# On the Mean Curvature Flow of Graphs of Symplectomorphisms of Kähler-Einstein Manifolds; Application to Complex Projective Spaces

Ivana Medoš

Submitted in partial fulfillment of the requirements

for the degree of Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2009

**Abstract**

On the Mean Curvature Flow of Graphs of Symplectomorphisms of Kähler-Einstein Manifolds; Application to Complex Projective Spaces

Ivana Medoš

The presented work is a study of the mean curvature flow of graphs of symplectomorphisms of Kähler-Einstein manifolds in general, and of complex projective spaces in particular. We establish properties of singular values of symplectic linear maps. Using these observations, we derive, in a general Kähler-Einstein setting, the evolution equation of the Jacobian of the projection from the graph of a symplectomorphism onto the domain manifold under the flow. Finally, we apply this result to the case when the domain and the image manifolds are complex projective spaces with the Fubini-Study metric: we formulate a pinching condition for the singular values of the initial symplectomorphism, sufficient for the flow to exist and converge to the graph of a biholomorphic isometry.

# Contents

**Acknowledgments**

This work would not have been possible without the supervision, guidance and support of my advisor, Prof. Mu-Tao Wang. I am thankful for the mathematical ideas to which he introduced me, as well as for his patience, generosity, kindness and attentiveness in all these years

Special thanks are also due to to Prof. Dusa McDuff for the exceptionally useful ideas and discussions, and the extraordinary support she provided on so many levels.

I am grateful to my dissertation committee members, Prof. Michael Weinstein, Prof. John Loftin and Prof. Daniela de Silva for the time, ideas and optimism they so generously shared, as well as to Prof. Panagiota Daskalopoulos and Prof. Richard Hamilton for their advice and encouragement.

My sincere appreciation goes to Terrance Cope, academic department administrator, and the administrative staff of the Department of Mathematics for facilitating the practical matters, and for doing it with such admirable efficiency and professionalism.

Finally, I would like to thank my parents and my sister for their enthusiasm and curiosity about my work, to Tristan for his unwavering patience and optimism, to Helena for sharing and understanding, and to Martin for adamantly defending the value of this endeavor.

To my grandparents, who are still in my thoughts.

# Chapter 1

# Introduction

On the space of immersed submanifolds of a Riemannian manifold, the area functional assigns to each immersion the area of its image in the ambient manifold. The gradient vector field of the area functional is the mean curvature vector field; it is of magnitude equal to the trace of the second fundamental form, and in the direction of the fastest decrease of the area of the immersed submanifold. Its associated flow – mean curvature flow – gives rise to an active area of study within the field of geometric flows. From the analytic point of view it represents a solution of a parabolic system of PDE. Its stationary cases are minimal surfaces.

It has been suggested by Wang [25] to apply mean curvature flow in the study of structure of the symplectomorphism groups of Kähler-Einstein manifolds. A diffeomorphism between Kähler manifolds is a symplectomorphism if and only if its graph is a Lagrangian submanifold in the product space of the domain and the image manifold. It follows from a result of Smoczyk [17] that when the domain and image manifolds are Kähler-Einstein, the property of being Lagrangian is preserved along mean curvature flow. Therefore if the flow of the graph of a symplectomorphism remains graphical, it necessarily remains a flow through symplectomorphisms. If it exists for all times and converges, it represents a symplectic isotopy between the initial symplectomorphism and the limit map.

This illustrates the applicability of methods arising from the study of geometric flows: analytic results about existence of solutions of certain PDE give rise, in a geometric context, to conclusions about the geometry and topology of the underlying spaces.

We approach the study of the mean curvature flow of graphs of symplectomorphisms of Kähler-Einstein manifolds by exploring conditions under which it remains graphical. We derive the evolution equations of certain geometric quantities along the flow, and apply these results to the case when the underlying manifolds are complex projective spaces with the Fubini-Study metric. We formulate a pinching condition for the initial symplectomorphism, under which the flow exists and remains graphical for all times. We further show that in that case it also converges to a biholomorphic isometry.

In the first chapter, we define the concepts. In the second one, we state the main results and discuss previous relevant research. In the third chapter, we provide the details of the proof. In the fourth chapter, we conclude the work by discussing possible generalizations of the result as well as certain technical details.

## 1.1 Mean curvature flow

Let $F : \Sigma \to \mathcal{M}$ be an immersion — a map of full rank at each point — of a smooth $n$-dimensional manifold $\Sigma$ into a smooth $m$-dimensional Riemannian manifold $\mathcal{M}$, where $m > n$. Let $\langle ., . \rangle$ denote the metric on $\mathcal{M}$.

**Definition 1.1.1** *The induced metric on $F(\Sigma) \subset \mathcal{M}$ is the symmetric positive-definite $(0,2)$-tensor field with coefficients:*

$$(g_0)_{ij} = \left\langle \frac{dF}{dx^i}, \frac{dF}{dx^j} \right\rangle$$

*with respect to the coordinate basis, where $x^i, i = 1, \ldots, n$, are the local coordinates on $\Sigma$, and $\frac{dF}{dx^i} \equiv DF(\frac{\partial}{\partial x^i})$.*

**Definition 1.1.2** *The second fundamental form of $F(\Sigma) \subset \mathcal{M}$ is the quadratic form on the tangent space $TF(\Sigma)$, with values in the normal bundle $NF(\Sigma)$, defined by:*

$$II(X,Y) = (\nabla^{\mathcal{M}}_X Y)^{\perp},$$

*for $X, Y \in TF(\Sigma)$.*

**Definition 1.1.3** *The mean curvature vector of $F(\Sigma)$ in $\mathcal{M}$ is the trace of the second fundamental form of $F(\Sigma)$:*

$$H = g_0^{ij} \left( \nabla^{\mathcal{M}}_{\frac{dF}{dx_i}} \frac{dF}{dx_j} \right)^{\perp},$$

*where $g_0^{ij}$ is the inverse matrix of the induced metric $(g_0)_{ij}$ on $F(\Sigma)$.*

**Remark 1.1.1** *$H$ represents a normal vector field to $F(\Sigma) \subset \mathcal{M}$ in the direction of the fastest decrease of the area of $F(\Sigma)$.*

Indeed, the volume of $F(\Sigma)$ is given by:

$$Vol(F(\Sigma)) = \int_{\Sigma} \sqrt{\det g_0} dx^1 \wedge \ldots \wedge dx^n. \tag{1.1.1}$$

Consider a family of immersions $F : \Sigma \times [0, S) \to \mathcal{M}$, $S > 0$. Then:

$$\begin{aligned}
&\frac{\partial}{\partial s} Vol(F(\Sigma, s)) \\
&= \int_{\Sigma} \frac{\sqrt{\det g_0}}{2} g_0^{ij} \left( \left\langle \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial s}, \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial F}{\partial x_i}, \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_j}} \frac{\partial F}{\partial s} \right\rangle \right) dx^1 \wedge \ldots \wedge dx^n \\
&= \int_{\Sigma} \frac{\sqrt{\det g_0}}{2} g_0^{ij} \left( \left\langle \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_i}} \left( \frac{\partial F}{\partial s} \right)^{\top}, \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial F}{\partial x_i}, \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_j}} \left( \frac{\partial F}{\partial s} \right)^{\top} \right\rangle \right) dx^1 \wedge \ldots \wedge dx^n \\
&+ \int_{\Sigma} \frac{\sqrt{\det g_0}}{2} g_0^{ij} \left( - \left\langle \left( \frac{\partial F}{\partial s} \right)^{\perp}, \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial x_j} \right\rangle - \left\langle \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_j}} \frac{\partial F}{\partial x_i}, \left( \frac{\partial F}{\partial s} \right)^{\perp} \right\rangle \right) dx^1 \wedge \ldots \wedge dx^n \\
&= \int_{\Sigma} \frac{\sqrt{\det g_0}}{2} g_0^{ij} \left( \left\langle \nabla^{\Sigma}_{\frac{\partial F}{\partial x_i}} \left( \frac{\partial F}{\partial s} \right)^{\top}, \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial F}{\partial x_i}, \nabla^{\Sigma}_{\frac{\partial F}{\partial x_j}} \left( \frac{\partial F}{\partial s} \right)^{\top} \right\rangle \right) dx^1 \wedge \ldots \wedge dx^n \\
&- \int_{\Sigma} \frac{\sqrt{\det g_0}}{2} g_0^{ij} \left\langle \left( \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_j}} \frac{\partial F}{\partial x_i} \right)^{\perp} + \left( \nabla^{\mathcal{M}}_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial x_j} \right)^{\perp}, \left( \frac{\partial F}{\partial s} \right)^{\perp} \right\rangle dx^1 \wedge \ldots \wedge dx^n \\
&= \int_{\Sigma} \operatorname{div} \left( \frac{\partial F}{\partial s} \right)^{\top} \sqrt{\det g_0} dx^1 \wedge \ldots \wedge dx^n
\end{aligned}$$

$$- \int_\Sigma \sqrt{\det g_0} \left\langle g_0^{ij} \left( \nabla^{\mathcal{M}}_{\frac{dF}{dx_i}} \frac{\partial F}{\partial x_j} \right)^\perp , \left( \frac{\partial F}{\partial s} \right)^\perp \right\rangle dx^1 \wedge \ldots \wedge dx^n$$
$$= - \int_\Sigma \left\langle H, \left( \frac{\partial F}{\partial s} \right)^\perp \right\rangle \sqrt{\det g_0} dx^1 \wedge \ldots \wedge dx^n.$$

**Definition 1.1.4** *The mean curvature flow of* $\Sigma$ *in* $\mathcal{M}$ *is a smooth family of immersions* $F : \Sigma \times [0, T) \to \mathcal{M}$*, for* $T > 0$*, satisfying:*

$$\left( \frac{\partial}{\partial t} F(x,t) \right)^\perp = H(F(x,t)). \tag{1.1.2}$$

**Remark 1.1.2** *In view of the discussion above, mean curvature flow is the gradient flow of the area functional* (1.1.1) *in the space of immersed submanifolds.*

**Remark 1.1.3** *The system of PDEs that defines mean curvature flow is equivalent to:*

$$\frac{\partial}{\partial t} F(p,t) = H(F(p,t))$$

*up to one-parameter family of diffeomorphisms of* $\Sigma$ *[2].*

To see why the statement is true, assume that $\tilde{F}$ is the mean curvature flow of $\Sigma$ in $\mathcal{M}$, and that $\phi : \Sigma \times [0, T) \to \Sigma$ is a family of diffeomorphisms such that:

$$D\tilde{F}(\phi(p,t), t) \frac{\partial \phi}{\partial t}(p,t) = - \left( \frac{\partial \tilde{F}}{\partial t}(\phi(p,t), t) \right)^\top .$$

Notice that, since $\tilde{F}$ is an immersion, $D\tilde{F}$ has a left inverse, so this system of ODEs reduces to:

$$\frac{\partial \phi}{\partial t}(p,t) = -(D\tilde{F}(\phi(p,t),t))^{-1}\left(\frac{\partial \tilde{F}}{\partial t}(\phi(p,t),t)\right)^{\top}.$$

The existence of such $\phi$ then follows from the theory of ODEs ([21], [2]).

Define $F(p,t) \equiv \tilde{F}(\phi(p,t),t)$. Then:

$$\begin{aligned}
\frac{\partial}{\partial t}F(p,t) &= D\tilde{F}(\phi(p,t),t)\frac{\partial \phi}{\partial t}(p,t) + \frac{\partial}{\partial t}\tilde{F}(\phi(p,t),t) \\
&= \left(\frac{\partial}{\partial t}\tilde{F}(\phi(p,t),t)\right)^{\perp} \\
&= H(\tilde{F}(\phi(p,t),t) \\
&= H(F(p,t)).
\end{aligned}$$

**Remark 1.1.4** *When $F$ satisfies the mean curvature flow equation* (1.1.2)*, then:*

$$\frac{d}{dt}d\mu_t = -|H|^2 d\mu_t, \tag{1.1.3}$$

*where $d\mu_t \equiv \sqrt{\det g_{ij}}dx^1 \wedge \ldots \wedge dx^n$ is the volume form of $\Sigma_t \equiv F(\Sigma,t)$ ([28]).*

This statement follows from the computation of the variation of volume functional above.

From the analytic point of view, mean curvature flow represents a second order quasilinear parabolic system of PDEs:

$$\frac{\partial F}{\partial t} = g^{ij}\left(\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \Gamma^A_{BC}\frac{\partial F^B}{\partial x^i}\frac{\partial F^C}{\partial x^j} - \tilde{\Gamma}^k_{ij}\frac{\partial F^A}{\partial x^k}\right)\frac{\partial}{\partial y^A},$$

where $x^i$, $i = 1, \ldots, n$, and $y^A$, $A = 1, \ldots, m$ are local coordinates on $\Sigma$ and $\mathcal{M}$ respectively, $\Gamma^A_{BC}$ are the Christoffel symbols of the metric on $\mathcal{M}$, and $\tilde{\Gamma}^k_{ij}$ are those of the induced metric on $\Sigma_t$ ([29]).

Short-time existence of a solution of such system of equations follows from the theory of second order quasilinear parabolic PDE [9]. Whether, and under what conditions, it exists for all times $t > 0$, and whether $\Sigma_t$ converge to a submanifold of $\mathcal{M}$ as $t \to \infty$, is the focus of research in this field.

## 1.2 Lagrangian mean curvature flow

A considerable amount of study has been done on the mean curvature flow of codimension-one surfaces ($m - n = 1$), but relatively little is known about higher codimension cases ($m - n > 1$). One promising area of research is the flow of Lagrangian submanifolds. They arise as submanifolds of symplectic manifolds, of codimension equal to their dimension (i.e. $m = 2n$).

**Definition 1.2.1** *Symplectic manifold is a pair $(M, \omega)$, where $M$ is a smooth manifold, and $\omega$ is a closed, non-degenerate 2-form on $M$. $\omega$ is called the symplectic form of $M$.*

Note: the nondegeneracy of symplectic form implies that the dimension of a symplectic manifold is always even.

**Definition 1.2.2** *Isotropic submanifold of a symplectic manifold is a submanifold on which the symplectic form vanishes.*

Due to nondegeneracy of $\omega$, an isotropic submanifold can be of dimension at most half that of the ambient manifold.

**Definition 1.2.3** *Lagrangian submanifold of a symplectic manifold is an isotropic submanifold of maximal dimension — half that of the ambient manifold.*

Lagrangian submanifolds are interesting because they satisfy the following important property.

**Remark 1.2.1** *When the ambient space is Kähler-Einstein, the property of being Lagrangian is preserved along mean curvature flow [17].*

Heuristically, the reasoning is the following: recall that a Kähler manifold $(\mathcal{M}, g, J)$ is Kähler-Einstein if the Ricci curvature form is a multiple of the symplectic form:

$$Ric = c\omega$$

for some constant $c$, where $\omega$ is the Kähler form of $\mathcal{M}$: $\omega(.,.) = g(., J.)$.

Now if $\Sigma$ is a submanifold of $\mathcal{M}$ with mean curvature vector $H$, let $\sigma = \omega(H, .)$. Then by the Codazzi equation:

$$d\sigma = Ric|_{\Sigma}.$$

If $\Sigma$ is a Lagrangian submanifold of $\mathcal{M}$, then:

$$d\sigma = Ric|_{\Sigma} = c\omega|_{\Sigma} = 0.$$

Thus $\sigma$ is closed. It follows that for $\omega' \equiv F^*\omega$:

$$\frac{d}{dt}\omega' = d(\omega'(H,.)) + (d\omega')(H,.) = 0,$$

(see [17]).

Two prominent classes of Lagrangian submanifolds are:

– graphs of symplectomorphisms

– graphs of one-forms

This work is focused on questions related to the first group. The geometrical setting is the following: if $(M,\omega)$ and $(\tilde{M},\tilde{\omega})$ are symplectic manifolds, the product space $M \times \tilde{M}$ with the form $\omega - \tilde{\omega}$ is also a symplectic manifold.

If $f : (M,\omega) \to (\tilde{M},\tilde{\omega})$ is a diffeomorphism, then the graph of $f$ is a Lagrangian submanifold of $(M \times \tilde{M}, \omega - \tilde{\omega})$ if and only if $f$ is a symplectomorphism. Indeed, for an arbitrary element $(X, df(X)) \in T\Sigma$, where $\Sigma$ is the graph of $f$:

$$(\omega - \tilde{\omega})(X, df(X)) = \omega(X) - \tilde{\omega}(df(X)) = \omega(X) - f^*\tilde{\omega}(X).$$

Thus:

$$(\omega - \tilde{\omega})|_{\Sigma} = 0 \Leftrightarrow \omega = f^*\tilde{\omega}.$$

Then from the discussion above, we can conclude the following.

**Remark 1.2.2** *If the mean curvature flow $\Sigma_t$ of the graph of a symplectomorphism $f$ remains graphical, at each $t$ it is necessary a graph of a symplectomorphism.*

*In that case, the flow represents a symplectic isotopy of the initial symplectomorphism $f$.*

The focus of this work is the mean curvature flow of graphs of symplectomorphisms of Kähler-Einstein manifolds. We derive the evolution equation of the Jacobian of the projection $\pi_1 : \Sigma_t \to M$ along mean curvature flow. We then apply it to the case when the underlying manifolds are complex projective spaces with the Fubini-Study metric, and we formulate an initial pinching condition that suffices for the flow to exist for all times $t > 0$ and remain graphical. Moreover, we combine these results with the evolution equation of the squared norm of the second fundamental form to show that, under the initial pinching assumption, the flow converges to a biholomorphic isometry.

# Chapter 2

# The Main Result and Previous Research

We develop a general method for addressing conditions under which the mean curvature flow of symplectomorphisms of Kähler-Einstein manifolds remains graphical, but the applications turn out to be particularly interesting in the case of complex projective spaces.

## 2.1 The main result

Recall that $\mathbb{CP}^n$ is compact and simply connected. The Fubini-Study metric $g$ makes it a Kähler-Einstein manifold with sectional curvature:

$$K(X,Y) = \frac{\frac{1}{4}||X \wedge Y||^2 + \frac{3}{4}\langle JX, Y\rangle^2}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

([10]), and Ricci curvature:

$$R_{ij} = \frac{n+1}{2} g_{ij}.$$

**Definition 2.1.1** *Let $\Lambda$ be a constant greater than 1. A symplectomorphism $f$ of a Riemannian symplectic manifold with metric $g$ is said to be $\Lambda$-pinched if*

$$\frac{1}{\Lambda} g \leq f^* g \leq \Lambda g. \tag{2.1.1}$$

Then the main result is the following.

**Theorem 1** *There exists a constant $\Lambda_0 > 1$, which depends only on $n$, such that, if $f : \mathbb{CP}^n \to \mathbb{CP}^n$ is a $\Lambda$-pinched symplectomorphism for any $\Lambda \in (1, \Lambda_0]$, then:*

*1) The mean curvature flow $\Sigma_t$ of the graph of $f$ in $\mathbb{CP}^n \times \mathbb{CP}^n$ exists smoothly for all $t \geq 0$.*

*2) $\Sigma_t$ is the graph of a symplectomorphism $f_t$ for each $t \geq 0$.*

*3)$f_t$ converges smoothly to a biholomorphic isometry of $\mathbb{CP}^n$ as $t \to \infty$.*

Therefore:

**Corollary 2.1.1** *There exists a constant $\Lambda_0 > 1$, which depends only on the dimension $n$, such that any $\Lambda$-pinched symplectomorphism $f : \mathbb{CP}^n \to \mathbb{CP}^n$, for $\Lambda \in (1, \Lambda_0]$, is symplectically isotopic to a biholomorphic isometry.*

## 2.2 Previous research

The main theorem generalizes a result by Wang ([23], [25]), who showed that the mean curvature flow of the graph of a symplectomorphism between compact Riemann surfaces of equal constant curvature exists for all times and converges to a minimal Lagrangian submanifold.

It also provides an alternative proof of a theorem of Smale [16] that states that the isometry group $SO(3)$ of $S^2 \simeq \mathbb{CP}^1$ is a deformation retract of the diffeomorphism group of $S^2$.

In the case of $\mathbb{CP}^2$, Gromov [4] has shown that its biholomorphic isometry group is a deformation retract of its symplectomorphism group. Using our method, we cannot omit the pinching assumption when $n = 2$. However, our results also apply to $\mathbb{CP}^n$ for $n > 2$, about which little had been previously known.

## 2.3 Outline of the proof

In Section 3.1 we make several observations about singular values of linear symplectic maps that prove important in the proof of the main results. Section 3.2 is devoted to deriving the evolution equation of the Jacobian of the projection $\pi_1 : \Sigma_t \to M$ along the mean curvature flow. By the Inverse Function Theorem, in order for the mean curvature flow to remain graphical, it suffices that this quantity remains positive. It is in this section that we then focus on the special case when the underlying manifolds are complex projective spaces, and establish the existence of a pinching constant that ensures that the Jacobian of the projection remains positive along the flow. In section 3.3 we prove, for the case of $\mathbb{CP}^n$, that the pinching condition is preserved along the flow. Long-time existence and convergence then follow from the standard blow-up analysis and asymptotic convergence results, as shown in Section 3.4 and 3.5.

# Chapter 3

# Proof of the Main Result

## 3.1 Singular values of symplectic linear maps between vector spaces

Let $(V, g)$ and $(\tilde{V}, \tilde{g})$ be $2n$-dimensional real inner product spaces, with almost complex structures $J$ and $\tilde{J}$, respectively, compatible with the corresponding inner products. Then $\omega(.,.) = g(J.,.)$, and $\tilde{\omega} = \tilde{g}(\tilde{J}.,.)$ are symplectic forms on $V$ and $\tilde{V}$. Recall that a linear map $L : (V, \omega) \to (\tilde{V}, \tilde{\omega})$ is said to be symplectic if $\omega(u, v) = \tilde{\omega}(L(u), L(v))$ for any $u, v \in V$.

For such $L$, we define $E : V \to \tilde{V}$ to be the map $E = L[L^*L]^{-\frac{1}{2}}$, where $L^* : \tilde{V} \to V$ is the adjoint operator of $L$ in the context of real inner product spaces.

In terms of adjoint operator, $L$ being symplectic is equivalent to:

$$L^*\tilde{J}L = J.$$

**Lemma 3.1.1** *$E$ is an isometry and it intertwines with $J$ and $\tilde{J}$, i.e.*

$$\tilde{J}E = EJ.$$

*In other words, $E$ is a symplectic isometry.*

*Proof:* $E$ is an isometry since:

$$\begin{aligned}
\tilde{g}(Eu, Ev) = \tilde{g}(L[L^*L]^{-\frac{1}{2}}u, L[L^*L]^{-\frac{1}{2}}v) &= g(L^*L[L^*L]^{-\frac{1}{2}}u, [L^*L]^{-\frac{1}{2}}v) \\
&= g([L^*L]^{\frac{1}{2}}u, [L^*L]^{-\frac{1}{2}}v) \\
&= g([L^*L]^{-\frac{1}{2}}[L^*L]^{\frac{1}{2}}u, v) \\
&= g(u, v)
\end{aligned}$$

for any $u, v \in V$.

Let $P = [L^*L]^{\frac{1}{2}}$, so that $E = LP^{-1}$. $-JP^{-1}J$ and $P$ are both positive definite ($-JP^{-1}J = J^{-1}P^{-1}J$ is positive definite since $P^{-1}$ is and since $J$ is an orthogonal operator), and, by the symplectic condition $L^*\tilde{J}L = J$, their squares are equal:

$$(-JP^{-1}J)^2 = -JL^{-1}(L^*)^{-1}J = -L^*\tilde{J}\tilde{J}L = P^2.$$

It follows that:

$$-JP^{-1}J = P.$$

By using the symplectic condition $L^*\tilde{J}L = J$ and the fact that $P = L^*LP^{-1}$, we obtain the desired result:

$$\begin{aligned}
-JP^{-1}J = P \Rightarrow -JP^{-1}J = L^*LP^{-1} &\Rightarrow -(L^*)^{-1}JP^{-1}J = LP^{-1} \\
&\Rightarrow -\tilde{J}LP^{-1}J = LP^{-1} \\
&\Rightarrow -\tilde{J}EJ = E.
\end{aligned}$$

Finally, the last equality implies $E^*\tilde{J}E = J$, so $E$ is in fact a symplectic isometry.

□

Let $(v_1, \ldots, v_{2n})$ be a basis of $V$ that diagonalizes $L^*L$. Since $L^*L$ is positive definite, it has the form:

$$L^*L = \begin{pmatrix} \lambda_1^2 & 0 & \ldots & & 0 \\ 0 & \lambda_2^2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{2n-1}^2 & 0 \\ 0 & & \ldots & 0 & \lambda_{2n}^2 \end{pmatrix}$$

with respect to this basis, for some $\lambda_i > 0$, $i = 1, \ldots, 2n$.

Then, by construction, $L(v_i) = \lambda_i E(v_i)$; in other words:

$$L = \begin{pmatrix} \lambda_1 & 0 & \dots & & 0 \\ 0 & \lambda_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{2n-1} & 0 \\ 0 & & \dots & 0 & \lambda_{2n} \end{pmatrix}$$

with respect to the bases $(v_1, \dots, v_{2n})$ and $(E(v_1), \dots, E(v_{2n}))$, and thus $\lambda_i$ are the singular values of $L$.

Note: Singular values of a linear map from one vector space to another are analogue to eigenvalues of a linear operator on a (single) vector space. Singular-value decomposition holds for any linear map $L$: there exist orthonormal bases of the domain and range of $L$ such that $L$ is diagonalized with respect to them. The diagonal values are unique.

**Lemma 3.1.2** *Let $\lambda_i$ be the singular values of $L$ and $v_i$ be the associated singular vectors, i.e. $L(v_i) = \lambda_i E(v_i)$. Then:*

$$(\lambda_i \lambda_j - 1) g(Jv_i, v_j) = 0.$$

*Proof:* By the symplectic condition and Lemma 3.1.1:

$$\begin{aligned} g(Jv_i, v_j) =& \tilde{g}(\tilde{J}L(v_i), L(v_j)) = \lambda_i \lambda_j \tilde{g}(\tilde{J}E(v_i), E(v_j)) \\ &= \lambda_i \lambda_j \tilde{g}(E(Jv_i), E(v_j)) \\ &= \lambda_i \lambda_j g(Jv_i, v_j). \end{aligned}$$

□

**Lemma 3.1.3** *If $\alpha$ is a singular value of $L$, then so is $\frac{1}{\alpha}$. Moreover, the singular values can be split into pairs whose product is 1: if $V(\alpha)$ denotes the subspace of singular vectors corresponding to a singular value $\alpha$, then*

$$\dim V(\alpha) = \dim V\left(\frac{1}{\alpha}\right),$$

*and $J$ restricts to an isomorphism between $V(\alpha)$ and $V\left(\frac{1}{\alpha}\right)$.*

*Proof:* The first statement is a consequence of Lemma 3.1.2. Indeed, let $(v_1, \ldots, v_{2n})$ be the basis described in the lemma. Then for each $i \in \{1, \ldots, 2n\}$ there exists some $j \in \{1, \ldots, 2n\}$ such that $\langle Jv_i, v_j \rangle \neq 0$ since $Jv_i$ is a nonzero vector. Then, by the lemma, it follows that $\lambda_i \lambda_j = 1$.

The second statement is trivial if $\alpha = 1$. Assume that $\alpha \neq 1$, and let $\dim V(\alpha) = k$, $\dim V\left(\frac{1}{\alpha}\right) = l$. Assume that $v_{i_1}, \ldots, v_{i_k}$ span $V(\alpha)$ (so that $\lambda_{i_1} = \ldots = \lambda_{i_k} = \alpha$). Then $Jv_{i_1}, \ldots, Jv_{i_k}$ belong to $V\left(\frac{1}{\alpha}\right)$. Indeed, by Lemma 3.1.2, $\lambda_i \lambda_j \neq 1 \Rightarrow \langle Jv_i, v_j \rangle = 0$ for all $i, j$. In other words, $Jv_i$ is orthogonal to each singular vector corresponding to a singular value not equal to $\frac{1}{\lambda_i}$. But $V = V(\alpha_1) \oplus \ldots \oplus V(\alpha_k)$, where $\alpha_1, \ldots, \alpha_k$ are distinct singular values of $L$, $k \leq 2n$, and thus $V = V(\frac{1}{\lambda_i}) \oplus V'$ where $V'$ is the subspace of singular vectors not corresponding to singular value $\frac{1}{\lambda_i}$. As stated above, $Jv_i$ is orthogonal to $V'$; it follows that $Jv_i \in V\left(\frac{1}{\lambda_i}\right)$.

Moreover, $Jv_{i_1}, \ldots, Jv_{i_k}$ are linearly independent because $v_{i_1}, \ldots, v_{i_k}$ are. It follows that $k \leq l$.

The same argument applies to $V\left(\frac{1}{\alpha}\right)$ as well: assume that $v_{j_1}, \ldots, v_{j_l}$ span it. Then $Jv_{j_1}, \ldots, Jv_{j_l}$ belong to $V(\alpha)$, and they are linearly independent. It follows that $k \geq l$.

We conclude that $k = l$, and that $J$ reduces to an isomorphism from $V(\alpha)$ to $V\left(\frac{1}{\alpha}\right)$.

□

**Remark 3.1.1** *The preceding lemma implies that $V$ splits into a direct sum of singular subspaces of the following form:*

$$V = V(1)^{k_0} \oplus V(\alpha_1)^{k_1} \oplus V\left(\frac{1}{\alpha_1}\right)^{k_1} \oplus \ldots \oplus V(\alpha_s)^{k_s} \oplus V\left(\frac{1}{\alpha_s}\right)^{k_s}, \quad (3.1.1)$$

*where $s+1$ is the total number of distinct singular values of $L$, $\alpha_i$ are distinct singular values of $L$ greater than 1, $i = 1, \ldots, s$, and the superscripts represent dimension, $k_0 \geq 0$, $k_j > 0$, for $j = 1, \ldots, s$.*

**Proposition 3.1.1** *Let $L : (V^{2n}, \omega) \to (\tilde{V}^{2n}, \tilde{\omega})$ be a symplectic linear map, where $V$ and $\tilde{V}$ are real vector spaces supplied with almost complex structures $J$ and $\tilde{J}$ and inner products $g$ and $\tilde{g}$ compatible with the complex structures; and where $\omega = g(J.,.)$, $\tilde{\omega} = \tilde{g}(\tilde{J}.,.)$.*
*Then there exists an orthonormal basis of $V$ with respect to which:*

$$J = \begin{pmatrix} 0 & -1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & -1 \\ 0 & \dots & 1 & 0 \end{pmatrix} \tag{3.1.2}$$

*and:*

$$L^*L = \begin{pmatrix} \lambda_1^2 & 0 & \dots & & 0 \\ 0 & \lambda_2^2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{2n-1}^2 & 0 \\ 0 & & \dots & 0 & \lambda_{2n}^2 \end{pmatrix}, \tag{3.1.3}$$

*where* $\lambda_{2i-1}\lambda_{2i} = 1$, *for* $i = 1, \dots, n$.

*Proof:* Lemma 3.1.3 and (3.1.1) imply that it is sufficient to find a basis satisfying (3.1.2) of the subspaces $V(\alpha) \oplus V(\frac{1}{\alpha})$ for each singular value $\alpha \neq 1$, as well as of $V(1)$ if 1 is a singular value of $L$.

Assume that there is a singular value $\alpha \neq 1$, and let $k = \dim V(\alpha)$. We choose an arbitrary basis $u_1, \dots, u_k$ of this space. Then $Ju_1, \dots, Ju_k$ is a basis of $V(\frac{1}{\alpha})$. Putting these bases together provides a basis of $V(\alpha) \oplus V(\frac{1}{\alpha})$ satisfying (3.1.2). Moreover, since $u_1, \dots, u_k$ are singular vectors of $L$ with singular velue $\alpha$, and $Ju_1, \dots, Ju_k$ are singular values of $L$ with singular value $\frac{1}{\alpha}$, it follows that $(u_1, Ju_1, u_2, Ju_2, \dots, u_k, Ju_k)$ is the desired basis.

If a singular value is equal to 1 (i.e. if $k_0 > 0$ in (3.1.1)), any basis of

$V(1)$ satisfying (3.1.2) suffices.

$\square$

Since the image of an orthonormal basis under an isometry is also an orthonormal basis, we obtain the following corollary.

**Corollary 3.1.1** *Let* $E : V \to \tilde{V}$ *be the isometry* $E = L[L^*L]^{-\frac{1}{2}}$. *If* $(a_1, \ldots, a_{2n})$ *is a basis of* $V$ *satisfying the properties of Proposition 3.1.1, and if* $(\tilde{a}_1, \ldots, \tilde{a}_{2n})$ *is the orthonormal basis* $(E(a_1), \ldots, E(a_{2n}))$ *of* $\tilde{V}$, *then:*

*(a)*

$$\tilde{J} = \begin{pmatrix} 0 & -1 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & -1 \\ 0 & \ldots & 1 & 0 \end{pmatrix}$$

*with respect to* $(\tilde{a}_1, \ldots, \tilde{a}_{2n})$;

*and:*

*(b) L is diagonalized with respect to these bases, with diagonal values ordered in pairs whose product is 1:*

$$L = \begin{pmatrix} \lambda_1 & 0 & \ldots & & 0 \\ 0 & \lambda_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{2n-1} & 0 \\ 0 & & \ldots & 0 & \lambda_{2n} \end{pmatrix}$$

*with* $\lambda_{2i-1}\lambda_{2i} = 1$, *for* $i = 1, \ldots, n$.

*Proof:* Part (a) follows from the Proposition 3.1.1 and Lemma 3.1.1. Part (b) follows from the fact that $L(a_i) = \lambda_i E(a_i)$.

$\square$

## 3.2 Evolution of symplectomorphisms of Kähler-Einstein manifolds under the mean curvature flow of their graphs

### 3.2.1 Geometric context

To prove the main theorem, we consider the evolution of the graph of $f$, $\Sigma \subset M \times \tilde{M}$, under mean curvature flow. Here $M \times \tilde{M}$ is the product space with product metric $G$. If $J$ and $\tilde{J}$ are almost complex structures of $M$ and $\tilde{M}$, respectively, then $\mathcal{J}(u, v) = (Ju, -\tilde{J}v)$ defines an almost complex structure on $M \times \tilde{M}$ parallel with respect to $G$. Let $\Sigma_t$ be the mean curvature flow of $\Sigma$ in $M \times \tilde{M}$.

Let $\Omega$ be the volume form of $M$ extended to $M \times \tilde{M}$ naturally (more precisely, let $\Omega$ be the pullback of the volume form of $M$ under the projection $\pi_1 : M \times \tilde{M} \to M$). Denote by $*\Omega$ the Hodge star of the restriction of $\Omega$ to $\Sigma_t$. At any point $q \in \Sigma_t$, $*\Omega(q) = \Omega(e_1, \ldots, e_{2n})$ for any oriented orthonormal basis of $T_q\Sigma$. $*\Omega$ is the Jacobian of the projection from $\Sigma_t$ onto $M$. The goal of this section is to prove that $*\Omega$ remains positive along the mean curvature flow. By the inverse function theorem, this implies that $\Sigma_t$ is a graph over $M$.

We will apply our result in the following section to choose a basis that simplifies the evolution equation of $*\Omega$.

Suppose $q \in \Sigma_t$ is of the form $q = (p, f(p))$ for $p \in M$ and $f(p) \in \tilde{M}$, and let $(a_1, \ldots, a_{2n})$ be the basis of $T_pM$ satisfying the properties listed in Proposition 3.1.1, for $L = Df_p : T_pM \to T_{f(p)}\tilde{M}$, with the inner products understood to be the metrics $g$ on $M$ at $p$ and $\tilde{g}$ on $\tilde{M}$ at $f(p)$. Define $E : T_pM \to T_{f(p)}\tilde{M}$ to be the isometry $E = Df_p[Df_p^* Df_p]^{-\frac{1}{2}}$ for $p \in M$. Let us also choose a basis of $T_{f(p)}\tilde{M}$ to be $(\tilde{a}_1, \ldots, \tilde{a}_{2n}) = (E(a_1), \ldots, E(a_{2n}))$, as per Corollary 3.1.1.

Then

$$e_i = \frac{1}{\sqrt{1 + |Df_p(a_i)|^2}}(a_i, Df_p(a_i)) = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i, \lambda_i E(a_i)) \tag{3.2.1}$$

and

$$\begin{aligned} e_{2n+i} = \mathcal{J}_{(x,f(x))} e_i &= \frac{1}{\sqrt{1+\lambda_i^2}}(J_p a_i, -\tilde{J}_{f(p)} \lambda_i E(a_i)) \\ &= \frac{1}{\sqrt{1+\lambda_i^2}}(J_p a_i, -\lambda_i E(J_p a_i)), \end{aligned} \tag{3.2.2}$$

for $i = 1, \ldots, 2n$, form an orthonormal basis of $T_q(M \times \tilde{M})$. By construction, $e_1, \ldots, e_{2n}$ span $T_q\Sigma$, and $e_{2n+1}, \ldots, e_{4n}$ span $N_q\Sigma$. In terms of this basis at each point $q \in \Sigma_t$:

$$*\Omega = \Omega(e_1, \ldots, e_{2n}) = \frac{1}{\sqrt{\prod_j (1+\lambda_j^2)}}.$$

The second fundamental form of $\Sigma_t$ is at each point $q \in \Sigma_t$ characterized by coefficients

$$h_{ijk} = G(\nabla^{M\times\tilde{M}}_{e_i} e_j, \mathcal{J} e_k). \tag{3.2.3}$$

Note that $h_{ijk}$ are completely symmetric with respect to $i, j, k$.

Before we prove the main result, we make a following remark.

**Remark 3.2.1** *A more general, but weaker result than Thereom 1 holds. Let $M$, $\tilde{M}$ and symplectomorphism $f$ satisfy the following property for a given constant $\Lambda > 1$:*

$$\sum_k \sum_{i\neq k} \frac{x_i}{(1+x_k^2)(x_i + x_i^{-1})}(R_{ikik} - x_k^2 \tilde{R}_{ikik}) \geq 0 \tag{3.2.4}$$

*whenever $\frac{1}{\sqrt{\Lambda}} \leq x_i \leq \sqrt{\Lambda}$, where $R_{ijkl} = R(a_i, a_j, a_k, a_l)$ and $\tilde{R}_{ijkl} = \tilde{R}(E(a_i), E(a_j), E(a_k), E(a_l))$ are the coefficients of curvature tensors on $M$ and $\tilde{M}$, respectively, with respect to the bases chosen as above.*

*Then the following general result holds.*

***Theorem 2*** *Let $\Sigma$ be the graph of a symplectomorphism $f : M \to M$ of a compact Kähler-Einstein manifold $M$. Then there exists a constant $\Lambda_0 > 1$,* *depending only on $n$, such that, when $f$ is $\Lambda$-pinched, for $\Lambda \in (1, \Lambda_0]$, and the property* (4.0.1) *is satisfied, then:*

*1) The mean curvature flow $\Sigma_t$ of the graph of $f$ in $M \times M$ exists smoothly for all $t \geq 0$.*

*2) $\Sigma_t$ is the graph of a symplectomorphism $f_t$ for each $t \geq 0$.*

*Unlike Theorem 1, this result does not establish convergence; the proof of convergence requires more refined curvature properties of $\mathbb{CP}^n$.*

In the rest of this section, we derive the evolution equation of $*\Omega$ under the mean curvature flow.

### 3.2.2 Evolution of $*\Omega$

**Proposition 3.2.1** *Let $\Sigma$ be the graph of a symplectomorphism $f : (M, \omega) \to (\tilde{M}, \tilde{\omega})$ between Kähler-Einstein manifolds $(M, g)$ and $(\tilde{M}, \tilde{g})$ of real dimension $2n$. At each point $q \in \Sigma_t$, $*\Omega$ satisfies the following equation:*

$$\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega\{Q(\lambda_i, h_{ijk}) + \sum_k \sum_{i \neq k} \frac{\lambda_i}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})}(R_{ikik} - \lambda_k^2 \tilde{R}_{ikik})\},$$

*where*

$$\begin{aligned} Q(\lambda_i, h_{ijk}) = & \sum_{i,j,k} h_{ijk}^2 - 2\sum_k \sum_{i \text{ odd}} (h_{iik}h_{i'i'k} - h_{ii'k}^2) \\ & - 2\sum_k \sum_{i<j} (-1)^{i+j} \lambda_i \lambda_j (h_{i'ik}h_{j'jk} - h_{i'jk}h_{j'ik}), \end{aligned} \tag{3.2.5}$$

*$R_{ijkl} = R(a_i, a_j, a_k, a_l)$ and $\tilde{R}_{ijkl} = \tilde{R}(E(a_i), E(a_j), E(a_k), E(a_l))$ are the coefficients of the curvature tensors $R$ and $\tilde{R}$ of $M$ and $\tilde{M}$ with respect to the chosen bases of $T_pM$ and $T_{f(p)}\tilde{M}$ as per Proposition 3.1.1 and Corollary 3.1.1, respectively, and $i' = i + (-1)^{i+1}$.*

Note: We use the following convention in defining Riemannian curvature tensor:

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Thus the covariant version is:

$$R(X,Y,Z,W) = \langle -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, W \rangle,$$

and sectional curvature is given by:

$$K(X,Y) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y \rangle^2}.$$

*Proof:* The evolution equation of $*\Omega$ under mean curvature flow is, by [24]:

$$\begin{aligned}
\frac{d}{dt} * \Omega &= \Delta * \Omega + *\Omega(\sum_{\imath,j,k} h_{ijk}^2) \\
&\quad - 2\sum_{p,q,k}\sum_{i<j} \Omega(e_1, \ldots, \mathcal{J}e_p, \ldots, \mathcal{J}e_q, \ldots, e_{2n}) h_{pik} h_{qjk} \\
&\quad - \sum_{p,k,i} \Omega(e_1, \ldots, \mathcal{J}e_p, \ldots, e_{2n}) \mathcal{R}(\mathcal{J}e_p, e_k, e_k, e_i) \\
&= \Delta * \Omega + \mathcal{A} + \mathcal{B},
\end{aligned}$$

where

$$\mathcal{A} = *\Omega(\sum_{\imath,j,k} h_{ijk}^2) - 2\sum_{p,q,k}\sum_{i<j}\Omega(e_1,\ldots,\mathcal{J}e_p,\ldots,\mathcal{J}e_q,\ldots,e_{2n})h_{pik}h_{qjk}$$

and

$$\mathcal{B} = -\sum_{p,k,i}\Omega(e_1,\ldots,\mathcal{J}e_p,\ldots,e_{2n})\mathcal{R}(\mathcal{J}e_p,e_k,e_k,e_i).$$

Here $\mathcal{R}$ is the curvature tensor of $M \times \tilde{M}$. All summation indices range from 1 to $2n$, unless stated otherwise.

Since $\Omega$ only picks up the $\pi_1$ projection part, and

$$\pi_1(\mathcal{J}e_p) = \frac{1}{\sqrt{1+\lambda_p^2}}Ja_p \tag{3.2.6}$$

by (3.2.1), it follows that:

$$\begin{aligned}\mathcal{A} = &* \Omega(\sum_{\imath,j,k} h_{ijk}^2)\\ &- 2(*\Omega)\sum_{p,q,k}\sum_{i<j}\frac{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}}\Omega(a_1,\ldots,Ja_p,\ldots,Ja_q,\ldots,a_{2n})h_{pik}h_{qjk}.\end{aligned}$$

Fixing $i < j$, we compute the term:

$$\Omega(a_1,\ldots,Ja_p,\ldots,Ja_q,\ldots,a_{2n}) =$$

$$= \Omega(a_1, \ldots, g(Ja_p, a_i)a_i + g(Ja_p, a_j)a_j, \ldots, g(Ja_q, a_i)a_i + g(Ja_q, a_j)a_j, \ldots, a_{2n})$$
$$= (J_{ip}J_{jq} - J_{jp}J_{iq}),$$

where $J_{rs} = g(Ja_s, a_r)$.

Now if $p = q$, the summation term in the second sum is 0. Therefore,

$$\mathcal{A} = *\Omega[\sum_{\imath,j,k} h_{ijk}^2 - 2\sum_k \sum_{p<q} \sum_{i<j} \frac{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}} (J_{ip}J_{jq} - J_{jp}J_{iq})(h_{pik}h_{qjk} - h_{pjk}h_{qik})].$$

The only cases when $J_{rs} \neq 0$ are when $r = s'$.

(Note that $(s')' = s$, $J_{ss'} = (-1)^s$, and $\lambda_s \lambda_{s'} = 1$.)

Therefore:

$$\begin{aligned}
\mathcal{A} = *\,\Omega[&\sum_{\imath,j,k} h_{ijk}^2 \\
&- 2\sum_k \sum_{i \text{ odd}} \frac{\sqrt{(1+\lambda_i^2)(1+\lambda_{i'}^2)}}{\sqrt{(1+\lambda_i^2)(1+\lambda_{i'}^2)}} (J_{ii}J_{i'i'} - J_{i'i}J_{ii'})(h_{iik}h_{i'i'k} - h_{ii'k}h_{i'ik}) \\
&- 2\sum_k \sum_{i<j} \frac{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}{\sqrt{(1+\lambda_{i'}^2)(1+\lambda_{j'}^2)}} (J_{ii'}J_{jj'} - J_{ji'}J_{ij'})(h_{i'ik}h_{j'jk} - h_{i'jk}h_{j'ik})] \\
= *\Omega[&\sum_{i,j,k} h_{ijk}^2 - 2\sum_k \sum_{i \text{ odd}} (h_{iik}h_{i'i'k} - h_{ii'k}^2) \\
&- 2\sum_k \sum_{i<j} (-1)^{i+j} \lambda_i \lambda_j (h_{i'ik}h_{j'jk} - h_{i'jk}h_{j'ik})].
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\mathcal{B} &= \sum_{p,k,i} \Omega(e_1 \ldots, \mathcal{J}e_p, \ldots, e_{2n}) \mathcal{R}(\mathcal{J}e_p, e_k, e_i, e_k) \\
&= *\Omega \sum_{p,k,i} \frac{\sqrt{1+\lambda_i^2}}{\sqrt{1+\lambda_p^2}} \Omega(a_1 \ldots, Ja_p, \ldots, a_{2n}) \mathcal{R}(\mathcal{J}e_p, e_k, e_i, e_k) \\
&= *\Omega \sum_{p,k,i} \frac{\sqrt{1+\lambda_i^2}}{\sqrt{1+\lambda_p^2}} g(Ja_p, a_i) \mathcal{R}(\mathcal{J}e_p, e_k, e_i, e_k) \\
&= *\Omega \sum_{k,i} \frac{\sqrt{1+\lambda_i^2}}{\sqrt{1+\lambda_{i'}^2}} (-1)^i \mathcal{R}(\mathcal{J}e_{i'}, e_k, e_i, e_k) \\
&= *\Omega \sum_{k,i} (-1)^i \lambda_i \mathcal{R}(\mathcal{J}e_{i'}, e_k, e_i, e_k) \\
&= *\Omega \sum_{k} \sum_{i \neq k} (-1)^i \lambda_i \mathcal{R}(\mathcal{J}e_{i'}, e_k, e_i, e_k).
\end{aligned}
$$

Denote by $R$ and $\tilde{R}$ the curvature tensors of $M$ and $\tilde{M}$, respectively. Then:

$$
\begin{aligned}
&\mathcal{R}(\mathcal{J}e_{i'}, e_k, e_i, e_k) \\
&= R(\pi_1(\mathcal{J}e_{i'}), \pi_1(e_k), \pi_1(e_i), \pi_1(e_k)) + \tilde{R}(\pi_2(\mathcal{J}e_{i'}), \pi_2(e_k), \pi_2(e_i), \pi_2(e_k)) \\
&= \frac{1}{(1+\lambda_k^2)\sqrt{(1+\lambda_i^2)(1+\lambda_{i'}^2)}} [R(Ja_{i'}, a_k, a_i, a_k) \\
&\qquad\qquad - \lambda_k^2 \lambda_i \lambda_{i'} R_2(\tilde{J}E(a_{i'}), E(a_k), E(a_i), E(a_k))] \\
&= \frac{1}{(1+\lambda_k^2)\sqrt{(1+\lambda_i^2)(1+\lambda_{i'}^2)}} [R(Ja_{i'}, a_k, a_i, a_k) \\
&\qquad\qquad - \lambda_k^2 \tilde{R}(E(Ja_{i'}), E(a_k), E(a_i), E(a_k))]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(1+\lambda_k^2)\sqrt{(1+\lambda_i^2)(1+\lambda_{i'}^2)}}[(-1)^i R(a_i,a_k,a_i,a_k) \\
&\qquad\qquad\qquad - (-1)^i \lambda_k^2 \tilde{R}(E(a_i),E(a_k),E(a_i),E(a_k))] \\
&= \frac{(-1)^i}{(1+\lambda_k^2)\sqrt{(1+\lambda_i^2)(1+\lambda_{i'}^2)}}(R_{ikik} - \lambda_k^2 \tilde{R}_{ikik}) \\
&= \frac{(-1)^i}{(1+\lambda_k^2)(\lambda_i+\lambda_{i'})}(R_{ikik} - \lambda_k^2 \tilde{R}_{ikik}).
\end{aligned}
$$

□

The ambient curvature term $\mathcal{B}$ can be further simplified when $M \simeq \tilde{M} \simeq \mathbb{CP}^n$.

**Corollary 3.2.1** *If $M \simeq \mathbb{CP}^n$ and $\tilde{M} \simeq \mathbb{CP}^n$, and the metric on each manifold is Fubini-Study, then:*

$$
\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega \left[ Q(\lambda_i, h_{ijk}) + \sum_{k\ odd} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} \right].
$$

*Proof:* On $\mathbb{CP}^n$ with Fubini-Study metric, the sectional curvature is: $K(X,Y) = \frac{\frac{1}{4}(||X \wedge Y||^2 + 3\langle JX, Y\rangle^2)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}$ ([10]). Therefore, with respect to the chosen orthonormal bases of $T_xM$ and $T_{f(x)}\tilde{M}$, the sectional curvatures $K$ and $\tilde{K}$ of $M$ and $\tilde{M}$ are:

$$
K(a_i, a_{i'}) = 1 \text{ and } K(a_r, a_s) = K(a_s, a_r) = \frac{1}{4}
$$

for all other $r, s$, and

$$\tilde{K}(E(a_i), E(a_{i'})) = 1 \text{ and } \tilde{K}(E(a_r), E(a_s)) = K(E(a_s), E(a_r)) = \frac{1}{4}$$

for all other $r, s$.

Therefore:

$$R_{ikik} = K(a_i, a_k) = \frac{1}{4}(1 + 3\delta_{ik'}),$$

$$\tilde{R}_{ikik} = \tilde{K}(E(a_i), E(a_k)) = \frac{1}{4}(1 + 3\delta_{ik'})$$

and:

$$\mathcal{R}(\mathcal{J}e_{i'}, e_k, e_i, e_k) = \frac{(-1)^i}{4} \frac{1 - \lambda_k^2}{(1 + \lambda_k^2)(\lambda_i + \lambda_{i'})}(1 + 3\delta_{ik'})$$

for any $i, k$. Plugging this into the sum above, we obtain:

$$\begin{aligned}
\mathcal{B} &= \frac{*\Omega}{4} \sum_k \sum_{i \neq k} \frac{\lambda_i(1 - \lambda_k^2)}{(1 + \lambda_k^2)(\lambda_i + \lambda_{i'})}(1 + 3\delta_{ik'}) \\
&= *\Omega \sum_k \frac{\lambda_{k'}(1 - \lambda_k^2)}{(1 + \lambda_k^2)(\lambda_k + \lambda_{k'})} + \frac{*\Omega}{4} \sum_k \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \sum_{i \neq k, k'} \frac{\lambda_i}{\lambda_i + \lambda_{i'}} \\
&= *\Omega \sum_k \frac{\lambda_{k'} - \lambda_k}{(1 + \lambda_k^2)(\lambda_k + \lambda_{k'})} + \frac{*\Omega}{4} \sum_k \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \sum_{i \text{ odd} \neq k, k'} \frac{\lambda_i + \lambda_{i'}}{\lambda_i + \lambda_{i'}} \\
&= *\Omega \sum_k \frac{\lambda_{k'} - \lambda_k}{\lambda_k(\lambda_k + \lambda_{k'})^2} + (n-1)\frac{*\Omega}{4} \sum_k \frac{\lambda_k(\lambda_{k'} - \lambda_k)}{\lambda_k(\lambda_{k'} + \lambda_k)} \\
&= *\Omega \sum_{k \text{ odd}} \left( \frac{\lambda_{k'} - \lambda_k}{\lambda_k(\lambda_k + \lambda_{k'})^2} + \frac{\lambda_k - \lambda_{k'}}{\lambda_{k'}(\lambda_k + \lambda_{k'})^2} \right) + (n-1)\frac{*\Omega}{4} \sum_k \frac{\lambda_{k'} - \lambda_k}{\lambda_{k'} + \lambda_k} \\
&= *\Omega \sum_{k \text{ odd}} \frac{\lambda_k - \lambda_{k'}}{\lambda_k \lambda_{k'}(\lambda_k + \lambda_{k'})^2}(\lambda_k - \lambda_{k'}) + (n-1)\frac{*\Omega}{4} \sum_{k \text{ odd}} \left( \frac{\lambda_{k'} - \lambda_k}{\lambda_{k'} + \lambda_k} + \frac{\lambda_k - \lambda_{k'}}{\lambda_{k'} + \lambda_k} \right)
\end{aligned}$$

$$= *\Omega \sum_{k \text{ odd}} \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})^2}$$
$$= *\Omega \sum_{k \text{ odd}} \frac{(1 - \lambda_k^2)^2}{(1 + \lambda_k^2)^2}.$$

□

Note: In this case $\mathcal{B} \geq 0$, with equality holding if and only if all the singular values of $f$ are equal (and thus necessarily equal to 1).

Moreover, $\frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} < 1$, so $\mathcal{B} < n(*\Omega) \leq \frac{n}{2^n}$.

We notice that $Q(\lambda_i, h_{ijk})$ is a quadratic form in $h_{ijk}$. In the next lemma, we rewrite it.

**Lemma 3.2.1**

$$Q(\lambda_i, h_{ijk}) = \sum_{i,j,k} h_{ijk}^2 - 2\sum_k \sum_{i \ odd} (h_{iik}h_{i'i'k} - h_{ii'k}^2)$$
$$- 2\sum_k \sum_{i \ odd < j \ odd} (\lambda_i - \lambda_{i'})(\lambda_j - \lambda_{j'})h_{i'ik}h_{j'jk}$$
$$- 2\sum_k \sum_{i \ odd < j \ odd} [-(\lambda_i\lambda_j + \lambda_{i'}\lambda_{j'})h_{i'jk}h_{j'ik} + (\lambda_{i'}\lambda_j + \lambda_i\lambda_{j'})h_{ijk}h_{j'i'k}].$$

**Lemma 3.2.2** *The smallest eigenvalue of $Q((1,\ldots,1), h_{ijk})$ is $3 - \sqrt{5}$.*

*Proof:* When $\lambda_i = 1$ for all $i$, $Q$ splits into smaller quadratic forms:

$$Q = \sum_i (h_{iii}^2 + 5h_{ii'i'}^2 - 2h_{iii}h_{ii'i'})$$
$$+ \sum_i \sum_{j \text{ odd} \neq i,i'} (3h_{ijj}^2 + 3h_{ij'j'}^2 + 8h_{i'jj'}^2 - 2h_{ijj}h_{ij'j'} + 4h_{ij'j'}h_{i'jj'} - 4h_{ijj}h_{i'jj'})$$

$$+ \sum_{i \text{ odd} < j \text{ odd} < k \text{ odd}} (6h_{ijk}^2 + 6h_{i'j'k}^2 + 6h_{i'jk'}^2 + 6h_{ij'k'}^2$$

$$- 4h_{ijk}h_{i'j'k} - 4h_{ijk}h_{i'jk'} - 4h_{ijk}h_{ij'k'} + 4h_{i'j'k}h_{i'jk'} + 4h_{i'j'k}h_{ij'k'} + 4h_{i'jk'}h_{ij'k'})$$

$$+ \sum_{i \text{ odd} < j \text{ odd} < k \text{ odd}} (6h_{i'j'k'}^2 + 6h_{ijk'}^2 + 6h_{ij'k}^2 + 6h_{i'jk}^2$$

$$- 4h_{i'j'k'}h_{ijk'} - 4h_{i'j'k'}h_{ij'k} - 4h_{i'j'k'}h_{i'jk} + 4h_{ijk'}h_{ij'k} + 4h_{ijk'}h_{i'jk} + 4h_{ij'k}h_{i'jk}).$$

The smallest eigenvalue of the quadratic form within the first sum is $3-\sqrt{5}$; of the quadratic form within the second sum (when $n \geq 2$) it is 2; and of the quadratic term in the third, as well as the fourth sum (when $n \geq 3$)it is 4.

These quadratic forms do not overlap, so the smallest eigenvalue of $Q$ is $3-\sqrt{5}$ (for all $n$).

$\square$

**Proposition 3.2.2** *Let $Q(\lambda_i, h_{jkl})$ be the quadratic form defined in Proposition 3.2.1. In each dimension $n$, there exist $\Lambda_0 > 1$ such that $Q(\lambda_i, h_{jkl})$ is non-negative whenever $\frac{1}{\sqrt{\Lambda_0}} \leq \lambda_i \leq \sqrt{\Lambda_0}$ for $i = 1, \ldots, 2n$. Moreover, for any $1 \leq \Lambda_1 < \Lambda_0$, there exists a $\delta > 0$ such that*

$$Q(\lambda_i, h_{jkl}) \geq \delta \sum_{i,j,k} h_{ijk}^2$$

*whenever $\frac{1}{\sqrt{\Lambda_1}} \leq \lambda_i \leq \sqrt{\Lambda_1}$ for $i = 1, \ldots, 2n$.*

*Proof:* By Lemma 3.2.2, $Q((1, \cdots, 1), h_{ijk})$ is a positive definite quadratic form in $h_{ijk}$. Since being a positive definite matrix is an open condition, there is an open neighborhood $U$ of $(\lambda_1, \ldots, \lambda_{2n}) = (1, \cdots, 1)$ such that $(\lambda_1, \ldots, \lambda_{2n}) \in U$ implies $Q(\lambda_i, h_{ijk})$ is positive definite. Let $\delta_{\vec{\lambda}}$ be the smallest eigenvalue of $Q$ at $\vec{\lambda} \equiv (\lambda_1, \ldots, \lambda_{2n})$. Note that $\delta_{\vec{\lambda}}$ is a continuous function in $\vec{\lambda}$ and set

$$\delta_\Lambda = \min\{\delta_{\vec{\lambda}} \,|\, \vec{\lambda} = (\lambda_1, \ldots, \lambda_{2n}) \text{ and } \frac{1}{\sqrt{\Lambda}} \leq \lambda_i \leq \sqrt{\Lambda} \text{ for } i = 1, \ldots, 2n\}.$$

Thus $\delta_1 = 3 - \sqrt{5}$. $\Lambda_0$ defined by

$$\Lambda_0 \equiv \sup\{\Lambda \,|\, \Lambda \geq 1 \text{ and } \delta_\Lambda \geq 0\}$$

has the desired property.

□

**Remark 3.2.2** *When $n = 1$: $\Lambda_0 = \infty$.*

*When $n = 2$: $\Lambda_0 = \frac{2}{5}\sqrt{10} + \frac{1}{5}\sqrt{15}$ (see Chapter 4).*

*In general, $\Lambda_0$ is computable.*

**Corollary 3.2.2** *Suppose $M$ and $\tilde{M}$ are both $\mathbb{CP}^n$, with Fubini-Study metrics, and let $|II|$ denote the norm of the second fundamental form of the symplectomorphism $f$. There exist constants $\Lambda_0 > 1$, depending only on $n$, such that for any $\Lambda_1$, $1 \leq \Lambda_1 < \Lambda_0$ there exists a $\delta > 0$ with*

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq \delta * \Omega |II|^2 + *\Omega \sum_{k\ odd} \frac{(1 - \lambda_k^2)^2}{(1 + \lambda_k^2)^2}, \tag{3.2.7}$$

*whenever* $\frac{1}{\sqrt{\Lambda_1}} \leq \lambda_i \leq \sqrt{\Lambda_1}$ *for every* $i$.

Recall that the norm of the second fundamental form is:

$$|\mathrm{II}| = \sqrt{\sum_{i,j,k,l} G^{ik}G^{jl}G(\mathrm{II}(w_i, w_j), \mathrm{II}(w_k, w_l))}$$
$$= \sqrt{\sum_{i,j,k,l,r,s} G^{ik}G^{jl}G^{rs}G(\nabla^{M\times\tilde{M}}_{w_i} w_j, \mathcal{J}w_r)G(\nabla^{M\times\tilde{M}}_{w_k} w_l, \mathcal{J}w_s)}$$

with respect to an arbitrary basis $w_1, \ldots, w_{2n}$ of $T_q\Sigma$ with $G_{ij} = G(w_i, w_j)$ and $G^{ij} = (G_{ij})^{-1}$. By (3.2.3),

$$|\mathrm{II}| = \sqrt{\sum_{i,j,k} h^2_{ijk}}$$

for the chosen basis (3.2.1).

*Proof:* The result follows from Corollary 3.2.1 and Proposition 3.2.2.

□

## 3.3 Preservation of the pinching condition under the mean curvature flow

Short-time existence of the mean curvature flow in question is guaranteed by the theory of parabolic PDE. In order to establish long-time existence and convergence, we need to show that, when an appropriate pinching holds initially, $f$ remains $\Lambda_0$-pinched along the flow, as well as that $*\Omega$ satisfies the differential inequality (3.2.7) along the flow, and that $\min_{\Sigma_t} *\Omega$ is non-decreasing in time.

To show this, we make several observations:

We consider $\frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}}$, for $\lambda_i > 0$, $\lambda_i \lambda_{i'} = 1$, where $i' = i + (-1)^{i+1}$, $i = 1, \ldots, 2n$ (in other words, $\lambda_{2k-1}\lambda_{2k} = 1$ for $k = 1, \ldots, n$). It can be rewritten as:

$$\frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}} = \frac{1}{\prod_i (\lambda_i + \lambda_{i'})}.$$

This expression always has an upper bound: $\lambda_i \lambda_{i'} = 1$ implies that $\lambda_i + \lambda_{i'} \geq 2$, so

$$\frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}} \leq \frac{1}{2^n},$$

with equality if and only if $\lambda_i = 1$ for all $i$.

If $\lambda_i$ are bounded, it also has a positive lower bound.

**Lemma 3.3.1** *If $\frac{1}{\sqrt{\Lambda}} \leq \lambda_i \leq \sqrt{\Lambda}$ for all $i$, where $\Lambda > 1$, then:*

$$\frac{1}{2^n} - \epsilon \leq \frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}},$$

*where $\epsilon = \frac{1}{2^n} - \frac{1}{(\sqrt{\Lambda} + \frac{1}{\sqrt{\Lambda}})^n} > 0$.*

*Proof:* The function $x + \frac{1}{x}$ is decreasing when $x > 1$. Therefore if

$\frac{1}{\sqrt{\Lambda}} \leq \lambda_i \leq \sqrt{\Lambda}$ for all $i$, where $\Lambda > 1$, then:

$$\lambda_i + \lambda_{i'} \leq \sqrt{\Lambda} + \frac{1}{\sqrt{\Lambda}}.$$

It follows that:

$$\frac{1}{2^n} - \epsilon \leq \frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}} \leq \frac{1}{2^n},$$

where $\epsilon = \frac{1}{2^n} - \frac{1}{(\sqrt{\Lambda}+\frac{1}{\sqrt{\Lambda}})^n}$.

(This bound is sharp: when $\lambda_i = \sqrt{\Lambda}$ for all odd $i$, and $\lambda_i = \frac{1}{\sqrt{\Lambda}}$ for all even $i$, $\frac{1}{2^n} - \epsilon = \frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}}$, so no better bound on $\frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}}$ holds.) □

Also, a positive lower bound on $\frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}}$ implies a bound on $\lambda_i$.

**Lemma 3.3.2** *If* $\frac{1}{2^n} - \epsilon \leq \frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}}$, *where* $0 < \epsilon < \frac{1}{2^n}$, *then:*

$$\frac{1}{\sqrt{\Lambda'}} \leq \lambda_i \leq \sqrt{\Lambda'}$$

*for all* $i = 1, \ldots, 2n$, *where* $\Lambda' = \left( \frac{\frac{1}{2^n}}{\frac{1}{2^n}-\epsilon} + \sqrt{\left(\frac{\frac{1}{2^n}}{\frac{1}{2^n}-\epsilon}\right)^2 - 1} \right)^2 > 1.$

*Proof:* If:

$$\frac{1}{2^n} - \epsilon \leq \frac{1}{\sqrt{\prod_i (1+\lambda_i^2)}} = \frac{1}{\prod_i (\lambda_i + \lambda_{i'})},$$

then:

$$\prod_i (\lambda_i + \lambda_{i'}) \leq \frac{2^n}{1 - 2^n \epsilon},$$

and thus:

$$\lambda_i + \lambda_{i'} \leq \frac{2^n}{(1 - 2^n \epsilon) \prod_{j \neq i} (\lambda_j + \lambda_{j'})}$$

for each $i$.

Since $\lambda_j + \lambda_{j'} \geq 2$ for each $j$, the inequality implies:

$$\lambda_i + \lambda_{i'} \leq 2 \frac{\frac{1}{2^n}}{\frac{1}{2^n} - \epsilon}$$

Since $\lambda_i \lambda_{i'} = 1$, it follows that:

$$\frac{1}{\sqrt{\Lambda'}} \leq \lambda_i \leq \sqrt{\Lambda'},$$

where $\Lambda' = \left( \frac{\frac{1}{2^n}}{\frac{1}{2^n} - \epsilon} + \sqrt{\left( \frac{\frac{1}{2^n}}{\frac{1}{2^n} - \epsilon} \right)^2 - 1} \right)^2$.

(This bound is sharp: when $\frac{1}{2^n} - \epsilon = \frac{1}{\sqrt{\prod_i (1 + \lambda_i^2)}}$, one possibility is that $\lambda_1 = \sqrt{\Lambda'}$, $\lambda_2 = \frac{1}{\sqrt{\Lambda'}}$, and $\lambda_3 = \ldots = \lambda_{2n} = 1$, so no better bound on $\lambda_i$ holds.)

$\square$

**Proposition 3.3.1** *Let $\Sigma_t$ be the graph at time $t$ of the mean curvature flow of the graph $\Sigma$ of a symplectomorphism $f : M \to \tilde{M}$, where $M \simeq \tilde{M} \simeq \mathbb{CP}^n$ with Fubini-Study metric. Let $*\Omega$ be the Jacobian of the projection*

$\pi_1 : \Sigma_t \to M$. *Let* $\Lambda_0$ *be the constants characterized by Proposition 3.2.2.*

*If* $*\Omega$ *has the initial lower bound:*

$$\frac{1}{2^n} - \epsilon \leq *\Omega$$

*for* $\epsilon = \frac{1}{2^n}\left(1 - \frac{2}{\sqrt{\Lambda_1}+\frac{1}{\sqrt{\Lambda_1}}}\right)$, *for some* $1 < \Lambda_1 < \Lambda_0$, *then* $*\Omega$ *satisfies:*

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq \delta * \Omega |II|^2 + *\Omega \sum_{k\ odd} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}$$

*along the mean curvature flow, where* $\delta$ *is given in Proposition 3.2.2. In particular,* $\min_{\Sigma_t} *\Omega$ *is nondecreasing as a function in* $t$.

*Proof:* Assume that the mean curvature flow exists for $t \in [0, T)$, for some $T > 0$ (possibly $\infty$). If initially $\frac{1}{2^n} - \epsilon \leq *\Omega$ for $\epsilon = \frac{1}{2^n}\left(1 - \frac{2}{\sqrt{\Lambda_1}+\frac{1}{\sqrt{\Lambda_1}}}\right)$, then, by Lemma 3.3.2, $f$ is $\Lambda_1$-pinched. That in turn implies that $*\Omega$ initially satisfies inequality (3.2.7), and thus:

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq 0. \tag{3.3.1}$$

Let $T' \in [0, T)$ be a time such that $*\Omega$ satisfies (3.3.1) for all times $t \in [0, T']$. Then, by the maximum principle, $\min_{\Sigma_t} *\Omega$ is nondecreasing for $t \in [0, T']$. Therefore the lower bound of $*\Omega$ is preserved for $t \in [0, T']$, and consequently $\Lambda_0$-pinching as well. It then follows by Corollary 3.2.2 that in fact $*\Omega$ satisfies inequality (3.2.7) for all $t \in [0, T']$.

In other words, $*\Omega$ satisfies the statement of this proposition as long as (3.3.1) is satisfied. Therefore what is left to verify is that (3.3.1) holds along the mean curvature flow (i.e. that the above is true for all $T' \in [0, T)$).

Assume the opposite: that there is a time between 0 and $T$ for which (3.3.1) does not hold. Let $T'$ be the maximum time such that (3.3.1) holds for all $t \in [0, T']$. Then there exists a point $(x', T') \in \Sigma_{T'}$ and time $T'' \in (T', T)$ such that:

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq 0$$

at $(x', T')$, and:

$$\left(\frac{d}{dt} - \Delta\right) * \Omega < 0$$

at $(x', t)$, for all $t \in (T', T'')$. From the discussion above it follows that $\Lambda_0$-pinching holds at $(x', T')$. On the other hand, $\left(\frac{d}{dt} - \Delta\right) * \Omega < 0$ implies that $Q(\lambda_i, h_{jkl}) < -\sum_{k \text{ odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} \leq 0$ for all $t \in (T', T'')$ (Corollary 3.2.1). In other words, $\max_i \lambda_i \leq \sqrt{\Lambda_1}$ at $(x', T')$, and $\max_i \lambda_i \geq \sqrt{\Lambda_0}$ at $(x', t)$, $t \in (T', T'')$. But $\max_i \lambda_i$ is a continuous function. Contradiction.

□

**Corollary 3.3.1** *If the initial symplectomorphism $f$ is $\Lambda_1$-pinched, for*

$$\Lambda_1 = \left( \left[ \frac{1}{2} \left( \sqrt{\Lambda_0} + \frac{1}{\sqrt{\Lambda_0}} \right) \right]^{\frac{1}{n}} + \sqrt{ \left[ \frac{1}{2} \left( \sqrt{\Lambda_0} + \frac{1}{\sqrt{\Lambda_0}} \right) \right]^{\frac{2}{n}} - 1 } \right)^2 < \Lambda_0,$$

*then it is also $\Lambda_0$-pinched, and $\Lambda_0$-pinching is preserved along the mean curvature flow.*

*Proof:* Note that:

$$\frac{1}{2}\left(\sqrt{\Lambda_1}+\frac{1}{\sqrt{\Lambda_1}}\right)=\left(\frac{1}{2}\left(\sqrt{\Lambda_0}+\frac{1}{\sqrt{\Lambda_0}}\right)\right)^{\frac{1}{n}}<\frac{1}{2}\left(\sqrt{\Lambda_0}+\frac{1}{\sqrt{\Lambda_0}}\right).$$

The last inequality holds since $\sqrt{\Lambda_0}+\frac{1}{\sqrt{\Lambda_0}}>2$. Since $\Lambda_0>1$ and $\Lambda_1>1$, it follows that $\Lambda_1<\Lambda_0$. Thus the initial $\Lambda_1$-pinching implies initial $\Lambda_0$-pinching.

For times $t>0$, $\Lambda_1$-pinching implies, by Lemma 3.3.1, that $*\Omega$ has initial lower bound:

$$\frac{1}{2^n}-\epsilon\leq *\Omega$$

for $\epsilon=\frac{1}{2^n}\left(1-\frac{2}{\sqrt{\Lambda_0}+\frac{1}{\sqrt{\Lambda_0}}}\right)$. Then, by Proposition 3.3.1, the lower bound is preserved. Lemma 3.3.2 then implies that $\Lambda_0$-pinching is preserved along the flow.

□

## 3.4 Long-time existence of the mean curvature flow

To prove long-time existence of the flow, we follow the method of Wang [24]. We isometrically embed $M\times\tilde{M}$ into $\mathbb{R}^N$. The mean curvature flow equation in terms of the coordinate function $F(x,t)\in\mathbb{R}^N$ is:

$$\frac{d}{dt}F(x,t)=H=\bar{H}+U,$$

where $H \in T(M \times \tilde{M})/T\Sigma$ is the mean curvature vector of $\Sigma_t$ in $M$, $\bar{H} \in T\mathbb{R}^N/T\Sigma$ is the mean curvature vector of $\Sigma_t$ in $\mathbb{R}^N$, and $U = -II_{M\times\tilde{M}}(e_a, e_a)$. Indeed:

$$\begin{aligned} H = \pi^{M\times\tilde{M}}_{N\Sigma}(\nabla^{M\times\tilde{M}}_{e_a} e_a) &= \nabla^{M\times\tilde{M}}_{e_a} e_a - \nabla^{\Sigma}_{e_a} e_a \\ &= \nabla^{\mathbb{R}^N}_{e_a} e_a - \pi^{\mathbb{R}^N}_{N(M\times\tilde{M})}(\nabla^{\mathbb{R}^N}_{e_a} e_a) - \nabla^{\Sigma}_{e_a} e_a \\ &= \nabla^{\mathbb{R}^N}_{e_a} e_a - \nabla^{\Sigma}_{e_a} e_a - II_{M\times\tilde{M}}(e_a, e_a) \\ &= \pi^{\mathbb{R}^N}_{N\Sigma}(\nabla^{\Sigma}_{e_a} e_a) - II_{M\times\tilde{M}}(e_a, e_a) \\ &= \bar{H} + U, \end{aligned}$$

Note that $U$ is bounded since $M$ and $\tilde{M}$ are assumed to be compact.

Following [24], we assume that there is a singularity at $(y_0, t_0) \in \Sigma \subset \mathbb{R}^N$. Huisken [6] introduced the $(2n)-$dimensional backward heat kernel $\rho_{y_0,t_0}$ at $(y_0, t_0)$:

$$\rho_{y_0,t_0}(y,t) = \frac{1}{4\pi(t_0 - t)^n} \exp\left(\frac{-|y - y_0|^2}{4(t_0 - t)}\right).$$

Let $d\mu_t$ denote the volume form of $\Sigma_t$. By Huisken's monotonicity formula, $\lim_{t\to t_0} \int \rho_{y_0,t_0} d\mu_t$ exists.

**Lemma 3.4.1** *The limit* $\lim_{t\to t_0} \int (1 - *\Omega)\rho_{y_0,t_0} d\mu_t$ *exists and:*

$$\frac{d}{dt}\int (1 - *\Omega)\rho_{y_0,t_0} d\mu_t \leq C - \delta \int *\Omega |II|^2 \rho_{y_0,t_0} d\mu_t$$

*for some constant $C > 0$.*

*Proof:* By [26]:

$$\frac{d}{dt}\rho_{y_0,t_0} = -\Delta\rho_{y_0,t_0} - \rho_{y_0,t_0}\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{F^\perp\cdot\bar{H}}{t_0-t} + \frac{F^\perp\cdot U}{2(t_0-t)}\right),$$

where $F^\perp \in T\mathbb{R}^N/T\Sigma_t$ is the orthogonal component of $F \in T\mathbb{R}^N$.

Using Equation (1.1.3):

$$\frac{d}{dt}d\mu_t = -|H|^2 d\mu_t = -\bar{H}\cdot(\bar{H}+U)d\mu_t.$$

Combining these results, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t \leq \int[\Delta(1-*\Omega) - \delta * \Omega|II|^2]\rho_{y_0,t_0}d\mu_t\\
&-\int(1-*\Omega)\left[\Delta\rho_{y_0,t_0} + \rho_{y_0,t_0}\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{F^\perp\cdot\bar{H}}{t_0-t} + \frac{F^\perp\cdot U}{2(t_0-t)}\right)\right]\\
&-\int(1-*\Omega)[\bar{H}\cdot(\bar{H}+U)]\rho_{y_0,t_0}d\mu_t\\
&=\int[\Delta(1-*\Omega)\rho_{y_0,t_0} - (1-*\Omega)\Delta\rho_{y_0,t_0}]d\mu_t - \delta\int *\Omega|II|^2\rho_{y_0,t_0}d\mu_t\\
&-\int(1-*\Omega)\rho_{y_0,t_0}\left[\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{F^\perp\cdot\bar{H}}{t_0-t} + \frac{F^\perp\cdot U}{2(t_0-t)}\right) + |\bar{H}|^2 + \bar{H}\cdot U\right]d\mu_t\\
&=-\delta\int *\Omega|II|^2\rho_{y_0,t_0}d\mu_t - \int(1-*\Omega)\rho_{y_0,t_0}\left|\frac{F^\perp}{2(t_0-t)} + \bar{H} + \frac{U}{2}\right|^2 d\mu_t\\
&+\int(1-*\Omega)\rho_{y_0,t_0}\left|\frac{U}{2}\right|^2 d\mu_t.
\end{aligned}$$

Since $U$ is bounded, and since $\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t \leq \int\rho_{(y_0,t_0)}d\mu_t < \infty$, it follows that:

$$\frac{d}{dt}\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t \leq C-\delta\int *\Omega|II|^2\rho_{y_0,t_0}d\mu_t$$

for some constant $C$. From this it follows that $\lim_{t\to t_0}\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t$ exists.

$\square$

For $\nu > 1$, the parabolic dilation $D_\nu$ at $(y_0, t_0)$ is defined by:

$$D_\nu : \mathbb{R}^N \times [0,t_0) \to \mathbb{R}^N \times [-\nu^2 t_0, 0),$$

$$(y,t) \mapsto (\nu(y-y_0), \nu^2(t-t_0)).$$

Let $\mathcal{S} \subset \mathbb{R}^N \times [0,t_0)$ be the total space of the mean curvature flow, and let $\mathcal{S}_\nu \equiv D_\nu(\mathcal{S}) \subset \mathbb{R}^N \times [-\nu^2 t_0, 0)$. If $s$ denotes the new time parameter, then $t = t_0 + \frac{s}{\nu^2}$.

Let $d\mu_s^\nu$ be the induced volume form on $\Sigma$ by $F_s^\nu \equiv \nu F_{t_0+\frac{s}{\nu^2}}$.

The image of $F_s^\nu$ is the $s-$slice of $\mathcal{S}_\nu$, denoted $\Sigma_s^\nu$.

**Remark 3.4.1** *Note that:*

$$\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t = \int(1-*\Omega)\rho_{0,0}d\mu_s^\nu,$$

*because $*\Omega$ and $\rho_{y_0,t_0}d\mu_t$ are invariant under parabolic dilation.*

**Lemma 3.4.2** *For any $\tau > 0$:*

$$\lim_{\nu\to\infty}\int_{-1-\tau}^{-1}\int *\Omega|II|^2\rho_{0,0}d\mu_s^\nu ds = 0.$$

*Proof:* From Remark 3.4.1:

$$\frac{d}{ds}\int(1-*\Omega)\rho_{0,0}d\mu_s^\nu = \frac{1}{\nu^2}\frac{d}{dt}\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t.$$

Then by Lemma 3.4.1:

$$\frac{d}{ds}\int(1-*\Omega)\rho_{0,0}d\mu_s^\nu \le \frac{C}{\nu^2} - \frac{\delta}{\nu^2}\int *\Omega|II|^2\rho_{y_0,t_0}d\mu_t$$

for some constant $C$. But $\frac{1}{\nu^2}\int *\Omega|II|^2\rho_{y_0,t_0}d\mu_t = \int *\Omega|II|^2\rho_{0,0}d\mu_s^\nu$ since the norm of the second fundamental form scales like the inverse of the distance, so:

$$\frac{d}{ds}\int(1-*\Omega)\rho_{0,0}d\mu_s^\nu \le \frac{C}{\nu^2} - \delta\int *\Omega|II|^2\rho_{0,0}d\mu_s^\nu.$$

Integrating this inequality with respect to $s$ from $-1-\tau$ to $-1$, we obtain:

$$\delta\int_{-1-\tau}^{-1}\int *\Omega|II|^2\rho_{0,0}d\mu_s^\nu ds \le -\int(1-*\Omega)\rho_{0,0}d\mu_{-1}^\nu + \int(1-*\Omega)\rho_{0,0}d\mu_{-1-\tau}^\nu + \frac{C}{\nu^2}.$$

By Remark 3.4.1 and the fact that $\lim_{t\to t_0}\int(1-*\Omega)\rho_{y_0,t_0}d\mu_t$ exists (Lemma 3.4.1), the right-hand side of the inequality above approaches zero as $\nu\to\infty$.

$\square$

We take a sequence $\nu_j \to \infty$. Then for a fixed $\tau$:

$$\int_{-1-\tau}^{-1} \int *\Omega |II|^2 \rho_{0,0} d\mu_s^{\nu_j} ds \leq C(j),$$

where $C(j) \to 0$.

Choose $\tau_j \to 0$ such that $\frac{C(j)}{\tau_j} \to 0$, and $s_j \in [-1-\tau_j, -1]$ so that:

$$\int *\Omega |II|^2 \rho_{0,0} d\mu_{s_j}^{\nu_j} \leq \frac{C(j)}{\tau_j}. \tag{3.4.1}$$

(Note that such $s_j$ always exist.)

Observe that:

$$\rho_{0,0}(F_{s_j}^{\nu_j}, s_j) = \frac{1}{(4\pi(-s_j)^2)^n} \exp\left(\frac{-|F_{s_j}^{\nu_j}|^2}{4(-s_j)}\right).$$

When $j$ is large enough, we may assume that $\tau_j \leq 1$, and thus that $s_j \in [-2, -1]$. For a ball centered at 0 of radius $R > 0$, $B_R(0) \in \mathbb{R}^N$, we have:

$$\int *\Omega |II|^2 \rho_{0,0} d\mu_{s_j}^{\nu_j} \geq C' \int_{\Sigma_{s_j}^{\nu_j} \cap B_R(0)} *\Omega |II|^2 d\mu_{s_j}^{\nu_j}$$

for a constant $C' > 0$, since $s_j$ are bounded and since $|F_{s_j}^{\nu_j}| \leq R$ on $\Sigma_{s_j}^{\nu_j} \cap B_R(0)$.

Then by inequality (3.4.1) and the fact that $*\Omega$ has a positive lower bound, we conclude the following.

**Lemma 3.4.3** *For any compact set $\mathcal{K} \subset \mathbb{R}^N$:*

$$\int_{\Sigma_{s_j}^{\nu_j} \cap \mathcal{K}} |II|^2 d\mu_{s_j}^{\nu_j} \to 0$$

*as $j \to \infty$.*

Then, as shown in [24], it follows that:

$$\lim_{t \to t_0} \int \rho_{y_0,t_0} d\mu_t \leq 1.$$

White's theorem [30] then implies that $(y_0, t_0)$ is a regular point whenever

$$\lim_{t \to t_0} \int \rho_{y_0,t_0} d\mu_t \leq 1 + \epsilon,$$

contradicting the initial assumption that $(y_0, t_0)$ is a singular point.

## 3.5 Convergence

In the preceding sections we have shown that the mean curvature flow $\Sigma_t$ of the graph of symplectomorphism $f : M \to \tilde{M}$ exists smoothly for all $t > 0$, and that $\Sigma_t$ is a graph of symplectomorphisms for each $t$. We conclude the proof of Theorem 1 by showing that $\Sigma_t$ converge to the graph of a biholomorphic isometry.

By Proposition 3.2.1:

$$\left(\frac{d}{dt} - \Delta\right) * \Omega = *\Omega \left[ Q(\lambda_i, h_{jkl}) + \sum_{k \text{ odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} \right]$$

along the mean curvature flow, where $Q \geq 0$ when $\frac{1}{\Lambda_0} \leq \lambda_i \leq \Lambda_0$.

We use this result to derive the evolution equation of $\ln *\Omega$, which we then apply to show that $\lim_{t\to\infty} *\Omega = \frac{1}{2^n}$.

**Proposition 3.5.1** *Let $\Sigma$ be the graph of a symplectomorphism $f : (M, \omega) \to (\tilde{M}, \tilde{\omega})$ between $2n$-dimensional Kähler-Einstein manifolds $(M, g)$ and $(\tilde{M}, \tilde{g})$. At each point $q \in \Sigma_t$, $\ln *\Omega$ satisfies the following equation:*

$$\frac{d}{dt} \ln *\Omega = \Delta \ln *\Omega + \tilde{Q}(\lambda_i, h_{jkl}) + \sum_k \sum_{i \neq k} \frac{\lambda_i}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})} (R_{ikik} - \lambda_k^2 \tilde{R}_{ikik}),$$

*where $R_{ijkl}$ and $\tilde{R}_{ijkl}$ are the coefficients of the curvature tensors of $M$ and $\tilde{M}$ with respect to the chosen bases (3.2.1) and (3.2.2), $i' = i + (-1)^{i+1}$, and*

$$\tilde{Q}(\lambda_i, h_{jkl}) = Q(\lambda_i, h_{jkl}) + \sum_k \sum_{i \text{ odd}} (\lambda_i^2 + \lambda_{i'}^2) h_{ii'k}^2 \tag{3.5.1}$$

*with $Q(\lambda_i, h_{jkl})$ given by Proposition 3.2.1 and Lemma 3.2.1.*

*Proof:*

$$\frac{d}{dt} \ln *\Omega = \frac{1}{*\Omega} \frac{d}{dt} * \Omega$$

and:

$$\Delta(\ln *\Omega) = \frac{*\Omega\Delta(*\Omega) - |\nabla * \Omega|^2}{(*\Omega)^2}.$$

It follows that:

$$\left(\frac{d}{dt} - \Delta\right) \ln *\Omega = Q(\lambda_i, h_{jkl})$$
$$+ \sum_k \sum_{i \neq k} \frac{\lambda_i}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})}(R_{ikik} - \lambda_k^2 \tilde{R}_{ikik}) + \frac{|\nabla * \Omega|^2}{(*\Omega)^2}$$

by Proposition 3.2.1. Now:

$$\begin{aligned}
(*\Omega)_k &= \sum_i \Omega(e_1, \ldots, (\nabla_{e_k}^{M\times\tilde{M}} - \nabla_{e_k}^{\Sigma})e_i, \ldots, e_{2n}) \\
&= \sum_i \Omega(e_1, \ldots, \langle \nabla_{e_k}^{M\times\tilde{M}} e_i, \mathcal{J}e_p\rangle \mathcal{J}e_p, \ldots, e_{2n}) \\
&= \sum_{p,i} \Omega(e_1, \ldots, \mathcal{J}e_p, \ldots, e_{2n}) h_{pik} \\
&= *\Omega \sum_{p,i} \frac{\sqrt{1+\lambda_i^2}}{\sqrt{1+\lambda_p^2}} J_{ip} h_{pik},
\end{aligned}$$

and thus:

$$\begin{aligned}
\frac{|\nabla * \Omega|^2}{(*\Omega)^2} &= \sum_k [\sum_{p,i} \frac{\sqrt{1+\lambda_i^2}}{\sqrt{1+\lambda_p^2}} J_{ip} h_{pik} \sum_{q,j} \frac{\sqrt{1+\lambda_j^2}}{\sqrt{1+\lambda_q^2}} J_{jq} h_{qjk}] \\
&= \sum_k \sum_{p,q} \sum_{i,j} \frac{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}} J_{ip} J_{jq} h_{pik} h_{qjk}
\end{aligned}$$

$$= \sum_k \sum_{p,q} \sum_{i<j} \frac{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}} (J_{ip}J_{jq}h_{pik}h_{qjk} - J_{jp}J_{iq}h_{pjk}h_{qik})$$
$$+ \sum_k \sum_{p,q} \sum_i \frac{1+\lambda_i^2}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}} J_{ip}J_{iq}h_{pik}h_{qik}.$$

In the first line, the summation term is a product of a symmetric and an anti-symmetric part in $p$ and $q$, so the terms corresponding to $p \neq q$ cancel out, while in the case when $p = q$ the term is 0. Therefore:

$$\frac{|\nabla * \Omega|^2}{(*\Omega)^2} = \sum_k \sum_{p,q} \sum_i \frac{1+\lambda_i^2}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}} J_{ip}J_{iq}h_{pik}h_{qik}$$
$$= 2 \sum_k \sum_{p<q} \sum_i \frac{1+\lambda_i^2}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}} J_{ip}J_{iq}h_{pik}h_{qik} + \sum_k \sum_p \sum_i \frac{1+\lambda_i^2}{1+\lambda_p^2} J_{ip}^2 h_{pik}^2.$$

But once again, with respect to the chosen bases, at least one of $J_{ip}$ and $J_{iq}$ is 0, so:

$$\frac{|\nabla * \Omega|^2}{(*\Omega)^2} = \sum_k \sum_p \sum_i \frac{1+\lambda_i^2}{1+\lambda_p^2} J_{ip}^2 h_{pik}^2 = \sum_k \sum_i \lambda_i^2 h_{ii'k}^2 = \sum_k \sum_{i \text{ odd}} (\lambda_i^2 + \lambda_{i'}^2) h_{ii'k}^2.$$

Thus:

$$\left(\frac{d}{dt} - \Delta\right) \ln *\Omega = \tilde{Q}(\lambda_i, h_{jkl}) + \sum_k \sum_{i \neq k} \frac{\lambda_i}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})} (R_{ikik} - \lambda_k^2 \tilde{R}_{ikik}),$$

where $\tilde{Q}(\lambda_i, h_{jkl}) = Q(\lambda_i, h_{jkl}) + \sum_k \sum_{i \text{ odd}} (\lambda_i^2 + \lambda_{i'}^2) h_{ii'k}^2$ is a new quadratic form in $h_{ijk}$, with coefficients depending on the singular values of $f$. $\square$

**Corollary 3.5.1** *If $M \simeq \mathbb{CP}^n$ and $\tilde{M} \simeq \mathbb{CP}^n$, and the metric on each manifold is Fubini-Study, then:*

$$\frac{d}{dt}\ln *\Omega = \Delta \ln *\Omega + \tilde{Q}(\lambda_i, h_{ijk}) + \sum_{k\ odd} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}.$$

*Proof:* This is a direct consequence of Proposition 3.5.1 and Corollary 3.2.1.

□

**Remark 3.5.1** *$\tilde{Q}$ is a positive definite quadratic form of $h_{ijk}$ whenever $Q$ is, and in fact it allows for an improvement of the pinching constant.*

We use the evolution equation of $\ln *\Omega$ to show that $\lim_{t\to\infty} *\Omega = \frac{1}{2^n}$. Note that:

$$\frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} = \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})^2} = \frac{x-4}{x},$$

where $x = (\lambda_k + \lambda_{k'})^2$.

Since $\lambda_k \lambda_{k'} = 1$, it follows that $\lambda_k + \lambda_{k'} \geq 2$, and thus $x \geq 4$. Moreover, the pinching condition implies that $x \leq \left(\sqrt{\Lambda_0} + \frac{1}{\sqrt{\Lambda_0}}\right)^2$.

Now:

$$\frac{x-4}{x} \geq c\left(\frac{1}{2}\ln x - \ln 2\right)$$

for $c = \frac{8}{(\sqrt{\Lambda_0} + \frac{1}{\sqrt{\Lambda_0}})^2}$.

To see this, let $f(x) = \frac{x-4}{x}$, $g(x) = c\left(\frac{1}{2}\ln x - \ln 2\right)$. Notice that:

$$f(4) = g(4) = 0.$$

Now:

$$f'(x) = \frac{x - x + 4}{x^2} = \frac{4}{x^2}$$

and:

$$g'(x) = \frac{c}{2x}.$$

Then:

$$\frac{f'(x)}{g'(x)} = \frac{4}{x^2}\frac{2x}{c} = \frac{8}{cx} \geq 1.$$

The last inequality follows from the choice of $c$ and the fact that $x \leq \left(\sqrt{\Lambda_0} + \frac{1}{\sqrt{\Lambda_0}}\right)^2$. Now since $f(4) = g(4)$ and $f'(x) \geq g'(x)$ for $4 \leq x \leq \left(\sqrt{\Lambda_0} + \frac{1}{\sqrt{\Lambda_0}}\right)^2$, it follows that $f(x) \geq g(x)$. Substituting back, we obtain:

$$\frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})^2} \geq c\left(\ln(\lambda_k + \lambda_{k'}) - \ln 2\right),$$

and thus:

$$\begin{aligned}\sum_{k \text{ odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} = \sum_{k \text{ odd}} \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})^2} &\geq c\left(-\ln \prod_{k \text{ odd}} \frac{1}{\lambda_k + \lambda_{k'}} - n \ln 2\right) \\ &= -c\left(\ln *\Omega - \ln \frac{1}{2^n}\right).\end{aligned}$$

Therefore under the pinching condition:

$$\left(\frac{d}{dt} - \Delta\right)\left(\ln *\Omega - \ln \frac{1}{2^n}\right) \geq -c\left(\ln *\Omega - \ln \frac{1}{2^n}\right).$$

The pinching condition holds along the mean curvature flow, so this holds for all times.

Then by the comparison principle for parabolic equations:

$$0 \geq \ln *\Omega - \ln \frac{1}{2^n} \geq K_0 e^{-ct}$$

for $K_0 = \min\limits_{\Sigma_0} \ln *\Omega - \ln \frac{1}{2^n}$. It follows that $\lim\limits_{t\to\infty} \min\limits_{\Sigma_t} \ln *\Omega - \ln \frac{1}{2^n} = 0$, and thus $\lim\limits_{t\to\infty} \min\limits_{\Sigma_t} *\Omega = \frac{1}{2^n}$. That in turn implies, by Lemma 3.3.2, that $\lambda_i \to 1$ as $t \to \infty$ for all $i$.

For the rest of the proof, we modify the method from [24] to show that the second fundamental form is bounded in $t$. Let $\epsilon > 0$ and let $\eta_\epsilon = *\Omega - \frac{1}{2^n} + \epsilon$. Note that $\min\limits_{\Sigma_t} \eta_\epsilon$ is nondecreasing, and $\eta_\epsilon \to \epsilon$ when $t \to \infty$. Let $T_\epsilon \geq 0$ be a time such that $\eta_\epsilon|_{T_\epsilon} > 0$ (so that for all $t \geq T_\epsilon$: $\eta_\epsilon > 0$).

Now for all $p \in M$, and all $t > T_\epsilon$:

$$\begin{aligned}
\frac{d}{dt}\eta_\epsilon &= \Delta\eta_\epsilon + *\Omega(Q+B) \\
&\geq \Delta\eta_\epsilon + \delta * \Omega|II|^2 = \Delta\eta_\epsilon + \frac{\delta}{\eta_\epsilon}\eta_\epsilon * \Omega|II|^2.
\end{aligned}$$

From [24], $|II|^2$ satisfies the following equation along the mean curvature flow:

$$\begin{aligned}
\frac{d}{dt}|II|^2 &= \Delta|II|^2 - 2|\nabla II|^2 \\
&\quad + [(\nabla^M_{\partial_k})\mathcal{R}(\mathcal{J}e_p, e_i, e_j, e_k) + (\nabla^M_{\partial_j}\mathcal{R})(\mathcal{J}e_p, e_k, e_i, e_k)]h_{pij} -
\end{aligned}$$

$$- 2\mathcal{R}(e_l, e_i, e_j, e_k)h_{plk}h_{pij} + 4\mathcal{R}(\mathcal{J}e_p, \mathcal{J}e_q, e_j, e_k)h_{qik}h_{pij}$$
$$- 2\mathcal{R}(e_l, e_k, e_i, e_k)h_{plj}h_{pij} + \mathcal{R}(\mathcal{J}e_p, e_k, \mathcal{J}e_q, e_k)h_{qij}h_{pij}$$
$$+ \sum_{p,r,i,m}(\sum_k h_{pik}h_{rmk} - h_{pmk}h_{rik})^2 + \sum_{i,j,m,k}(\sum_p h_{pij}h_{pmk})^2.$$

Since $M \times \tilde{M}$ is a symmetric space, the curvature tensor $\mathcal{R}$ of $M \times \tilde{M}$ is parallel, and thus $|II|^2$ satisfies:

$$\frac{d}{dt}|II|^2 \leq \Delta|II|^2 - 2|\nabla II|^2 + K_1|II|^4 + K_2|II|^2$$

for positive constants $K_1$ and $K_2$ that depend only on $n$. Therefore:

$$\frac{d}{dt}(\eta_\epsilon^{-1}|II|^2) \leq -\eta_\epsilon^{-2}|II|^2(\Delta\eta_\epsilon + \delta * \Omega|II|^2)$$
$$+ \eta_\epsilon^{-1}(\Delta|II|^2 - 2|\nabla II|^2 + K_1|II|^4 + K_2|II|^2)$$
$$= -\eta_\epsilon^{-2}\Delta\eta_\epsilon|II|^2 + \eta_\epsilon^{-1}\Delta|II|^2 - 2\eta_\epsilon^{-1}|\nabla II|^2 + \eta_\epsilon^{-2}(\eta_\epsilon K_1 - \delta * \Omega)|II|^4$$
$$+ \eta_\epsilon^{-1}K_2|II|^2$$
$$= \Delta(\eta_\epsilon^{-1})|II|^2 - 2\eta_\epsilon^{-3}|\nabla\eta_\epsilon|^2|II|^2 + \eta_\epsilon^{-1}\Delta|II|^2 - 2\eta_\epsilon^{-1}|\nabla II|^2$$
$$+ \eta_\epsilon^{-2}(\eta_\epsilon K_1 - \delta * \Omega)|II|^4 + \eta_\epsilon^{-1}K_2|A|^2$$
$$= \Delta(\eta_\epsilon^{-1})|II|^2 - 2\eta_\epsilon|\nabla(\eta_\epsilon^{-1})|^2|II|^2 + \eta_\epsilon^{-1}\Delta|II|^2 - 2\eta_\epsilon^{-1}|\nabla II|^2$$
$$+ \eta_\epsilon^{-2}(\eta_\epsilon K_1 - \delta * \Omega)|II|^4 + \eta_\epsilon^{-1}K_2|II|^2$$
$$= \Delta(\eta_\epsilon^{-1}|II|^2) - 2\nabla(\eta_\epsilon^{-1}) \cdot \nabla(|II|^2) - 2\eta_\epsilon|\nabla(\eta_\epsilon^{-1})|^2|II|^2 - 2\eta_\epsilon^{-1}|\nabla II|^2$$
$$+ \eta_\epsilon^{-2}(\eta_\epsilon K_1 - \delta * \Omega)|II|^4 + \eta_\epsilon^{-1}K_2|II|^2$$

$$= \Delta(\eta_\epsilon^{-1}|II|^2) - \eta_\epsilon \nabla(\eta_\epsilon^{-1})\nabla(\eta_\epsilon^{-1}|II|^2) - \nabla(\eta_\epsilon^{-1}) \cdot \nabla(|II|^2)$$
$$- \eta_\epsilon|\nabla(\eta_\epsilon^{-1})|^2|II|^2 - 2\eta_\epsilon^{-1}|\nabla II|^2 + \eta_\epsilon^{-2}(\eta_\epsilon K_1 - \delta * \Omega)|II|^4 + \eta_\epsilon^{-1}K_2|II|^2$$
$$\leq \Delta(\eta_\epsilon^{-1}|II|^2) - \eta_\epsilon \nabla(\eta_\epsilon^{-1})\nabla(\eta_\epsilon^{-1}|II|^2) + \eta_\epsilon^{-2}(\eta_\epsilon K_1 - \delta * \Omega)|II|^4$$
$$+ \eta_\epsilon^{-1}K_2|II|^2.$$

The last inequality follows from the fact that:

$$\begin{aligned}
&\nabla(\eta_\epsilon^{-1}) \cdot \nabla(|II|^2) + \eta_\epsilon|\nabla(\eta_\epsilon^{-1})|^2|II|^2 + 2\eta_\epsilon^{-1}|\nabla II|^2 \\
&\quad = \frac{1}{2}|\sqrt{2\eta_\epsilon}|II|\nabla(\eta_\epsilon^{-1}) + \frac{1}{\sqrt{2\eta_\epsilon}|II|} \cdot \nabla(|II|^2)|^2 - \frac{1}{4\eta_\epsilon|II|^2}|\nabla(|II|^2)|^2 \\
&\quad + 2\eta_\epsilon^{-1}|\nabla II|^2 \\
&\quad = \frac{1}{2}|\sqrt{2\eta_\epsilon}|II|\nabla(\eta_\epsilon^{-1}) + \frac{1}{\sqrt{2\eta_\epsilon}|II|} \cdot \nabla(|II|^2)|^2 - \frac{1}{\eta_\epsilon}|\nabla|II||^2 \\
&\quad + 2\eta_\epsilon^{-1}|\nabla II|^2 \\
&\quad = \frac{1}{2}|\sqrt{2\eta_\epsilon}|II|\nabla(\eta_\epsilon^{-1}) + \frac{1}{\sqrt{2\eta_\epsilon}|II|} \cdot \nabla(|II|^2)|^2 \\
&\quad + \eta_\epsilon^{-1}(2|\nabla II|^2 - |\nabla|II||^2) \\
&\quad \geq 0,
\end{aligned}$$

since, by Hölder's inequality:

$$\begin{aligned}
|\nabla|II||^2 &= \sum_i(\sum_{j,k,l}\frac{h_{jkl}}{|II|}\partial_i h_{jkl})^2 \\
&\leq \sum_i(\sum_{j,k,l}\frac{h_{jkl}^2}{|II|^2}\sum_{j,k,l}(\partial_i h_{jkl})^2) = \sum_{i,j,k,l}(\partial_i h_{jkl})^2 = |\nabla II|^2.
\end{aligned}$$

Therefore the function $\psi = \eta_\epsilon^{-1}|II|^2$ satisfies:

$$\begin{aligned}\frac{d}{dt}\psi &\leq \Delta\psi - \eta_\epsilon \nabla \eta_\epsilon^{-1} \cdot \nabla\psi + (\eta_\epsilon K_1 - \delta * \Omega)\psi^2 + K_2\psi \\ &\leq \Delta\psi - \eta_\epsilon \nabla \eta_\epsilon^{-1} \cdot \nabla\psi + (\epsilon K_1 - \delta C_0)\psi^2 + K_2\psi,\end{aligned}$$

where $C_0 = \min_{\Sigma_0} *\Omega$, since $\min_{\Sigma_t} *\Omega$ is nondecreasing and $\eta_\epsilon \leq \epsilon$.

$\epsilon$ can be chosen small enough so that $\epsilon K_1 - \delta C_0 < 0$. Then by the comparison principle for parabolic PDE:

$$\psi \leq y(t)$$

for all $t \geq T_\epsilon$, where $y(t)$ is the solution of the ODE:

$$\frac{d}{dt}y = -(\delta C_0 - \epsilon K_1)y^2 + K_2 y$$

satisfying the initial condition $y(T_\epsilon) = \max_{\Sigma_{T_\epsilon}} \psi$.

Explicitly:

$$y(t) = \begin{cases} \frac{K_2}{\delta C_0 - \epsilon K_1}, & \text{if } \max_{\Sigma_{T_\epsilon}} \psi = \dfrac{K_2}{\delta C_0 - \epsilon K_1} \\ \frac{K_2}{\delta C_0 - \epsilon K_1} \frac{K e^{K_2 t}}{K e^{K_2 t} - 1}, & \text{otherwise} \end{cases},$$

where $K$ is a constant satisfying $K > 1$ if $\max_{\Sigma_{T_\epsilon}} \psi > \dfrac{K_2}{\delta C_0 - \epsilon K_1}$, and $K < 0$ if $\max_{\Sigma_{T_\epsilon}} \psi < \dfrac{K_2}{\delta C_0 - \epsilon K_1}$.

Thus:

$$|II|^2 \leq \eta_\epsilon y(t) \leq \epsilon y(t)$$

for all $t \geq T_\epsilon$.

Sending $t \to \infty$ and $\epsilon \to 0$, we conclude that $\max_{\Sigma_t} |II|^2 \to 0$ as $t \to \infty$.

Finally, the induced metric has analytic dependence on $F$, so by Simon's theorem [15] the flow coverges to a unique limit at infinity.

Since $\lambda_i \to 1$ for all $i$ as $t \to \infty$, the limit map is an isometry. Denote it by $f_\infty$. Being symplectic is a closed property, so $f_\infty$ is symplectic.

Then at every $p \in M$:

$$Df_\infty J = \tilde{J} Df_\infty.$$

In other words, $f_\infty$ is holomorphic.

The same is true for the inverse of $f_\infty$, and thus the map $f_\infty$ is biholomorphic.

# Chapter 4

# Conclusions and Directions

The proof of the preservation of the pinching condition and the long-time existence of the mean curvature flow (Sections 3.3 and 3.4) never used the assumption that the underlying manifolds are complex projective spaces. Thus in fact all the implications of Theorem 1 but the convergence result hold for general Kähler-Einstein manifolds, as long as the following, rather technical, property is satisfied.

For $\Lambda \in (1, \Lambda_0]$:

$$\sum_k \sum_{i \neq k} \frac{x_i}{(1+x_k^2)(x_i + x_i^{-1})}(R_{ikik} - x_k^2 \tilde{R}_{ikik}) \geq 0 \tag{4.0.1}$$

whenever $\frac{1}{\sqrt{\Lambda}} \leq x_i \leq \sqrt{\Lambda}$, where $R_{ijkl} = R(a_i, a_j, a_k, a_l)$ and $\tilde{R}_{ijkl} = \tilde{R}(E(a_i), E(a_j), E(a_k), E(a_l))$ are the coefficients of curvature tensors on $M$ and $\tilde{M}$, respectively, with respect to the bases chosen as before.

Note that whether the condition is satisfies depends both on the underlying manifolds (through curvature tensors) and the symplectomorphism (through the choice of bases of tangent spaces). At this point it is not very clear what the geometric implications of such a condition are.

The evolution equation of $*\Omega$ took particularly nice form in the case of $\mathbb{CP}^n$ because the holomorphic sectional curvature $K(X, JX) = \frac{R(X,JX,X,JX)}{|X|^2|JX|^2-\langle X,JX\rangle^2}$ (on holomorphic planes) was constant, as was non-holomorphic sectional curvature (on all other planes), so a question arises of whether there are other compact Kähler-Einstein manifolds of the same or similar property.

A case analogous to complex projective space is that of hyperbolic projective space with Bergman metric. Its sectional curvature is exactly the negative of the sectional curvature of $\mathbb{CP}^n$. The evolution equation of $*\Omega$ then becomes:

$$\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega \left[ Q(\lambda_i, h_{ijk}) - \sum_{k \text{ odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} \right].$$

It can be shown that $*\Omega$ remains positive as long as the appropriate pinching condition holds. However, the pinching condition is not necessarily preserved along the flow: the negative part of the expression on the right-hand side, arising from negative sectional curvature, may lead to a decrease of $\min_{\Sigma_t} *\Omega$.

To avoid such difficulty, one may want to focus on manifolds with positive sectional curvature. It turns out that if an $n$-dimensional compact Kähler manifold has constant positive holomorphic sectional curvature, then it is biholomorphically isometric to $\mathbb{CP}^n$ [12]. Under a weaker assumption that the holomorphic bisectional curvature $K_{bisec}(X,Y) = \frac{R(X,JX,Y,JY)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}$ be positive (not necessarily constant), a compact connected Kähler-Einstein manifold is isometric to $\mathbb{CP}^n$ [3].

However, it is not known whether there exist manifolds of different diffeomorphism type with positive holomorphic sectional curvature (but not necessarily positive holomorphic bisectional curvature). It is also not clear which other assumptions may be needed for the property (4.0.1) to be satisfied. It would be interesting to see whether in fact our method may shed light to these questions in an indirect way - by possible implications it would have if the answers were nontrivial.

For the end, we discuss some properties of the pinching region in the space of $\lambda_i$.

## 4.1 The pinching region

We already remarked that when $n = 1$ no pinching is required, as $Q(\lambda_i, h_{jkl})$ is always positive definite.

When $n > 1$, the quadratic form decomposes into smaller forms:

$$Q = \sum_r Q_{1,r} + \sum_{p<q<r \text{ odd}} Q_{2,pqr} + \sum_{p<q<r \text{ odd}} Q_{3,pqr},$$

where:

$$
\begin{aligned}
Q_{1,r} = {} & h_{rrr}^2 + 5h_{rr'r'}^2 + 3\sum_{i \neq r,r'} h_{rii}^2 + 8\sum_{i \text{ odd} \neq i,i'} h_{r'ii'}^2 + 2\sum_{i \text{ odd}} h_{rii}h_{1i'i'} \\
& - 2\sum_{i \text{ odd} < j \text{ odd}} (\lambda_i - \lambda_{i'})(\lambda_j - \lambda_{j'})h_{r'i'i}h_{r'j'j} \\
& + 2\sum_{j \text{ odd} \neq r,r'} (\lambda_r\lambda_j + \lambda_{r'}\lambda_{j'})h_{rj'j'}h_{r'jj'} \\
& - 2\sum_{j \text{ odd} \neq r,r'} (\lambda_{r'}\lambda_j + \lambda_1\lambda_{j'})h_{rjj}h_{r'jj'},
\end{aligned}
$$

$$
\begin{aligned}
Q_{3,pqr} = {} & 6h_{pqr}^2 + 6h_{p'q'r}^2 + 6h_{pq'r'}^2 + 6h_{p'qr'}^2 \\
& + 2(\lambda_p\lambda_q + \lambda_{p'}\lambda_{q'})h_{p'qr'}h_{pq'r'} + 2(\lambda_p\lambda_r + \lambda_{p'}\lambda_{r'})h_{p'rq'}h_{pr'q'} \\
& + 2(\lambda_q\lambda_r + \lambda_{q'}\lambda_{r'})h_{q'rp'}h_{qr'p'} - 2(\lambda_{p'}\lambda_q + \lambda_p\lambda_{q'})h_{pqr}h_{p'q'r} \\
& - 2(\lambda_{p'}\lambda_r + \lambda_p\lambda_{r'})h_{prq}h_{p'r'q} - 2(\lambda_{q'}\lambda_r + \lambda_q\lambda_{r'})h_{qrp}h_{q'r'p}
\end{aligned}
$$

and:

$$
\begin{aligned}
Q_{3,pqr} = {} & 6h_{p'q'r'}^2 + 6h_{p'qr}^2 + 6h_{pq'r}^2 + 6h_{pqr'}^2 \\
& + 2(\lambda_p\lambda_q + \lambda_{p'}\lambda_{q'})h_{p'qr}h_{pq'r} + 2(\lambda_p\lambda_r + \lambda_{p'}\lambda_{r'})h_{p'rq}h_{pr'q} \\
& + 2(\lambda_q\lambda_r + \lambda_{q'}\lambda_{r'})h_{q'rp}h_{qr'p} - 2(\lambda_{p'}\lambda_q + \lambda_p\lambda_{q'})h_{pqr'}h_{p'q'r'} \\
& - 2(\lambda_{p'}\lambda_r + \lambda_p\lambda_{r'})h_{prq'}h_{p'r'q'} - 2(\lambda_{q'}\lambda_r + \lambda_q\lambda_{r'})h_{qrp'}h_{q'r'p'}.
\end{aligned}
$$

When $n = 2$, the quadratic form has the following matrix representation:

$$Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 & \frac{x}{y} + \frac{y}{x} - xy - \frac{1}{xy} \\ 0 & 0 & 3 & -1 & -\frac{x}{y} - \frac{y}{x} \\ 0 & 0 & -1 & 3 & xy + \frac{1}{xy} \\ \\ 0 & \frac{x}{y} + \frac{y}{x} - xy - \frac{1}{xy} & -\frac{x}{y} - \frac{y}{x} & xy + \frac{1}{xy} & 8 \end{pmatrix},$$

where $x = \lambda_1$, $y = \lambda_3$. It is positive definite if and only if each diagonal minor has positive determinant. Since all but the largest ones do have positive determinant, the value of $\Lambda_0$ featured in Theorem 1is determined by the region when the determinant of the whole matrix is positive.

It can be shown that the region is bound by a curve $\mathfrak{C}$ satisfying the following:

- if $(x, y) \in \mathfrak{C}$, then $(y, x) \in \mathfrak{C}$;

- if $(x, y) \in \mathfrak{C}$, then $(\frac{1}{x}, y) \in \mathfrak{C}$.

It can be shown that in this case $\Lambda_0 = \frac{2}{5}\sqrt{10} + \frac{1}{5}\sqrt{15}$.

Similar analysis can be done for cases when $n > 2$.

# Bibliography

[1] S. Brendle, *Minimal Lagrangian diffeomorphisms between domains in the hyperbolic plane.* J. Differential Geom. 80 (2008), 1–20.

[2] K. Ecker, *Regularity Theory for Mean Curvature Flow.* Birkhäuser, Boston, MA (2004).

[3] S. I. Goldberg, S. Kobayashi, *Holomorphic bisectional curvature.* J. Diff. Geom. 1 (1967), 225–233.

[4] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds* Invent. Math. 82 (1985), 307–347.

[5] R. Harvey and H. B. Lawson, *Calibrated geometries.* Acta Math. 148 (1982), 47–157.

[6] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow.* J. Differential Geom. 31 (1990), no. 1, 285–299.

[7] T. Ilmanen, *Singularities of mean curvature flow of surfaces.* Preprint, 1995.

[8] D. Joyce, *Lectures on special Lagrangian geometry.* Global theory of minimal surfaces, Clay Math. Proc., 2, Amer. Math. Soc., Providence, RI (2005), 667–695.

[9] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasilinear equations of parabolic type.* Transl. Amer. Math. Soc. 23 (1968).

[10] H. B. Lawson, J. Simons, *On stable curves and their application to global problems in real and complex geometry* Ann. Math. 98 (1973), no. 3, 427–250.

[11] Y.-I. Lee, *Lagrangian minimal surfaces in Kähler-Einstein surfaces of negative scalar curvature.* Comm. Anal. Geom. 2 (1994), no. 4, 579–592.

[12] M. de Leon, J. C. Marrero, *Compact cosymplectic manifolds of positive constant $\varphi$-sectional curvature.* Extr. Math. 9 (1994), no. 1, 28–31.

[13] D. McDuff, D. Salomon, *Introduction to Symplectic Geometry.* Oxford University Press, Oxford, UK (1995).

[14] R. Schoen, *The role of harmonic mappings in rigidity and deformation problems.* Complex geometry (Osaka, 1990), 179–200, Lecture Notes in Pure and Appl. Math., 143, Dekker, New York, NY, 1993.

[15] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems.* Ann. of Math. (2) 118 (1983), no. 3, 525–571.

[16] S. Smale, *Diffeomorphisms of the 2-sphere.* Proc. Amer. Math. Soc. 10 (1959), 621–626.

[17] K. Smoczyk, *A canonical way to deform a Lagrangian submanifold.* preprint, dg-ga/9605005.

[18] K. Smoczyk, *Angle theorems for the Lagrangian mean curvature flow.* Math. Z. 240 (2002), no. 4, 849–883.

[19] K. Smoczyk, *Longtime existence of the Lagrangian mean curvature flow.* Calc. Var. Partial Differential Equations 20 (2004), no. 1, 25–46.

[20] K. Smoczyk and M.-T. Wang, *Mean curvature flows of Lagrangians submanifolds with convex potentials.* J. Differential Geom. 62 (2002), no. 2, 243–257.

[21] M. Spivak, *A Comprehensive Introduction to Differential Geometry.* 3rd ed., Vol. I, Publish or Perish, Houston, TX (1999).

[22] R. P. Thomas and S.-T. Yau, *Special Lagrangians, stable bundles and mean curvature flow.* Comm. Anal. Geom. 10 (2002), no. 5, 1075–1113.

[23] M.-T. Wang, *Deforming area preserving diffeomorphism of surfaces by mean curvature flow.* Math. Res. Lett. 8 (2001), no. 5-6, 651-662.

[24] M.-T. Wang, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension.* Invent. Math. 148 (2002), no. 3, 525–543.

[25] M-T. Wang, *Mean curvature flow in higher codimenison.* in the Proceedings of the second International Congress of Chinese Mathematicians, 2002, math.DG/0204054.

[26] M-T. Wang, *Mean curvature flow of surfaces in Einstein four-manifolds.* J. Differentioal Geom. 57 (2001), no. 2, 301–338.

[27] M.-T. Wang, *A convergence result of the Lagrangian mean curvature flow.* in the Proceedings of the third International Congress of Chinese Mathematicians. arXiv:math/0508354.

[28] M.-T. Wang, *Some recent developments in Lagrangian mean curvature flow.* To appear in Surveys in differential geometry, Vol. XII, Surv. Differ. Geom. IX, Int. Press, Somerville, MA, 2008.

[29] M.-T. Wang and M.-P. Tsui; *Mean curvature flows and isotopy maps between spheres.* Comm. Pure App. Math. 57 (2004), no. 8, 1110-1126.

[30] B. White, *A local regularity theorem for classical mean curvature flow.* Ann. of Math. (2) 161 (2005), no. 3, 1487–1519.

[31] J. Wolfson, *Minimal Lagrangian diffeomorphisms and the Monge-Ampère equation.* J. Differential Geom. 46 (1997), 335–373.