REMARKS ON SOME NON-LINEAR HEAT FLOWS IN KÄHLER GEOMETRY

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Abstract

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In this thesis, we clarify or simplify certain aspects of the Calabi flow and of the Donaldson heat flow.

In particular, in [33], the Calabi flow is studied as a flow of conformal factors $g_{ij}(t) \equiv e^{2u(t)}\hat{g}_{ij}(0)$,

$$\dot{u}(t) = \frac{1}{2}\Delta R \tag{0.0.1}$$

and the convergence of the conformal factors $u(t)$ in the Sobolev norm $\| \cdot \|_{(2)}$ is obtained. Although the convergence of the conformal factors established by Struwe [33] is only in the $\| \cdot \|_{(2)}$ norm, he states clearly that the convergence in arbitrary Sobolev norms, and hence in C^∞, should follow in the same way. In the first part of this thesis, we confirm that this is indeed the case.

Next we discuss the Donaldson heat flow. We shall show directly the C^0 boundedness of the full curvature tensor $F^{\alpha}_{\bar{k}j\beta}$ on $[0, \infty)$. Once again, our main technique is differential inequalities for the L^2 norms of the derivatives of $F_{\bar{k}j}{}^{\alpha}{}_{\beta}$, in analogy with the methods of [22, 34] and the treatment of the Calabi flow that we used in the previous section.

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1 Introduction

The classic uniformization theorem asserts that, on any compact Riemann surface X, there exists a unique metric of constant curvature. This is the starting point for a modern viewpoint to geometry, which is to try and characterize each geometric structure by a "canonical metric", that is, a metric with optimal curvature properties. Major successes in this viewpoint have been the existence of Kähler-Einstein metrics on Kähler manifolds X with $c_1(X) < 0$ [40, 1] or $c_1(X) = 0$ [40], and the existence of Hermitian-Einstein metrics on stable vector bundles [11, 12, 38]. There is at the present time a great deal of activity in addressing the existence of canonical metrics in a wide variety of contexts in complex and algebraic geometry, in particular Kähler-Einstein metrics when $c_1(X) > 0$ under suitable stability conditions [41, 36, 13, 14], extremal metrics [3], metrics of constant scalar curvature in a given Kähler polarization [13, 14] (see also the survey in [23] for a more extensive list of references), as well as singular Kähler-Einstein metrics when the first Chern class is not definite [15, 37, 10, 19, 30, 31, 32].

The equation for optimal curvature, e.g., constant scalar curvature or constant Ricci curvature, is usually a non-linear partial differential equation in the metric. For example, let $L \to X$ be a positive line bundle over an n-dimensional compact complex manifold X. Let h_0 be a metric on L with strictly positive curvature $\omega_0 = -\frac{i}{2}\partial\bar\partial \log h_0$. The problem of finding a Kähler metric ω in the class $c_1(L)$ with constant scalar curvature $\bar R$ can be viewed as the problem of finding a metric $h = e^{-\phi}h_0$ on L, with positive curvature $\omega = \omega_0 + \frac{i}{2}\partial\bar\partial\phi$, and satisfying the differential equation

$$-g^{j\bar k}\partial_j\partial_{\bar k} \log \omega^n = \bar R, \tag{1.0.1}$$

where $g_{\bar k j}$ is the metric corresponding to the Kähler form ω. This is a 4th-order non-linear partial differential equation in the potential ϕ. In the particular case where $L = K_X^{\pm 1}$, it is not difficult to see that the condition of constant scalar curvature is actually equivalent to the seemingly stronger condition of constant Ricci curvature,

$$R_{\bar{k}j} = \mu g_{\bar{k}j}, \tag{1.0.2}$$

where μ is a constant. This is the defining condition of Kähler-Einstein metrics. In a different set-up, let $E \to (X, \omega)$ be a holomorphic bundle of rank $r \geq 1$ over an n-dimensional compact Kähler manifold $(X, \omega = \frac{i}{2} g_{\bar{k}j} dz^j \wedge d\bar{z}^k)$. Let $H_{\bar{\alpha}\beta}$ denote metrics on E, $1 \leq \alpha, \beta \leq r$. Let $F_{\bar{k}j}{}^{\alpha}{}_{\beta} = -\partial_{\bar{k}}(H^{\alpha\bar{\gamma}}\partial_j H_{\bar{\gamma}\beta}$ denote the curvature of $H_{\bar{\alpha}\beta}$. Then the metric $H_{\bar{\alpha}\beta}$ is said to be a Hermitian-Einstein metric if

$$g^{j\bar{k}} F_{\bar{k}j}{}^{\alpha}{}_{\beta} = \mu \delta^{\alpha}{}_{\beta} \tag{1.0.3}$$

where μ is a constant. Although there has been considerable progress in the study of all these equations in recent years, many important questions about the existence of solutions, their regularity, and their geometric implications remain open.

One general approach to the problem of existence of canonical metrics is through the study of geometric flows with these canonical metrics as fixed points. Thus, corresponding to the three notions of canonical metrics we have just discussed are the Calabi flow,

$$\dot{g}_{\bar{k}j}(t) = \partial_j \partial_{\bar{k}} R, \tag{1.0.4}$$

whose fixed points are the Kähler metrics of constant scalar curvature, the Kähler-Ricci flow,

$$\dot{g}_{\bar{k}j}(t) = -(R_{\bar{k}j} - \mu g_{\bar{k}j}) \tag{1.0.5}$$

which is a special case of Hamilton's Ricci flow [16] and whose fixed points are Kähler-Einstein metrics, and the Donaldson heat flow

$$H^{-1}(t)\dot{H}(t) = -(g^{j\bar{k}} F_{\bar{k}j} - \mu I), \tag{1.0.6}$$

whose fixed points are the Hermitian-Einstein metrics. All these geometric flows are parabolic, and their short-time existence is guaranteed by the general theory of parabolic partial differential equations. So the main issue is their long-time existence, the description of the singularities if they develop, and their convergence. More specifically, the Donaldson heat flow [12] and the Kähler-Ricci flow (for definite Chern classes) exist for all time [4], and the issue is to determine the asymptotic behavior of the flow with or without stability conditions. As for the Calabi flow, it has been shown by Chen-He [6] and Szekelyhidi [35] that the flow will continue to exist as long as the Ricci curvature remains bounded, but many other natural questions remain unanswered.

In this thesis, we clarify or simplify certain aspects of the Calabi flow and of the Donaldson heat flow. We discuss now the relevant aspects of these flows in greater detail.

The Calabi flow has received a lot of attention very recently. Especially noteworthy are the criteria for long-time existence which we have already discussed above, and the analysis of the flow in various important special cases, such as ruled surfaces and toric varieties, for initial metrics with certain symmetry properties [34, 7]. Perhaps the most complete results to date are in the case of compact Riemann surfaces, thanks to the works of Chen [5], Chrusciel [8], and Struwe [33]. In particular, in [33], the Calabi flow is studied as a flow of conformal factors $g_{ij}(t) \equiv e^{2u(t)}\hat{g}_{ij}(0)$,

$$\dot{u}(t) = \frac{1}{2}\Delta R \qquad (1.0.7)$$

and the convergence of the conformal factors $u(t)$ in the Sobolev norm $\|\cdot\|_{(2)}$ is obtained by ruling out concentration-compactness phenomena, using an estimate of Brezis-Merle for the two-dimensional Laplace equation when studying the Liouville equation. Although the convergence of the conformal factors established by Struwe [33] is only in the $\|\cdot\|_{(2)}$ norm, he states clearly that the convergence in arbitrary Sobolev norms, and hence in C^∞, should follow in the same way. In the first part of this thesis, we confirm that this is indeed the case. An important step along the way is to establish the uniform boundedness of the scalar curvature R. To do this, we derive explicit differential inequalities for the L^2 norms of R and its derivatives, and apply the results of [33]. The method of exploiting such differential inequalities is a well-known method in the study of geometric flows (see e.g. [17], but our analysis is perhaps closest in spirit to those of [22, 35, 18, 20, 24].

Next, we discuss the Donaldson heat flow. The flow was originally introduced by Donaldson [12], who established its long-time existence, and who showed that, if the bundle $E \to X$ is stable in the sense of Mumford-Takemoto, a subsequence of the flow will converge to a Hermitian-Einstein metric. That Mumford-Takemoto stability implies the existence of Hermitian-Einstein metrics was also shown independently by Uhlenbeck and Yau [38], using elliptic rather than parabolic PDE methods. An important component of the approach of Uhlenbeck and Yau is a separate meromorphicity theorem, the proof of which was subsequently simplified by Shiffman [27] (see also [26] for a more recent proof). A detailed exposition of Donaldson's results is provided in the lecture notes of Siu [29]. Subsequently, Simpson [28] extended the Donaldson heat flow approach to Higgs bundles over non-compact manifolds, incorporating also the ideas of Uhlenbeck-Yau of how to obtain a coherent destabilizing bundle from $\| \cdot \|_{(1)}$ estimates. In the process, he also showed how to derive directly Donaldson's inequality from the Mumford-Takemoto condition, without establishing first the existence of Hermitian-Einstein metrics, as was shown first by Donaldson [12]. Although these results have essentially clarified the picture completely for the Donaldson heat flow when the bundle E is Mumford-Takemoto stable, we note that it is much less understood when the bundle E is not stable. For the case of complex surfaces X, the convergence of the flow, in a suitable sense, to the Harder-Narasimhan-Seshadri filtration of E has been established by G. Daskalopoulos and R. Wentworth [9], but again, many questions remain open in the general case.

The theory of geometric flows has been developed considerably since the time when the Kähler-Ricci flow and the Donaldson heat flow were first introduced. In particular, it is useful to reduce the (subsequential) convergence of geometric flows to a single (or at most a few) uniform bounds for a key geometric quantity along the flow. For example, the convergence of the Kähler-Ricci flow was known from early days, following the work of Yau [40], to reduce to a single C^0 estimate

$$\sup_{t\geq 0}\|\phi\|_{C^0} \leq A_0 < \infty, \tag{1.0.8}$$

along the flow, where $g_{\bar{k}j}(t) = g_{\bar{k}j}(0) + \partial_j\partial_{\bar{k}}\phi$, and ϕ is suitably normalized (see e.g. [20, 24, 23], where seemingly weaker criteria are also given). In the second part of this thesis, we establish a similar criterion for the Donaldson heat flow, namely, we show that, if $H_{\bar{\alpha}\beta}(t)$ is the metric evolving under the Donaldson heat flow and $h(t)^\alpha{}_\beta \equiv H(0)^{\alpha\bar{\gamma}}H_{\bar{\gamma}\beta}(t)$, then the estimate

$$\sup_{t\geq 0}\|\mathrm{Tr}\, h\|_{C^0} \leq A_0 < \infty \tag{1.0.9}$$

implies the uniform boundedness of all the derivatives of $h(t)$

$$\sup_{t\geq 0}\|h\|_{C^k} \leq A_k < \infty. \tag{1.0.10}$$

Properties of this type are formulated more precisely in the elliptic version of [38]. In the study of the Donaldson heat flow, as carried out in [12] and explained in greater detail in [29], they are more implicit. We provide a precise statement and complete derivation. Perhaps the main technical innovation in our approach is in the proof of the uniform boundedness of the relative connection $\nabla h\, h^{-1}$. The basic observation here is to exploit the analogy of this estimate with that of the C^3 estimate for the Monge-Ampère equation, once this latter estimate has been formulated in terms of the relative endomorphism $h(t)$, instead of the Kähler potential ϕ (see [20, 25]). Once this observation has been made, we can adapt the proof of the Calabi identity, as formulated in [20]. As in [12, 29], the boundedness of the relative connection $\nabla h\, h^{-1}$ implies then the uniform boundedness of Δh, and hence the L^p boundedness of the full curvature $F_{\bar{k}j}{}^\alpha{}_\beta$ for any $1 \leq p < \infty$. In [12, 29], it is shown that this L^p uniform boundedness is sufficient to lead to the desired conclusions. However, it would be simpler if the C^0 boundedness of the full curvature tensors can be derived directly from the L^p uniform boundedness. For finite time intervals $[0,T)$, $T < \infty$, it was shown in [12, 29] that the C^0 boundedness on $[0,T)$ follows from the L^p estimates for p large enough and standard estimates for the heat hernel. In this thesis, we shall show directly the C^0 boundedness of the full curvature tensor $F_{\bar{k}j}{}^{\alpha}{}_{\beta}$ on $[0,\infty)$. Once again, our main technique is differential inequalities for the L^2 norms of the derivatives of $F_{\bar{k}j}{}^{\alpha}{}_{\beta}$, in analogy with the methods of [22, 34] and the treatment of the Calabi flow that we used in the previous section.

2 The Calabi Flow

2.1 Background

Throughout this chapter, X represents a Riemann surface with an initial metric g_0. The Calabi flow generates a family of metrics, denoted by $g(t)$, with scalar curvatures R. We use $\hat{g}$ to denote a fixed metric in the Kähler class $[g(t)]$ with constant scalar curvature $\widehat{R}$. We denote the $(1,1)$-forms corresponding to g and $\hat{g}$ by ω and $\hat{\omega}$. Also, we denote their laplacians by Δ and $\widehat{\Delta}$, with the sign convention that $\Delta f := g^{z\bar{z}}\nabla_z\nabla_{\bar{z}} f$. All constants, $C, k, \ldots$, are positive and independent of time, and all norms are taken with respect to the evolving metric, unless otherwise indicated by a subscript.

Definition 1. *The Calabi energy, $\mathcal{C}(\omega)$, is defined by the integral*

$$\mathcal{C}(\omega) := \int_X \left| R - \widehat{R} \right|^2 \omega. \tag{2.1.1}$$

2.1.1 Previous Results

If we let u be a function satisfying $g(t) = e^{2u(t)}\hat{g}$, then the Calabi flow on a surface,

$$\begin{cases} \dot{g} &= g\Delta R; \\ g(0) &= g_0, \end{cases} \tag{2.1.2}$$

can be rewritten in terms of u as

$$\begin{cases} \dot{u} &= \frac{1}{2}\Delta R; \\ u(0) &= u_0. \end{cases} \tag{2.1.3}$$

In [33], Struwe proves that the metrics given by the Calabi flow are uniformly equivalent. He also proves that for some constants C and k,

$$\|u\|_{H^2(\hat{\omega})} \leq Ce^{-kt}, \tag{2.1.4}$$

and that

$$\mathcal{C}(\omega) \leq Ce^{-kt}. \tag{2.1.5}$$

From the relation between R, $\widehat{R}$, and u given by $R = e^{-2u}\widehat{R} - 2\Delta u$, we calculate the derivative of R with respect to time to be

$$\dot{R} = -R\Delta R - \Delta^2 R. \tag{2.1.6}$$

2.2 Exponential Decay of the Calabi Flow

We show u converges to 0 smoothly and exponentially with respect to $\hat{g}$ in five steps. First, in Lemma 2, we show that for positive genus surfaces, $\|\Delta^n R\|_{L^2} \leq Ce^{-kt}$ for all n. Next, in Lemma 3, we show that in the case of the sphere, $\|\Delta^n R\|_{L^2} \leq Ce^{-kt}$ for all n. Then, in Lemma 4, we show this implies $\|\Delta^n u\|_{L^2} \leq Ce^{-kt}$ for all n. Next, in Lemma 5, we show this implies $\|\widehat{\Delta}^n u\|_{L^2(\hat{\omega})} \leq Ce^{-kt}$ for all n. Then by the Sobolev imbedding theorem, we can prove the main theorem.

Theorem 1. *For all $r \geq 0$, $t > 0$, and some constants C and k depending on r,*

$$\|u\|_{C^r(X,\hat{\omega})} \leq Ce^{-kt}. \tag{2.2.1}$$

Before we prove the convergence of $\Delta^n R$, we state and prove some preliminary lemmas, claims, and corollaries that we use throughout this chapter.

Lemma 1. *For all functions f defined on X with $\dim X = 2$ and for all $p > 2$,*

$$\|f\|_{L^p(\hat{\omega})} \leq C_p \left(\|\widehat{\nabla} f\|_{L^2(\hat{\omega})} + \|f\|_{L^2(\hat{\omega})} \right). \tag{2.2.2}$$

Corollary 1.

$$\|f\|_{L^p} \leq C_p \left(\|\nabla f\|_{L^2} + \|f\|_{L^2} \right). \tag{2.2.3}$$

Proof. By the equivalence of the evolving metrics and the fixed metric([33], Thm 3.2.),

$$\|f\|_{L^p} \leq C\|f\|_{L^p(\hat{\omega})} \leq C_p \left(\|\widehat{\nabla} f\|_{L^2(\hat{\omega})} + \|f\|_{L^2(\hat{\omega})} \right) \leq C_p \left(\|\nabla f\|_{L^2} + \|f\|_{L^2} \right). \qquad \square$$

Corollary 2. *Moreover, for all tensors $\nabla^k f$ on X with $\dim X = 2$, for all $k \in \mathbb{N}$, and for all $p > 2$,*

$$\|\nabla^k f\|_{L^p} \leq C \left(\|\nabla^{k+1} f\|_{L^2} + \|\nabla^k f\|_{L^2} \right). \tag{2.2.4}$$

Proof. We have for any integer k and any function $f \in C^{k+1}(X)$,

$$|\nabla |\nabla^k f|| \leq |\nabla^{k+1} f|, \tag{2.2.5}$$

(See [2], Prop 2.11). Now we apply Corollary 1 to the function $|\nabla^k f|$.

$$\|\nabla^k f\|_{L^p} \leq C \left(\|\nabla |\nabla^k f|\|_{L^2} + \|\nabla^k f\|_{L^2} \right) \leq C \left(\|\nabla^{k+1} f\|_{L^2} + \|\nabla^k f\|_{L^2} \right). \qquad \square$$

Claim 1. *For all functions f defined on X, there exists a constant C, such that*

$$\|\nabla f\|_{L^2} \leq C\|\Delta f\|_{L^2}, \tag{2.2.6}$$

$$\|\nabla^2 f\|_{L^2} \leq C\|\Delta f\|_{L^2}, \tag{2.2.7}$$

$$\|\Delta f\|_{L^2} \leq C\|\nabla \Delta f\|_{L^2}. \tag{2.2.8}$$

Proof. Letting λ_1 be the first positive eigenvalue of $-\widehat{\Delta}$, we use the Poincaré inequality and the fact that $-c \leq u \leq 0$ ([33] p. 251) to obtain inequality (2.2.6).

$$\|\nabla f\|^2_{L^2} = \|\widehat{\nabla} f\|^2_{L^2(\hat{\omega})} \leq C\|\widehat{\Delta} f\|^2_{L^2(\hat{\omega})} = C\|e^u \Delta f\|^2_{L^2} \leq C\|\Delta f\|^2_{L^2}. \tag{2.2.9}$$

To establish (2.2.7), first, we integrate by parts and commute derivatives to get

$$\|\nabla^2 f\|^2_{L^2} = \|\Delta f\|^2_{L^2} + \int_X (2\Delta u - e^{-2u}\widehat{R})|\nabla f|^2 \omega. \tag{2.2.10}$$

Then by (2.2.6), Hölder's inequality, Corollary 2, and the fact that

$$\|\Delta u\|_{L^2} \leq Ce^{-kt}, \tag{2.2.11}$$

we have inequality (2.2.7) for sufficiently large t.

Lastly, for t sufficiently large, (2.2.8) follows from the Poincaré inequality, Hölder's inequality, and Corollary 1. $\square$

Corollary 3. *For t sufficiently large,*

$$\|\nabla f\|_{L^2} \leq C\|\Delta f\|_{L^2} \leq C\|\nabla\Delta f\|_{L^2} \leq C\|\Delta^2 f\|_{L^2} \leq C\|\nabla\Delta^2 f\|_{L^2} \leq \ldots. \quad (2.2.12)$$

Now we have the necessary tools to prove $\|\Delta^n R\|_{L^2} \leq Ce^{-kt}$. We treat the cases of positive genus surfaces and the sphere separately.

2.2.1 Surfaces with positive genus.

Lemma 2. *If $\widehat{R} \leq 0$, then for all $n \in \mathbb{Z}^+$ and $t > 0$, $\|\Delta^n R\|_{L^2} \leq Ce^{-kt}$.*

Proof. The key step is proving $\|\nabla R\|_{L^2} \leq Ce^{-kt}$. After we have this estimate, we can bound the L^2 norms of all higher derivatives of R by induction.

Step 1: We show $\|\nabla R\|_{L^2} \leq Ce^{-kt}$ by calculating $\frac{d}{dt}\|\nabla R\|^2_{L^2}$, integrating by parts, and applying Hölder's inequality. We get

$$\begin{aligned} \frac{d}{dt}\|\nabla R\|^2_{L^2} &= -2\|\nabla\Delta R\|^2_{L^2} + 2\widehat{R}\|\Delta R\|^2_{L^2} + 2\int_X (R-\widehat{R})|\Delta R|^2\omega & (2.2.13)\\ &\leq -2\|\nabla\Delta R\|^2_{L^2} + 2\widehat{R}\|\Delta R\|^2_{L^2} + 2\mathcal{C}(\omega)^{\frac{1}{2}}\|\Delta R\|^2_{L^4}. & (2.2.14)\end{aligned}$$

Then by Corollaries 1 and 3, $\mathcal{C}(\omega) \leq Ce^{-kt}$, and for large enough t,

$$\frac{d}{dt}\|\nabla R\|^2_{L^2} \leq -2\|\nabla\Delta R\|^2_{L^2} + \varepsilon\left(\|\nabla\Delta R\|^2_{L^2} + \|\Delta R\|^2_{L^2}\right) \leq -k\|\nabla R\|^2_{L^2}. \quad (2.2.15)$$

Step 2: To prove Lemma 2 for $n = 1$, we use the method and results of Step 1.

$$\begin{aligned} \frac{d}{dt}\|\Delta R\|^2_{L^2} &\leq -2\|\Delta^2 R\|^2_{L^2} + 2\widehat{R}\|\nabla\Delta R\|^2_{L^2} + 2\mathcal{C}(\omega)^{\frac{1}{2}}\|\nabla\Delta R\|^2_{L^4} & (2.2.16)\\ &\quad +4\|\nabla R\|_{L^2}\|\Delta R\|_{L^4}\|\nabla\Delta R\|_{L^4} & (2.2.17)\\ &\leq -2\|\Delta^2 R\|^2_{L^2} + \varepsilon\|\nabla\Delta R\|_{L^2}\|\Delta^2 R\|_{L^2} + \varepsilon\|\Delta^2 R\|^2_{L^2} & (2.2.18)\\ &\leq -k\|\Delta R\|^2_{L^2}. & (2.2.19)\end{aligned}$$

Step 3: We assume for $n \leq m$, $|\alpha| = 2n-1$, and $|\beta| = 2n$

$$\|\nabla^\alpha f\|_{L^2} \leq C\|\nabla\Delta^{n-1}f\|_{L^2}, \tag{2.2.20}$$

$$\|\nabla\Delta^{n-1}R\|_{L^2} \leq Ce^{-kt}, \tag{2.2.21}$$

$$\|\nabla^\beta f\|_{L^2} \leq C\|\Delta^n f\|_{L^2}, \tag{2.2.22}$$

$$\|\Delta^n R\|_{L^2} \leq Ce^{-kt}. \tag{2.2.23}$$

Step 4: If $|\alpha| = 2m+1$, we commute derivatives, integrate by parts, and apply the triangle inequality. Then

$$\|\nabla^\alpha f\|_{L^2} \leq \|\nabla^\lambda \Delta f\|_{L^2} + C\|R\nabla^\lambda f\|_{L^2} + C\sum_{\gamma,\delta}\|\nabla^\gamma R\nabla^\delta f\|_{L^2}, \tag{2.2.24}$$

where $|\lambda| \leq 2m-1$, and $|\gamma|+|\delta| = 2m-1$. It follows from Hölder's inequality, Corollaries $1-3$, and our assumptions in Step 3 that

$$\|\nabla^\alpha f\|_{L^2} \leq C\|\nabla\Delta^m f\|_{L^2} + C\|\Delta^m R\|_{L^2}\|\Delta^m f\|_{L^2} \leq C\|\nabla\Delta^m f\|_{L^2}. \tag{2.2.25}$$

Step 5: To prove (2.2.21) for $n = m+1$, first, we calculate

$$\begin{aligned}\frac{d}{dt}\|\nabla\Delta^m R\|^2_{L^2} &= -2\|\nabla\Delta^{m+1}R\|^2_{L^2} + 2\int_X R\Delta R\Delta^{2m+1}R\omega \\ &\quad +2\sum_{i=1}^m\int_X \Delta R\Delta^i R\Delta^{2m+1-i}R\omega. \end{aligned}\tag{2.2.26}$$

We integrate by parts repeatedly and apply Hölder's inequality, Corollaries $1-3$, and the assumptions in Step 3. Then for sufficiently large t,

$$\begin{aligned}\frac{d}{dt}\|\nabla\Delta^m R\|^2_{L^2} &\leq -2\|\nabla\Delta^{m+1}R\|^2_{L^2} + 2\widehat{R}\|\Delta^{m+1}R\|^2_{L^2} \\ &\quad +2\mathcal{C}(\omega)^{\frac{1}{2}}\|\nabla\Delta^{m+1}R\|^2_{L^2} \\ &\quad +C\|\Delta^m R\|_{L^2}\|\nabla\Delta^{m+1}R\|^2_{L^2} \qquad (2.2.27) \\ &\quad +C\|\Delta^m R\|^2_{L^2}\|\nabla\Delta^{m+1}R\|_{L^2} \\ &\leq -k\|\nabla\Delta^m R\|^2_{L^2}. \qquad (2.2.28)\end{aligned}$$

Step 6: If $|\beta| = 2m+2$, the results of Steps 4 and 5 and the method of Step 4 yields

$$\|\nabla^{\beta} f\|_{L^2} \leq C\|\Delta^{m+1} f\|_{L^2} + C\|\nabla\Delta^m R\|_{L^2}\|\nabla\Delta^m f\|_{L^2} \leq C\|\Delta^{m+1} f\|_{L^2}. \quad (2.2.29)$$

Step 7: To prove (2.2.23) for $n = m+1$, first, we calculate

$$\begin{aligned} \frac{d}{dt}\|\Delta^{m+1}R\|_{L^2}^2 &= -2\|\Delta^{m+2}R\|_{L^2}^2 - 2\int_X R\Delta R\Delta^{2m+2}R\omega \\ &\quad -2\sum_{i=1}^{m}\int_X \Delta R\Delta^i R\Delta^{2m+2-i}R\omega - \int_X \Delta R|\Delta^{m+1}R|^2\omega. \end{aligned} \quad (2.2.30)$$

We integrate by parts repeatedly and apply Hölder's inequality, Corollaries 1−3, the assumptions in Step 3, and the results of Steps 4 and 5. Then for sufficiently large t,

$$\begin{aligned} \frac{d}{dt}\|\Delta^{m+1}R\|_{L^2}^2 &\leq -2\|\Delta^{m+2}R\|_{L^2}^2 + 2\widehat{R}\|\nabla\Delta^{m+1}R\|_{L^2}^2 + 2\mathcal{C}(\omega)^{\frac{1}{2}}\|\Delta^{m+2}R\|_{L^2}^2 \\ &\quad + C\|\nabla\Delta^m R\|_{L^2}\|\nabla\Delta^{m+2}R\|_{L^2}^2 + C\|\nabla\Delta^m R\|_{L^2}^2\|\Delta^{m+2}R\|_{L^2} \\ &\leq -k\|\Delta^{m+1}R\|_{L^2}^2. \end{aligned}$$

□

2.2.2 The sphere.

When $\widehat{R} > 0$, we are dealing with the case of the sphere (i.e $\widehat{R} = 2$). The result of Lemma 2 is still valid, however the proof requires more bootstrapping and some additional results of Stuwe's. First, let $G = \text{Möb}^+(2)$ be the Möbius group of oriented conformal diffeomorphisms on S^2. For any element $\varphi \in G$, let $\varphi^* g$ be the pull-back of the metric g under φ. We define a new function, v, by

$$v := u \circ \varphi + \frac{1}{2}\log(\det d\varphi). \quad (2.2.31)$$

Then $\varphi^* g = e^{2v}\hat{g}$. We denote the $(1,1)$-form with respect to $\varphi^* g$ by $\tilde{\omega}$ and its Laplacian by $\widetilde{\Delta}$. We also define

$$\widetilde{R} := R_{\varphi^* g} = R \circ \varphi, \quad (2.2.32)$$

$$\widetilde{\widehat{R}} := R_{\varphi^* \hat{g}} = \widehat{R} = 2. \quad (2.2.33)$$

This implies that $\mathcal{C}(\tilde{\omega}) = \mathcal{C}(\omega)$. Moreover, for any function f on S^2,

$$\widetilde{\Delta}(f \circ \varphi) = (\Delta f) \circ \varphi. \tag{2.2.34}$$

It follows that, for any $n \geq 0$,

$$\|\widetilde{\Delta}^n \widetilde{R}\|_{L^2(\tilde{\omega})} = \|\Delta^n R\|_{L^2} \tag{2.2.35}$$

$$\|\widetilde{\nabla}\widetilde{\Delta}^n \widetilde{R}\|_{L^2(\tilde{\omega})} = \|\nabla \Delta^n R\|_{L^2}. \tag{2.2.36}$$

Then we also know

$$\mathcal{C}(\tilde{\omega}) \leq C \|\widetilde{\nabla} R\|^2_{L^2(\tilde{\omega})}. \tag{2.2.37}$$

In [33], Struwe proves

$$\|v\|_{H^2(\hat{\omega})} \leq C e^{-kt}. \tag{2.2.38}$$

Next, we let $\{\varphi_i\}$ be an L^2 orthonormal basis of eigenfunctions of $-\widehat{\Delta}$ on $(S^2, \hat{g})$, with eigenvalues $\lambda_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 2 < \lambda_4 \leq \lambda_5 \leq \ldots$. We expand

$$\widetilde{R} - \widehat{R} = \sum \widetilde{R}^i \varphi_i, \tag{2.2.39}$$

and it follows that

$$-\widehat{\Delta}\widetilde{R} = \sum_{i=0}^{\infty} \lambda_i \widetilde{R}^i \varphi_i. \tag{2.2.40}$$

Furthermore, in [33], Struwe proves

$$(\widetilde{R}^i)^2 \leq \varepsilon \mathcal{C}(\tilde{\omega}) \text{ for } i = 1, 2, 3. \tag{2.2.41}$$

We use the following two equations relating v and $\widetilde{R}$.

$$\widehat{\Delta} v = \frac{-1}{2} e^{2v} \widetilde{R} + 1, \tag{2.2.42}$$

$$\widetilde{\Delta} v = \frac{-1}{2} \widetilde{R} + e^{-2v}. \tag{2.2.43}$$

Claim 2. *There exists a constant C, such that for all functions f defined on X,*

$$\begin{aligned} \|\widetilde{\nabla} f\|_{L^2(\tilde{\omega})} &\leq C\|\widetilde{\Delta} f\|_{L^2(\tilde{\omega})}, &(2.2.44)\\ \|\widetilde{\nabla}^2 f\|_{L^2(\tilde{\omega})} &\leq C\|\widetilde{\Delta} f\|_{L^2(\tilde{\omega})}, &(2.2.45)\\ \|\widetilde{\Delta} f\|_{L^2(\tilde{\omega})} &\leq C\|\widetilde{\nabla}\widetilde{\Delta} f\|_{L^2(\tilde{\omega})}. &(2.2.46)\end{aligned}$$

Proof. The proof is done using the same method as the proof of Claim 1. □

Corollary 4. *For t sufficiently large,*

$$\|\widetilde{\nabla} f\|_{L^2(\tilde{\omega})} \leq C\|\widetilde{\Delta} f\|_{L^2(\tilde{\omega})} \leq C\|\widetilde{\nabla}\widetilde{\Delta} f\|_{L^2(\tilde{\omega})} \leq C\|\widetilde{\Delta}^2 f\|_{L^2(\tilde{\omega})} \leq \dots . \qquad (2.2.47)$$

Lemma 3. *If $\widehat{R} > 0$, then for all $n \in \mathbb{Z}^+$ and $t > 0$, $\|\Delta^n R\|_{L^2} \leq Ce^{-kt}$.*

Proof. We use the bounds we have on $\|v\|_{H^2(\hat{\omega})}$ to get bounds on $\|\nabla R\|_{L^2}$. Then we use those bounds to get bounds on higher derivatives of v. Then we use those to get bounds on higher derivatives of R. We can then prove the lemma by induction.

Step 1: We prove $\|\widetilde{\nabla}\widetilde{\Delta} f\|_{L^2(\tilde{\omega})} \approx \|\widehat{\nabla}\widehat{\Delta} f\|_{L^2(\hat{\omega})}$ by applying the triangle inequality, Hölder's inequality, Lemma 1, Claim 2, and the exponential decay of $\|v\|_{H^2(\hat{\omega})}$.

Step 2: We show $\|\nabla R\|_{L^2} \leq Ce^{-kt}$. Recall from Step 1 of the proof of Lemma 2 that

$$\frac{d}{dt}\|\nabla R\|^2_{L^2} \leq 2\widehat{R}\|\Delta R\|^2_{L^2} - 2\|\nabla\Delta R\|^2_{L^2} + 2\mathcal{C}(\omega)^{\frac{1}{2}}\|\Delta R\|^2_{L^4}. \qquad (2.2.48)$$

Due to (2.2.35) and (2.2.36), it suffices to show that for some C,

$$\widehat{R}\|\widetilde{\Delta}\widetilde{R}\|^2_{L^2(\tilde{\omega})} - \|\widetilde{\nabla}\widetilde{\Delta}\widetilde{R}\|^2_{L^2(\tilde{\omega})} \leq -C\|\widetilde{\nabla}\widetilde{\Delta}\widetilde{R}\|^2_{L^2(\tilde{\omega})} = -C\|\nabla\Delta R\|^2_{L^2}. \qquad (2.2.49)$$

After writing the left hand side in terms of the fixed metric, we must show

$$F(\widetilde{R}) := 2\int_X e^{-2v}|\widehat{\Delta}\widetilde{R}|^2\hat{\omega} - \int_X e^{-4v}|\widehat{\nabla}\widehat{\Delta}\widetilde{R}|^2_{\hat{\omega}}\hat{\omega} \leq -C\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})}. \quad (2.2.50)$$

Since $|v| \leq Ce^{-kt}$, for all $\varepsilon > 0$, we can take t large enough such that

$$2e^{-2v} \leq 2+\varepsilon, \quad (2.2.51)$$

$$-e^{-4v} \leq \varepsilon - 1. \quad (2.2.52)$$

We choose ε such that $\varepsilon < \lambda_4 - 2$. Then by (2.2.40) and (2.2.41),

$$\begin{aligned}
F(\widetilde{R}) &\leq (2+\varepsilon)\int_X |\widehat{\Delta}\widetilde{R}|^2\hat{\omega} + \int_x \widehat{\Delta}\widetilde{R}\widehat{\Delta}^2\widetilde{R}\hat{\omega} + \varepsilon\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})} && (2.2.53)\\
&= \sum_{i=4}^{\infty}\lambda_i^3\left(\frac{2+\varepsilon}{\lambda_i} - 1\right)(\widetilde{R}^i)^2 + \varepsilon\sum_{i=1}^{3}\lambda_i^2(\widetilde{R}^i)^2 + \varepsilon\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})} && (2.2.54)\\
&\leq \left(\frac{2+\varepsilon}{\lambda_4} - 1\right)\sum_{i=4}^{\infty}\lambda_i^3(\widetilde{R}^i)^2 + 4\varepsilon\sum_{i=1}^{3}(\widetilde{R}^i)^2 + \varepsilon\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})} && (2.2.55)\\
&= -k\sum_{i=1}^{\infty}\lambda_i^3(\widetilde{R}^i)^2 + (k+4\varepsilon)\sum_{i=1}^{3}(\widetilde{R}^i)^2 + \varepsilon\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})} && (2.2.56)\\
&= (-k+\varepsilon)\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})} + (k+4\varepsilon)\sum_{i=1}^{3}(\widetilde{R}^i)^2 && (2.2.57)\\
&\leq -C\|\widehat{\nabla}\widehat{\Delta}\widetilde{R}\|^2_{L^2(\hat{\omega})}. && (2.2.58)
\end{aligned}$$

The result follows from (2.2.36), Step 1, and the remainder of the argument in Step 1 of Lemma 2, which shows that all other terms can be absorbed by $-C\|\nabla\Delta R\|^2_{L^2}$.

Step 3: We apply $\widetilde{\nabla}$ to (2.2.43), the triangle inequality, and Steps 1 and 2 to show

$$\|\widetilde{\nabla}\widetilde{\Delta}v\|_{L^2(\tilde{\omega})} \leq Ce^{-kt} \quad (2.2.59)$$

$$\|\widehat{\nabla}\widehat{\Delta}v\|_{L^2(\tilde{\omega})} \leq Ce^{-kt} \quad (2.2.60)$$

Step 4: $\|\widetilde{\Delta}^2 f\|_{L^2(\tilde{\omega})} \approx \|\widehat{\Delta}^2 f\|_{L^2(\hat{\omega})}$ follows from the triangle inequality, Hölder's inequality, Lemma 1, Claim 2, and Step 3.

Step 5: Using the same method as in Step 2 and applying the results of Step 4, we prove Lemma 3 for $n = 1$, $\|\Delta R\|_{L^2} \le Ce^{-kt}$.

Step 6: The proof is completed using an inductive argument similar to the one used in the proof of Lemma 2.

□

2.2.3 Any closed compact surfaces

Observe that by Lemmas 2 and 3, we know for any closed compact surface, multi indices α and β with $|\alpha| = 2n$ and $|\beta| = 2n + 1$, for all $n \ge 0$, $t > 0$, and some constants C and k

$$\|\Delta^n R\|_{L^2} \le Ce^{-kt} \tag{2.2.61}$$

$$\|\nabla^\alpha u\|_{L^2} \le \|\Delta^n u\|_{L^2} \tag{2.2.62}$$

$$\|\nabla^\beta u\|_{L^2} \le \|\nabla\Delta^n u\|_{L^2} \tag{2.2.63}$$

Lemma 4. $\|\Delta^n u\|_{L^2} \le Ce^{-kt}$ *for all* $n \ge 0$ *and* $t > 0$.

Proof. We know this is true for $n = 1$ and

$$\Delta u = \frac{-1}{2}R + \frac{1}{2}\widehat{R}e^{-2u}. \tag{2.2.64}$$

Assuming true for n, we apply Δ^n to both sides of the equation. The result is

$$\Delta^{n+1}u = \frac{-1}{2}\Delta^n R - \widehat{R}e^{-2u}\Delta^n u + \text{lower order derivatives of } u. \tag{2.2.65}$$

We take the L^2 norm with respect to the evolving metric and apply the triangle inequality, Hölder's inequality, Corollary 1, and inequalities (2.2.61)-(2.2.63) to get

$$\|\Delta^{n+1}u\|_{L^2} \le C\|\Delta^n R\|_{L^2} + C\|\Delta^n u\|_{L^2} \le Ce^{-kt}.$$ □

Corollary 5. $\|\widehat{\Delta}^n u\|_{L^2(\hat{\omega})} \leq Ce^{-kt}$ *for all* $n \geq 0$ *and* $t > 0$.

Proof. We write $\|\widehat{\Delta}^n u\|_{L^2(\hat{\omega})}$ in terms of the evolving metric, then we apply the triangle inequality, Hölder's inequality, Corollary 1, inequalities (2.2.61)-(2.2.63), and Lemma 4.

$$\begin{aligned} \|\widehat{\Delta}^n u\|_{L^2(\hat{\omega})} &= \|e^u \Delta(e^{2u}\Delta(\ldots(e^{2u}\Delta u)\ldots)\|_{L^2} &(2.2.66)\\ &= \|e^{(2n-1)u}\left[\Delta^n u + \text{lower order derivatives of } u\right]\|_{L^2} &(2.2.67)\\ &\leq C\|\Delta^n u\|_{L^2} \leq Ce^{-kt}. \qquad \square \end{aligned}$$

Now we prove the main theorem.

Proof of Theorem 1. By Corollary 5, we know that for all $s > 0$ and $t > 0$

$$\|u\|_{H^{(s)}(X,\hat{\omega})} \leq Ce^{-kt}. \tag{2.2.68}$$

Then by the Sobolev imbedding theorem for all $r \geq 0$ and $t > 0$

$$\|u\|_{C^r(X,\hat{\omega})} \leq Ce^{-kt}. \qquad \square$$

3 The Donaldson Heat Flow

3.1 Background

Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X, equipped with a Kähler metric $\omega = \frac{i}{2} g_{\bar{k}j} dz^j \wedge d\bar{z}^k$. Our notation is as follows. Locally, the sections ϕ of E are viewed as vector-valued functions $\phi^\alpha(z)$, $1 \leq \alpha \leq r$, where r is the rank of E. A Hermitian metric on E is locally a positive definite Hermitian form $H_{\bar{\alpha}\beta}$. We denote by $H^{\alpha\bar{\gamma}}$ its inverse, and by ${F_{\bar{k}j}}^\alpha{}_\beta$ its curvature,

$$ {F_{\bar{k}j}}^\alpha{}_\beta = -\partial_{\bar{k}}(H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta}), \qquad H^{\alpha\bar{\gamma}} H_{\bar{\gamma}\beta} = \delta^\alpha{}_\beta. \tag{3.1.1} $$

It is also convenient to use matrix notation, in which case $H_{\bar{\alpha}\beta}$ is denoted by H, $H^{\alpha\bar{\gamma}}$ is denoted by H^{-1}, and $F_{\bar{k}j} = -\partial_{\bar{k}}(H^{-1}\partial_j H)$. We note that $F_{\bar{k}j} dz^j \wedge d\bar{z}^k$ is a (1,1)-form valued in the bundle $End(E)$ of endomorphisms of E.

Fix now a Hermitian metric $K_{\bar{\alpha}\beta}$ on the bundle E, with $1 \leq \alpha, \beta \leq r$, where r is the rank of E. The Donaldson heat flow is the following flow of metrics $H_{\bar{\alpha}\beta}(t)$ on E,

$$ H^{-1}\dot{H} = (\Lambda F - \mu I), \qquad H(0) = K, \tag{3.1.2} $$

where Λ is the standard Hodge operator $\Lambda F = g^{j\bar{k}} F_{\bar{k}j}$, and μ is given by

$$ \mu = \frac{\int_X \mathrm{Tr} F \wedge \frac{\omega^{n-1}}{(n-1)!}}{r \int_X \frac{\omega^n}{n!}}. \tag{3.1.3} $$

It is easily seen that μ depends only on the class $[\omega]$ of ω and on the first Chern class $c_1(E)$ of E, and not on the metrics $g_{\bar{k}j}$ and $H_{\bar{\alpha}\beta}$ themselves.

It is sometimes convenient to use the endomorphism h defined by

$$ h^\alpha{}_\beta = K^{\alpha\bar{\gamma}} H_{\bar{\gamma}\beta}, \tag{3.1.4} $$

instead of the metric $H_{\bar{\gamma}\beta}$ itself. In terms of h, the Donaldson heat flow can be readily verified to be equivalent to the following flow of endomorphisms,

$$ h^{-1}\dot{h} = -(\Lambda F - \mu I), \qquad h(0) = I. \tag{3.1.5} $$

3.1.1 The flow of the curvature

Our main goal is to discuss some qualitative properties of the flow. In order to do so, we shall need the evolutions under the flow of key quantities associated with the curvature. It is convenient to derive all these evolutions here in this preliminary subsection.

3.1.2 General variational formulas

The curvature $F_{\bar{k}j}$ can be defined by the following equation on endomorphisms,

$$[\nabla_{\bar{k}}, \nabla_j] = -F_{\bar{k}j}. \tag{3.1.6}$$

Under a variation $\delta H_{\bar{\alpha}\beta}$ of metrics, $\delta\nabla_{\bar{k}} = 0$, and

$$\delta\nabla_j = \delta(H^{-1}\partial_j H) = H^{-1}(\partial_j \delta H - \delta H) = H^{-1}\nabla_j(\delta H). \tag{3.1.7}$$

This implies

$$\begin{aligned} \delta F_{\bar{k}j} &= \delta[\nabla_j, \nabla_{\bar{k}}] = [\delta\nabla_j, \nabla_{\bar{k}}] + [\nabla_j, \delta\nabla_{\bar{k}}] \\ &= -\nabla_{\bar{k}}(H^{-1}\nabla_j(\delta H)). \end{aligned} \tag{3.1.8}$$

3.1.3 The flow of the curvature

Specializing now to the Donaldson heat flow (3.1.2), we find

$$\dot{F}_{\bar{k}j} = -\nabla_{\bar{k}}(H^{-1}\partial_j \nabla \dot{H}) = \nabla_{\bar{k}}\nabla_j(\Lambda F - \mu I) = \nabla_{\bar{k}}\nabla_j(\Lambda F), \tag{3.1.9}$$

and, upon contracting with Λ,

$$(\Lambda F)^{\dot{}} = g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j(\Lambda F) = \bar{\Delta}(\Lambda F), \tag{3.1.10}$$

where $\Delta \equiv g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}$. However, on endomorphisms $W^\alpha{}_\beta$, we have the following commutation rules

$$\begin{aligned} g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j W^\alpha{}_\beta &= g^{j\bar{k}}(\nabla_j\nabla_{\bar{k}}W^\alpha{}_\beta - F_{\bar{k}j}{}^\alpha{}_\gamma W^\gamma{}_\beta + W^\alpha{}_\gamma F_{\bar{k}j}{}^\gamma{}_\beta) \\ &= \Delta W^\alpha{}_\beta - (\Lambda F)^\alpha{}_\gamma W^\gamma{}_\beta + W^\alpha{}_\gamma(\Lambda F)^\gamma{}_\beta, \end{aligned} \tag{3.1.11}$$

or, in simpler matrix notation,

$$\bar{\Delta} W = \Delta W - (\Lambda F)W + W(\Lambda F). \tag{3.1.12}$$

In particular, taking $W = \Lambda F$, we obtain

$$(\Lambda F)^{\dot{}} = \bar{\Delta}(\Lambda F) = \Delta(\Lambda F). \tag{3.1.13}$$

We recast now the evolution equation for the full curvature tensor $F_{\bar{k}j}$ as a non-linear heat equation for $F_{\bar{k}j}$. For this we apply the Bianchi identity to the right hand side $\nabla_{\bar{k}}\nabla_j(\Lambda F)$,

$$\begin{aligned} \nabla_{\bar{k}}\nabla_j(\Lambda F) &= g^{p\bar{q}}\nabla_{\bar{k}}\nabla_p F_{\bar{q}j}{}^\alpha{}_\beta \\ &= \Delta F_{\bar{k}j}{}^\alpha{}_\beta \\ &\quad + g^{p\bar{q}}\left(-R_{\bar{k}p\bar{q}}{}^{\bar{\ell}}F_{\bar{\ell}j}{}^\alpha{}_\beta + R_{\bar{k}p}{}^m{}_j F_{\bar{q}m}{}^\alpha{}_\beta - F_{\bar{k}p}{}^\alpha{}_\gamma F_{\bar{q}j}{}^\gamma{}_\beta + F_{\bar{k}p}{}^\gamma{}_\beta F_{\bar{q}j}{}^\alpha{}_\gamma\right). \end{aligned} \tag{3.1.14}$$

Substituting in the evolution equation for $F_{\bar{k}j}$, we obtain, in matrix notation,

$$\dot{F}_{\bar{k}j} = \Delta F_{\bar{k}j} - R_{\bar{k}}{}^{\bar{\ell}}F_{\bar{\ell}j} + R_{\bar{k}}{}^{\bar{q}m}{}_j F_{\bar{q}m} - F_{\bar{k}}{}^{\bar{q}}F_{\bar{q}j} + F_{\bar{q}j}F_{\bar{k}}{}^{\bar{q}}. \tag{3.1.15}$$

When the detailed expression for the multilinear terms on the right hand side does not play any significant role, it is convenient to abbreviate this type of equation by

$$\dot{F} = \Delta F + R \star F + F \star F. \tag{3.1.16}$$

3.1.4 The flow of curvature densities

As in [12, 29], we introduce the following densities and derive their evolution,

$$\begin{aligned} e &= |F|^2 \quad (\equiv F_{\bar{k}j}{}^\alpha{}_\beta \overline{F_{\bar{\ell}m}{}^\gamma{}_\delta} g^{\ell\bar{k}} g^{j\bar{m}} H_{\bar{\gamma}\alpha} H^{\beta\bar{\delta}}) \\ \hat{e} &= |\Lambda F|^2 \\ e_k &= |\nabla^k F|^2. \end{aligned} \tag{3.1.17}$$

Consider first the flow of $\hat{e}$,

$$\begin{aligned}(\partial_t - \Delta)\hat{e} &= (\partial_t - \Delta)(\Lambda F, \Lambda F) \\ &= ((\partial_t - \Delta)(\Lambda F), \Lambda F) + (\Lambda F, (\partial_t - \bar{\Delta})(\Lambda F)) - |\nabla(\Lambda F)|^2 - |\bar{\nabla}(\Lambda F)|^2 \\ &= -|\nabla(\Lambda F)|^2 - |\bar{\nabla}(\Lambda F)|^2 \leq 0, \qquad (3.1.18)\end{aligned}$$

in view of the equation (3.1.13. This implies the following important property of the Donaldson heat flow:

Lemma 5. *We have the following uniform estimate*

$$\sup_{t\geq 0}\|\Lambda F\|_{C^0} < \infty. \qquad (3.1.19)$$

Next, consider the flow for e,

$$\begin{aligned}(\partial_t - \Delta)e &= (\partial_t - \Delta)(F, F) \\ &= ((\partial_t - \Delta)F, F) + (F, (\partial_t - \bar{\Delta})F) - |\nabla F|^2 - |\bar{\nabla} F|^2. \qquad (3.1.20)\end{aligned}$$

Substituting in the equation (3.1.16), and noting that $\bar{\Delta}F = \Delta F + R \star F + F \star F$, we find $(\partial_t - \Delta)e \leq C(|F|^2 + |F|^3)$, or

$$(\partial_t - \Delta)e \leq C\left(e + e^{\frac{3}{2}}\right), \qquad (3.1.21)$$

where C is a constant independent of time.

Similarly, as shown in [12, 29], we have

$$(\partial_t - \Delta)e_k \leq C_k\, e_k^{\frac{1}{2}}\left(\sum_{i+j=k} e_i^{\frac{1}{2}}(e_j^{\frac{1}{2}} + 1)\right). \qquad (3.1.22)$$

3.1.5 Normalization of the flow

The Donaldson heat flow is a parabolic flow, so its existence for some time interval $[0, T)$ is guaranteed by the general theory of parabolic partial differential equations. It is well-known that the flow can be normalized as follows:

Lemma 6. *There exists an initial metric K so that for all endomorphisms h in the maximal existence interval $[0, T)$, we have*

$$\det h = 1. \tag{3.1.23}$$

Proof. First, we show that there exists a metric $K_{\bar{\alpha}\beta}$ on E with

$$\mathrm{Tr}(\Lambda F_K - \mu I) = 0, \tag{3.1.24}$$

where F_K denotes the curvature of $K_{\bar{\alpha}\beta}$. To see this, let $\hat{K}_{\bar{\alpha}\beta}$ be any metric on E, and consider the metric $K_{\bar{\alpha}\beta} = e^{\phi}\hat{K}_{\bar{\alpha}\beta}$. Clearly,

$$(F_K)_{\bar{k}j} = -\partial_{\bar{k}}(e^{-\phi}\hat{K}^{-1}\partial_j(e^{\phi}\hat{K})) = -\partial_j\partial_{\bar{k}}\phi\, I + (F_{\hat{K}})_{\bar{k}j}, \tag{3.1.25}$$

and hence

$$\mathrm{Tr}(\Lambda F_K) = -(\Delta\phi)r + \mathrm{Tr}(\Lambda F_{\hat{K}}). \tag{3.1.26}$$

Thus, to insure that $\mathrm{Tr}(\Lambda F_K - \mu I) = 0$, it suffices to choose ϕ so that

$$\Delta\phi = \frac{1}{r}\mathrm{Tr}(\Lambda F_{\hat{K}} - \mu I). \tag{3.1.27}$$

The definition of the constant μ is precisely the value which guarantees that the right hand side integrates to 0 with respect to the measure ω^n. Thus the equation admits a C^∞ solution ϕ.

Consider now the Donaldson heat flow (3.1.4) where the initial metric K satisfies the condition $\mathrm{Tr}(\Lambda F_K - \mu I) = 0$. However, in view of the equation (3.1.13) derived earlier, we have

$$(\partial_t - \Delta)\mathrm{Tr}(\Lambda F - \mu I) = 0. \tag{3.1.28}$$

By the uniqueness of solutions of the heat equation, and the fact that $\mathrm{Tr}(\Lambda F - \mu I) = 0$ at time $t = 0$, it follows that $\mathrm{Tr}(\Lambda F - \mu I) = 0$ for all times t.

This implies that $\det h$ is constant, since

$$(\log\, \det h)^{\dot{}} = h^{\alpha\bar{\beta}}\dot{h}_{\bar{\beta}\alpha} = \mathrm{Tr}(h^{-1}\dot{h}) = -\mathrm{Tr}(\Lambda F - \mu I) = 0. \tag{3.1.29}$$

Since $\det h = 1$ at time $t = 0$, the lemma is proved. ∎

3.2 C^k Estimates for the Donaldson Heat Flow

It is one of the basic results of the theory [12] that the flow exists for all times, so that $T = \infty$. Here we shall establish the following theorem:

Theorem 2. *Consider the Donaldson heat flow (3.1.2), with an initial metric chosen as in Lemma 6, so that* $\det h = 1$ *for all times. Then the hypothesis*

$$\sup_{0 \leq t < T} \|\mathrm{Tr}\, h\|_{C^0} \leq A_{0,T} < \infty \tag{3.2.1}$$

implies that for any $k \in \mathbf{N}$*, we have*

$$\sup_{0 \leq t < T} \|\mathrm{Tr}\, h\|_{C^k} \leq A_{k,T} < \infty. \tag{3.2.2}$$

In particular, for $T = \infty$*, we have*

$$\sup_{0 \leq t} \|\mathrm{Tr}\, h\|_{C^0} < \infty \Rightarrow \sup_{0 \leq t} \|\mathrm{Tr}\, h\|_{C^k} < \infty, \qquad k \in \mathbf{N}. \tag{3.2.3}$$

3.2.1 The C^1 estimate

The first step is to show that

$$\sup_{0 \leq t < T} \|\mathrm{Tr}\, h\|_{C^0} < \infty \Rightarrow \sup_{0 \leq t < T} \|\nabla h\, h^{-1}\|^2_{C^0} < \infty. \tag{3.2.4}$$

This C^1 estimate for h is established in the same way as the Calabi identity for C^3 estimate for the potential in complex Monge-Ampère equations [40]. We follow the derivation given in [20], which adapts particularly well in our case, since it is formulated in terms of endomorphisms. Thus set

$$W^\alpha_{j\beta} = (\nabla_j h\, h^{-1})^\alpha{}_\beta, \qquad S = |W|^2. \tag{3.2.5}$$

Then the same formula as in [20] gives

$$\begin{aligned}(\Delta - \partial_t)S &= g^{j\bar k} H_{\bar\gamma\alpha} H^{\beta\bar\delta} \left((\Delta - \partial_t)W^\alpha_{j\beta}\overline{W^\gamma_{k\delta}}\right) + W^\alpha_{j\beta}\overline{(\Delta - \partial_t)W^\gamma_{k\delta}} + |\bar\nabla W|^2 + |\nabla W|^2 \\ &\quad + g^{j\bar k} H_{\bar\gamma\alpha} H^{\beta\bar\delta} W^\alpha_{j\beta} R_{\bar k}{}^{\bar\ell}\overline{W^\gamma_{\ell\delta}} \\ &\quad + g^{j\bar k}(h^{-1}\dot h + \Lambda F)^{\beta\bar\delta} W^\alpha_{j\beta}\overline{W^\gamma_{k\delta}} H_{\bar\gamma\alpha} + g^{j\bar k}(h^{-1}\dot h + \Lambda F)_{\bar\gamma\alpha} W^\alpha_{j\beta}\overline{W^\gamma_{k\delta}} H^{\beta\bar\delta}\end{aligned} \tag{3.2.6}$$

More succinctly, we can express this identity as

$$\begin{aligned}(\Delta-\partial_t)S &= \langle(\Delta-\partial_t)W,W\rangle+\langle W,(\bar\Delta-\partial_t)W\rangle+|\bar\nabla W|^2+|\nabla W|^2\\ &\quad +g^{j\bar k}\mathrm{Tr}\,W_j(h^{-1}\dot h+\Lambda F)\overline{W_k}+g^{j\bar k}\mathrm{Tr}\,\overline{W_k}(h^{-1}\dot h+\Lambda F)W_j+R^{j\bar\ell}\mathrm{Tr}\,W_j\overline{W_{\bar\ell}}.\end{aligned} \tag{3.2.7}$$

We shall show that

$$|(\Delta-\partial_t)W|\le C_1|W|+C_2. \tag{3.2.8}$$

To see this, we begin by evaluating the left hand side.

Recalling the formula $(F_K)_{\bar kj}-F_{\bar kj}=\nabla_{\bar k}(\nabla_j h\,h^{-1})$, we obtain

$$\begin{aligned}\Delta(\nabla_j h\,h^{-1}) &= g^{m\bar k}\nabla_m((F_K)_{\bar kj}-F_{\bar kj})\\ &= g^{m\bar k}\nabla_m(F_K)_{\bar kj}-\nabla_j(\Lambda F),\end{aligned} \tag{3.2.9}$$

in view of the Bianchi identity. Since we also have

$$\partial_t(\nabla_j h\,h^{-1})=\nabla_j(h^{-1}\dot h), \tag{3.2.10}$$

we find

$$(\Delta-\partial_t)(\nabla_j h\,h^{-1}) = g^{m\bar k}\nabla_m(F_K)_{\bar kj}-\nabla_j(\Lambda F+h^{-1}\dot h). \tag{3.2.11}$$

Since h flows by the Donaldson heat flow (3.1.4), we arrive at

$$(\Delta-\partial_t)(\nabla_j h\,h^{-1})=g^{m\bar k}\nabla_m(F_K)_{\bar kj}-\nabla_j(\mu I)=g^{m\bar k}\nabla_m(F_K)_{\bar kj}. \tag{3.2.12}$$

Since the connection is of size $O(|W|)$, this establishes the desired estimate.

With this, we can conclude that

$$(\Delta-\partial_t)S\ge|\bar\nabla W|^2+|\nabla W|^2-C_3S-C_4. \tag{3.2.13}$$

We need to find now a counterterm to take care of the expression $-C_3S$ on the right hand side of the previous inequality. For this, we begin by calculating

$$\Delta\mathrm{Tr}\,h=\mathrm{Tr}((\Lambda F_K-\Lambda F)h)+g^{p\bar q}\mathrm{Tr}(\nabla_p h\,h^{-1}\nabla_{\bar q}h), \tag{3.2.14}$$

which implies that

$$(\Delta - \partial_t)\mathrm{Tr}\, h = \mathrm{Tr}((\Lambda F_K - \mu I)h) + g^{p\bar q}\mathrm{Tr}(\nabla_p h\, h^{-1}\nabla_{\bar q}h). \qquad (3.2.15)$$

We now need the following version of the classical inequality of Yau [40], formulated in terms of endomorphisms:

Lemma 7. *We have for some positive constant c,*

$$g^{p\bar q}\mathrm{Tr}(\nabla_p h\, h^{-1}\nabla_{\bar q}h) \geq c\,|W|^2. \qquad (3.2.16)$$

Proof. To see this, we rewrite the left hand side as follows

$$g^{p\bar q}\mathrm{Tr}(\nabla_p h\, h^{-1}\nabla_{\bar q}h) = g^{p\bar q}\mathrm{Tr}((\nabla_p h\, h^{-1})(\nabla_{\bar q}h^{-1})h) = g^{p\bar q}W^{\alpha}_{p\beta}W^{\beta}_{\bar q\delta}\,h^{\delta}{}_{\alpha}. \qquad (3.2.17)$$

This can be re-written in turn as

$$g^{p\bar q}\mathrm{Tr}(\nabla_p h\, h^{-1}\nabla_{\bar q}h) = g^{p\bar q}W^{\alpha}_{p\beta}\overline{W^{\nu}_{q\rho}}\,H^{\beta\bar\rho}H^{\vee}_{\bar\nu\alpha} \qquad (3.2.18)$$

where $H^{\vee}$ is defined by $H^{\vee}_{\bar\nu\alpha} = H_{\bar\nu\delta}\hat H^{\delta\bar\mu}H_{\bar\mu\alpha}$. Since the metrics $H_{\bar\alpha\beta}$ are uniformly bounded from above and below, the desired statement follows.

In view of the hypothesis $\|\mathrm{Tr}\,h\|_{C^0} \leq C_5$, we can apply the preceding lemma and obtain

$$(\Delta - \partial_t)\mathrm{Tr}\, h \geq -C_6 + C_7 S, \qquad (3.2.19)$$

for some strictly positive constant C_7. It follows that for a constant A large enough, we have

$$(\Delta - \partial_t)(S + A\mathrm{Tr}\, h) \geq |\bar\nabla W|^2 + |\nabla W|^2 + C_8 S - C_9. \qquad (3.2.20)$$

An easy application of the maximum principle implies now that $\|S\|_{C^0}$ is bounded by a constant. $\square$

3.2.2 The $W^{2,p}$ estimate for h

We show next that

$$\sup_{t\geq 0}\|\nabla h\, h^{-1}\|_{C^0} < \infty \Rightarrow \|h\|_{W^{2,p}} < \infty \tag{3.2.21}$$

for any $1 \leq p < \infty$. This follows from the same arguments as in [12, 29]. Recall the following classic formula relating the curvatures of two metrics $H_{\bar{\alpha}\beta}$ and $K_{\bar{\alpha}\beta}$,

$$F_{\bar{k}j} - (F_K)_{\bar{k}j} = -\partial_{\bar{k}}(h^{-1}\hat{\nabla}_j h) = -h^{-1}\hat{\nabla}_{\bar{k}}\hat{\nabla}_j h + h^{-1}\hat{\nabla}_{\bar{k}} h\, h^{-1}\hat{\nabla}_j h \tag{3.2.22}$$

where $\hat{\nabla}$ denotes covariant derivatives with respect to the metric $K_{\bar{\alpha}\beta}$. Thus

$$\hat{\Delta} h = h(\Lambda F_K - \Lambda F) + g^{j\bar{k}}\hat{\nabla}_{\bar{k}} h\, h^{-1}\, \hat{\nabla}_j h. \tag{3.2.23}$$

Since h is uniformly bounded by hypothesis, and since we can apply Lemma 5, we see that the right hand side is in L^∞. By the general L^p theory of elliptic PDE's, it follows that for any $1 \leq p < \infty$, we have

$$\|h\|_{W^{2,p}} < \infty. \tag{3.2.24}$$

3.2.3 The L^p boundedness of $F_{\bar{k}j}$

The uniform boundedness in C^0 and $W^{2,p}$ of h and the uniform boundedness of $\nabla h\, h^{-1}$ imply that for any $1 \leq p < \infty$,

$$\sup_{0\leq t}\|F\|_{L^p} < \infty. \tag{3.2.25}$$

3.2.4 The C^0 boundedness of $F_{\bar{k}j}$ for finite time T

We can now quote the following lemma from [12, 29], which shows that the C^0 boundedness of the curvature $F_{\bar{k}j}$ is a consequence of the L^p boundedness and the differential inequality (3.1.21):

Lemma 8. *Let p be any value strictly greater than $p_0 = 3n$, and let $F_{\bar{k}j}$ be the curvature of the metric $H_{\bar{\alpha}\beta}$ evolving according to the Donaldson heat flow. Then*

$$\sup_{0\leq t<T}\|F\|_{L^p} < \infty \Rightarrow \sup_{0\leq t<T}\|F\|_{C^0} \leq C_T < \infty. \tag{3.2.26}$$

This lemma does not extend to the case $T = \infty$.

3.2.5 The L^p boundedness for $\nabla^k F$

Our boundedness arguments will make use of the following elementary claims.

Claim 3. *For any function $f \geq 0$, if $\dot{f} \leq -kf + c$, then f is bounded for $t \in [0, \infty)$.*

Proof. Rearranging the given inequality, we have

$$\begin{aligned} kf + \dot{f} &\leq c && (3.2.27)\\ ke^{kt} f + e^{kt}\dot{f} &\leq ce^{kt} && (3.2.28)\\ \frac{d}{dt}[e^{kt} f] &\leq ce^{kt} && (3.2.29)\\ e^{kt} f &\leq \frac{c}{k}e^{kt} + A && (3.2.30)\\ 0 \leq f &\leq \frac{c}{k} + Ae^{-kt} \leq C. \end{aligned}$$

□

Claim 4. *For any real numbers a and b, and for all $\varepsilon > 0$*

$$ab \leq \varepsilon a^2 + C_\varepsilon b^2. \tag{3.2.31}$$

In [35], the following lemma is derived from the basic interpolation inequality in [16].

Lemma 9. *Let $E \to X$ be a holomorphic vector bundle over a compact complex Kähler manifold X, and T be any (1,1)-form on X valued in the bundle $End(E)$. For $0 < i < k$ and $1 \leq q < k/i$ we have*

$$\|\nabla^i T\|_{L^{2q}} \leq C\|\nabla^k T\|_{L^2}^{\frac{i}{k}} \|T\|_{L^p}^{\frac{p}{2}\left(\frac{1}{q}-\frac{i}{k}\right)}, \tag{3.2.32}$$

where

$$p = 2q\frac{k-i}{k-qi}, \tag{3.2.33}$$

and the constant C depends only on the dimension of X, i, k, and q.

Proof. (Adapted from [35]). The basic interpolation inequality in [16] is

$$\|\nabla T\|_{L^{2r}} \leq C\|\nabla^2 T\|_{L^p}^{\frac{1}{2}}\|T\|_{L^q}^{\frac{1}{2}}, \tag{3.2.34}$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{3.2.35}$$

We choose an increasing sequence of numbers $r_0 > r_1 > \ldots > r_k$ such that

$$\frac{2}{r_i} = \frac{1}{r_{i-1}} + \frac{1}{r_{i+1}}. \tag{3.2.36}$$

Then we let

$$f(i) = \|\nabla^i T\|_{L^{r_i}}. \tag{3.2.37}$$

Then the basic interpolation inequality says that

$$f(i) \leq Cf(i-1)^{\frac{1}{2}} f(i+1)^{\frac{1}{2}}. \tag{3.2.38}$$

By a lemma in [16], it follows that

$$f(i) \leq C' f(k)^{\frac{i}{k}} f(0)^{1-\frac{i}{k}}. \tag{3.2.39}$$

If we choose r_i such that $\frac{1}{r_i}$ are in an arithmetic progression, $r_k = 2$, and $r_i = 2q$, then

$$r_0 = 2q\frac{k-i}{k-qi}, \tag{3.2.40}$$

and we obtain the desired inequality. □

This interpolation inequality together with the L^p boundedness of F implies

Claim 5. *For $k \in \mathbb{N}$,*

$$\|\bar{\nabla}\nabla^k F\|_{L^2}^2 \leq C\|\nabla^{k+1} F\|_{L^2}^2 + C\|\nabla^k F\|_{L^2}^2, \tag{3.2.41}$$

$$\|\bar{\nabla}\nabla^k (\Lambda F)\|_{L^2}^2 \leq C\|\nabla^{k+1} (\Lambda F)\|_{L^2}^2 + C. \tag{3.2.42}$$

Proof. The proof follows from integration by parts, Hölder's inequality, and Lemma 9.

$$\begin{aligned}\|\bar\nabla\nabla^k F\|_{L^2}^2 &= \|\nabla^{k+1}F\|_{L^2}^2 + \int_X R\star\nabla^k F\star\bar\nabla^k F\omega^n + \int_X F\star\nabla^k F\star\bar\nabla^k F\omega^n & (3.2.43)\\ &\leq \|\nabla^{k+1}F\|_{L^2}^2 + \|R\|_{L^\infty}\|\nabla^k F\|_{L^2}^2 + \|F\|_{L^p}\|\nabla^k F\|_{L^{2+\delta}}^2 & (3.2.44)\\ &\leq C\|\nabla^{k+1}F\|_{L^2}^2 + C\|\nabla^k F\|_{L^2}^2. & (3.2.45)\end{aligned}$$

Also,

$$\begin{aligned}\|\bar\nabla\nabla^k(\Lambda F)\|_{L^2}^2 &= \|\nabla^{k+1}(\Lambda F)\|_{L^2}^2 + \int_X F\star\nabla^k(\Lambda F)\star\bar\nabla^k(\Lambda F)\omega^n & (3.2.46)\\ &\leq \|\nabla^{k+1}(\Lambda F)\|_{L^2}^2 + \|F\|_{L^p}\|\nabla^k(\Lambda F)\|_{L^{2+\delta}}^2 & (3.2.47)\\ &\leq C\|\nabla^{k+1}(\Lambda F)\|_{L^2}^2 + C & (3.2.48)\end{aligned}$$

□

From this claim, we deduce the following claim.

Claim 6. *For $k\in\mathbb{N}$,*

$$\begin{aligned}\|\nabla^k F\|_{L^2}^2 &\leq \varepsilon\|\nabla^{k+1}F\|_{L^2}^2 + C_\varepsilon\|\nabla^{k-1}F\|_{L^2}^2, & (3.2.49)\\ \|\nabla^k(\Lambda F)\|_{L^2}^2 &\leq \varepsilon\|\nabla^{k+1}(\Lambda F)\|_{L^2}^2 + C_\varepsilon\|\nabla^{k-1}(\Lambda F)\|_{L^2}^2 + C\|\nabla^{k-1}(\Lambda F)\|_{L^2}. & (3.2.50)\end{aligned}$$

Proof. From integration by parts, Hölder's inequality, and Claims 4 and 5, we have

$$\begin{aligned}\|\nabla^k F\|_{L^2}^2 &= -\int_X \bar\nabla\nabla^k F\bar\nabla^{k-1}F\omega^n & (3.2.51)\\ &\leq \|\bar\nabla\nabla^k F\|_{L^2}\|\nabla^{k-1}F\|_{L^2} & (3.2.52)\\ &\leq C\|\nabla^{k+1}F\|_{L^2}\|\nabla^{k-1}F\|_{L^2} + \|\nabla^k F\|_{L^2}\|\nabla^{k-1}F\|_{L^2} & (3.2.53)\\ &\leq \varepsilon\|\nabla^{k+1}F\|_{L^2}^2 + \delta\|\nabla^k F\|_{L^2}^2 + C_\varepsilon\|\nabla^{k-1}F\|_{L^2}^2 & (3.2.54)\\ &\leq \varepsilon\|\nabla^{k+1}F\|_{L^2}^2 + C_\varepsilon\|\nabla^{k-1}F\|_{L^2}^2. & (3.2.55)\end{aligned}$$

Also,

$$\begin{aligned}\|\nabla^k(\Lambda F)\|_{L^2}^2 &= -\int_X \bar\nabla\nabla^k(\Lambda F)\bar\nabla^{k-1}(\Lambda F)\omega^n & (3.2.56)\\ &\leq \|\bar\nabla\nabla^k(\Lambda F)\|_{L^2}\|\nabla^{k-1}(\Lambda F)\|_{L^2} & (3.2.57)\\ &\leq C\|\nabla^{k+1}(\Lambda F)\|_{L^2}\|\nabla^{k-1}(\Lambda F)\|_{L^2} + C\|\nabla^{k-1}(\Lambda F)\|_{L^2} & (3.2.58)\\ &\leq \varepsilon\|\nabla^{k+1}F\|_{L^2}^2 + C_\varepsilon\|\nabla^{k-1}F\|_{L^2}^2 + C\|\nabla^{k-1}(\Lambda F)\|_{L^2}. \square\end{aligned}$$

By induction, we have

Claim 7. *For $k \in \mathbb{N}$,*

$$\begin{aligned} \|\nabla^k F\|_{L^2}^2 &\leq \varepsilon \|\nabla^{k+1} F\|_{L^2}^2 + C_\varepsilon \|F\|_{L^2}^2 && (3.2.59)\\ \|\nabla^k (\Lambda F)\|_{L^2}^2 &\leq \varepsilon \|\nabla^{k+1}(\Lambda F)\|_{L^2}^2 + C_\varepsilon \|\Lambda F\|_{L^2}^2 + C\|\Lambda F\|_{L^2}. && (3.2.60)\end{aligned}$$

Using the interpolation inequality, we will establish the L^p boundedness for $\nabla^k F$.

Lemma 10. *Let $F_{\bar{k}j}$ be the curvature of the metric $H_{\bar{\alpha}\beta}$ evolving according to the Donaldson heat flow. For any $1 \leq p < \infty$ and any $k \in \mathbb{N}$,*

$$\sup_{0 \leq t} \|\nabla^k F\|_{L^p} < \infty. \tag{3.2.61}$$

Proof. The key step is proving the L^2 boundedness of ∇F. After we know this bound, we can bound the L^2 norms of all of the higher derivatives of F. We will then be able to get L^p bounds on ∇F by Lemma 9. Proceeding inductively, we will be able to complete the proof.

Step 1: The first step is to show that

$$\sup_{0 \leq t} \|\nabla \Lambda F\|_{L^2} < \infty. \tag{3.2.62}$$

This is shown by first calculating

$$\begin{aligned}\frac{d}{dt}\|\nabla(\Lambda F)\|_{L^2}^2 &= -2\|\nabla^2(\Lambda F)\|_{L^2}^2 + \int_X F \star \nabla(\Lambda F) \star \bar{\nabla}(\Lambda F)\omega^n \\ &\quad + \int_X \Lambda F \star \nabla(\Lambda F) \star \bar{\nabla}(\Lambda F)\omega^n. && (3.2.63)\end{aligned}$$

For the second term, we apply Hölder's inequality and then apply Lemma 9 with $i = 1$, $k = 2$, and $q = 3/2$.

$$\begin{aligned}\int_X F \star \nabla(\Lambda F) \star \bar{\nabla}(\Lambda F)\omega^n &\leq \|F\|_{L^3}\|\nabla(\Lambda F)\|_{L^3}^2 && (3.2.64)\\ &\leq C\|\nabla^2(\Lambda F)\|_{L^2}\|\Lambda F\|_{L^6}^6 && (3.2.65)\\ &\leq C\|\nabla^2(\Lambda F)\|_{L^2} && (3.2.66)\\ &\leq \varepsilon\|\nabla^2(\Lambda F)\|_{L^2}^2 + C. && (3.2.67)\end{aligned}$$

For the third term, we also apply Hölder's inequality and then apply Lemma 9 with $i = 1$, $k = 2$, and $q = 3/2$.

$$\begin{aligned}\int_X \Lambda F \star \nabla(\Lambda F) \star \bar{\nabla}(\Lambda F)\omega^n &\leq \|\Lambda F\|_{L^3}\|\nabla(\Lambda F)\|_{L^3}^2 && (3.2.68)\\ &\leq C\|\nabla^2(\Lambda F)\|_{L^2}\|\Lambda F\|_{L^6}^6 && (3.2.69)\\ &\leq C\|\nabla^2(\Lambda F)\|_{L^2} && (3.2.70)\\ &\leq \varepsilon\|\nabla^2(\Lambda F)\|_{L^2}^2 + C. && (3.2.71)\end{aligned}$$

Combining these estimates and Claim 7, we have

$$\begin{aligned}\frac{d}{dt}\|\nabla(\Lambda F)\|_{L^2}^2 &= -C\|\nabla^2(\Lambda F)\|_{L^2}^2 + C && (3.2.72)\\ &\leq -C\|\nabla(\Lambda F)\|_{L^2}^2 + C, && (3.2.73)\end{aligned}$$

which means that the L^2 norm of $\nabla(\Lambda F)$ is bounded for all $t \geq 0$.

Step 2: Next, we show that

$$\sup_{0\leq t}\|\nabla F\|_{L^2} < \infty, \quad (3.2.74)$$

by calculating

$$\begin{aligned}\frac{d}{dt}\|\nabla F\|_{L^2}^2 &= -2\|\nabla^2 F\|_{L^2}^2 + \int_X R\star\nabla F\star\bar{\nabla}F\omega^n + \int_X \nabla R\star F\star\bar{\nabla}F\omega^n\\ &\quad + \int_X F\star\nabla F\star\bar{\nabla}F\omega^n + \int_X F\star\nabla F\star\bar{\nabla}(\Lambda F)\omega^n. && (3.2.75)\end{aligned}$$

Then we use Hölder's inequality, the L^p boundedness of F, Lemma 9, and Claim 7 to control all of the terms by the first one. Since R is fixed

$$\begin{aligned}\int_X R\star\nabla F\star\bar{\nabla}F\omega^n &\leq \|R\|_{L^\infty}\|\nabla F\|_{L^2}^2 && (3.2.76)\\ &\leq C\|\nabla F\|_{L^2}^2 && (3.2.77)\\ &\leq \varepsilon\|\nabla^2 F\|_{L^2}^2 + C_\varepsilon\|F\|_{L^2}^2 && (3.2.78)\\ &\leq \varepsilon\|\nabla^2 F\|_{L^2}^2 + C. && (3.2.79)\end{aligned}$$

Also,

$$\begin{aligned}\int_X \nabla R\star F\star\bar{\nabla}F\omega^n &\leq \|\nabla R\|_{L^\infty}\|F\|_{L^2}\|\nabla F\|_{L^2} && (3.2.80)\\ &\leq C\|\nabla F\|_{L^2} \leq \varepsilon\|\nabla^2 F\|_{L^2}^2 + C. && (3.2.81)\end{aligned}$$

For the fourth term, we apply Hölder's inequality and then apply Lemma 9 with $i = 1$, $k = 2$, and $q = 3/2$.

$$\begin{aligned} \int_X F \star \nabla F \star \bar{\nabla} F \omega^n &\leq \|F\|_{L^3} \|\nabla F\|_{L^3}^2 && (3.2.82) \\ &\leq C \|\nabla^2 F\|_{L^2} \|F\|_{L^6}^6 && (3.2.83) \\ &\leq C \|\nabla^2 F\|_{L^2} && (3.2.84) \\ &\leq \varepsilon \|\nabla^2 F\|_{L^2}^2 + C. && (3.2.85) \end{aligned}$$

For the last term, we also apply Hölder's inequality, and then we use the results of Step 1 and apply Lemma 9 with $i = 1$, $k = 2$, and $q = 3/2$.

$$\begin{aligned} \int_X F \star \nabla F \star \bar{\nabla}(\Lambda F) \omega^n &\leq \|F\|_{L^6} \|\nabla F\|_{L^3} \|\nabla(\Lambda F)\|_{L^2} && (3.2.86) \\ &\leq C \|\nabla^2 F\|_{L^2} \|F\|_{L^6}^6 && (3.2.87) \\ &\leq C \|\nabla^2 F\|_{L^2} && (3.2.88) \\ &\leq \varepsilon \|\nabla^2 F\|_{L^2}^2 + C. && (3.2.89) \end{aligned}$$

Combining these estimates and Claim 7, we have

$$\frac{d}{dt} \|\nabla F\|_{L^2}^2 = -C \|\nabla^2 F\|_{L^2}^2 + C \leq -C \|\nabla F\|_{L^2}^2 + C, \qquad (3.2.90)$$

which means that the L^2 norm of ∇F is bounded for all $t \geq 0$.

Step 3: The next step is to show that if

$$\begin{aligned} \sup_{0 \leq t} \|\nabla^k F\|_{L^2} &< \infty && (3.2.91) \\ \sup_{0 \leq t} \|\nabla^k (\Lambda F)\|_{L^2} &< \infty && (3.2.92) \end{aligned}$$

for $k \leq m - 1$, then for any multi-indice δ with $|\delta| \leq m + 1$ and some constant C independent of time, we have the estimate

$$\|\nabla^\delta (\Lambda F)\|_{L^2} \leq C \|\nabla^{m+1} (\Lambda F)\|_{L^2} + C. \qquad (3.2.93)$$

After integrating by parts and commuting derivatives, we have

$$\begin{aligned}
\|\nabla^{\delta}(\Lambda F)\|_{L^2}^2 &= \|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 - \int_X \nabla^{m-2}(F \star \nabla(\Lambda F)) \star \nabla\bar{\nabla}^m(\Lambda F)\omega^n \\
&\quad + \int_X \nabla^{m-2}(F \star \nabla(\Lambda F)) \star \bar{\nabla}^{m-2}(F \star \nabla(\Lambda F))\omega^n \qquad (3.2.94) \\
&= \|\nabla^{m+1} F\|_{L^2}^2 \\
&\quad - \sum_{k=0}^{m-2} \int_X \nabla^{m-1-k} F \star \nabla^{k+1}(\Lambda F) \star \nabla\bar{\nabla}^m(\Lambda F)\omega^n \qquad (3.2.95) \\
&\quad + \sum_{j,k=0}^{m-2} \int_X \nabla^{m-2-k} F \star \nabla^{k+1}(\Lambda F) \star \nabla^{m-2-j} F \star \nabla^{j+1}(\Lambda F)\omega^n
\end{aligned}$$

After applying Hölder's inequality to the second term, it is bounded by

$$\sum_{k=0}^{m-2} \|\nabla^{m-2-k} F\|_{L^{2q}} \|\nabla^{k+1}(\Lambda F)\|_{L^{2p}} \|\bar{\nabla}\nabla^m(\Lambda F)\|_{L^2}, \qquad (3.2.96)$$

with

$$\frac{1}{p} + \frac{1}{q} = 1. \qquad (3.2.97)$$

Case 1: $k < m-2$. Let

$$q = \frac{m-1-\varepsilon}{m-2-k}. \qquad (3.2.98)$$

Then

$$p = \frac{m-1-\varepsilon}{1+k-\varepsilon}. \qquad (3.2.99)$$

If we choose ε such that

$$0 < \varepsilon < \frac{2k+2}{m-k}, \qquad (3.2.100)$$

then we will have the following bounds on p and q

$$\begin{aligned}
1 &\leq q < \frac{m-1}{m-2-k}, \\
1 &\leq p < \frac{m-1}{k+1}. \qquad (3.2.101)
\end{aligned}$$

Then by Lemma 9 and Claim 4

$$\begin{aligned}\|\nabla^{m-2-k}F\|_{L^{2q}}\|\nabla^{k+1}(\Lambda F)\|_{L^{2p}} &\le C\|\nabla^{m-1}F\|_{L^2}^{\frac{m-2-k}{m-1}}\|\nabla^{m+1}(\Lambda F)\|_{L^2}^{\frac{k+1}{m+1}} \\ &\le \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}+C_\varepsilon. \qquad (3.2.102)\end{aligned}$$

Case 2: $k = m-2$ **.** We can let $p = 1+\varepsilon$ for ε sufficiently small. Then q will be large, but we know that the L^p norm of F is bounded for all $1 \le p < \infty$. Also, by Claim 5, we have

$$\|\bar{\nabla}\nabla^m(\Lambda F)\|_{L^2} \le C\|\nabla^{m+1}(\Lambda F)\|_{L^2} + \|\nabla^m(\Lambda F)\|_{L^2} \qquad (3.2.103)$$

Then the second term

$$-\int_X \nabla^{m-2}[F\star\nabla(\Lambda F)]\star\nabla\bar{\nabla}^m F\omega^n \le \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C. \qquad (3.2.104)$$

After applying Hölder's inequality to the third term, it is bounded by

$$\sum_{j,k=0}^{m-2}\|\nabla^{m-2-k}F\|_{L^{2q}}\|\nabla^{k+1}(\Lambda F)\|_{L^{2p}}\|\nabla^{m-2-j}F\|_{L^{2s}}\|\nabla^{j+1}(\Lambda F)\|_{L^{2r}}, \qquad (3.2.105)$$

with

$$\frac{1}{p}+\frac{1}{q}=1, \qquad \frac{1}{r}+\frac{1}{s}=1. \qquad (3.2.106)$$

We choose the same p, q, and ε as we did for the second term, and we choose r and s in the same way, then the third term is also bounded

$$\int_X \nabla^{m-2}(F\star\nabla(\Lambda F))\star\bar{\nabla}^{m-2}(F\star\nabla(\Lambda F))\omega^n \le \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C_\varepsilon. \qquad (3.2.107)$$

Step 4: The next step is to show that if

$$\begin{aligned}\sup_{0\le t}\|\nabla^k F\|_{L^2} &< \infty \qquad (3.2.108)\\ \sup_{0\le t}\|\nabla^k(\Lambda F)\|_{L^2} &< \infty \qquad (3.2.109)\end{aligned}$$

for $k \leq m-1$, then for any multi-indice δ with $|\delta| \leq m+1$ and some constant C independent of time, we have the estimate

$$\|\nabla^\delta F\|_{L^2} \leq C\|\nabla^{m+1}F\|_{L^2} + C. \tag{3.2.110}$$

After integrating by parts and commuting derivatives, we have

$$\begin{aligned} \|\nabla^\delta F\|^2_{L^2} &= \|\nabla^{m+1}F\|^2_{L^2} + \int_X \nabla^m(R\star F + F\star F)\star \bar{\nabla}^m F\omega^n \\ &\quad + \int_X \nabla^{m-1}(R\star F + F\star F)\star \bar{\nabla}^{m-1}(R\star F + F\star F)\omega^n \\ &= \|\nabla^{m+1}F\|^2_{L^2} + \sum_{a,b}\int_X (\nabla^a R + \nabla^a F)\star\nabla^b F\star\nabla^m F\omega^n \qquad (3.2.111) \\ &\quad + \sum_{i,j,k,\ell}\int_X \left(\nabla^i R + \nabla^i F\right)\star\nabla^j F\star\left(\nabla^k R + \nabla^k F\right)\star\nabla^\ell F\omega^n, \end{aligned}$$

where a, b, i, j, k, and ℓ are integers that satisfy

$$\begin{aligned} &0 \leq a,b \leq m, \quad 0 \leq i,j \leq m-1, \quad 1 \leq k,\ell \leq m-1, \\ &a+b = m, \quad i+j = m-1, \quad k+\ell = m-1. \end{aligned} \tag{3.2.112}$$

After carefully applying Hölder's inequality and Lemma 9, we can bound all of the terms by $C\|\nabla^{m+1}F\|^2_{L^2} + C$. Since R is fixed, the most difficult term to bound is

$$\int_X \nabla^i F\star\nabla^j F\star\nabla^k F\star\nabla^\ell F\omega^n. \tag{3.2.113}$$

We can assume

$$0 \leq i,k < \frac{m}{2}, \qquad \frac{m}{2} \leq j,\ell \leq m-1. \tag{3.2.114}$$

We apply Hölder's inequality

$$\int_X \nabla^i F\star\nabla^j F\star\nabla^k F\star\nabla^\ell F\omega^n \leq \|\nabla^i F\|_{L^{2p}}\|\nabla^j F\|_{L^{2q}}\|\nabla^k F\|_{L^{2r}}\|\nabla^\ell F\|_{L^{2s}}, \tag{3.2.115}$$

with

$$\frac{1}{p}+\frac{1}{q}=1, \qquad \frac{1}{r}+\frac{1}{s}=1. \tag{3.2.116}$$

Let

$$q=\frac{m+1-\varepsilon}{j}. \tag{3.2.117}$$

Then

$$p=\frac{m+1-\varepsilon}{2+i-\varepsilon}. \tag{3.2.118}$$

If we choose ε such that

$$0<\varepsilon<\frac{2m-2-2i}{m-1-i}, \tag{3.2.119}$$

then we will have the following bounds on p and q

$$\begin{aligned} 1 \leq q &< \frac{m+1}{j}, \\ 1 \leq p &< \frac{m-1}{i}. \end{aligned} \tag{3.2.120}$$

Then by Lemma 9 and Claim 4

$$\begin{aligned} \|\nabla^i F\|_{L^{2p}} \|\nabla^j F\|_{L^{2q}} &\leq C\|\nabla^{m-1}F\|_{L^2}^{\frac{i}{m-1}} \|\nabla^{m+1}F\|_{L^2}^{\frac{j}{m+1}} \\ &\leq \varepsilon\|\nabla^{m+1}F\|_{L^2}+C_\varepsilon. \end{aligned} \tag{3.2.121}$$

Next, let

$$s=\frac{m+1-\sigma}{\ell}. \tag{3.2.122}$$

Then

$$r=\frac{m+1-\sigma}{2+k-\sigma}. \tag{3.2.123}$$

If we choose σ such that

$$0<\sigma<\frac{2m-2-2k}{m-1-k}, \tag{3.2.124}$$

then we will have the following bounds on r and s

$$1 \leq r < \frac{m+1}{\ell},$$
$$1 \leq s < \frac{m-1}{k}. \qquad (3.2.125)$$

Then again by Lemma 9 and Claim 4

$$\begin{aligned} \|\nabla^k F\|_{L^{2r}} \|\nabla^\ell F\|_{L^{2s}} &\leq C \|\nabla^{m-1} F\|_{L^2}^{\frac{k}{m-1}} \|\nabla^{m+1} F\|_{L^2}^{\frac{\ell}{m+1}} \\ &\leq \varepsilon \|\nabla^{m+1} F\|_{L^2} + C_\varepsilon. \end{aligned} \qquad (3.2.126)$$

Then

$$\int_X \nabla^i F \star \nabla^j F \star \nabla^k F \star \nabla^\ell F \omega^n \leq \varepsilon \|\nabla^{m+1} F\|_{L^2}^2 + C_\varepsilon. \qquad (3.2.127)$$

We bound the remaining terms in a similar way.

Step 5: It follows from Steps 3 and 4 that if

$$\sup_{0 \leq t} \|\nabla^k F\|_{L^2} < \infty \qquad (3.2.128)$$
$$\sup_{0 \leq t} \|\nabla^k (\Lambda F)\|_{L^2} < \infty \qquad (3.2.129)$$

for $k \leq m-1$, then

$$\sup_{0 \leq t} \|\nabla^m (\Lambda F)\|_{L^2} < \infty. \qquad (3.2.130)$$

This is shown by calculating

$$\begin{aligned} \frac{d}{dt} \|\nabla^m (\Lambda F)\|_{L^2}^2 &= -2 \|\nabla^{m+1} (\Lambda F)\|_{L^2}^2 - \int_X \nabla^{m-2} [F \star \nabla (\Lambda F)] \star \nabla \bar{\nabla}^m (\Lambda F) \omega^n \\ &\quad + \int_X \nabla^m (\Lambda F \star \Lambda F) \star \bar{\nabla}^m (\Lambda F) \omega^n. \end{aligned} \qquad (3.2.131)$$

Now we must apply Hölder's inequality carefully, in order to use Lemma 9 to control the second and third terms by the first one.

From (3.2.104), we know

$$-\int_X \nabla^{m-2}[F \star \nabla(\Lambda F)] \star \nabla\bar{\nabla}^m(\Lambda F)\omega^n \leq \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C. \quad (3.2.132)$$

Next, we bound the third term.

$$\int_X \nabla^m[(\Lambda F) \star (\Lambda F)]\bar{\nabla}^m(\Lambda F)\omega^n = \sum_{k=0}^{m} \int_X \nabla^k(\Lambda F) \star \nabla^{m-k}(\Lambda F) \star \bar{\nabla}^m(\Lambda F)\omega^n. \quad (3.2.133)$$

Here we can assume $k \leq \frac{m}{2}$.

Case 1: $k = 0$. We make use of the boundedness of the L^p norm of (ΛF) for all $p < \infty$.

$$\begin{aligned}\int_X (\Lambda F) \star \nabla^m(\Lambda F) \star \bar{\nabla}^m(\Lambda F)\omega^n &\leq \|(\Lambda F)\|_{L^p}\|\nabla^m(\Lambda F)\|_{L^{2+\delta}}^2 \\ &\leq \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C. \quad (3.2.134)\end{aligned}$$

Case 2: $k < \frac{m}{2}$. We apply Hölder's inequality as follows

$$\begin{aligned}\int_X \nabla^k(\Lambda F) \star \nabla^{m-k}(\Lambda F) \star \bar{\nabla}^m(\Lambda F)\omega^n &\leq \|\nabla^k(\Lambda F)\|_{L^{2p}}\|\nabla^{m-k}(\Lambda F)\|_{L^{2q}} \cdot \\ &\cdot\|\nabla^m(\Lambda F)\|_{L^2}, \quad (3.2.135)\end{aligned}$$

with

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (3.2.136)$$

Let

$$q = \frac{m+1-\varepsilon}{m-k}. \quad (3.2.137)$$

Then

$$p = \frac{m+1-\varepsilon}{k+1-\varepsilon}. \quad (3.2.138)$$

If we choose ε such that

$$0 < \varepsilon < \frac{m-2k}{m-k-1}, \quad (3.2.139)$$

then we will have

$$1 \leq q < \frac{m+1}{m-k}, \tag{3.2.140}$$
$$1 \leq p < \frac{m-1}{k}. \tag{3.2.141}$$

Then by Lemma 9

$$\begin{aligned}\|\nabla^k(\Lambda F)\|_{L^{2p}}\|\nabla^{m-k}(\Lambda F)\|_{L^{2q}} &\leq C\|\nabla^{m-1}(\Lambda F)\|_{L^2}^{\frac{k}{m-1}}\|\nabla^{m+1}(\Lambda F)\|_{L^2}^{\frac{m-k}{m+1}} \\ &\leq \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2} + C_\varepsilon. \end{aligned}\tag{3.2.142}$$

Also by Lemma 9

$$\|\nabla^m(\Lambda F)\|_{L^2} \leq \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2} + C_\varepsilon. \tag{3.2.143}$$

Then it follows that

$$\int_X \nabla^k(\Lambda F) \star \nabla^{m-k}(\Lambda F) \star \bar{\nabla}^m(\Lambda F)\omega^n \leq \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C_\varepsilon. \tag{3.2.144}$$

Case 3: When m is even and $k = \frac{m}{2}$. By Hölder's inequality

$$\int_X \nabla^{\frac{m}{2}}(\Lambda F) \star \nabla^{\frac{m}{2}}(\Lambda F) \star \bar{\nabla}^m(\Lambda F)\omega^n \leq \|\nabla^{\frac{m}{2}}(\Lambda F)\|_{L^{2p}}^2\|\nabla^m(\Lambda F)\|_{L^{2q}}, \tag{3.2.145}$$

with

$$\frac{1}{p} + \frac{1}{2q} = 1. \tag{3.2.146}$$

Let

$$q = \frac{m+1-\varepsilon}{m}. \tag{3.2.147}$$

Then

$$p = \frac{m+1-\varepsilon}{\frac{m}{2}+1-\varepsilon}. \tag{3.2.148}$$

For any $0 < \varepsilon < 1$, we have

$$1 \leq q < \frac{m+1}{m}, \tag{3.2.149}$$
$$1 \leq p < \frac{m+1}{\frac{m}{2}}. \tag{3.2.150}$$

Then by Lemma 9

$$\begin{aligned}\int_X \nabla^{\frac{m}{2}}(\Lambda F)\star\nabla^{\frac{m}{2}}(\Lambda F)\star\bar{\nabla}^m(\Lambda F)\omega^n &\leq \|\nabla^{\frac{m}{2}}(\Lambda F)\|_{L^{2p}}^2\|\nabla^m(\Lambda F)\|_{L^{2q}} \\ &\leq C\|\nabla^{m+1}(\Lambda F)\|_{L^2}^{\frac{2m}{m+1}} \qquad (3.2.151) \\ &\leq \varepsilon\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C_\varepsilon.\end{aligned}$$

Now that all of the terms are controlled by the first term, we have

$$\frac{d}{dt}\|\nabla^m(\Lambda F)\|_{L^2}^2 \leq -C\|\nabla^{m+1}(\Lambda F)\|_{L^2}^2 + C \qquad (3.2.152)$$

$$\leq -C\|\nabla^m(\Lambda F)\|_{L^2}^2 + C. \qquad (3.2.153)$$

Then the L^2 norm of $\nabla^m(\Lambda F)$ is bounded.

Step 6: It follows from Steps 3-5 that if

$$\sup_{0\leq t}\|\nabla^k F\|_{L^2} < \infty \qquad (3.2.154)$$

$$\sup_{0\leq t}\|\nabla^k(\Lambda F)\|_{L^2} < \infty \qquad (3.2.155)$$

for $k \leq m-1$, then

$$\sup_{0\leq t}\|\nabla^m F\|_{L^2} < \infty. \qquad (3.2.156)$$

This is shown by calculating

$$\begin{aligned}\frac{d}{dt}\|\nabla^m F\|_{L^2}^2 &= -2\|\nabla^{m+1}F\|_{L^2}^2 - \int_X \nabla^{m-2}[\nabla(\Lambda F)\star F]\star\nabla\bar{\nabla}^m F\omega^n \\ &\quad + \int_X \nabla^m(R\star F + F\star F)\star\bar{\nabla}^m F\omega^n. \qquad (3.2.157)\end{aligned}$$

Now we must apply Hölder's inequality carefully, in order to use Lemma 9 to control all of the terms by the first one.

First we control the terms involving R,

$$\begin{aligned}\int_X \nabla^m(R\star F)\bar{\nabla}^m F\omega^n &= \sum_{0\le k\le m}\int_X \nabla^{m-k}R\star\nabla^k F\star\bar{\nabla}^m F\omega^n & (3.2.158)\\ &\le \sum \|\nabla^{m-k}R\|_{L^\infty}\|\nabla^k R\|_{L^2}\|\nabla^m F\|_{L^2} & (3.2.159)\\ &\le C\|\nabla^m F\|_{L^2}^2 & (3.2.160)\\ &\le \varepsilon\|\nabla^{m+1}F\|_{L^2}^2 + C. & (3.2.161)\end{aligned}$$

Next, we bound

$$\int_X \nabla^m(F\star F)\bar{\nabla}^m F\omega^n = \sum_{0\le k\le m}\int_X \nabla^k F\star\nabla^{m-k}F\star\bar{\nabla}^m F\omega^n. \quad (3.2.162)$$

Here we can assume $k \le \frac{m}{2}$.

Case 1: $k = 0$. We make use of the boundedness of the L^p norm of F for all $p < \infty$.

$$\begin{aligned}\int_X F\star\nabla^m F\star\bar{\nabla}^m F\omega^n &\le \|F\|_{L^P}\|\nabla^m F\|_{L^{2+\delta}}^2 & (3.2.163)\\ &\le \varepsilon\|\nabla^{m+1}F\|_{L^2}^2 + C. & (3.2.164)\end{aligned}$$

Case 2: $k < \frac{m}{2}$. We apply Hölder's inequality as follows

$$\int_X \nabla^k F\star\nabla^{m-k}F\star\bar{\nabla}^m F\omega^n \le \|\nabla^k F\|_{L^{2p}}\|\nabla^{m-k}F\|_{L^{2q}}\|\nabla^m F\|_{L^2}, \quad (3.2.165)$$

with

$$\frac{1}{p}+\frac{1}{q}=1. \quad (3.2.166)$$

Let

$$q = \frac{m+1-\varepsilon}{m-k}. \quad (3.2.167)$$

Then

$$p = \frac{m+1-\varepsilon}{k+1-\varepsilon}. \quad (3.2.168)$$

If we choose ε such that

$$0 < \varepsilon < \frac{m-2k}{m-k-1}, \tag{3.2.169}$$

then we will have

$$\begin{aligned} 1 &\leq q < \frac{m+1}{m-k}, &(3.2.170)\\ 1 &\leq p < \frac{m-1}{k}. &(3.2.171)\end{aligned}$$

Then by Lemma 9

$$\begin{aligned} \|\nabla^k F\|_{L^{2p}}\|\nabla^{m-k}F\|_{L^{2q}} &\leq C\|\nabla^{m-1}F\|_{L^2}^{\frac{k}{m-1}}\|\nabla^{m+1}F\|_{L^2}^{\frac{m-k}{m+1}} \\ &\leq \varepsilon\|\nabla^{m+1}F\|_{L^2} + C_\varepsilon. &(3.2.172)\end{aligned}$$

Also by Lemma 9

$$\|\nabla^m F\|_{L^2} \leq \varepsilon\|\nabla^{m+1}F\|_{L^2} + C_\varepsilon. \tag{3.2.173}$$

Then it follows that

$$\int_X \nabla^k F \star \nabla^{m-k}F \star \bar{\nabla}^m F\omega^n \leq \varepsilon\|\nabla^{m+1}F\|_{L^2}^2 + C_\varepsilon. \tag{3.2.174}$$

Case 3: When m is even and $k = \frac{m}{2}$. By Hölder's inequality

$$\int_X \nabla^{\frac{m}{2}}F \star \nabla^{\frac{m}{2}}F \star \bar{\nabla}^m F\omega^n \leq \|\nabla^{\frac{m}{2}}F\|_{L^{2p}}^2\|\nabla^m F\|_{L^{2q}}, \tag{3.2.175}$$

with

$$\frac{1}{p} + \frac{1}{2q} = 1. \tag{3.2.176}$$

Let

$$q = \frac{m+1-\varepsilon}{m}. \tag{3.2.177}$$

Then

$$p = \frac{m+1-\varepsilon}{\frac{m}{2}+1-\varepsilon}. \tag{3.2.178}$$

For any $0 < \varepsilon < 1$, we have

$$1 \leq q < \frac{m+1}{m}, \quad (3.2.179)$$

$$1 \leq p < \frac{m+1}{\frac{m}{2}}. \quad (3.2.180)$$

Then by Lemma 9

$$\begin{aligned}\int_X \nabla^{\frac{m}{2}} F \star \nabla^{\frac{m}{2}} F \star \bar{\nabla}^m F \omega^n &\leq \|\nabla^{\frac{m}{2}} F\|^2_{L^{2p}} \|\nabla^m F\|_{L^{2q}} &(3.2.181)\\ &\leq C\|\nabla^{m+1} F\|^{\frac{2m}{m+1}}_{L^2} &(3.2.182)\\ &\leq \varepsilon\|\nabla^{m+1} F\|^2_{L^2} + C_\varepsilon. &(3.2.183)\end{aligned}$$

Lastly, we bound

$$\begin{aligned}\int_X \nabla^{m-2}[\nabla(\Lambda F) \star F] \star \nabla\bar{\nabla}^m F \omega^n \star &= \sum_{k=0}^{m-2} \int_X \nabla^k F \star \nabla^{m-1-k} \Lambda F\\ &\quad \star \nabla\bar{\nabla}^m F \omega^n\\ &\leq \sum \|\nabla^k F\|_{L^{2p}} \|\nabla^{m-1-k} \Lambda F\|_{L^{2q}} \cdot\\ &\quad \cdot \|\nabla\bar{\nabla}^m F\|_{L^2},\end{aligned}$$

with

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (3.2.184)$$

Let

$$q = \frac{m-\varepsilon}{m-1-k}. \quad (3.2.185)$$

Then

$$p = \frac{m-\varepsilon}{k+1-\varepsilon}. \quad (3.2.186)$$

If we choose ε such that

$$0 < \varepsilon < \frac{m}{m-k}, \quad (3.2.187)$$

then we will have

$$1 \leq q < \frac{m}{m-1-k}, \quad (3.2.188)$$

$$1 \leq p < \frac{m}{k}. \quad (3.2.189)$$

Then by Lemma 9

$$\|\nabla^k F\|_{L^{2p}}\|\nabla^{m-1-k}\Lambda F\|_{L^{2q}} \leq C\|\nabla^m F\|_{L^2}^{\frac{k}{m}}\|\nabla^m \Lambda F\|_{L^2}^{\frac{m-1-k}{m}} \quad (3.2.190)$$

$$\leq \varepsilon\|\nabla^{m+1}F\|_{L^2} + C_\varepsilon. \quad (3.2.191)$$

We also know that

$$\|\nabla\bar{\nabla}^m F\|_{L^2} \leq C\|\nabla^{m+1}F\|_{L^2} + C \quad (3.2.192)$$

Then all of the terms are controlled by the first term, and we have

$$\frac{d}{dt}\|\nabla^m F\|_{L^2}^2 \leq -k\|\nabla^{m+1}F\|_{L^2}^2 + C \quad (3.2.193)$$

$$\leq -k\|\nabla^m F\|_{L^2}^2 + C. \quad (3.2.194)$$

Then the L^2 norm of $\nabla^m F$ is bounded.

Step 7: Steps 3-6 imply that for $|\gamma| \leq m$,

$$\sup_{0\leq t}\|\nabla^\gamma F\|_{L^2} < \infty. \quad (3.2.195)$$

Step 8: By an inductive argument, we have for any $k \geq 1$,

$$\sup_{0\leq t}\|\nabla^k F\|_{L^2} < \infty. \quad (3.2.196)$$

Step 9: Now, consider

$$\|\nabla^i F\|_{L^{2q}}. \quad (3.2.197)$$

By Step 8, for any $i \in \mathbb{N}$ and $1 \leq q < \infty$, there exists a $k > qi$ such that

$$\|\nabla^k F\|_{L^2} \leq C. \quad (3.2.198)$$

Then by Lemma 9,

$$\|\nabla^i F\|_{L^{2q}} \leq \|\nabla^k F\|_{L^2}^{\frac{i}{k}}\|F\|_{L^p}^{\frac{p}{2}(\frac{1}{q}-\frac{i}{k})} \leq C. \qquad \square$$

The following corollary follows immediately from the Sobolev imbedding theorem.

Corollary 6. *For any $k \geq 0$*

$$\sup_{0\leq t}\|F\|_{C^k} < \infty. \tag{3.2.199}$$

We note that, once the C^0 uniform boundedness of the full curvature tensor is established, the higher order C^k estimates also follow from the general theory (see e.g. [39]).

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