomposition) of a graph manifold. In fact it turns out that the only additional information we need to determine the decomposition graph $G(M)$ from the splice diagram $\Gamma(M)$ is $|H_1(M)|$.

The underlying graph of $G(M)$ is obtained from $\Gamma(M)$ by removing all leaves, i.e. by removing all vertices of valence one and the edges leading to them. By definition of $\Gamma(M)$, its nodes correspond to Seifert fibered pieces in the JSJ-decomposition of M, and each edge between two nodes corresponds to a gluing along a torus. Hence $\Gamma(M)$ without leaves has the shape of the decomposition graph.

We start to recover the information encoded in $G(M)$ by giving a formula for the orbifold euler characteristic.

Proposition 5.1. *Let v be a node in the splice diagram $\Gamma(M)$ of the manifold M.*

Then

$$\chi_v^{orb} = 2 - n(v) + \sum_e \frac{1}{d_{ve}} \tag{5.1}$$

where $n(v)$ is the valence of v and the sum is taken over all edges leading to leaves.

Proof. The node v corresponds to a Seifert fibered piece M_v, and each leaf of v corresponds to a singular fiber of M_v. Thus taking the sum in (2.6) over singular fibers is the same as taking the sum over edges at v leading to leaves.

The negative intersection matrix $-A(\Delta(M_{ve}))$ has numbers $b_i \geq 2$ on the diagonal and -1 adjacent to diagonal entries and 0 elsewhere since e leads to a leaf. We use the following algorithm to diagonalize $-A(\Delta(M_{ve}))$. If the matrix is $n \times n$, we clear the -1 at the $(n, n-1)$ entry by adding $-\frac{1}{a_{nn}}$ times the n'th row to the $(n-1)$'st row. Then we clear the -1 at $(n-1, n)$ by adding $-\frac{1}{a_{nn}}$ times the n'th column to the $(n-1)$'st column. In the n'th row and n'th column, only the diagonal entry remains. We then proceed to clear the $(n-1, n-2)$ and $(n-2, n-1)$ entries the same way. This then continues until the matrix is diagonal. Call this matrix D.

The (ii)'th entry of D is $[b_i, b_{i-1}, \ldots, b_1]$, which is defined to be the continued fraction

$$[b_i, b_{i-1}, \ldots, b_1] = b_i - \cfrac{1}{b_{i-1} - \cfrac{1}{b_{i-2} - \ldots}}. \tag{5.2}$$

Then $d_{ve} = |\det(\Delta(M_{ve}))| = |[b_n, b_{n-1} \ldots, b_1][b_{n-1}, b_{n-2}, \ldots, b_1] \cdots [b_1]|$. The denominator of the reduced fraction expression of $[b_i, , b_{i-1}, \ldots, b_1]$ is the numerator of the reduced fraction expression of $[b_{i-1}, , b_{i-2}, \ldots, b_1]$. This implies that d_{ve} is equal to the numerator of the reduced fraction expression of $|[b_n, , b_{n-1}, \ldots, b_1]|$. We call this algorithm for diagonalizing the intersection matrix *diagonalizing by continued fractions.*

It follows from Theorem 2.23 that the numerator of $[b_n, b_{n-1}, \ldots, b_1]$ is equal to α, where α is the first part of the Seifert invariant of the singular fiber corresponding to the leaf at e. So $d_{ve} = \alpha$ since $\alpha > 0$.

We now have that

$$\chi_v^{orb} = \chi_v - \sum_e (1 - \frac{1}{d_{ve}}) = \chi_v - l(v) + \sum_e \frac{1}{d_{ve}}. \tag{5.3}$$

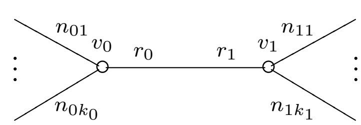
where $l(v)$ is the number of singular fibers, which is the same as the number of leaves at v. The base surface is a sphere since M is a rational homology sphere, so $\chi_v = 2 - r(v)$ where $r(v)$ is the number of boundary components, which is the same as the the number of edges leading to other nodes. The formula then follows since $l(v) + r(v) = n(v)$. □

We next prove a lemma relating the fiber intersection number to the edge determinant.

Lemma 5.2 (Unnormalized edge determinant equation)**.** *Assume that we have an edge in our splice diagram between two nodes. Let T be the torus corresponding to the edge and p the intersection number in T of Seifert fibers from each of the sides of T. Let $d = \det(\Delta(M))$. Then*

$$p = \frac{\widetilde{D}}{d}. \tag{5.4}$$

Proof. Let the numbers on the edges be as in



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and let $N_i = \prod_{j=1}^{k_i} n_{ij}$ for $i \in \{0, 1\}$.

We start by proving the formula under the additional assumption that there is no edge of weight 0 adjacent to the nodes, except possibly r_0 and r_1.

Let H_i be a fiber at the i'th node for $i = 0, 1$. Let $L_i \subset T^2$ be a simple curve which generates $\ker(H_1(T^2, \mathbb{Q}) \hookrightarrow H_1(M_i, \mathbb{Q}))$ where M_i is the piece of M gotten by cutting along the torus corresponding to the edge, including M_{v_i}. Since M is a rational homology, sphere the Meyer Vietoris sequence gives us that $H_1(T^2, \mathbb{Q}) \cong$

$H_1(M_0,\mathbb{Q}) \bigoplus H_1(M_1,\mathbb{Q})$. $H_1(M_i,\mathbb{Q}) \cong H_1(T^2,\mathbb{Q})/L_i$ by the long exact sequence. Finally $H_1(M_i/T^2) = H_1(M/M_{i+1})$ is a finite group, since $H_1(M)$ is finite. This implies that L_0 and L_1 are linearly independent, so $L_0 \cdot L_1 \neq 0$, where $\cdot$ denotes the intersection product in T^2.

We have the following relation

$$a_i H_i = b_{i0} L_0 + b_{i1} L_1 \tag{5.5}$$

for some $a_i, b_{i0}, b_{i1} \in \mathbb{Z}$, since L_0, L_1 are linearly independent in $H^1(T^2,\mathbb{Q}) = \mathbb{Q}^2$ and hence a basis. We also note that $L_i \cdot L_i = 0$.

We now want to compute the linking numbers $\mathrm{lk}(H_i, H_j)$ for $i, j \in \{0, 1\}$. Let $C_i \subset M_i$ be such that $\partial C_i = L_i$. This implies that $a_i H_i = b_{i0}\partial C_0 + b_{i1}\partial C_1$. Then one can compute $\mathrm{lk}(H_i, H_j)$ as $\mathrm{lk}(H_i, \frac{1}{a_j}(b_{j0}\partial C_0 + b_{j1}\partial C_1))$, but this is the same as computing $H_i \bullet (\frac{1}{a_j}(b_{j0}C_0 + b_{j1}C_1))$, where $\bullet$ denotes the intersection number in M. Now C_0 lives in M_0 and C_1 in M_1, so when one computes $H_0 \bullet (\frac{1}{a_j}(b_{j0}C_0 + b_{j1}\frac{1}{c_1}C_1))$, it is only the C_0 part that matters, since H_0 is in M_0, and therefore does not intersect surfaces in M_1. This means that $\mathrm{lk}(H_0, H_j) = H_0 \bullet (\frac{1}{a_j} b_{j0} C_0)$.

T^2 has a collar neighborhood in M_0, so when we want to compute $\mathrm{lk}(H_0, H_0)$ we can assume that the push-off of one of the copies of H_0 in T^2 lives in this collar neighborhood. In other words if the collar neighborhood is $(0, 1] \times T^2$, then the push off is $s \times H_0$ for some $s \in (0, 1]$. Over the collar neighborhood C_0 is just $(0, 1] \times L_0$, so

$$\begin{aligned}
H_0 \bullet (\frac{1}{a_0} b_{00} C_0) &= (s \times H_0) \bullet (\frac{1}{a_0} b_{00}((0,1] \times c_0 L_0)) \\
&= H_0 \cdot (\frac{1}{a_0} b_{00} L_0) \\
&= \frac{1}{a_0}(b_{00} L_0 + b_{01} L_1) \cdot (\frac{1}{a_0} b_{00} L_0) \\
&= \frac{1}{a_0^2} b_{01} b_{00} (L_1 \cdot L_0).
\end{aligned}$$

So we get that $\mathrm{lk}(H_0, H_0) = \frac{1}{a_0^2} b_{01} b_{00} (L_1 \cdot L_0)$. By a similar calculation one gets that $\mathrm{lk}(H_0, H_1) = \frac{1}{a_0 a_1} b_{01} b_{10} (L_1 \cdot L_0)$ and $\mathrm{lk}(H_1, H_1) = \frac{1}{a_1^2} b_{10} b_{00} (L_1 \cdot L_0)$.

Recall that the linking number of two fibers is also given by the inverse intersection matrix as in the proof of Lemma 4.7. Then by Corollary 4.12 and Theorem 4.11 we get that $\mathrm{lk}(H_i, H_j) = \frac{l_{ij}}{\det(\Delta(M))}$.

Returning to our situation, we then get the following equations for the linking numbers using the notation from above. $\mathrm{lk}(H_0, H_0) = \frac{N_0 r_0}{d}$, $\mathrm{lk}(H_1, H_1) = \frac{N_1 r_1}{d}$ and $\mathrm{lk}(H_0, H_1) = \frac{N_0 N_1}{d}$. Combining this with our other equations for the linking numbers we get

$$\frac{N_0 r_0}{d} = \frac{1}{a_0^2} b_{01} b_{00} (L_1 \cdot L_0) \tag{5.6}$$

$$\frac{N_1 r_1}{d} = \frac{1}{a_1^2} b_{10} b_{11} (L_1 \cdot L_0) \tag{5.7}$$

$$\frac{N_0 N_1}{d} = \frac{1}{a_0 a_1} b_{01} b_{10} (L_1 \cdot L_0). \tag{5.8}$$

It follows from (5.8) that the $b_{ij} \neq 0$, since $N_i \neq 0$ by our assumptions. So we can divide the product of (5.6) and (5.7) by (5.8), this gives us.

$$\frac{r_0 r_1}{d} = \frac{1}{a_0 a_1} b_{00} b_{11} (L_1 \cdot L_0) \tag{5.9}$$

Let us now compute p, which is equal to $H_0 \cdot H_1$ by definition

$$\begin{aligned}
H_0 \cdot H_1 &= \frac{1}{a_0}(b_{00} L_0 + b_{01} L_1) \cdot \frac{1}{a_1}(b_{10} L_0 + b_{11} L_1) \\
&= \frac{1}{a_0 a_1}(b_{01} b_{10} L_1 L_0 + b_{00} b_{11} L_0 L_1) \\
&= \frac{r_0 r_1 - N_0 N_1}{d} \\
&= \frac{\widetilde{D}}{d}
\end{aligned}$$

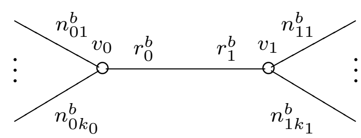
Here we use (5.8), (5.9) and the definition of $\widetilde{D}$.

We have now proved the the equality

$$dp = r_0 r_1 - \prod_{i=1}^{k_0} n_{0i} \prod_{j=1}^{k_1} n_{1j} \tag{5.10}$$

whenever $n_{ij} \neq 0$. Now this is an equation concerning minors of the negative intersection matrix $-A(\Delta(M))$ of M. We want to see what happens if we vary the diagonal entries of $-A(\Delta(M))$. We are especially interested in what happens when we change n_{ij}. Let a be an entry on the diagonal of $-A(\Delta(M))$ which lies in the minor n_{0l}. Replacing a with any integer b, we get a new matrix $-A(\Delta(M_b))$, which is the intersection matrix of the graph manifold M_b one gets from a plumbing diagram with the weight a replaced by b. Computing $d = \det(\Delta(M_b))$ by expanding by the row which include b, gives the equation $d = bA + B$. Then d can only be zero for at most one value of b, since $d = bA + B$ is a nonzero linear equation in b, and has a nonzero value for $b = a$. Hence M_b is a rational homology sphere for almost all b.

M_b has splice diagram which locally look like



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The only weights of the splice diagram of M_b which are different from the weights from the splice diagram of M, are n_{0l} and r_1, since none of the others see the the entry of $-\Delta(M)_b$ which we have changed. Again $n^b_{0l} = bA_{01} + B_{01}$ and $n_{01} = aA_{01} + B_{01}$, so $n^b_{0l} = 0$ for at most one value of b. So for almost all b, we have the following equation

$$d_b p = r_0 r^b_1 - n^b_{0l} \prod_{\substack{i=1 \\ i \neq l}}^{k_0} n_{0i} \prod_{j=1}^{k_1} n_{1j}. \tag{5.11}$$

Let $\widetilde{N}_l = \prod_{\substack{i=1 \\ i \neq l}}^{k_0} n_{0i} \prod_{j=1}^{k_1} n_{1j}$. We get that $r^b_1 = bA_1 + B_1$ and the above equation becomes

$$(bA + B)p = r_0(bA_1 + B_1) - (bA_{01} + B_{01})\widetilde{N}_l. \tag{5.12}$$

This is equivalent to

$$b(Ap - A_1 r_0 + A_{01}\widetilde{N}_j) = Bp - B_1 r_0 + B_{01}\widetilde{N}_j. \tag{5.13}$$

Since this is true for almost all b, it implies that

$$Ap - A_1 r_0 + A_{01}\widetilde{N}_j = Bp - B_1 r_0 + B_{01}\widetilde{N}_j = 0. \tag{5.14}$$

But this implies that equation (5.13) holds for any value of b. So the equation $dp = D$ (5.10) holds even if we change the diagonal entries of $-\Delta(M)$, and in particular, it holds if some $n_{ij} = 0$. Since we are only interested in rational homology spheres, for which $d \neq 0$, we divide (5.10) by d to get our result.

We call this method to derive the general result from a result with restrictions on the splice diagram, by working with the equation as an equation of determinant of matrices, *variation of diagonal entries* □

Corollary 5.3 (Edge determinant equation)**.** *For an edge between nodes in the splice diagram for a rational homology sphere graph manifold M, we get*

$$p = \frac{|D|}{|H_1(M)|}, \tag{5.15}$$

where p is the intersection number in the torus corresponding to the edge of a fiber from each of the sides of the torus, and D is the edge determinant associated to that edge.

Proof. This result follows from the previous result and the relation between D and $\widetilde{D}$ and that $|\det(\Delta(M))| = |H_1(M)|$ by taking absolute value. □

A consequence of the edge determinant equation is that no node in the splice diagram can have more than one adjacent edge weight of value 0. This is because we know that no leaf has edge weight 0, so if we have a node with at least two adjacent edge weights of value 0, the edge determinant of an edge with 0 on would be $0r_1 - \varepsilon_0\varepsilon_1 0N_1 = 0$, and then the edge determinant equation implies that $p = 0$. But $p = 0$ means that the fibers from each side of the torus corresponding to the edge have intersection number 0, so we could extend the fibration over T^2. The nodes v_0 and v_1 correspond to a single Seifert fibered piece, which would not be cut in the JSJ decomposition.

Next we need a formula for computing the rational euler class of the Seifert fibered pieces of our graph manifold, using only information from the splice diagram and the order of the first homology group. Consider a node in our splice diagram, as in Fig. 1 below, where everything to the left is leaves

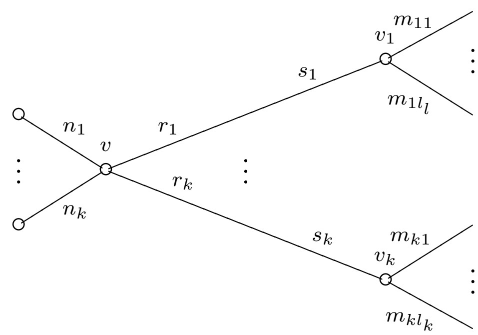


Figure 1

We let $N = \prod_{j=1}^{k} n_j$ and let $M_i = \prod_{j=1}^{l_i} m_{ij}$.

Proposition 5.4. *Let v be a node in a splice diagram decorated as in Fig. 1 above with $r_i \neq 0$ for $i \neq 1$, and let e_v be the rational euler number of M_v. Then*

$$e_v = -d\left(\frac{\varepsilon s_1}{ND_1 \prod_{j=2}^{k} r_k} + \sum_{i=2}^{k} \frac{\varepsilon_i M_i}{r_i D_i}\right) \tag{5.16}$$

where $d = |H_1(M)|$ and D_i is the edge determinant associated to the edge between v and v_i.

Proof. We start by proving a formula for e_v using an unnormalized splice diagram, and then show that the relation between unnormalized and normalized splice diagram will give us our result.

We first assume that $r_1 \neq 0$ and prove the formula under that hypothesis.

Let $\Gamma(M)$ be a unnormalized splice diagram, looking like the above. It is constructed from the plumbing diagram Δ, which looks like Fig. 2 below around the node v.

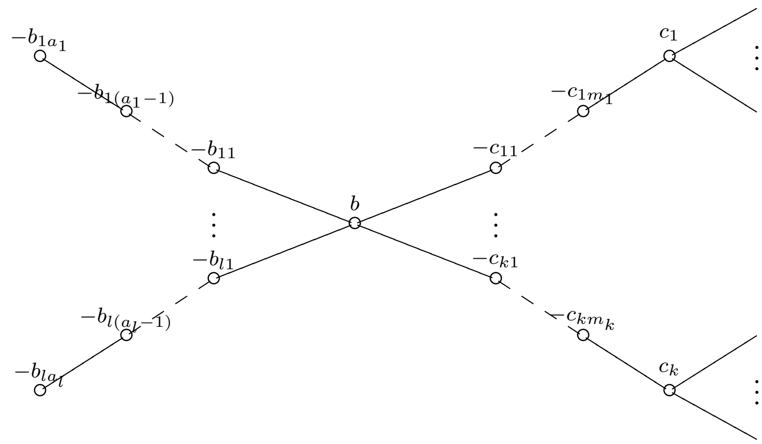


Figure 2 ,

where $b_{ij}, c_{ij} \geq 2$.

We want to compute $\det(-\Delta)$, so we look at the negative intersection matrix $-A(\Delta)$ of Δ, which we can write like

$$-A(\Delta) = \begin{pmatrix} b & -1 & 0 & \dots & -1 & \dots & -1 & \dots & -1 & \dots \\ -1 & b_{11} & -1 & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ 0 & -1 & b_{12} & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \\ -1 & 0 & 0 & \dots & b_{21} & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & \\ -1 & 0 & 0 & \dots & 0 & \dots & c_{11} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots & \\ -1 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & c_{21} & \dots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \ddots \end{pmatrix}$$

Let us describe the matrix more explicitly. If we delete the b-weighted vertex $v \in \Delta$ we get l components on the left and k components on the right. Let B_i be the negative intersection matrix of the i'th component to the left. It is of the form

$$B_i = \begin{pmatrix} b_{i1} & -1 & \dots & 0 & 0 \\ -1 & b_{i2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{ia_{i-1}} & -1 \\ 0 & 0 & \dots & -1 & b_{ia_i} \end{pmatrix}.$$

Likewise we let C_i be the negative intersection matrix of the i'th component to the right.

$$C_i = \begin{pmatrix} c_{i1} & -1 & \dots & 0 & 0 & \dots \\ -1 & c_{i2} & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_{im_i} & -1 & 0 \\ 0 & 0 & \dots & -1 & \multicolumn{2}{c}{\multirow{2}{*}{D_i}} \\ \vdots & \vdots & \dots & 0 & & \end{pmatrix}$$

Where the D_i is the matrix of the intersection form of the plumbing diagram consisting of the vertex weighted c_i and every vertex to the right of this vertex connected to it. We do not care how D_i looks.

$-A(\Delta)$ has b in its upper left corner, followed by the B_i's and the C_i's along the diagonal. The first row and column has a -1 in the column/row corresponding to the upper left corner of a B_i or C_i, and all other entries 0.

$$-A(\Delta) = \begin{pmatrix} b & -1 & 0 & \cdots & -1 & -1 & & -1 \\ -1 & B_1 & & & & & & \\ 0 & & & & & & & \\ \vdots & & & \ddots & & & & \\ -1 & & & & B_l & & & \\ -1 & & & & & C_1 & & \\ & & & & & & \ddots & \\ -1 & & & & & & & C_k \end{pmatrix}$$

We will diagonalize the matrix to compute $\det(-\Delta) = \det(-A(\Delta))$. We first diagonalize everything except for the first row and column. We use the diagonalization by continued fractions we used in the proof of 5.1 to diagonalize each B_i. By starting with the with the last column of B_i we assures that we never use the first row or column of B_i. This assures us that it does not change the matrix outside the B_i block, since the rows and columns we use have zeros outside B_i. The first entry of the block after diagonalising it will then be

$$\beta_i = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \dots}}$$

We can also diagonalize the C_i's in the same way. We will denote the first entry of the diagonalization of C_i by γ_i

To get the matrix completely diagonal we have to remove the copies of -1 in the first row and the first column corresponding to each B_i (or C_i). We remove it by adding $\frac{1}{\beta_i}$ $(\frac{1}{\gamma_i})$ times the first row of B_i (C_i) to the first row. This changes the the first entry by subtracting $\frac{1}{\beta_i}$ $(\frac{1}{\gamma_i})$. Notice that $\gamma_i \neq 0$ by the assumption that $r_i \neq 0$, since $r_i = \det(C_i)$. After doing this for all B_i and C_i, we get that the first entry of the diagonalized matrix is

$$b - \sum_{i=1}^{l} \frac{1}{\beta_i} - \sum_{i=1}^{k} \frac{1}{\gamma_i}. \tag{5.17}$$

We are not interested in γ_i itself but only part of it, so we make the following definition

$$\xi_i = \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}} - \frac{1}{\gamma_i}$$

We then get that first entry of the diagonalized matrix is

$$b - \sum_{i=1}^{l} \cfrac{1}{b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \dots}}} - \sum_{i=1}^{k} \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}} + \sum_{i=1}^{k} \xi_i.$$

Now we know that

$$-e_v = b - \sum_{i=1}^{l} \cfrac{1}{b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \dots}}} - \sum_{i=1}^{k} \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}}$$

by arguments of Walter Neumann in the proof of theorem 4.1 in [Neu97]. So if $d = \det(A(\Delta))$ we get that

$$d = (-e_v + \sum_{i=1}^{k} \xi_i) \prod_{i=1}^{l} \det(B_1) \prod_{i=1}^{k} \det(C_i). \tag{5.18}$$

But we also know that $n_i = \det(B_i)$ and $r_i = \det(C_i)$, so we get the following formula.

$$d = (-e_v + \sum_{i=1}^{k} \xi_i) \prod_{i=1}^{l} n_i \prod_{i=1}^{k} r_i = (-e(v) + \sum_{i=1}^{k} \xi_i) N \prod_{i=1}^{k} r_i. \tag{5.19}$$

Let e_j denote the edge in $\Gamma(M)$ between v and v_j, then $s_j = |H_1(M_{v_je_j})|$ by definition. The plumbing diagram $\Delta_{v_je_j}$ of this $M_{v_je_j})$ corresponds to Δ with everything after c_{jm_j} removed. We can now calculate $\det(\Delta')$ as above, and get that

$$\det(\Delta)_{v_je_j} = p_j(-e_v + \sum_{\substack{i=1 \\ i\neq j}}^{k} \xi_i) N \prod_{\substack{i=1 \\ i\neq j}}^{k} r_i \tag{5.20}$$

where

$$p_j = \det \begin{pmatrix} c_{i1} & -1 & \dots & 0 \\ -1 & c_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{im_i} \end{pmatrix}$$

But it follows from the proof of theorem 4.1 in [Neu97] that p_j is the fiber intersection number for the edge. By definition $\det(\Delta_{v_je_j}) = s_j$. So by combining (5.19) and (5.20) we get that

$$-\xi_j = \frac{s_j}{p_j N \prod_{\substack{i=1 \\ i\neq j}}^{k} r_i} - \frac{d}{N \prod_{i=1}^{k} r_i} = \frac{s_j r_j - d p_j}{p_j N \prod_{i=1}^{k} r_i}.$$

By using that $p_j = \frac{\widetilde{D}}{d}$ we get

$$-\xi_j = d \frac{s_j r_j - \widetilde{D}_j}{\widetilde{D}_j N \prod_{i=1}^{k} r_i} = d \frac{N M_j \prod_{\substack{i=1 \\ i\neq j}}^{k} r_i}{\widetilde{D}_j N \prod_{i=1}^{k} r_i} = \frac{d M_j}{r_j \widetilde{D}_j}$$

So plugging this into (5.19) we get

$$e_v = -d\Big(\frac{1}{N \prod_{j=1}^{k} r_k} + \sum_{i=1}^{k} \frac{M_i}{r_i \widetilde{D}_i}\Big) \tag{5.21}$$

$$= -d\Big(\frac{\widetilde{D}_1}{N \widetilde{D}_1 \prod_{j=1}^{k} r_k} + \sum_{i=1}^{k} \frac{M_i}{r_i \widetilde{D}_i}\Big) \tag{5.22}$$

$$= -d\Big(\frac{r_1 s_1 - N \prod_{j=2}^{k} r_k M_1}{N \widetilde{D}_1 \prod_{j=1}^{k} r_k} + \sum_{i=2}^{k} \frac{M_i}{r_i \widetilde{D}_i}\Big) \tag{5.23}$$

$$= -d\Big(\frac{s_1}{N \widetilde{D}_1 \prod_{j=2}^{k} r_k} + \sum_{i=2}^{k} \frac{M_i}{r_i \widetilde{D}_i}\Big) \tag{5.24}$$

This proves the formula if $r_1 \neq 0$.

For the $r_1 = 0$ case we use the variation of diagonal entries we used in the proof of 5.2 on (5.24) multiplied by $\prod_{i=1}^{k} \widetilde{D}_i \prod_{i=2}^{k} r_k$, the only difference is that the equations are now polynomial instead of linear, but they still only have finitely many solutions which is what we use. One gets the equation which holds for all values of b. We get our formula by dividing this equation by $\prod_{i=1}^{k} \widetilde{D}_i \prod_{i=2}^{k} r_i$, which is not 0 by our assumption on the r_i's.

We saw earlier that $\varepsilon = \operatorname{sign}(d) \prod_{i=1}^{l} \operatorname{sign}(n_i) \prod_{i=1}^{k} \operatorname{sign}(r_i)$, we get that $\frac{d}{N \prod_{i=1}^{k} r_i} = \frac{\varepsilon |d|}{|N| \prod_{i=1}^{k} |r_i|}$.

Using that $D_i = \operatorname{sign}(r_i) \operatorname{sign}(s_i) \widetilde{D}$ and $\varepsilon_i = \operatorname{sign}(M_i) \operatorname{sign}(s_i) \operatorname{sign}(d)$, so we also get that $d \frac{M_i}{r_i \widetilde{D}_i} = |d| \frac{\varepsilon_j |M_i|}{|r_i| D_i}$, which proves the proposition.

□

From Corollary 5.3 and Propositions 5.1 and 5.4 we get the information needed to make the decomposition graph.

5.2 Proof of the First Main Theorem

In the previous section we saw that the splice diagram and the order of the first homology group contains enough information to construct the decomposition graph of M. It therefore also lets us construct the decomposition matrix (as defined in section 2.3). By Theorem 3.7, M will be a singularity link if the decomposition matrix is negative definite.

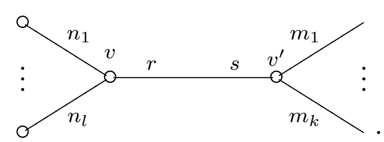
Theorem 5.5. *Let M be a rational homology sphere graph manifold, with splice diagram Γ. Then M is a singularity link if and only if all edge determinants are positive and Γ has no negative decorations at nodes.*

Proof. We start by proving the only if direction. Let $\Delta(M)$ be the normal form plumbing diagram for M. Since M is a singularity link, $A(\Delta(M))$ is negative definite as we saw in Theorem 3.5, and hence $-A(\Delta(M))$ is positive definite. This implies that that $\det(\Delta) = \det(-A(\Delta)) > 0$. Furthermore, all the edge weights in the unnormalized splice diagram $\widetilde{\Gamma}(M)$ from $\Delta(M)$ are positive, since they are determinants of $-A$ restricted to a subspace. Then from Lemma 4.13, we have that Γ is the same as $\widetilde{\Gamma}(M)$; in particular, all signs at nodes are positive. Since all edge weights of $\widetilde{\Gamma}(M)$ and $\det(\Delta(M)) > 0$, the unnormalized edge determinant equation (5.4) shows that each unnormalized edge determinant $\widetilde{D}_e$ is positive. It then follows that the edge determinant D_e is positive, since $\widetilde{D}_e = D_e$ because $\widetilde{\Gamma}(M)$ and Γ are the same.

For the other implication, first notice that no edge weight is 0. If we had an edge weight of 0, then it had to be on a edge between nodes, but the edge determinant of this edge would be $0r_1 - \varepsilon_0\varepsilon_1 N_0 N_1 = -N_0 N_1 < 0$.

Let $d = |H_1(M)|$. We proceed by induction in the number of nodes of Γ. If Γ only has one node, then M is Seifert fibered and the reduced plumbing matrix is a 1×1 matrix, with the rational euler number e of M as its entry. By Proposition 5.4, $e = -d\frac{\varepsilon}{N\prod_{j=0}^{k} r_k}$. But N, r_k, d are all greater than 0 by definition, and $\varepsilon = 1$ by assumption, so e is negative. Hence the reduced plumbing matrix is negative definite.

Assume that there are n nodes in Γ. Let v be an *end node* of Γ, meaning a node of the form



Such nodes always exist since Γ is a tree. Then the reduced plumbing matrix is of the form

$$\begin{pmatrix} e_v & \frac{1}{p} & 0 & \dots \\ \frac{1}{p} & e_{v'} & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

If we set $N = \prod_{i=0}^{l} n_i$ and $M = \prod_{i=0}^{k} m_i$ we get by Proposition 5.4 that

$$e_v = -d\frac{\varepsilon s}{DN}$$

where D is the edge determinant of the edge between v and v'. This means that the matrix look like

$$\begin{pmatrix} \frac{-\varepsilon sd}{DN} & \frac{1}{p} & 0 & \dots \\ \frac{1}{p} & e_{v'} & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

By a row and column operation we get the matrix to the form

$$\begin{pmatrix} \frac{-\varepsilon sd}{DN} & 0 & 0 & \dots \\ 0 & e_{v'} + \frac{\varepsilon DN}{p^2 sd} & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix} = \begin{pmatrix} \frac{-\varepsilon sd}{DN} \end{pmatrix} \oplus \begin{pmatrix} e_{v'} + \frac{\varepsilon Nd}{Ds} & \\ & \ddots \\ & \end{pmatrix},$$

since $\frac{1}{p^2} = \frac{d^2}{D^2}$. Since s, d, N are positive by definition and D, ε are positive by assumption, the reduced plumbing matrix is negative definite if the matrix

$$\begin{pmatrix} e_{v'} + \frac{\varepsilon Nd}{Ds} & \\ & \ddots \\ & \end{pmatrix}$$

is negative definite. Now

$$e_{v'} + \frac{\varepsilon Nd}{Ds} = -\frac{d}{Ms} - \sum_{i=0}^{k} \frac{\varepsilon_i M_i}{r_i D_i} - \frac{\varepsilon Nd}{Ds} + \frac{\varepsilon Nd}{Ds} = -\frac{d}{Ms} - \sum_{i=0}^{k} \frac{\varepsilon_i M_i}{r_i D_i} = \tilde{e}_{v'}$$

But $\tilde{e}_{v'}$ is the rational euler number of the Seifert fibered piece corresponding to v' in the manifold M' which is the manifold one gets by cutting M along the edge between v and v', and gluing in a solid tori in the piece containing v'. Then the matrix

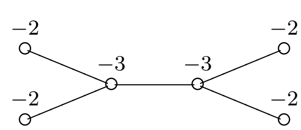
$$\begin{pmatrix} \tilde{e}_{v'} & & \\ & \ddots & \\ & & \end{pmatrix}$$

is the reduced plumbing matrix for the manifold M'. But since the splice diagram of M' is the same as Γ except that it has a leaf instead of the node v, it only has $n-1$ nodes. By induction, the reduced plumbing matrix of M' is negative definite, so the reduced plumbing matrix of M is negative definite, and then by Theorem 3.7, M is the link of a complex surface singularity.

$\square$

5.3 Graph orbifolds

To prove the second main theorem about universal abelian covers we need to extend our notions and results about graph manifolds to graph orbifolds. In the proof of the theorem we will do induction on the size of the splice diagram of a graph manifold M. Topologically, this means we have to cut our manifold along a torus and glue in solid tori to get some smaller manifolds, for which the statement holds by induction and whose universal abelian cover contribute pieces to the universal abelian cover of M. The problem is that we do not always get manifolds when we glue in the solid torus. Already in the simple case of the plumbing diagram



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the spaces one gets by gluing in the solid tori if one cuts along the central edge are not manifolds.

Fortunately the space we get when glue in the solid torus is not that bad; it happens to be a graph orbifold.

Definition 5.6. Let M be a 3-dimensional orbifold. We call M a *graph orbifold* if there exist a collection of disjoint smoothly embedded tori $T_i \subset M$, such that each connected component of $M - \bigcup T_i$ is an S^1 orbifold bundle over an orbifold surface.

It is clear that if a connected component of $M - \bigcup T_i$ is smooth, then it is a Seifert fibered manifold. Hence if M is smooth it is a graph manifold.

We want to define the splice diagram of a rational homology sphere graph orbifold. But to do this we have to consider which homology theory we are going to use. Remember that if M is smooth then $\pi_1^{orb}(M) = \pi_1(M)$, where $\pi_1^{orb}(M)$ is the orbifold fundamental group defined by Thurston (see e.g. [Sco83]). So in the case of smooth manifolds, orbifold coverings and coverings are the same. We need to study abelian orbifold coverings, for which the interesting homology group is then $H_1^{orb}(M)$. For our purposes it is enough to define $H_1^{orb}(M)$ as the abelianization of $\pi_1^{orb}(M)$; to make it clear that it governs the abelian covers. It should be mentioned that there exists a de Rham theorem for orbifold cohomology with rational coefficients, stating that $H^*_{orb}(X;\mathbb{Q}) \cong H^*(X;\mathbb{Q})$. So an orbifold is a rational homology sphere as an orbifold if and only if its underlying space is a rational homology sphere. Orbifolds also satisfy Poincare duality with rational coefficients. See e.g. [ALR07] for these results.

Next we look at the decomposition of a graph orbifold M into fibered pieces. To have a unique decomposition we do it the following way. Consider *orbifold curves*, K_j, that is curves along which M is not a manifold. Remove a solid torus neighborhood N_j of each orbifold curve K_j, such that $M' = M - \bigcup_{j=1}^m N_j$ is a graph manifold with m torus boundary components. We then take the JSJ decomposition of M', and glue the N_j's back in the pieces of the JSJ decomposition of M. This gives us our decomposition of M into fibered pieces. It is unique since the JSJ decomposition of M' is unique.

To define the splice diagram $\Gamma(M)$ of a graph orbifold M, take a node for each connected component of $M - \bigcup_{i=1}^{n} T_i$, where the set $\{T_i\}$ comes from the decomposition we defined above. Connect two nodes in $\Gamma(M)$ if the corresponding connected components of $M - \bigcup_{i=1}^{n} T_i$ are glued along a torus. Add a leaf at a node for each singular fiber of the S^1-orbifold bundle over the orbifold surface Σ, this is the same as adding a leaf for each point in Σ which does not have trivial isotropy group.

To add decorations to $\Gamma(M)$ we do the same as in the splice diagram, except that where previously we used the first singular homology group, we now use the first orbifold homology group. That is, to get decorations on an edge we cut M along the corresponding torus, glue in a solid torus in the same way as for manifolds, and take the order of the first orbifold homology group. At nodes we put the sign of the linking number of two non singular fibers of the S^1 fibration corresponding to the node.

Let us take a closer look at the orbifold curves. In any 3-dimensional orbifold M, an orbifold curve $K \subset M$ is a embedding of S^1 such that there exists a neighborhood N_K of K with $N_K - K$ is smooth. Now N_K can be chosen to be topologically a solid torus, and in this case $H_1^{orb}(N_K) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. We call p the orbifold degree of K. Another way to view N_K is as a S^1 fibration over a disk D_α with one orbifold point where the isotropy group is $\mathbb{Z}/\alpha\mathbb{Z}$. Then N_K is defined by a integer β which tells you how many times the fibers over the non orbifolds points wrap around the singular fiber. If $\gcd(\alpha, \beta) = 1$ then N_K is in fact smooth, and K is not an orbifold curve, but a singular fiber of the Seifert fibration in a neighborhood of K. In general, a calculation shows that if K is a orbifold curve of degree p, then $\gcd(\alpha, \beta) = p$. In general if α and β are we will given denote N_K by $T_{(\alpha,\beta)}$, and by abuse of notation T_p will denote a solid torus neighbourhood of a degree p orbifold curve.

Next consider how N_K is glued to $M' = \overline{M - N_K}$. We have a collar neighborhood $U = (0, 1] \times T^2$ of $\partial(M')$. The fibration of $N_K \to D_\alpha$ gives a fibration $\partial N_K \to S^1$, and this again gives a fibration of ∂U which can be extended to all of U. The image of a meridian N_K in ∂U defines a simple closed curve transverse to the fibration. By a *meridian* of N_K we mean a simple closed curve of the boundary that is transverse to the fibration and has homology class of finite order in $H_1^{orb}(N_K)$. The fibration on U and the simple closed curve transverse to the boundary uniquely describe a way to glue in a solid torus in the boundary of U to make it a Seifert fibered manifold, we call this manifold M_K. Note that the gluing maps $\varphi\colon \partial M' \to \partial N_K$ and $\varphi'\colon \partial M' \to \partial(S^1 \times D^2)$ are the same, and it is therefore also clear that as topological spaces M and M_K are the same.

Proposition 5.7. *Let $K \subset M$ be an orbifold curve of degree p in a rational homology sphere orbifold M. Then $|H_1^{orb}(M)| = p|H_1^{orb}(M_K)|$.*

Proof. The Meyer-Vietoris sequence of the cover of M by M' and N_K yields

$$0 \to \mathbb{Z}^2 \xrightarrow{i_*} H_1^{orb}(M') \oplus \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to H_1^{orb}(M) \to 0 \tag{5.25}$$

after observing that $H_1^{orb}(N_K) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. The initial zero occurs since M is a rational homology sphere, hence $H_2^{orb}(M)$ has to be finite, and therefore has zero image in $\mathbb{Z}^2$. We likewise get the exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{i'_*} H_1^{orb}(M') \oplus \mathbb{Z} \to H_1^{orb}(M_K) \to 0 \tag{5.26}$$

from the Meyer-Vietoris sequence of M_K by the cover of M' and $S^1 \times D^2$. Now $i_* = (i'_*, g)$ where the image of g is in $\{0\} \times \mathbb{Z}/p\mathbb{Z}$ from the way we constructed M_K above. We have the maps $\pi\colon H_1^{orb}(M') \oplus \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to H_1^{orb}(M') \oplus \mathbb{Z}$ given by $\pi(a, b, c) = (a, b)$ and $f_*\colon H_1^{orb}(M) \to H_1^{orb}(M_K)$, the map induced on orbifold homology groups by the homeomorphism $f\colon M \to M_K$ which is the identity on the complement of N_K. Note that $f|_{M'}\colon M' \to M'$ is the identity and $f|_{N_K}\colon N_K \to S^1 \times D^2$ induces the map from $\mathbb{Z} \bigoplus \mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}$ given by $(b, c) = b$, so we have the following map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "i_*"] \arrow[d, "\cong"] & H_1^{orb}(M') \oplus \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\pi"] & H_1^{orb}(M) \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "i'_*"] & H_1^{orb}(M') \oplus \mathbb{Z} \arrow[r] & H_1^{orb}(M_K) \arrow[r] & 0
\end{tikzcd} \tag{5.27}$$

Using the snake lemma on (5.27) we get the following short exact sequence.

$$0 \to \mathbb{Z}/p\mathbb{Z} \to H_1^{orb}(M) \to H_1^{orb}(M_K) \to 0 \tag{5.28}$$

and since this is a short exact sequence of finite abelian groups, the order of the group in the middle is the product of the order of the other two groups. $\square$

With the *underlying topological manifold* of a graph orbifold M, we mean the the unique manifold structure on the underlying topological space of M.

Corollary 5.8. *Let M be a rational homology sphere graph orbifold and $\overline{M}$ be its underlying topological manifold. Then $|H_1^{orb}(M)| = P|H_1(\overline{M})|$, where P is the product of the degrees of all orbifold curves in M.*

Remember how we defined that edge weight sees a vertex of a splice diagram in 4.19.

Corollary 5.9. *The splice diagram $\Gamma(M)$ is equal to the splice diagram $\Gamma(\overline{M})$ except if an edge weight sees a leaf corresponding to an orbifold curve of M, it is multiplied by the degree of the orbifold curve.*

Proof. If an edge weight r sees a leaf then the orbifold curve corresponding to that leaf is in the orbifold piece whose order of the first homology group gives r. $\square$

Corollary 5.10. *Assume that we have an edge in $\Gamma(M)$ between two nodes. Let T be the torus corresponding to the edge and p the intersection number in T of non-singular fibers from each of the sides of T. Let $d = |H_1^{orb}(M)|$, then*

$$p = \frac{|D|}{d} \tag{5.29}$$

where D is the edge determinant of that edge.

Proof. Since T is in the smooth part of M, the fiber intersection number is the same in M and M', and hence the same in $\overline{M}$. The equation holds in $\overline{M}$ by Proposition 5.3 and, since each term of D sees each orbifold curve once, it holds in M. $\square$

Corollary 5.11. *Let v be a node in a splice diagram decorated as in Fig. 1 with $r_i \neq 0$ for $i \neq 1$. Let e_v be the rational euler number of the S^1 fibered orbifold piece corresponding to v. Then*

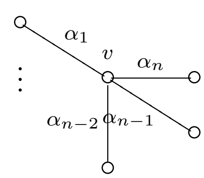
$$e_v = -d\big(\frac{\varepsilon s_1}{N D_1 \prod_{j=2}^{k} r_k} + \sum_{i=2}^{k} \frac{\varepsilon_i M_i}{r_i D_i}\big) \tag{5.30}$$

where $d = |H_1^{orb}(M)|$ and D_i is the edge determinant associated to the edge between v and v_i.

Proof. Since the rational euler number associated to the node is the same in M and $\overline{M}$ and the formula holds in $\overline{M}$ by proposition 5.4, it follows for M by noticing that the same orbifold degrees show up in the numerator and the denominator. $\square$

5.4 Proof of the Second Main Theorem

Let $\Gamma(M)$ be the splice diagram for a Seifert fibered manifold M, which has Seifert invariant $(0; (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$. $\Gamma(M)$ look likes



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and by Proposition 5.4 the sign of the rational euler number is equal to minus the sign at the node of the splice diagram. So from the splice diagram, we can read off the α_i's and the sign of the rational euler number. But this is exactly the information that determines the universal abelian cover of M. By Theorem 3.12 the universal abelian cover of M is homeomorphic to the Brieskorn complete intersection $\sum(\alpha_1, \ldots, \alpha_n)$, provided $e < 0$. If $e > 0$ one has to compose the universal abelian cover of $-M$ with a orientation reversing map, since reversing orientation multiplies e by -1. The case $e = 0$ does not occur for a rational homology Seifert fibered manifold. We will generalize this result to graph manifolds, but to do so we need to prove it for graph orbifolds.

An S^1-fibered orbifold will also have a splice diagram of the above form. In this case it still true that the universal abelian cover is $\sum(\alpha_1, \ldots, \alpha_n)$. The proof in [Neu83b] for Seifert fibered manifolds also holds in the case of S^1-fibered orbifolds, since it does not rely on the fact that $\gcd(\alpha_i, \beta_i) = 1$.

This will prove the base case of our induction, i.e. splice diagram with 1 node if $\alpha \neq 0$. To prove it in general we need the following lemma.

Lemma 5.12. *Let $\pi_1 \colon M \to M_1$ and $\pi_2 \colon M \to M_2$ be universal abelian orbifold covers such that $\deg(\pi_1) = \deg(\pi_2) = d$ and both M_1 and M_2 have an orbifold curve of degree p. Let $L(n, m_1)$ and $L(n, m_2)$ be orbifold quotients of S^3 by $\mathbb{Z}/n\mathbb{Z}$ which contain orbifold curves of degree p. Then the universal abelian cover of $L(n, m_1)\#_p M_1$ is homeomorphic to $L(n, m_2)\,\#_p\, M_2$, where $\#_p$ means taking connected sum along a B^3 that intersects the orbifold curve of degree p. The degree of the cover is nd/p.*

Proof. We are going to prove the lemma by constructing the universal abelian cover of $L(n, m_i)\,\#_p\, M_i$, and observing that it is determined by M, n, d and p.

Let $B^3_p \subset M_i$ be the ball which we are going to remove to take connected sum. The subscript p indicates that an orbifold curve of degree p passes through it. Let $M_i' = M_i - B^3_p$ and $S^2_p = \partial M_i'$. Then $\pi^{-1}(M_i') = \widetilde{M}_i$ is connected submanifold of M and $\pi_1|_{\widetilde{M}_i} \colon \widetilde{M}_i \to M_i$ is an abelian cover. Now π_i restricted to a connected component of $\partial\widetilde{M}_i$ is the p-fold cyclic branched cover of S^2, hence the number of boundary components of $\widetilde{M}_i$ is d/p. So clearly $\widetilde{M}_i$ is homeomorphic to M with d/p balls removed, and hence does not depend on M_i and π_i. If we then look at $S^2_p = \partial(L(n, m_i) - B^3_p)$ then the preimage under the universal abelian cover of $p_i \colon S^3 \to L(n, m_i)$ of $L(n, m_i) - B^3_p$ is S^3 with n/p balls removed. Let $\widetilde{M}$ be the manifold constructed in the following way. Take n/p copies of $\widetilde{M}_i$ and d/p copies of S^3 with n/p balls removed. Then glue each of the $\widetilde{M}_i$ to each of the S^3's exactly once, to form $\widetilde{M}$. Since π_i and the universal abelian cover map from S^3 to $L(n, m_i)$ agree on boundary components, we get an abelian cover $\tilde{\pi}_i \colon \widetilde{M} \to L(n, m_i)\,\#_p\, M_i$ of degree nd/p, by letting $\tilde{\pi}_i$ be equal to π_i on each of the $\widetilde{M}_i$ components and to the p_i on the

S^3 components.

Using the Meyer-Vietoris sequence we get that $|H_1^{orb}(L(n, m_i) \#_p M_i)| = nd/p$ hence $\tilde{\pi}_i \colon \widetilde{M} \to L(n, m_i) \#_p M_i$ is the universal abelian cover. This proves the lemma since the homeomorphism type of $\widetilde{M}$ only depends on M, n, d and p. □

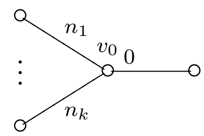
Remark 5.13. The above lemma is not true if we took connected sum along spheres with different degrees of the orbifold points, e.g $L(6, 3) \#_3 L(6, 3)$ has universal abelian cover $S^1 \times S^2$, but the universal abelian cover of $L(6, 3) \# L(6, 3)$ has first homology group of rank 25. The splice diagram can not see which orbifold curve we are going to take connected sum along, which forces us to make an additional assumption on our graph orbifolds in the theorem below. (Alternatively, one could make a new definition of splice diagram, where at leaves of weight zero one specifies the degree of the orbifold curve in the solid torus corresponding to the leaf. The theorem would then hold for all graph orbifolds with this alternative splice diagram).

Theorem 5.14. *Let M and M' be two rational homology sphere graph orbifolds having the same splice diagram Γ, and assume that all solid tori corresponding to leaves of weight zero do not have orbifold curves. Then $\widetilde{M}$ is homeomorphic to $\widetilde{M}'$, where $\pi \colon \widetilde{M} \to M$ and $\pi' \colon \widetilde{M}' \to M'$ are the universal abelian orbifold covers.*

Proof. We will prove the theorem by inductively constructing $\widetilde{M}$ only using information from the splice diagram.

For the case with one node, this almost follows from Theorem 3.12, since every one-node graph orbifold is an S^1 fibered orbifold, if there is no edge weight of 0. So we have to consider the case of a one-node splice diagram with an edge weight of 0. We saw in the discussion after Corollary 5.3 that a node can have at most one edge weight of 0.

Let M be a orbifold with the following splice diagram



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We will show that this orbifold is S^3 connect-summed along smooth S^2's with the orbifold quotients of S^3 by $\mathbb{Z}/n_i\mathbb{Z}$ acting as subgroups of $O(4)$. We denote the quotients by $L(n_1, q_1), \ldots, L(n_k, q_k)$, where the pair (n_i, q_i) is the Seifert invariant of the i'th singular fiber.

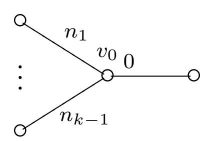
The criterion of a leaf with edge weight zero means that the fibers of the piece corresponding to the central node bound a meridional disc in the solid torus Z corresponding to that leaf. Take two fibers F_1 and F_2 and a simple path p between them in the boundary of Z. Then the region $B \subset Z$ bounded by all the fibers intersecting p and the meridional discs bounded by F_1 and F_2 is a ball. We can now extend B to the boundary of the solid torus L corresponding to the leaf of weight n_i. So by this, part of the boundary of B is an annulus of fibers in ∂L. Now $L \bigcup B$ is a solid torus glued to a ball along a $[0, 1] \times k$ where k is a knot which is a representative of a non trivial homology class of the boundary of L. Clearly $L \bigcup B$ has boundary S^2. Another way to see this is that the boundary is an annulus together with 2 discs. $L \bigcup B$ includes a singular fiber, so it is not a ball, therefore, the S^2 is a separating sphere, and $M = (L \bigcup B)\#((M - L) \bigcup B)$.

We will investigate what $L \bigcup B$ is. The complement of B in Z is a ball. So gluing this ball to $L \bigcup B$ we get the same orbifold as if we glued L to Z, and since Z does not have any singular fibers it is a quotient of S^3 with a orbifold curve having Seifert invariant (n_i, q_i).

By doing this for each leaf with non zero weight, we get that M is a connected sum of the various S^3 quotients $L(n_1, q_1), \ldots, L(n_k, q_k)$ and a central piece M'. It remains to show that M' is homeomorphic to S^3. If we glue a ball into $M - L \bigcup B$ to make the closed manifold M', we see that M' is Z glued to a solid torus, and the gluing map is the same as when we glued Z to $M - Z$. Since the weight of the leaf corresponding to Z was zero, the manifold which the edge weight where defined to be the order of the first homology group of is $S^1 \times S^2$, since it is the gluing of two solid tori. This implies that a fiber of $T^2 = \partial(M - Z)$ is a generator of $H_1(T^2)$ and is glued to a meridian of Z, and that a simple closed curve c corresponding to the other generator is glued to a longitude of Z. But gluing two solid tori according to the gluing of M and Z described above creates a S^3.

To show that the universal abelian cover of M is determined by the splice diagram, we do induction in the number of S^3 quotients in M, i.e. the number of leaves of the splice diagram of M. If there is only one S^3 quotient, then S^3 connect sum $L(n, q)$ is just $L(n, q)$, so the universal abelian cover of M is just S^3 and the covering map has degree n, hence determined by the splice diagram.

Let M' be the connected sum of S^3 with $L(n_1, q_1), \ldots, L(n_{k-1}, q_{k-1})$. Then M' has splice diagram as follows

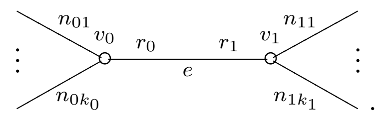


So the induction assumption is that the universal abelian cover $\widetilde{M}'$ of M' and the degree for this universal abelian cover d is determined by the splice diagram. We also have that $M = L(n_k, q_k) \# M'$ so by Lemma 5.12 the universal abelian cover of M is determined by $\widetilde{M}'$, n_k and the degree of the cover $\widetilde{M}' \to M'$ (remember in this case $p = 1$). But all this information is given by the splice diagram, since the splice diagram of M' is determined by the splice diagram of M.

This completes the one node case. For more than one node we will reduce our case to one with fewer nodes by cutting along a torus in M corresponding to an edge joining two nodes in Γ. There are additional complications arising from the fact that in a covering map $\pi \colon \widetilde{M} \to M$, the restriction to a connected component of $\pi^{-1}(T) \to T$ may not be trivial (unlike the a sphere, which is simply connected). Moreover, the gluing of T^2 boundary components is not trivial, as it was with S^2.

Let us assume that M has splice diagram Γ with $n > 1$ nodes. We look at an edge

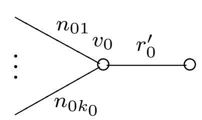
e between two nodes of the form



Let $T^2 \subset M$ be the separating torus, corresponding e. Let $M_i^\circ \subset M - T^2$ be the connected component containing M_{v_i}, and let $M_i = M_i^\circ \cup T^2$. M_0 and M_1 are graph orbifolds with one boundary torus each. $\pi|_{\pi^{-1}(M_i)} \colon \pi^{-1}(M_i) \to M_i$ is a (possibly disconnected) abelian covering. Let $\widetilde{M}_i \subset \pi^{-1}(M_i)$ be a connected component. Then $\pi|_{\widetilde{M}_i} \colon \widetilde{M}_i \to M_i$ is a connected abelian covering.

To describe the covering $\pi|_{\widetilde{M}_i} \colon \widetilde{M}_i \to M_i$ we are going to construct a closed graph orbifold M_i', with $M_i \subset M_i'$, such that if $p \colon \widetilde{M}_i' \to M_i'$ is the universal abelian cover, then $p|_{p^{-1}(M_i)} \colon p^{-1}(M_i) \to M_i$ is equal to $\pi|_{\widetilde{M}_i} \colon \widetilde{M}_i \to M_i$, i.e. $\widetilde{M}_i = p^{-1}(M_i)$ and the maps $p|_{\widetilde{M}_i}$ and $\pi|_{\widetilde{M}_i}$ agree.

We first look at M_0'. We will construct it from M_0 by gluing a solid torus in the boundary of M_0, in a way we will now explain. M_0' has splice diagram



where every weight on the left is as in the splice diagram of M if it does not see e. We want to determine r_0' and the other weights that see e, such that the universal abelian cover has the desired properties.

The components of a non-connected abelian cover are always determined by the map from H_1^{orb} of the base space to the abelian group which determines the non-connected cover. So in our case, $\widetilde{M}_0$ is determined by $H_1^{orb}(M_0) \to H_1^{orb}(M)$. One constructs M_0' by gluing in a T_p, such that the generator of $\ker\big(H_1^{orb}(M_0) \to H_1^{orb}(M)\big)$ is the curve that got killed, i.e. $\ker\big(H_1^{orb}(M_0) \to H_1^{orb}(M)\big) = \ker\big(H_1^{orb}(M_0) \to H_1^{orb}(M_0')\big)$. More precisely, let α be the primitive element such that $p\alpha$ generates $\ker\big(H_1^{orb}(M_0) \to H_1^{orb}(M)\big)$. Then M_0' is given by gluing α to a meridian of T_p and a simple closed curve with intersection 1 with the generator to a longitude. This ensures that $\widetilde{M}_0$ embeds into the universal abelian cover of M_0'. We also get that

$$\begin{aligned} H_1(M_0') &= \mathrm{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M_0')\right) = H_1^{orb}(M_0)/\ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M_0')\right) \\ &= H_1^{orb}(M_0)/\ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right) = \mathrm{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right). \end{aligned}$$

We need to know the order of $H_1^{orb}(M_o')$ to calculate the splice diagram of M_0', so we need to determine $\mathrm{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)$.

We first determine $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M))$, by looking at the Meyer Vietoris sequence of the covering of M by M_0 and M_1.

$$\cdots \to H_2^{orb}(M) \to H_1(T^2) \to H_1^{orb}(M_0) \oplus H_1^{orb}(M_1) \to H_1^{orb}(M) \to \dots \tag{5.31}$$

Since we have Poincare duality with rational coefficients, it follows that $H_2^{orb}(M)$ is finite, so $H_1^2(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ injects into $H_1^{orb}(M_0) \oplus H_1^{orb}(M_1)$. Hence $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M))$ is equal to the intersection of $H_1^{orb}(M_0)$ with $\mathbb{Z} \oplus \mathbb{Z}$. Since the rational orbifold homology of M_0 is the same as the rational homology of the underlying topological manifold, it follows that $H_1^{orb}(M_0)$ is rank one. Therefore $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M)) = \mathbb{Z}$, and is generated by a class in the boundary of M_0.

Let $Q_0 \in H_1(M_0)$ be a representative of the homology class of a section of the fibration on T^2, and let $F_0 \in H_1(M_0)$ be a representative of the class of the fiber of the Seifert fibered piece corresponding to the node v_0 in M_0. Then some homology class T^2 given by $r_0'Q_0 + s_0F_0$ is the class that gets killed when we glue in M_1, so it represents the generator of $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M))$.

We now have that $|H_1^{orb}(M_1)/\langle F_0\rangle| = r_0$ by the definition of splice diagram, since $H_1^{orb}(M_1)/\langle F_0\rangle = H_1^{orb}(M_1/F_0)$. Let $|H_1^{orb}(M_1)/\langle F_0, Q_0\rangle| = d_0$. Since we have that $H_1^{orb}(M_1)/\langle F_0, Q_0\rangle = H_1^{orb}(M_1/\partial)$, Theorem 4.17 implies that d_0 is equal to the ideal generator as defined in Definition 4.5, which is an invariant of the splice diagram. The proof of Theorem 4.17 also works for graph orbifolds. This implies that the order of Q_0 in $H_1^{orb}(M_1)/\langle F_0\rangle$ is r_0/d_0. Since $r_0'Q_0 + s_0F_0 = 0$ in $H_1^{orb}(M_1)$ we get that $r_0'Q_0 = 0$ in $H_1^{orb}(M_1)/\langle F_0\rangle$, so $r_0' \mid r_0/d_0$.

We also have the following map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\langle F_0\rangle \arrow[r] & H_1^{orb}(M_1) \arrow[r] & H_1^{orb}(M_1)/\langle F_0\rangle \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}\langle F_0\rangle \arrow[r] \arrow[u, "\cong"] & (\mathbb{Z}\times\mathbb{Z})/\langle r_0'Q_0 + s_0F_0\rangle \arrow[r] \arrow[u] & \mathbb{Z}/(r_0') \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and since the left map is an isomorphism and middle map is injective, it follows that the right map is injective too. Hence $r_0' = r_0/d_0$. Also note that $H_1^{orb}(M_1, \partial) = H_1^{orb}(M, M_0) = H_1^{orb}(M)/\operatorname{Im}\big(H_1^{orb}(M_0) \to H_1^{orb}(M)\big)$, so by taking the order of the groups, we get that $d_1 = d/|\operatorname{Im}\big(H_1^{orb}(M_0) \to H_1^{orb}(M)\big)|$. Since $H_1^{orb}(M_0') = \operatorname{Im}\big(H_1^{orb}(M_0) \to H_1^{orb}(M)\big)$, one gets that $|H_1^{orb}(M_0')| = d/d_0$.

Now for the other edge weights which see e, we start by determining the edge weight on an edge e_j which connects v_0. The fiber intersection number corresponding to the edge is the same in M and M_0', this gives by Theorem 5.2 the following equations

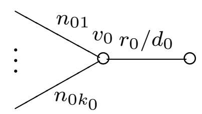
$$\frac{|D|}{|H_1^{orb}(M)|} = \frac{|D'|}{|H_1^{orb}(M_0')|} \tag{5.32}$$

where D is the edge determinant corresponding to e_j in M and D' the edge determinant corresponding e_j in M'. Since $|H_1^{orb}(M_0')| = |H_1^{orb}(M)|/d_0$ by the above calculation, we get that $|D'| = |D|/d_0$. The calculation also gave that $r_0' = r_0/d_0$, so the definition of edge determinants implies that the new edge weight has also been divided by d_0. Continuing inductively we see that all edge weights that see e are divided by d_0.

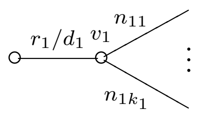
Suppose the curve killed by gluing in the solid torus into M_0 is a multiple of a primitive element. Then we get an orbifold curve in the glued in torus. This is not a problem if the weight on the edge is non-zero, so let us consider the case when the edge weight $r_0 = 0$. The curve we kill is the curve that bounds in M_1, but $r_0 = 0$ implies that a fiber in ∂M_0 bounds in M_1, so the curve we kill is a fiber. But fibers are primitive elements, so we do not get a orbifold curve in this case. This insures that M_1' satisfies the hypothesis of the theorem.

By a similar argument $r_1' = r_1/d_1$ where $d_1 = |H_1^{orb}(M_0, \partial)| = |H_1(M, M_1)|$.

The previous discussion implies that the splice diagram Γ_0 of M_0' looks like



and the splice diagram Γ_1 for M_1' is



All edge weights which see e are gotten from the old weight by dividing by d_0 in the first case and d_1 in the second case.

Since d_0 and d_1 are the ideal generators defined in Definition 4.5, they are completely determined by Γ. The splice diagrams Γ_0 and Γ_1 have at most $n-1$ nodes each, hence by induction the universal abelian cover $\widetilde{M_0'}$ of M_o' is determined by Γ_0 and the universal abelian cover $\widetilde{M_1'}$ of M_1' is determined by Γ_1. Since Γ_0 and Γ_1 are determined by Γ, this implies that $\widetilde{M_0'}$ and $\widetilde{M_1'}$ are determined by Γ, and therefore by the construction of M_0' and M_1', we have that $\widetilde{M_0}$ and $\widetilde{M_1}$ are determined by Γ.

Next we want to see that the splice diagram determines the number of components of $\pi^{-1}(M_0)$ and $\pi^{-1}(M_1)$, and which components are glued to which. The deck transformation group of $\widetilde{M}$ is $H_1(M)$ and its action restrict to deck transformation of the abelian cover of M_i, given by $\pi|_{\pi^{-1}(M_i)}\colon \pi^{-1}(M_i) \to M_i$. The group of permutations of the set of components of $\pi^{-1}(M_i)$ is given by $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(M_i) \to H_1^{orb}(M))$. But $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(M_i) \to H_1^{orb}(M)) = H_1^{orb}(M, M_i)$, so the number of components of $\pi^{-1}(M_i)$ is the order of $H_1^{orb}(M, M_i)$, which we have seen before is d_i, and only depends on the splice diagram.

The action of $H_1(M)$ on $\widetilde{M}$ also restricts to $\pi^{-1}(T^2)$, and we get that the group of permutations of components of $\pi^{-1}(T^2)$ is given by $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(T^2) \to H_1(M))$. This is the same as $H_1^{orb}(M, T^2)$, and by excision this is $H_1^{orb}(M/T^2)$. Now $M/T^2 = M_1/T^2 \vee M_2/T^2$, so the group of permutations of components $\pi^{-1}(T^2)$, is given by $H_1^{orb}(M_1, T^2) \oplus H_1^{orb}(M_2, T^2)$. Let t be a component of $\pi^{-1}(T^2)$, which is glued to a component M_{0i} of $\pi^{-1}(M_0)$ and a component M_{1j} of $\pi^{-1}(M_1)$. An element of the form $(1, \sigma) \in H_1(M/T^2)$ then sends t to another t', which is glued to M_{0i} and $\sigma(M_{1i})$. Since $H_1(M/M_1)$ acts transitively on the components $\pi^{-1}(M_1)$, we see that M_{0i} is glued to each component of $\pi^{-1}(M_1)$. Now the argument is symmetric, a component of $\pi^{-1}(M_1)$ is also glued to each of the components of $\pi^{-1}(M_0)$. By counting the number of tori in $\pi^{-1}(T^2)$ we get that a component on the one side is glued to each component on the other side exactly once.

Finally we need to specify the gluings of components of $\pi^{-1}(M_0)$ to components of $\pi^{-1}(M_1)$ from the splice diagram. Let $\widetilde{M}_0$ and $\widetilde{M}_1$ be components on each of the sides. To specify the gluing, we show that the splice diagram determines two distinct essential curves up (to homotopy) in each component T^2_{0i} of $\partial\widetilde{M}_0$ and in each component T^2_{1j} of $\partial\widetilde{M}_1$.

We first notice that $\widetilde{M}_i$ is a covering space of a graph orbifold (with boundary), and hence itself a graph orbifold, Moreover if F_i is a fiber of the base, then $\pi^{-1}(F_i)$ is a union of fibers of $\widetilde{M}_i$. So we take the first simple closed curve to be a fiber of T_{ij}, hence it is a connected component of the restriction of $\pi^{-1}(\widetilde{F}_i)$ to T_{ij}, where $\widetilde{F}_i$ is a fiber of ∂M_i. This insures that the fibers are specified by Γ. Moreover, the F_i in T^2_{ij} is identified with the F_i in $T^2_{ij'}$ under the action of the deck transformation group.

Given fibers $F_0 \subset T^2_{oi}$ and $F_1 \subset T^2_{1i}$, to specify the gluing we need only to know which curve C_1 in T^2_{1j} is identified with F_0, and which curve C_0 in T^2_{01} is identified with F_1. The rest of the proof consist of showing that C_0 and C_1 are determined by Γ.

Let $\widetilde{N}_i \subset \widetilde{M}_i$ be the Seifert fibered piece sitting over the M_{v_i}. $\widetilde{N}_i$ is a piece of the JSJ decomposition of $\widetilde{M}$ and therefore has a rational euler number $\widetilde{e}_i$. This is the rational euler number of the closed Seifert fibered manifold one gets by gluing solid tori into $\partial\widetilde{N}_i$, in a way specified by the gluings of $\widetilde{N}_i$ to the other pieces of the JSJ decomposition as described in beginning of Section 5.1. We are going compute $\widetilde{e}_i$, and then use this to specify the other curve in each T^2_{ij}. To do this we can assume by induction that a simple closed curve transverse to the fibration is determined in all the boundary components of $\widetilde{N}_i$ except the boundary components lying over e. But the fibers and the simple closed curves in all the boundary components lying over e will be the same. Since we know $\widetilde{F}_i$, the rational euler number $\widetilde{e}_i$ of $\widetilde{N}_i$ determines a simple closed curve transverse to the fibration in each of the boundary components of $\widetilde{M}_i$.

To compute $\widetilde{e}_0$ we will use the relation between $\widetilde{e}_0$ and the rational euler number e_{v_0} of M_{v_0}. This relation is given by Theorem 2.15, which says in our situation that $\widetilde{e}_0 = \frac{b_0}{f_0} e_{v_0}$, where b_0 is the degree of the covering map restricted to the base of the Seifert fibered pieces and f_0 is the degree of π restricted to the fibers. Now $\deg(\pi|_{\widetilde{M}_0}) = b_0 f_0$ and $\deg(\pi|_{\widetilde{M}_0}) = \frac{d}{d_0}$ since d_0 is the number of components of $\pi^{-1}(M_0)$. This implies that $\widetilde{e}_0 = \frac{d}{d_0 f_0^2} e_{v_0}$.

To calculate f_0 notice that $f_0 = |\mathrm{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$, and since

$$|H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))| = |H_1^{orb}(M)|/|\mathrm{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$$

we get that $f_0 = |H_1^{orb}(M)|/|H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$. We want to calculate $|H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$.

Now $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))$ is equal to $H_1^{orb}(M, F_0) = H_1^{orb}(M/F_0)$, so we need to find $|H_1^{orb}(M/F_0)|$. M/F_0 is M with the fibers collapsed at the piece M_{v_0}; this is the same as gluing in disks in the fibers of M_{v_0}. So we glue a disk to a simple closed curve transverse to the fibration of all the S^1–fibered pieces glued to M_{v_0} along a torus. This implies that $H_1^{orb}(M/F_0) = \left(A_1 \times A_2 \times \cdots A_{k_0} \times G\right)/\ (a_1, a_2, \ldots, a_{k_0}, g)$, where the group $A_j = H_1^{orb}(M_{v_0 e_j})$, where e_i is the edge between v_0 and the j'th piece to the right. In particular $|A_j| = n_{0j}$. Likewise $G = H_1^{orb}(M_{v_0 e})$. Hence $|G| = r_0$. The $a_j \in A_j$ are the elements that correspond to the singular fibers over the disk we just glued in; in particular $A_j/\langle a_j \rangle$ is the first orbifold homology group of M with everything except the part across the edge $0j$ collapsed. Thus $|A_j/\langle a_j\rangle|$ is the ideal generator d_{0j} corresponding to the edge $0j$. The same holds for g and G, especially that $|G/\langle g\rangle| = d_0$.

If none of the A_j and G is infinite, i.e., $n_{0j}, r_0 \neq 0$, then we get that $|H_1^{orb}(M/F_0)| =$

$(r_0 \prod_{j=0}^{k_0} n_{0j})/\operatorname{lcm}(n_{01}/d_{01}, \dots, n_{0k_0}/d_{0k_0}, r_0/d_0)$. Instead assume that A_l is infinite, so that all the other groups are finite. We have the following exact sequence

$$0 \to G \times \prod_{\substack{j=1\\ j\neq l}}^{k_0} A_j \to \big(A_1 \times \cdots A_{k_0} \times G\big)/\ (a_1, \dots, a_{k_0}, g) \to A_l/\langle a_l\rangle \to 0,$$

where $\big(A_1 \times \cdots A_{k_0} \times G\big)/\ (a_1, \dots, a_k, g) \to A_l/\langle a_l\rangle$ is projection, and $G \times \prod_{\substack{j=1\\ j\neq l}}^{k_0} A_j$ is the kernel of this map. This implies that $|H_1^{orb}(M/F_0)| = d_{0l} r_0 \prod_{\substack{j=1\\ j\neq l}}^{k_0} n_{0j}$. If G is infinite we use the same method to show that $|H_1^{orb}(M/F_0)| = d_0 \prod_{j=1}^{k_0} n_{0j}$. In all cases we see that $|H_1^{orb}(M/F_0)| = \lambda_0$ only depends on the splice diagram, since the ideal generators only depend on the splice diagram.

So $f_0 = |H_1^{orb}(M)|/|H_1^{orb}(M/F_0)| = \frac{d}{\lambda_0}$, hence $\widetilde{e}_0 = \frac{\lambda_0^2}{d_0 d} e_{v_0}$. But the value of e_{v_0} is given by Proposition 5.11, and the formula given there shows that e_{v_0} is d times a number given by the splice diagram, hence $\widetilde{e}_0$ only depends on Γ. We can in the same way calculate $\widetilde{e}_1$ and see that it also only depends on Γ. This implies that the splice diagram specifies a simple closed curve C_i transverse to the fibration of the T_{ij}^2's, which in particular is not null homologous.

Now the gluing of $\widetilde{M}_0$ to $\widetilde{M}_1$ is specified by identifying F_0 with C_1 and F_1 with C_0. But since the F_i's and the C_i's are determined by Γ, the gluing is determined by Γ, and hence $\widetilde{M}$ is determined by Γ. $\square$

Chapter 6

Some Corollaries of the Main Theorems

6.1 Determining when the universal abelian cover is a rational homology sphere

In this chapter we will give some corollaries of the proof of the second main theorem. We determine from the splice diagram when the universal abelian cover is a rational or an integer homology sphere. We begin by recognizing manifolds with integer homology sphere universal abelian covers.

Corollary 6.1. *Let M be a rational homology sphere graph manifold with splice diagram $\Gamma(M)$, such that around any node in $\Gamma(M)$ the edge weights are pairwise coprime. Then the universal abelian cover of M is an integer homology sphere.*

Proof. It is shown in [EN85] that a splice diagram with pairwise coprime edge weights at nodes is the splice diagram for an integer homology sphere. So given a splice diagram satisfying the assumption there is a integer homology sphere M' with splice diagram $\Gamma(M)$, hence M' is the universal abelian cover of M by Theorem 5.14. □

This actually gives a way to construct the universal abelian cover for any rational homology sphere graph manifold that has a splice diagram with pairwise coprime edge weights at nodes, since [EN85] describes how to construct the integral homology sphere by splicing, and to construct a plumbing diagram for it in the case where we have positive decorations at nodes. If we want a construction of the plumbing diagram when there are negative decorations at nodes, Theorem 3.1 in [Neu89a] contains a method whereby given a splice diagram $\Gamma(M)$ as above, one can construct an unimodular tree $\Delta(M)$ which will be the plumbing diagram.

To determine (at least necessary) conditions on the splice diagram for the universal abelian cover to be a rational homology sphere, we investigate the construction of the universal abelian cover in Theorem 5.14. Since we construct the universal abelian cover by induction, there are two places where obstructions to being a rational homology sphere can arise: in the inductive step, and in the base case.

We start by looking at the base case, that is a splice diagram with one node. We distinguish between diagrams with an edge weight of 0 and those without. In the case of an edge weight of 0, we never get rational homology sphere universal abelian covers. The universal abelian cover X of $L(p,q) \# L(p',q')$ is p copies of S^3 with p' balls removed, glued to p' copies of S^3 with p balls removed, where the former pieces are glued to the latter pieces exactly once each. Then a Meyer-Vietoris argument shows that the rank of the first homology group is $(p-1)(p'-1)$. Since the universal abelian covers of iterated connected sums of lens spaces will contain several copies of X as connected summands, it is clear that a connected sum of lens space can not have rational homology sphere universal abelian covers.

This leaves the second case, determining which Seifert fibered, or more precisely, which S^1 orbifold bundles have rational homology sphere universal abelian covers. By Theorem 3.12 this is the same as determining which links of Brieskorn complete intersections are rational homology spheres.

Proposition 6.2. $\Sigma(\alpha_1, \dots, \alpha_n)$ *is a rational homology sphere if and only if one of* *the three following conditions holds.*

1. $\gcd(\alpha_i, \alpha_j) = 1$ *for all* $i \neq j$.

2. *There exist a single pair* k, l, *such that* $\gcd(\alpha_k, \alpha_l) \neq 1$.

3. *There exist a single triple* k, l, m *such that* $\gcd(\alpha_l, \alpha_k) = \gcd(\alpha_l, \alpha_m) = \gcd(\alpha_m, \alpha_k) = 2$; *for all other indices* $\gcd(\alpha_i, \alpha_j) = 1$.

The first condition is of course the case where $\Sigma(\alpha_1, \ldots, \alpha_n)$ is a integer homology sphere, as we saw earlier.

Proof. The if direction follows from [Ham72], where Hamm proves a sufficient condition for the link of Brieskorn complete intersections of any dimension to be rational homology spheres. He could only prove the other direction if the number of variables was at most twice the dimension plus two. We will give a different proof in the case of surfaces, using the description of the Seifert invariants given in Theorem 3.10.

It follows from Corollary 2.12 that a Seifert fibered manifold is a rational homology sphere if and only if the rational euler number e is nonzero, and the genus g is zero. From the formulas of Theorem 3.10 we see that $e(\Sigma(\alpha_1, \ldots, \alpha_n)) \neq 0$, so we need only show that the conditions above are equivalent to the genus being 0. In other words it is enough to show that the following equation holds if and only if one of the three conditions does:

$$0 = 2 + (n-2)\frac{\prod_i \alpha_i}{\mathrm{lcm}_i(\alpha_i)} - \sum_{i=1}^{n} \frac{\prod_{j\neq i} \alpha_j}{\mathrm{lcm}_{j\neq i}(\alpha_j)}. \tag{6.1}$$

Let $A = \frac{\prod_i \alpha_i}{\mathrm{lcm}_i(\alpha_i)}$ and $A_i = \frac{\prod_{j\neq i} \alpha_j}{\mathrm{lcm}_{j\neq i}(\alpha_j)}$.

We start by proving the "if" direction. Assume condition 1 holds, then $A = 1$ and $A_i = 1$ for all $i \in 1, 2, \ldots, n$, and we get

$$g = 2 + (n-2)A - \sum_{i=1}^{n} A_j = 2 + (n-2) + \sum_{i=1}^{n} 1 = 2 + (n-2) - n = 0 \tag{6.2}$$

Assume that condition 2 holds, and let $\gcd(\alpha_k, \alpha_l) = B$. Then $A = B$, $A_k = A_l = 1$ and $A_i = B$ if $i \neq k, l$. We get

$$g = 2 + (n-2)B - 1 - 1 - \sum_{i \neq k,l} B = 2 + (n-2)B - 2 - (n-2)B = 0 \tag{6.3}$$

Finally for condition 3, $A = 4$, $A_k = A_l = A_m = 2$ and $A_j = 4$ if $j \neq k, l, m$. The genus is

$$g = 2 + (n-2)4 - 2 - 2 - 2 - \sum_{i \neq k,l,m} 4 = (n-2)4 - 4 - (n-3)4 = 0. \tag{6.4}$$

This conclude the "if" direction.

For the "only if" direction we start by assuming the equation (6.1) holds. Suppose we have $\alpha_j, \alpha_k, \alpha_l, \alpha_m$, such that $\gcd(\alpha_j, \alpha_k) = B$ and $\gcd(\alpha_l, \alpha_m) = C$. Notice that $BC \mid A$, $B \mid A_l, A_m$, $C \mid A_j, A_k$ and $BC \mid A_i$ for $i \neq j, k, l, m$. Let $A' = \frac{A}{BC}$, $A'_j = \frac{A_j}{C}$, $A'_k = \frac{A_k}{C}$, $A'_l = \frac{A_l}{B}$, $A'_m = \frac{A_m}{B}$ and $A'_i = \frac{A_i}{BC}$ if $i \neq j, k, l, m$. $A \geq A_i$ for all i so clearly $A' \geq A'_i$ for $i \neq j, k, l, m$. If $i = j, k$ then $B \mid \frac{A}{A_i}$, and if $i = l, m$ then $C \mid \frac{A}{A_i}$, so we also get $A' \geq A'_i$.

Hence

$$\begin{aligned} 0 &= 2 + (n-2)BCA' - CA'_j - CA'_k - BA'_l - BA'_m - \sum_{s \neq j,k,l,m} BCA'_s \\ &\geq 2 + (n-2)BCA' - 2CA' - 2BA' - \sum_{s \neq j,k,l,m} BCA' = 2 + 2A'(BC - C - B). \end{aligned} \tag{6.5}$$

Since $A' \geq 1$ this implies that $BC - C - B < 0$ and hence either $B = 1$ or $C = 1$.

We have now proved that $\gcd(\alpha_i, \alpha_j) = 1$ except that there might be $\alpha_k, \alpha_l, \alpha_m$ such that $\gcd(\alpha_k, \alpha_l) = B$ and $\gcd(\alpha_k, \alpha_m) = C$ and $\gcd(\alpha_l, \alpha_m) = D$. Notice that $A = \frac{\alpha_k \alpha_l \alpha_m}{\operatorname{lcm}(\alpha_k, \alpha_l, \alpha_m)}$, $A_k = \frac{\alpha_l \alpha_m}{\operatorname{lcm}(\alpha_l, \alpha_m)}$, $A_l = \frac{\alpha_k \alpha_m}{\operatorname{lcm}(\alpha_k, \alpha_m)}$, $A_m = \frac{\alpha_k \alpha_l}{\operatorname{lcm}(\alpha_k, \alpha_l)}$, and $A_i = A$ for $i \neq k, l, m$. The equation becomes

$$\begin{aligned} 0 &= 2 + (n-2)A - A_k - A_l - A_m - \sum_{s \neq k,l,m} A_j \\ &= 2 + (n-2)A - A_k - A_l - A_m - \sum_{s \neq k,l,m} A = 2 + A - A_k - A_l - A_m, \end{aligned} \tag{6.6}$$

which is excaltly the same equation as if $n = 3$. But it is known in this case that either $B = C = D = 2$ or two of B, C, D is 1, from the article of Hamm [Ham72]. One can also see this directly, if $\alpha_1 = ds_2s_3t_1$, $\alpha_1 = ds_1s_3t_2$, and $\alpha_3 = ds_1s_2t_3$ where $\gcd(s_i, s_j) = 1$ and $\gcd(t_i, t_j) = 1$, then the equation becomes $0 = 2 + d^2s_1s_2s_3 - d(s_1 - s_2 - s_3)$. It is clear that $d = 1$ or 2. If $d = 2$ then the only solution is $s_1 = s_2 = s_3 = 1$ since the righthand side is increasing in s_i. If $d = 1$ then the only solution is if two of the s_i's are zero, since the righthand side increases if we increase two of the s_i's.

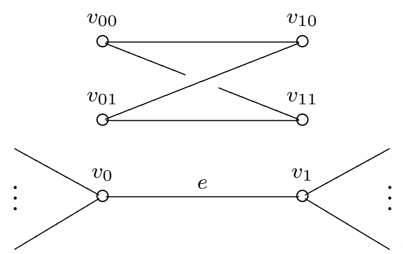
□

Combining this result with an investigation of the inductive step yields a necessary condition on the splice diagram for the universal abelian cover to be a rational homology sphere. We remember how we defined the notation $r_{v'}(v)$ in 4.20.

Corollary 6.3. *Let Γ be the the splice diagram of a manifold M, where the universal abelian cover of M is a rational homology sphere. Then all edge weights are nonzero, and there is a special node $v \in \Gamma$, with the following properties. For all other nodes $v' \in \Gamma$, the weights other than $r_{v'}(v)$ are pairwise coprime, and at most one of these edge weights is not coprime with $r_{v'}(v)/d_{v'}(v)$. At v all the edge weights satisfy one of the conditions from Proposition 6.2.*

Proof. What we are going to show is that the condition on the splice diagram given above is equivalent to the absence of cycles in the decomposition graph (or a plumbing graph) of the universal abelian cover $\widetilde{M}$, and all the pieces of the decomposition having a base of genus 0. The corollary then follows by Proposition 2.25. That the decomposition graph must also have no cycles and bases of genus 0 follows from the relation between plumbing graphs and decomposition graph given in [Neu97].

We saw that, when we cut along an edge e between nodes v_0 and v_1 in the inductive construction of $\widetilde{M}$ given in the proof of Theorem 5.5, we took d_0 pieces v_0 and glued to d_1 pieces above v_1, where d_i is the ideal generator at e associated to v_i. Each piece on the one side is glued exactly once to each piece on the other side. Each of these pieces has a Seifert fibered piece sitting above the corresponding M_{v_i}. If $d_0, d_1 > 1$ then a piece v_{00} over M_{v_0} is glued to a piece v_{10} sitting over M_{v_1}, then v_{10} is glued to a piece v_{01} sitting over M_{V_0}, and v_{01} is glued to a piece v_{11} sitting over M_{v_1}. Finally v_{11} is glued to v_{00}. We have now constructed a cycle in the decomposition graph of $\Delta(\widetilde{M})$ since each of the v_{ij} represent a vertex of $\Delta(\widetilde{M})$. If one of the d_i's is 1, then we do not get cycles, since we will have only one piece above the appropriate end of e.



So we now proved that a cycle in the decomposition graph for $\widetilde{M}$ occurs if an edge e in the splice diagram has ideal generators d_0 and d_1 (associated to each end), such that both d_0 and d_1 are not equal to one.

Let M_0 and M_1 be graph manifolds with universal abelian covers $\widetilde{M}_0$ and $\widetilde{M}_1$, and assume that there are no cycles in $\widetilde{M}_i$. Let $\widetilde{M}_{01}$ be the universal abelian cover of M_{01} which is M_0 glued to M_1 after removing a solid torus from each. Assume that $\widetilde{M}_{01}$ has cycles in its decomposition graph. $\widetilde{M}_{01}$ is a number of $\widetilde{M}_0$ with n_0 solid tori removed glued to $\widetilde{M}_1$ with n_1 solid tori removed, such that each of the first type is glued to each of the second type. If one of the n_i is 1, then $\widetilde{M}_{01}$ has no cycles, so $n_0, n_1 > 1$. But $n_i = d_i$ so we are in the situation above.

So there are cycles in the decomposition graph of $\widetilde{M}$ if and only if there is an edge which has both associated ideal generators different from 1.

We need to show that the conditions we stated on Γ are equivalent to the statement that for each edge one of the ideal generators associated to an end of it is 1.

Suppose there were two nodes v and w of Γ, such that the edge weights at v that do not see w are not pairwise coprime, and the same with v and w exchanged. On any edge e on the string between v and w, the ideal generator associated to either end of e is then greater than 1 by Proposition 4.22, so we a have cycle in the decomposition graph. This implies that there can be at least be one node v, such that at all other nodes, edge weights that do not see v are pairwise coprime. On the other hand, if Γ satisfies this, then it is not hard to see that all ideal generators that do not see v are 1, since all the edge weight they see at a node are pairwise coprime.

We have so far shown that there are no cycles in the decomposition graph of $\widetilde{M}$ if and only if there is a special node v such that at all other nodes the edge weights that do not see v are pairwise coprime. Next we have to see that our condition on Γ also gives that all the pieces of the decomposition have genus 0.

Remember that when we do the induction in the proof of Theorem 5.5 and cut along an edge e between v_0 and v_1, for be any node v' in Γ not equal to v_0 or v_1, the weight $r_{v'}(v_i)$ gets replaced by $r_{v'}(v_i)/d_{v'}(v_i)$, where v_i ($i = 0$ or 1) is the node not in the same piece as v' after cutting. When we cut Γ along its edges, we do it in the following way. Always choose an edge e to an end node w, that is not the special node v to cut along. Then after the cutting we get two new pieces. The first corresponds to the end node w and has a one node splice diagram with as many edges as w had in Γ, and the edges have the same weights, except $r_w(v)$ is divided by $d_w(v)$. The splice diagram of the other piece Γ_e looks like Γ with the node w replaced by a leaf, and no edge weight is changed since all the $d_{v'}(w) = 1$ for any node v'. We then find an end node of Γ_e which is not v to cut along, and repeat until we have cut along all the edges between nodes.

We have now cut Γ into a collection of one-node splice diagrams. Each of these will contribute at least one Seifert fibered piece to $\widetilde{M}$, (the same one-node splice diagram may of course contribute with the same Seifert fibered piece of $\widetilde{M}$ more than once). We distinguish the piece corresponding to our special node v. The pieces not corresponding to v have splice diagrams with the same weights as in Γ, except $r_w(v)$ is replaced by $r_w(v)/d_w(v)$. Our assumptions on the Γ then imply that all the weights are pairwise coprime, except possibly two weights who are pairwise coprime with the rest, but might have a common divisor. Since the Seifert fibered pieces corresponding to each of the nodes are the Brieskorn complete intersections defined by the edge weights, so condition one or two of Proposition 6.2 holds. Then the Seifert fibered pieces of the decomposition of $\widetilde{M}$ corresponding to these nodes are rational homology spheres.

The special piece of the decomposition of $\widetilde{M}$ (corresponding to v, there will in fact only be one), has genus 0, since the assumption on Γ are equivalent to the Brieskorn complete intersection being genus 0, by proposition 6.2.

Hence the assumptions on Γ are equivalent to the decomposition graph of $\widetilde{M}$ having no cycles, and all the pieces of the decomposition having a base of genus 0. $\square$

Remark 6.4. The converse to the corollary does not immediately follow, since having no cycles and having genus 0 pieces are only two of the three conditions for a graph manifold to be a rational homology sphere. The last one (as we saw in proposition 2.25) is that the intersection matrix I must have non zero determinant. Proving that $\det(I) \neq 0$, reduces to a simpler problem since Neumann showed in [Neu97] that, by doing row and column additions, I becomes the direct sum of the decomposition matrix and a number of 1×1 matrices with non zero entries. Hence it is enough to show that the determinant of the decomposition matrix is non zero. Unfortunately, I do not have a proof of this yet. Fortunately, in the case of singularity links this is not needed.

Corollary 6.5. *If M is a rational homology sphere singularity link, then the universal abelian cover of M is a rational homology sphere if and only if the splice diagram Γ satisfies the conditions of corollary 6.3.*

Proof. We only have to prove that the intersection matrix I, satisfies that $\det(I) \neq 0$. Note that the universal abelian cover of a singularity link is also a singularity link, and hence by Theorem 3.5, I is negative definite. Therefore $\det(I) \neq 0$. $\square$

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