Finite atomic lattices and their relationship to resolutions of monomial ideals

Sonja Mapes

Advisor: David Bayer

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2009

Abstract

Finite atomic lattices and their relationship to resolutions of monomial ideals

Sonja Mapes

This thesis studies monomial ideals and their resolutions by using combinatorial methods. In the study of cellular resolutions of monomial ideals it is often useful to consider the LCM lattice of the given monomial ideal. It has been shown that all finite atomic lattices can realized as the LCM lattice of some monomial ideal, and that the parameter space of these lattices, $\mathcal{L}(n)$, is itself a finite atomic lattice. This thesis focuses on exploring this notion that finite atomic lattices are abstract monomial ideals and aims to use the structure of $\mathcal{L}(n)$ as a tool to provide new insights into concepts such as deformation of exponents. The main results of this thesis fall into three categories: structural results about $\mathcal{L}(n)$, results relating to deformation of exponents, and results relating these constructions to those found in recent work by Fløystad.

I also include two appendices describing computer packages written to aid in my research. One is an implementation in Haskell which uses reverse search to enumerate $\mathcal{L}(n)$, and the other is a package for Macaulay2 which introduces posets as a new data type.

Contents

Acknowledgements

Finally I am here at the end of my time as a graduate student here at Columbia and I get to thank all of the people who have helped and supported me along this journey!

To Dave Bayer, thank you for being my advisor; for teaching me and pushing me to think about things in new ways; and for always believing in me. Whether you realized it or not, your ability to at least never appear judgmental was crucial to my success since as I am my own harshest critic I always really needed someone to point out the directions forward.

To Mike Stillman, Hal Schenck, Bernd Sturmfels, Diane Maclagan, Irena Peeva, Greg Smith, Jessica Sidman, Amelia Taylor, Julius Ross, Henry Pinkham, Johan de Jong, and Cathy O'Neil, thank you for teaching me both about mathematics and about being a mathematician. My conversations with all of you over the years were invaluable and without your kind words and encouragement I would not be here today.

To Josephine Yu, Sonja Petrović, Mauricio Velasco, Jeff Mermin, and Christine Berkesch, thank you for all of our mathematical conversations as well as your friendship. Many of my favorite memories from graduate school

are my time spent with all of you at various conferences over the years. I look forward to the days when we can be the "Mike and Dave" of conferences and tell stories that begin with "back when we were in graduate school...." to groups of youngsters.

To my fellow Columbia students Debbie Yuster, Dave Swinarski, Lindsay Piechnik, Jeff Phan, Johan Martens, Alp Atici, Matt Hedden, PJ Lamberson, Xander Faber, Helena Kauppila, and Donovan McFeron, thank you too for teaching me and for your friendship and support. And to Mirela Ciperiani thanks for always including me in the "postdoc" outings and for all of your encouragement.

To my math siblings, Debbie and Lindsay thank you for countless informal talks between all of us; and Jeff thank you for being an awesome math big brother and for introducing the world to the ideas which inspired this thesis.

To all of the staff in the Columbia math department especially Terrance Cope and Mary Young, thank you for all the little things that you do which made my life functional. I will miss chatting with you and blowing off steam about various ridiculous things.

To my defense committee, thank you for taking the time to be on my committee. Your comments on my thesis were greatly appreciated.

To Gábor Székelyhidi, thank you for being my resident LaTex expert during the last push of typing up this thesis. Without you, nothing would be formatted nicely and I would have spent many more hours scouring the web trying and failing to figure out how to do even the simplest things.

To Emily, Sarah, Lisa, Jenna, Debbie, Sharon, Kristina, Debbie, Dave, Lindsay, and Gabor, thank you for being my friends, for supporting me and

inspiring me, and for pushing me to finish. Especially for those of you who endured my “math-attacks” (you know who you were) thank you for patiently listening to me and then helping me find the small thing that I could do to move forward.

To my parents, Herbert and Ida Mapes, thank you so much for everything that you’ve ever done for me. You’ve always believed in me and taught me that it is best to be myself. I owe all of my success to you. Thank you!

To my parents, Herbert and Ida Mapes and

to my grandmother Margaret Neese.

Chapter 1

Introduction

The study of commutative algebra, in particular free resolutions of modules has become inextricably linked to combinatorics. There are many classes of algebraic objects that have been shown to possess nice combinatorial structures; examples include monomial ideals, and toric or lattice ideals. This thesis focuses in particular on the combinatorial structure of free resolutions of monomial ideals.

Finding the free resolution of a module is an important step in the computation of many interesting algebraic and geometric invariants. This is because free resolutions play a central role in computations of sheaf or local cohomology. Thus, finding general descriptions of minimal resolutions for classes of modules is a very active area of research in commutative algebra. The problem of finding minimal resolutions entails finding both the Betti numbers of a module as well as a description of the maps in the resolution.

For monomial ideals, finding the multigraded Betti numbers is well understood. There are a number of different formulas computing Betti numbers, most of which use the homology of certain simplicial complexes, all of which can be found in [MS05]. However, finding a closed form description of the maps in a minimal resolution remains an open problem. The theory of cellular resolutions, first introduced by Bayer and Sturmfels in [BS98], provides a framework for computing both the multigraded Betti numbers as well as a description of the possible maps in such a resolution. In particular it relates resolutions of monomial ideals to chain complexes of regular cell complexes. The issue is that cellular resolutions do not always yield minimal resolutions, thus a description of the maps remains open.

In an attempt to further understand these resolutions, Gasharov, Peeva and Welker proved in [GPW99] that the combinatorial type of the minimal resolution of a monomial ideal is determined by its LCM lattice. This introduction of the LCM lattice prompted Phan to prove in [Pha06] that all finite atomic lattices can be realized as LCM lattices, thus establishing the notion that finite atomic lattices are abstract monomial ideals. An appropriate analogy here is that just as one studies both abstract and embedded varieties in algebraic geometry, one should also consider abstract monomial ideals in the study of free resolutions of monomial ideals. Phan also proves that the space of all finite atomic lattices on n atoms is itself a finite atomic lattice. Thus much in the same way that one seeks to understand families of schemes and how they relate to each other by studying the Hilbert scheme, I aim to understand resolutions of monomial ideals by studying this parameter space of lattices.

In chapter 2, I give all of the necessary background information to understand the main ideas in this thesis. It should be noted that the sections in both this chapter and the next covering the material in [Pha06] either include a sketch of his original proof or a more general statement is proven in chapter 3. The presentation of this material is meant to stand alone since Phan's thesis has not been published.

In chapter 3, I first demonstrate how to recover the coordinatization of a finite atomic lattice given a specific monomial ideal. This allows me to give a generalization of the main construction in [Pha06] to describe all monomial ideals with a given LCM lattice. This allows more freedom when coordinatizing finite atomic lattices, and will be of use in other sections.

The fact that the set of finite atomic lattices on n atoms, denoted $\mathcal{L}(n)$, is itself a finite atomic lattice motivates the question: what is the relationship between the minimal resolutions of coordinatizations of lattices in $\mathcal{L}(n)$? The answer, due to a result in [GPW99], is that the total Betti numbers are weakly monotonic along chains in $\mathcal{L}(n)$. This is the motivation for understanding the structure of $\mathcal{L}(n)$, as it gives greater insight into understanding concepts such as deformation of exponents introduced in [BPS98] which will be discussed in chapter 5.

In chapter 4, I provide an alternate proof that the Betti numbers increase as the lattices increase. Many points in this proof are used in chapter 5, and they motivate the two main ways I will represent lattices in $\mathcal{L}(n)$ as explained in the rest of chapter 4. These two methods of representing allow me to prove some structural results about $\mathcal{L}(n)$. The first method is used to give a description of the meet-irreducible elements of $\mathcal{L}(n)$. The second

method is used to give a description of the covering relations in $\mathcal{L}(n)$. It is also used in the reverse search algorithm which computes all of the elements of $\mathcal{L}(n)$ explained in appendix A. Moreover, this description of the covering relations leads to the main structural result in this thesis which states that $\mathcal{L}(n)$ is a graded lattice of rank $2^n - n - 2$.

The increasing nature of Betti numbers in $\mathcal{L}(n)$ nicely mirrors the upper-semicontinuity of Betti numbers which is known for deformation of exponents. In chapter 5, I show that for some coordinatization every two lattices comparable in $\mathcal{L}(n)$ can be related via a deformation of exponents. This implies that for each lattice, there exists a coordinatization such that its entire filter in $\mathcal{L}(n)$ corresponds to all possible deformations of exponents of that coordinatization.

Notice, if one fixes total Betti numbers then $\mathcal{L}(n)$ breaks up into strata of lattices with the same Betti number. Deformation of exponents does not guarantee that it will yield a monomial ideal in the same stratum as the original ideal. In particular, if the minimal resolution of a given ideal cannot be supported on a simplicial complex, then a total deformation of exponents always yields an ideal in a higher stratum. This follows from the fact that the intention of such a deformation is to move to an ideal that is resolved by a simplicial complex. The rest of chapter 5 focuses on first understanding the concepts involved with deformation of exponents for monomial ideals whose minimal resolution is supported on a simplicial complex. Here I build upon the known fact that all acyclic simplicial complexes can be realized as the Scarf complex supporting a minimal resolution of some ideal [Pha06]. I show that in fact all lattices greater than the augmented face lattice of such

a complex yet in the same Betti strata are minimally resolved by the same complex. I also show that monomial ideals whose LCM lattice is graded of maximal rank are strongly generic. This means that their minimal resolution is known to be the Scarf complex, and implies that deformation of exponents is likely to increase Betti numbers. The rest of chapter 5 focuses on trying to generalize these ideas to monomial ideals whose minimal resolution is supported on a regular CW-complex. Generalizing the result from [Pha06], I show that for appropriately chosen regular cell complexes one can always find a monomial ideal whose minimal resolution is supported on that cell complex. Moreover, for certain lattices which cover the augmented face lattice of these complexes I can show that their minimal resolution is supported on the same complex.

The point of view introduced by Phan and continued in this thesis rests on the idea of associating monomial ideals with certain properties to finite atomic lattices. It should be noted however, that there is other work which associates monomial ideals to certain cell complexes which support their resolutions. One goal this thesis is to demonstrate that all such constructions can be rephrased in terms of the constructions in this thesis introduced by Phan. Unfortunately, it is not actually true that all constructions associating monomial ideals to cell complexes which support their resolution can be phrased this way, see the constructions in [NPS02]. For others though, this can be done: see the references to [Vel08], [PV] in chapter 3.

Chapter 6 focuses on one such construction associating monomial ideals to cell complexes found in [Fløy09]. Fløystad defines the notion of a "maximal" Cohen-Macaulay monomial ideal. Moreover, for certain simplicial and

polyhederal cell complexes he gives constructions for how to find such an ideal whose resolution is supported on the given complex. I show that Fløystad's description of these maximal ideals easily translates into conditions on a coordinatization of a finite atomic lattice. Additionally, I show that in the case where the simplicial complex is a tree that for an appropriate choice of lattice his construction is equivalent to Phan's original squarefree construction.

Chapter 2

Preliminaries

2.1 Posets and Finite Atomic Lattices

A *poset* $(P, <)$ is a set P with an order relation $<$ which is transitive and antisymmetric. If P has a maximum element it is denoted $\hat{1}$, and likewise minimum elements are denoted as $\hat{0}$. The *join* of two elements a and b in P is denoted as $a \vee b$ and is the least upper bound of the two elements. Similarly the *meet* of a and b is denoted as $a \wedge b$ and is the greatest lower bound of the two elements. Note that if a join or a meet exists, it is unique. Then P is a *lattice* if every pair of elements has a meet and a join. Moreover, P is a *meet-semilattice* if every pair of elements has a meet, and equivalently a *join-semilattice* if every pair has a join. The following proposition from [Sta97] describes when either of these is actually a lattice.

Proposition 2.1.1. *If P is a meet-semilattice with a unique maximal el-*

ement then P is a lattice. Equivalently, if P is a join-semilattice with a unique minimal element then P is a lattice.

If P is a lattice, then we define elements $x \in P$ to be *meet-irreducible* if $x \neq a \wedge b$ for some $a > x, b > x$. Equivalently, the *join-irreducible* elements are the elements $x \neq a \vee b$ for some $a < x, b < x$. Given an element $x \in P$, the *order ideal* of x is defined to be the set $\lfloor x \rfloor = \{a \in P | a \leqslant x\}$. Similarly, we define the *filter* of x to be $\lceil x \rceil = \{a \in P | x \leqslant a\}$. We can also speak of intervals in P which will be defined as

$$(a, b) = \{x \in P | a < x \text{ and } x < b\}$$

or

$$[a, b] = \{x \in P | a \leqslant x \text{ and } x \leqslant b\}.$$

Moreover, we define the following posets

$$P_{\leqslant a} = [\hat{0}, a] = \lfloor a \rfloor$$

and

$$P_{<a} = [\hat{0}, a).$$

Define a *covering relation* in a poset P as follows, a *covers* b if $a > b$ and there is no element c such that $a > c > b$. We define an *atom* of a lattice P to be an element $x \in P$ such that x covers $\hat{0}$. We will denote the set of atoms as $\text{atoms}(P)$. If every element in $P - \{\hat{0}\}$ is the join of atoms, then

P is an *atomic lattice*. Furthermore, if P is finite, then it is a *finite atomic lattice*. One of the main objects of study in this thesis will be finite atomic lattices.

Often it will be useful for us to consider certain simplicial complexes which can be associated to a poset. Define $\Delta(P)$ to be the order complex of a poset P, where the vertices are the elements of P and the facets correspond to maximal chains of P. In the special case where P is a finite atomic poset we can define a special case of the cross cut complex $\Gamma(P)$ where the atoms correspond to vertices and faces correspond to subsets of atoms which have a join or meet in P. It is known that $\Delta(P)$ is homotopy equivalent to $\Gamma(P)$ [Bjö95]. Moreover, it should be noted that when P is a finite atomic lattice on n atoms that $\Gamma(P)$ will be the $n-1$-simplex.

If P and Q are two atomic lattices then $f : P \to Q$ is a *join-preserving map* if $f(a \vee b) = f(a) \vee f(b)$. We will need the following proposition from [Pha06] so I will state the relevant portions here.

Proposition 2.1.2. *Let P and Q be finite atomic lattices. Let $f : P \to Q$ and $g : Q \to P$ be maps of sets which are bijections on the atoms of P and Q defined as*

$$f(p) = \bigvee_{\text{supp}(p)} a_i \text{ and } g(q) = \bigvee_{\text{supp}(q)} a_i$$

where $\text{supp}(p) = \{a_i \,|\, a_i \leqslant p\}$. *Then the following are equivalent:*

1. *f preserves joins*

2. *g preserves meets*

If either of these conditions hold then f is an isomorphism if and only if it is bijective.

Finally, posets can be represented by a *Hasse diagram* defined to be a graph where the vertices are the elements of P and the edges are the covering relations in P.

2.2 Regular CW-complexes and reduced homology

Let B^n, U^n, and S^{n-1} denote the closed unit ball, the open unit ball and the unit sphere in $\mathbb{R}^n$, respectively.

A a *(finite) CW-complex* is a topological space X is constructed in the following way (using finitely many steps):

1. X^0 is a finite discrete set.

2. For $n > 0$ and any finite collection of continuous maps $\phi_\alpha : S^{n-1} \to X^{(n-1)}$, $X^{(n)} = X^{(n-1)} \sqcup_\alpha B^n_\alpha / \sim$ where $x \sim \phi_\alpha(x)$ for all $x \in S^{n-1}_\alpha$. Endow this space with the quotient topology.

3. $X = \cup_n X^{(n)}$ with the weak topology.

Every map ϕ_α can naturally be extended to a *characteristic map*, denoted $\Phi_\alpha : B^n_\alpha \to X^{(n)}$ which is a homeomorphism between U^n_α and its image F^n_α. Call F^n_α an *n-cell*. A CW- complex is a *(finite) regular cell complex* if all of its characteristic maps are homeomorphisms.

A cell F' is a face of the cell F if they are not equal and if $F' \subset \bar{F}$. Also we denote $\mathcal{X}^d$ as the set of all d-dimensional cells in $\mathcal{X}$. A consequence of the above conditions is that for any $F \in \mathcal{X}^d$ and $F' \in \mathcal{X}^{d-2}$ such that F' is a face of F then there exist exactly two cells $E_1, E_2 \in \mathcal{X}^{d-1}$ such that F' is a face of E_i and E_i is a face of F. Another fundamental property of regular cell complexes is that for any n-cell F, $\bar{F} - F$ is the union of the closures of $(n-1)$-cells. Moreover, two regular cell complexes with isomorphic face posets are homeomorphic.

A good description CW-complexes can be found in [Hat02] and [Mas78], the later also gives a good treatment of regular cell complexes. Examples of regular cell complexes include simplicial complexes and polyhederal complexes.

While cellular homology can be defined for any CW-complex, in the case where X is regular the description of the homology can be described combinatorially. The function "sign" is an incidence function on $\mathcal{X}$ if it satisfies the following properties:

1. to each pair (F, G) such that $F \in \mathcal{X}^d$ and $G \in \mathcal{X}^{d-1}$ for some $d \geqslant 0$, sign assigns a number from $\{0, \pm 1\}$ to the pair.

2. $\operatorname{sign}(F, G) \neq 0$ if and only if G is a face of F

3. $\operatorname{sign}(F, \emptyset) = 1$ for all $F \in \mathcal{X}^0$

4. if $F \in \mathcal{X}^d$ and $G \in \mathcal{X}^{d-2}$ is a face of F then $\operatorname{sign}(F, E_1)\operatorname{sign}(E_1, G) + \operatorname{sign}(F, E_2)\operatorname{sign}(E_2, G) = 0$ where E_1 and E_2 are as above.

Note that any two incidence functions on $\mathcal{X}$ differ only by a a function

$\delta : \mathcal{X} \to \{\pm 1\}$ where $\delta(\emptyset)$ and $\delta(F)$ where F is a 0-cell are all equal to 1.

Using this incidence function, we can define the maps in the augmented chain complex of $\mathcal{X}$. The chain complex is

$$\mathcal{C}_{\mathcal{X}} : 0 \to C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to 0$$

where

$$C_i = \bigoplus_{F \in \mathcal{X}^i} kF \text{ and,}$$

$$\partial_i(F) = \sum_{G \in \mathcal{X}^{i-1}} \text{sign}(F, G)G \text{ for } F \in \mathcal{X}^i.$$

Then the reduced homology of X is defined as $\tilde{H}_i(X, k) = \ker \partial_i / \operatorname{im} \partial_{i+1}$

2.3 Free Resolutions of Modules

Let $R = \mathbb{C}[x_0, \ldots, x_n]$, and let I be an ideal of R. Then the free resolution of R/I is an exact sequence of maps between free R-modules:

$$\mathcal{F} : 0 \to F_t \xrightarrow{d_t} \cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F \to R/I \to 0$$

We call F_i the *i-th syzygy module* of R/I. We say that $\mathcal{F}$ is a *minimal* resolution if each module F_i is generated by minimal syzygies, and if each map d_i has no entries which are units. If $\mathcal{F}$ is minimal we say $F_i = R^{\beta_i}$, and we call β_i the i-th Betti number.

Note also that if we want the R-graded maps in $\mathcal{F}$ to be degree 0 then

we let

$$F_i = \bigoplus_d R(-d)^{\beta_{i,d}}$$

where $R(-d)_e = R_{e-d}$ and $\beta_{i,d}$ is called a graded Betti number. Moreover, if the ring R is graded by $\mathbb{Z}^d$ for some d (i.e. a mulitgrading) then one can also define multigraded Betti numbers. We will see instances of this in the next section since monomial ideals are multigraded.

2.4 Cellular Resolutions of Monomial Ideals

In the special case where I is a monomial ideal, there are combinatorial descriptions of resolutions of I or S/I. The construction explained below was first done for regular cell complexes in [BS98] and later extended to cover CW-complexes in [BW02].

Let X be a cell complex whose vertices are labeled by the generators of a monomial ideal I and whose faces σ are labeled by the lcms, m_σ, of the verticies contained in the face. Then define

$$\mathcal{F}_X : 0 \to F_t \xrightarrow{d_t} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_0} F_0$$

using the reduced chain complex of X. It will be a complex of free R-modules where

$$F_i = \bigoplus_{\dim \sigma = i-1} R(-m_\sigma)$$

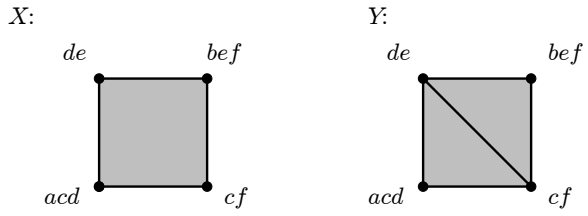
and the maps are defined as

$$d_i(F) = \sum_{\gamma \text{ facet of } \sigma} \operatorname{sign}(\gamma, \sigma) \frac{m_\sigma}{m_\gamma}.$$

For $b \in \mathbb{N}^n$, define the complex $X_{\leqslant b} = \{\sigma \in X | x^b \text{ divides } m_\sigma\}$, and $X_{\leqslant b}$ is acyclic if it is either empty or has no reduced homology. We state the result of Bayer and Sturmfels which gives the condition for the complex $\mathcal{F}_X$ associated to a monomial ideal to be exact [BS98]. It should be noted that the results of Bayer and Sturmfels in [BS98] builds upon previous methods which associated resolutions of monomial ideals to certain simplicial complexes such as the Taylor complex introduced in [Tay60] and the Scarf complex introduced in [BPS98].

Theorem 2.4.1. *$\mathcal{F}_X$ is a resolution of R/I if and only if $X_{\leqslant b}$ is acyclic over k for all degrees $b \in \mathbb{N}^n$.*

Example 2.4.2. The following figure depicts two possible cell complexes that may support the monomial ideal $M = (de, bef, cf, acd) \subset k[a, b, c, d, e, f]$ with the vertices labeled.



Notice that $X_{\leqslant dcef}$ consists of two vertices whereas $Y_{\leqslant dcef}$ is the same two vertices with the diagonal edge between them. This shows that X does

not support a resolution of M. By checking the other degrees it is easy to see that Y does support the following resolution of M.

$$\mathcal{F}_Y : 0 \to \begin{matrix} R(-bcdef) \\ \oplus R(-acdef) \end{matrix} \xrightarrow{\begin{pmatrix} c & 0 \\ -d & 0 \\ 0 & e \\ -b & a \\ 0 & -f \end{pmatrix}} \begin{matrix} R(-bdef) \\ \oplus R(-bcef) \\ \oplus R(-acdf) \\ \oplus R(-dcef) \\ \oplus R(-acde) \end{matrix} \xrightarrow{\begin{pmatrix} -bf & 0 & 0 & -cf & -ac \\ d & c & 0 & 0 & 0 \\ 0 & -be & -ad & de & 0 \\ 0 & 0 & f & 0 & e \end{pmatrix}}$$

$$\begin{matrix} R(-de) \\ \oplus R(-bef) \\ \oplus R(-cf) \\ \oplus R(-acd) \end{matrix} \xrightarrow{\begin{pmatrix} de & bef & cf & acd \end{pmatrix}} R \to R/M \to 0$$

One can easily see here that since no entries in the any of the maps are units, that this resolution is minimal.

Note that the theory of cellular resolutions provides a criterion for when a complex is a resolution, but it does not provide an algorithm for finding minimal resolutions. For some classes of ideals though, the minimal resolution is known. I include a discussion of those results here.

Before discussing when the minimal resolution is known, I will first explain what that resolution is.

Definition 2.4.3. Given a monomial ideal M, let $\{m_1, \ldots, m_t\}$ be a minimal generating set. Then the *Scarf complex*, as introduced in [BPS98], is the simplicial complex whose faces are all subsets $\sigma \subset \{1, \ldots, t\}$ such that the $\operatorname{lcm}\{m_i \,|\, i \in \sigma\}$ is unique in the set of all possible lcms. We will denote it here as $\operatorname{scarf}(M)$.

The complex $\mathcal{F}_{\operatorname{scarf}(M)}$ is usually not a resolution (i.e. it does not satisfy 2.4.1), but if it is a resolution it is guaranteed to be minimal[BPS98]. Moreover, it is known that in the following cases $\operatorname{scarf}(M)$ satisfies the conditions of 2.4.1.

In [BPS98], the authors define the notion of a generic monomial ideal as follows:

Definition 2.4.4. A monomial ideal M is *strongly generic* [1] if no variable appears with the same exponent in any of the generators.

In [MSY00], the authors loosen the definition of generic monomial ideal appearing in [BPS98] to the current standard definition as follows:

Definition 2.4.5. A monomial ideal M is *generic* if whenever two distinct monomial generators m_i and m_j have the same positive (nonzero) degree in some variable, a third generator m_k divides $\operatorname{lcm}(m_i, m_j)/x_l$ for all x_l.

And they give the following characterization:

[1] In [MS05] the adverb "strongly" is added to distinguish it from the definition 2.4.5

Theorem 2.4.6. *A monomial ideal M is generic if and only if the following two conditions hold:*

1. *$\mathcal{F}_{\text{scarf}(M)}$ equals the minimal free resolution of R/M.*

2. *No variable x_t appears with the same non-zero exponent in m_i and m_j for any edge $\{i, j\}$ of the Scarf complex.*

The interest in studying generic monomial ideals is that their minimal resolutions are always given by the Scarf complex. Neither of these characterizations though cover all monomial ideals whose minimal resolution is the Scarf resolution as we will see in 5.1.1.

The final idea that I need to introduce from [BPS98] is the notion of "deformation of exponents." Naively, this is just a process by which one deforms the exponent vectors of the monomial generators in a small neighborhood with the intention of obtaining a monomial ideal whose minimal resolution is known.

Definition 2.4.7. A *deformation* of a monomial ideal $M = (m_1, \ldots, m_t)$ is a choice of vectors $\{\epsilon_1, \ldots, \epsilon_t\}$ where each $\epsilon_i \in \mathbb{R}^n$ (where n is the number of variables) and the following condition is satisfied:

$$m_{is} < m_{js} \text{ implies } m_{is} + \epsilon_{is} < m_{js} + \epsilon_{js}, \text{ and}$$

$$m_{is} = 0 \text{ implies } \epsilon_{is} = 0.$$

Where by abuse of notation, m_{is} is the exponent on x_s in the monomial m_i.

Then we can form the monomial ideal (in a polynomial ring with real exponents) M_ϵ with generators $m_i * \mathbf{x}^{\epsilon_i}$.

Note the need to work with real exponents in this definition. In actuality, we are only interested in the combinatorics of the deformation which amounts to looking only at the coordinatewise order on the resulting exponent vectors. For any set of ϵ_i vectors, there is a choice of vectors with integer values which yields a deformation of exponents with the same combinatorics.

The main result concerning deformation of exponents appears in [BPS98] and says the following:

Theorem 2.4.8. *If $\mathcal{F}$ is a minimal free resolution of R/M_ϵ, then it is a ((not necessarily minimal) resolution of R/M.*

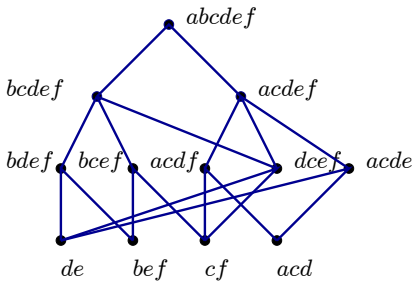
The idea then is that if one can obtain a generic monomial ideal via deformation of exponents (i.e. a generic deformation) and if the Betti numbers do not increase under this deformation then the minimal resolution of the original ideal is known. Note however, that since this minimal resolution will be the Scarf complex of the deformed ideal, that it will be simplicial. Thus, any monomial ideal whose minimal resolution is not simplicial will necessarily be in the situation where under any generic deformation Betti numbers increase.

2.5 LCM lattices and their relation to resolutions of monomial ideals

A useful tool in the study of cellular resolutions of monomial ideals is the LCM lattice associated to the generators (or a generating set) of the ideal. This link between resolutions of ideals and LCM lattices was explored by Gasharov, Peeva, and Welker in [GPW99].

Definition 2.5.1. The LCM lattice, LCM(I), of a monomial ideal I is the set of least common multiples of the minimal generators of I, partially ordered by divisibility.

Example 2.5.2. For the monomial ideal $M = (de, bef, cf, acd) \subset k[a, b, c, d, e, f]$ the Hasse diagram of the LCM lattice of M is shown in the following figure (note the minimal element of the lattice has been left off, as will often be the case).



One conclusion of their work is that for monomial ideals the minimal resolution is completely dependent on the information in the LCM lattice. Specifically, one can compute multigraded Betti numbers using the LCM

lattice LCM(I) and all ideals with a given LCM lattice have isomorphic minimal free resolutions. We state those results here without proof.

Theorem 2.5.3. *For $i \geqslant 1$ and $m \in \mathrm{LCM}(I) = P$ we have*

$$b_{i,m}(R/I) = \dim \tilde{H}_{i-2}(\Delta(\hat{0}, m); k),$$

and

$$b_i(R/I) = \sum_{m \in P} \dim \tilde{H}_{i-2}(\Delta(\hat{0}, m); k)$$

Note that because of the homotopy equivalence between the order complex of a poset and the cross-cut complex that the above theorem can be rephrased entirely in terms of $\Gamma(\hat{0}, m)$. The next theorem states that the combinatorial type of a resolution depends only on its LCM lattice.

Theorem 2.5.4. *If I and I' are both monomial ideals in polynomial rings R and R' respectively. Let $P_I = \mathrm{LCM}(I)$ and let $f : P_I \to P_{I'}$ be a map which is a bijection on the atoms and preserves joins. Denote by $\mathcal{F}_I$ the minimal free resolution of R/I. Then $f(\mathcal{F}_I)$ is defined as in [GPW99] and is a free resolution of R'/I'. If f is an isomorphism of lattices then $f(\mathcal{F}_I)$ is the minimal free resolution of R'/I'.*

2.6 Associating monomial ideals to finite atomic lattices: Minimal Monomial Ideals

The point of view of this thesis relies heavily on the ideas presented in Phan's thesis [Pha06] . In summary the main idea of Phan's thesis influencing this work is that all finite atomic lattices P can be realized as the LCM lattice of some monomial ideal M. He gives a construction which is motivated by the observation that for any coordinatization of an atomic lattice as a monomial ideal the set of lattice elements for which a given variable has a given degree bound is an order ideal. Essentially, he identifies which order ideals are necessary and labels them with variables.

Phan's construction of a square free monomial ideal is as follows.

1. Denote $\mathrm{mi}(P)$ as the set of meet-irreducible elements in $P - \{\hat{0}, \hat{1}\}$. Let $R(P) = k[x_1, ..x_N]$ where $N = |\,\mathrm{mi}(P)|$.

2. To each atom in P assign the following monomial:

$$x(a) = \prod_{l \in (\mathrm{mi}(P) - \lceil a \rceil)} x_l.$$

3. M_P is the monomial ideal generated by $\{x(a) | a \in \mathrm{atoms}(P)\}$.

This is a specific monomial ideal whose LCM lattice is P which in [Pha06] is called the "minimal squarefree monomial ideal associated to P." It is called minimal because its generators have the smallest possible degree (i.e. if any one of the generators had smaller degree the ideal could not have the correct

LCM lattice). Phan also explains how to construct non-squarefree monomial ideals whose generators are of the same degree, in other words depolarizations of the square-free minimal monomial ideal constructed above. I will forego that discussion and replace it instead with a construction of how to obtain any monomial ideal with a given LCM lattice P in section 3.

2.7 Paramatrization of finite atomic lattices on n atoms

Consider the set $\mathcal{L}(n)$ of finite atomic lattices on n atoms. It is shown in Phan's thesis [Pha06] that one can partially order $\mathcal{L}(n)$ as follows, $Q \leqslant P$ if and only if there exists a join-preserving map which is a bijection on atoms from P to Q (note that such a map will also be surjective). Most surprising is the following nice result.

Theorem 2.7.1. *With the partial order $\leqslant$, $\mathcal{L}(n)$ is a finite atomic lattice with $2^n - n - 2$ atoms.*

Roughly this theorem is proved by showing that this poset is a meet-semilattice. Then by proposition 2.1.1 because the boolean lattice B_n is the unique maximal element, we can conclude that $\mathcal{L}(n)$ is a finite atomic lattice. To show that it is a meet-semilattice Phan shows that the meet of any two lattices is given by embedding them into B_n and then taking the intersection of their images.

Figure 2.1 shows all of $\mathcal{L}(3)$. It is important to note that in general $\mathcal{L}(n)$

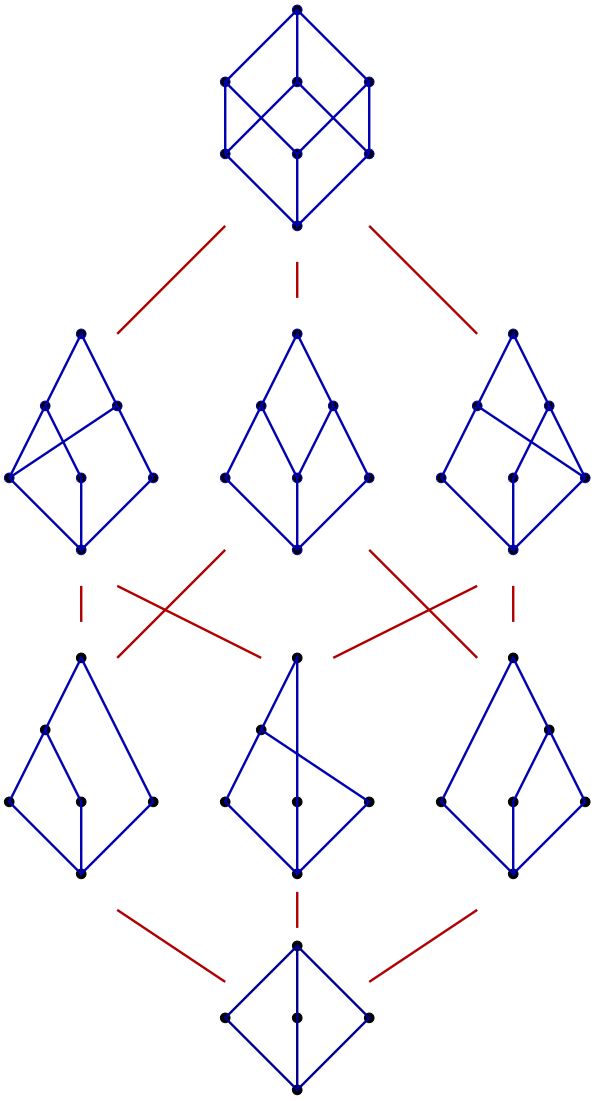


Figure 2.1: $\mathcal{L}(3)$

will not be B_n. For $n = 4$, there are 545 elements thus the picture cannot be shown here.

Chapter 3

Characterizing all monomial ideals with a given LCM lattice

The goal of this chapter is to give a description of how to find all monomial ideals with a given finite atomic lattice P. Rather than just providing the answer, I will include the ideas which motivate proposition 3.2.1. Moreover, proposition 3.2.1 includes the depolarizations discussed in [Pha06] and so this chapter serves to cover the details left out in section 2.6.

3.1 Deficit labelings

We begin by examining an example shown in figure 3.1. Let P be the following finite atomic lattice with monomial ideal $M_P = (def, ade, abe, abcd)$.

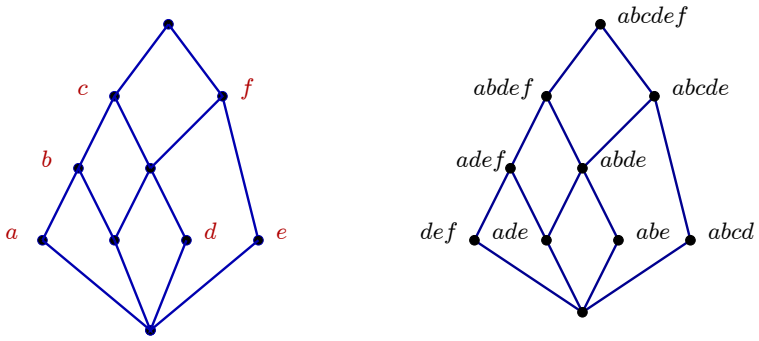


Figure 3.1: A lattice P shown with two labelings

Here P is shown twice, in one case labeled with the variables that correspond to each meet-irreducible element and in the other labeled with the lcms at each node. The purpose of looking at this example is to illustrate that in Phan's construction, the product of the variables corresponding to the meet-irreducibles in a principal filter of the lattice P is the monomial that does not divide the lcm at the generator of that filter. This observation motivates the following.

I want to introduce the notion of a *deficit labeling*. Given a monomial ideal M we can construct its LCM lattice P_M. For each element in a poset there are several ways that we can refer to it, so we will fix some notation here that will hopefully alleviate confusion. Let P be a finite atomic lattice (whose elements are just atoms and joins of atoms) where the map $\psi : P_M \to P$ defined by $\psi(\tilde{q}) = q$ is an isomorphism. (i.e. we've just dropped the lcm labeling of each element in P_M). Henceforth when we refer to Q we will always be referring to $\psi^{-1}(q) = \tilde{q}$ for the appropriate $q \in P$ where $\tilde{q}$ is the lcm of the atoms of which it is the join.

A deficit labeling of P can be obtained as follows, first each element

$q \in P$ can be labeled with the monomial $d_q = (\psi^{-1}(\hat{1})/\tilde{q})$. Then the *deficit label at* q is the monomial $D_q = d_q/(\text{lcm}\{d_t | t \in \lceil q \rceil, t \neq q\})$. Note that if we are thinking of finite atomic lattices as abstract or non-embedded monomial ideals then a deficit labeling is the embedding data of a given monomial ideal.

Proposition 3.1.1. *Any deficit labeling of an LCM lattice* P_M *will label each element of* $\text{mi}(P)$ *with a nontrivial monomial.*

Proof. All we need to prove is that if q is a meet irreducible, then $D_q \neq 1$.

It is obvious that $d_q \neq 1$ since q is not the maximal element in P. So we just need to show that $\text{lcm}\{d_t | t \in \lceil q \rceil, t \neq q\} \neq d_q$.

First note that

$$\text{lcm}\{d_t | t \in \lceil q \rceil, t \neq q\} = \psi^{-1}(\hat{1}) / \text{gcd}\{\tilde{t} | t \in \lceil q \rceil, t \neq q\}.$$

Since q is meet irreducible, this means that every element $t \in \lceil q \rceil$ can be written as $q \vee a_i \vee b$ where a_i is the atom that specifically gives the only element that covers q. This means that

$$gcd\{\tilde{t} | t \in \lceil q \rceil, t \neq q\} = \tilde{q} * \tilde{a}_i * \underset{a_j \in \cup \text{supp}(b)}{\text{gcd}} \tilde{a}_j.$$

It follows both that $\text{lcm}\{d_t | t \in \lceil q \rceil, t \neq q\} \neq d_q$ (as needed), and that $D_q = \text{gcd}\{\tilde{t} | t \in \lceil q \rceil, t \neq q\}/\tilde{q}$ (note q need not be meet irreducible for this formula to hold). Since $\tilde{q}$ divides the gcd that this also proves that D_q is a monomial with non-negative exponents. □

Proposition 3.1.2. *If* $\gcd\{D_{q_1}, .., D_{q_r}\} \neq 1$ *for a subset of elements* $\{q_1, ..., q_r\}$ *in* P *then* $\{q_1, ..., q_r\}$ *must lie in a chain in* P.

Proof. In order to prove this, first we must note that that if two elements q and q' do not lie in a chain then $\text{lcm}\{\tilde{q}, \tilde{q}'\} = \psi^{-1}(\hat{1})$. In particular what we must show here is that every pair of elements $\{q_i, q_j\}$ is comparable, i.e. that $\text{lcm}\{\tilde{q}_i, \tilde{q}_j\} \neq \psi^{-1}(\hat{1})$ for all $i \neq j$ between 1 and r.

Since $\gcd\{D_{q_1}, .., D_{q_r}\} \neq 1$, we can say $\gcd\{D_{q_1}, .., D_{q_r}\} = C$ for some monomial C. Then there exists monomials B_i such that $C * B_i = D_{q_i}$, so we can rewrite $\tilde{q}_i = \gcd\{t \in \lceil q_i \rceil, t \neq q_i\}/CB_i$. We are interested in showing that

$$\text{lcm}\{\frac{\gcd\{t \in \lceil q_i \rceil, t \neq q_i\}}{CB_i}, \frac{\gcd\{t \in \lceil q_j \rceil, t \neq q_j\}}{CB_j}\} \neq \psi^{-1}(\hat{1}).$$

It is easy to see that even in a best case scenario where $\gcd\{t \in \lceil q_j \rceil, t \neq q_j\}$ or $\gcd\{t \in \lceil q_i \rceil, t \neq q_i\}$ equal $\psi^{-1}(\hat{1})$ they are both being divided by C. Thus for any x_i that divides C, its exponent in the lcm will be less than that for the same variable in $\psi^{-1}(\hat{1})$. Thus all pairs of q_i are comparable which means they must lie in a chain. □

3.2 Labelings and Coordinatizations

The conditions that the deficit labelings satisfy motivate the following definitions and proposition which characterize which monomial ideals can be associated to a given lattice.

I define a *labeling* of P, to be any assignment of monomials $\mathcal{M} = \{m_{p_1}, ..., m_{p_t}\}$ to some set of elements $p_i \in P$. Then a labeling is a *co-*

ordinatization if the monomial ideal $M_{P,\mathcal{M}}$ which is generated by monomials

$$x(a) = \prod_{p \in \lceil a \rceil^c} m_p$$

for each $a \in \text{atoms}(P)$ has LCM lattice isomorphic to P.

The above description of deficit labelings motivates the following characterization of possible coordinatizations given a lattice P.

Proposition 3.2.1. *Any labeling $\mathcal{M}$ of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of the lattice P.*

- *If $p \in \text{mi}(P)$ then $m_p \neq 1$. (i.e. all meet-irreducibles are labeled)*
- *If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$ then p and q must be comparable. (i.e. each variable only appears in monomials along one chain in P.)*

Note: This proof is an adaptation of Phan's original proof in his thesis that his specific labeling yielded a coordinatization of the lattice P

Proof. Let P' be the LCM lattice of $M_{P,\mathcal{M}}$. We just need to show that P' is isomorphic to P. Let $f : P \to P'$ by 2.1.2 it is only necessary to show that f is either join-preserving or meet-preserving and is a bijection.

For $b \in P$ define f to be the map such that

$$f(b) = \prod_{l \in \lceil b \rceil^c} m_p.$$

So obviously f is a bijection on atoms. Note also, that

$$\lceil b \rceil^c = \bigcup_{a_i \in \mathrm{supp}(b)} \lceil a_i \rceil^c.$$

In order to show that f is join-preserving and a surjection, we need to show that, $f(b) = \mathrm{lcm}\{f(a_i) | a_i \in \mathrm{supp}(b)\}$. By the two remarks above, we know that

$$f(b) = \prod m_p$$

where $m_p \in \lceil a_i \rceil^c$ for at least one $a_i \in \mathrm{supp}(b)$. Since,

$$\mathrm{lcm}\{f(a_i) | a_i \in \mathrm{supp}(b)\} = \prod {x_i}^{n_i}$$

for $n_i = \max_j n_{ij}$ where n_{ij} is the exponent on x_i in $f(a_j)$, we just need to show that $x_i^{n_i} | f(b)$ and that no higher powers of x_i divide $f(b)$.

This follows from the fact that the x_i only divides monomials that label elements in a chain of P. This ensures that $x_i^{n_i}$ appears as the highest power of x_i for a unique product of monomials m_p because if $p \in \lceil a_i \rceil^c$ then all p' such that $p' \leqslant p$ are also elements of $\lceil a_i \rceil^c$. Moreover, this unique product of monomials appears in the product of monomials forming $f(b)$. Thus, $x_i^{n_i}$ divides $f(b)$. Moreover, no higher powers of x_i divide $f(b)$ since we chose n_i to be the $\max_j n_{ij}$.

It follows that $f(a \vee b) = \mathrm{lcm}(f(a), f(b)) = f(a) \vee f(b)$ so, f is join preserving and surjective. It remains to show that f is injective.

Clearly, if $a \leqslant b$ then $\lceil a \rceil^c \subset \lceil b \rceil^c$ so $f(a) \leqslant f(b)$. It remains to show

that $f(a) \leqslant f(b)$ implies that $a \leqslant b$. We know that every $a \in P$ equals the meet of those $c \in \mathrm{mi}(P)$ such that $a \leqslant c$. This implies that $a \leqslant b$ if and only if $\mathrm{mi}(P) \cap \lceil b \rceil \subset \mathrm{mi}(P) \cap \lceil a \rceil$ if and only if $\mathrm{mi}(P) - \lceil a \rceil \subset \mathrm{mi}(P) - \lceil b \rceil$. Since we required that all of the meet-irreducibles be assigned a monomial, then the product over these sets are contained in $\lceil a \rceil^c$, and $\lceil b \rceil^c$ (respectively). Thus, $f(a) \leqslant f(b)$ implies that $a \leqslant b$ and so f is injective. □

In [Pha06], he shows that if one labels meet-irreducibles along chains with the same variable that this will yield a depolarized version of the "minimal monomial ideal" associated to P. This clearly satisfies the conditions of proposition 3.2.1, thus this proves that result as well. Coordinatizations of lattices have appeared in several other places as instances of associating monomial ideals to cell complexes. A nice example of this are the "nearly Scarf" ideals introduced by Peeva and Velasco in [PV], [Vel08] can easily be seen as a coordinatization of augmented face lattices of simplicial complexes. Their construction associates to every face of a simplicial complex a variable, and defines a monomial at vertex to be the product of all the variables on faces not touching that vertex. This corresponds to labeling every element of the augmented face lattice of the simplicial complex with a different variable, clearly this satisfies the conditions of proposition 3.2.1. Other examples will be addressed in chapter 6.

Note that it possible for a labeling which does not satisfy the second condition can be a coordinatization. For example in figure 3.2 one sees that both labelings yield the monomial ideal $M = (y^3z^2, xy^2z, x^2y, x^3z)$ which has the correct LCM lattice thus these are both coordinatizations. However, the one on the right is the only one of the two which satisfies all of the conditions

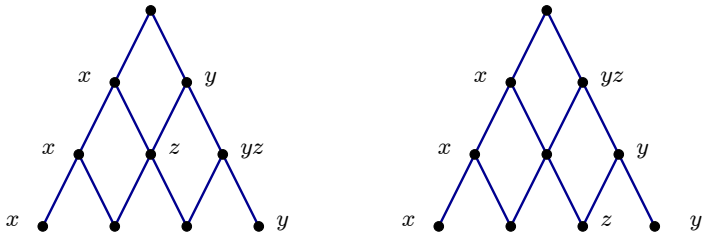


Figure 3.2: A lattice P shown with two equivalent labelings

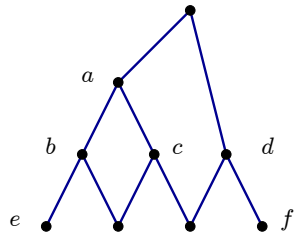
of proposition 3.2.1.

Although, using the following "moves" one can always rearrange such a labeling to one that does satisfy the conditions of 3.2.1. In particular the "move" is that if $\gcd(m_p, m_q) = m$ then label both $p \vee q$ and $p \wedge q$ with m and relabel p, q with $m_p/m, m_q/m$ respectively. This relabeling will satisfy condition two of 3.2.1 and will also yield the same monomial ideal as is shown in figure 3.2.

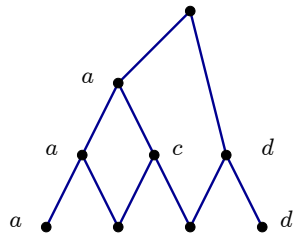
3.3 Specific coordinatizations

It will be useful for us to discuss several specific coordinatizations of lattices L in the subsequent sections of this thesis. I will give a description of them here.

1. **Minimal Squarefree** The description of this is given above in 2.6. This obviously satisfies the conditions of 3.2.1 since only meet-irreducibles are labeled and each variable is used only once. An example is shown below, the monomial ideal given by this coordinatization is $M = (cdf, def, bef, abce)$.



2. **Minimal Depolarized** Label every meet irreducible, using the same variable along chains when possible. This obviously satisfies the conditions of 3.2.1. In the example below, we see that this is a depolarization of the minimal squarefree example above. The monomial ideal here is $M = (cd^2, ad^2, a^2d, a^3c)$. Note that since there will be multiple ways of using a variable along a chain, that this coordinatization is in no way unique.



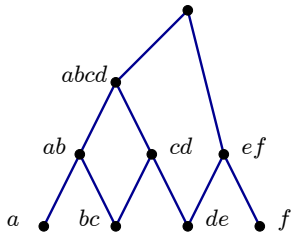
3. **Greedy** Let $\{c_1, \ldots, c_t\}$ be the set of all maximal chains in P. Then for variables in the ring $R = k[x_1, \ldots x_t]$ define the following labeling,

$$\mathcal{M} = \{m_p = \prod_{\substack{i: \\ p \in c_i}} x_i | p \in P\}.$$

Every meet-irreducible is covered since every element of P is covered and each variable appears only along one chain by definition, so the

conditions of 3.2.1 are satisfied. The example below shows such a coordinatization, the monomial ideal is

$$M = (bc^2d^2e^2f^2, ade^2f^2, a^2b^2cf, a^3b^3c^3d^3e).$$



Chapter 4

Structure of $\mathcal{L}(n)$

As discussed in section 2.5 we can compute the Betti numbers of a monomial ideal using its LCM lattice P, so from now on we will denote $\beta_i(R/I) = \beta_i(P)$. For any given finite atomic lattice $P \in \mathcal{L}(n)$, we define $\mathbf{b}_P = (\beta_0, \beta_1, ..., \beta_{n-2})$ as the Betti vector associated to P. We then can define a map

$$\phi : \mathcal{L}(n) \to \beta(n) \cong \mathbb{N}^{n-1}$$

which takes P to $\mathbf{b}_P$ and we will call $\beta(n)$ the space of Betti vectors. Following theorem 2.5.4 one can observe that if $P \geqslant Q \in \mathcal{L}(n)$ then the minimal resolution of P is a resolution of Q. In other words, $\phi(P)$ is coordinatewise greater than or equal to $\phi(Q)$.

We can define an equivalence relation on $\mathcal{L}(n)$ by saying $P \equiv Q$ if $\phi(P) = \phi(Q)$. This breaks $\mathcal{L}(n)$ up into strata where total Betti numbers are constant in each strata. Most of the main ideas in this thesis focus on

my larger goal of understanding the boundaries of these strata and how they fit together.

A challenge to doing this is that other than Phan's theorem 2.7.1, little is known about the structure of $\mathcal{L}(n)$ and as n increases $|\mathcal{L}(n)|$ increases rapidly. For instance, $|\mathcal{L}(3)| = 8$, $|\mathcal{L}(4)| = 545$, $|\mathcal{L}(5)| = 702{,}525$, and $|\mathcal{L}(6)| = 66{,}960{,}965{,}307$.[1] In the original proof that $\mathcal{L}(n)$ is a lattice meets are defined via embedding each lattice into B_n and then intersecting the images. There is however, no "nice" description of joins or covering relations.

This weakly monotonic nature of Betti numbers is the central idea guiding the work in this thesis. What follows is an alternate proof of this fact which explicitly shows how the multigraded Betti numbers change as one moves around in $\mathcal{L}(n)$ rather than using theorem 2.5.4. This may seem out of place, but the methods used in this alternate proof provide motivation for the content of the subsequent sections. In particular the key observation is that as one moves around in $\mathcal{L}(n)$ it is important to keep track of how relations between joins of atoms vary.

4.1 Motivation: Alternate proof of increasing Betti numbers

As alluded to above, when we are discussing elements in a given finite atomic lattice P there is a constant point of ambiguity concerning the "names" of elements in P. The problem is any given element can usually be described

[1] The computations for $n = 5, 6$ were made using a program given in Appendix A.

by several different joins of atoms. To allow ourselves to have all equivalent "names" of a given element $m \in P$ at our disposal, we define the following set

$$\text{equiv}_P(m) = \left\{ \sigma_i \subseteq \text{atoms}(P) | \bigvee_{a_i \in \sigma_i} a_i = m \right\} = f^{-1}(m),$$

where $f : B_n \to P$ is the join preserving map which is a bijection on atoms. Note that one of these σ_i will always be equal to

$$\text{supp}(m) = \{a_i \in \text{atoms}(P) | a_i \leqslant m\},$$

and all the rest will satisfy $\sigma_i \subset \text{supp}(m)$.

The following is a technical lemma that allows us to see precisely which subcomplexes of $\Gamma(P)$ are candidates for having homology thus indicating that a syzygy exists.

Lemma 4.1.1. *If $\tilde{h}_i(\Gamma(P_{<m}); k) = t$ then there exists a subset E in $\text{equiv}_P(m)$ such that $|E| \geqslant t$. Moreover, $\tilde{h}_i(\Gamma(P_{<m})|_{\sigma_j}); k) = 1$ for all $\sigma_j \in E$.*

Proof. First, note that since $P_{\leqslant m}$ is a lattice, it is acyclic. So

$$\tilde{H}_i(\Gamma(P_{\leqslant m}); k) = \frac{\ker \partial_i : C_i \to C_{i-1}}{\text{im}\, \partial_{i+1} : C_{i+1} \to C_i} = 0.$$

Thus, $\ker \partial_i = \text{im}\, \partial_{i+1}$. Now we want to consider

$$\tilde{H}_i(\Gamma(P_{<m}); k) = \frac{\ker \partial'_i : C'_i \to C'_{i-1}}{\text{im}\, \partial'_{i+1} : C'_{i+1} \to C'_i}$$

where ∂_i' is just the restriction of ∂_i and a basis for C_i' is $I \subset \text{supp}(m)$ such that $|I| = i+1$. Note that $\ker \partial_i' \subseteq \ker \partial_i$ and $\text{im}\, \partial_{i+1}' \subseteq \partial_{i+1}$.

Since $\tilde{h}_i(\Gamma(P_{<m}); k) = t$, we can find a basis $\{[\gamma_1], ..., [\gamma_k]\}$ for $\tilde{H}_i(\Gamma(P_{<m}); k)$. Consider, γ_j which is an element of $\ker \partial_i'$. Since $\ker \partial_i' \subseteq \ker \partial_i = \text{im}\, \partial_{i+1}$ we know that there exists some $c \in C_{i+1}$ such that $\partial_{i+1}(c) = \gamma_j$ for all j. However, $\gamma_j \notin \text{im}\, \partial_{i+1}'$ meaning that γ_j is the boundary of some $(i+1)$-cell c which is not in C_{i+1}'. Let σ_j be the vertex set of this missing $(i+1)$-cell c. Clearly then, if we restrict $\Gamma(P_{<m})$ to the vertex set described by σ_j then $\tilde{h}_i(\Gamma(P_{<m})|_{\sigma_j}); k) = 1$.

□

We can define a partial order on Betti vectors as follows $B_P > B_Q$ if

$$\beta_i(P) \geqslant \beta_i(Q)$$

for all i, and obviously if $\beta_i(P) = \beta_i(Q)$ for all i we say $B_P = B_Q$.

Recall that if $P \geqslant Q$ in $\mathcal{L}(n)$ there exists a join preserving map $\psi : P \to Q$. Observe that if one takes $q \in Q$ and looks at the pre-image $\psi^{-1}(q) \subset P$ then because ψ is join-preserving we see:

$$\text{equiv}_Q(q) = \bigcup_{p \in \psi^{-1}(q)} \text{equiv}_P(p),$$

and that $\text{equiv}_P(p) \cap \text{equiv}_P(p')$ for any two $p, p' \in \psi^{-1}(q)$.

Observe that for $p \in \psi^{-1}(q)$ if $\text{supp}(p) = \text{supp}(q)$, that faces in $\Gamma(P_{<p})$ are by definition are subsets $J \subset \text{supp}(p) = \text{supp}(q)$ such that $J \notin \text{equiv}_P(p)$

which implies that $J \notin \text{equiv}_Q(q)$. So faces of $\Gamma(P_{<p})$ correspond to elements in $\text{equiv}_Q(q) - \text{equiv}_P(p)$. So the concern as one moves up chains in $\mathcal{L}(n)$ is that by adding faces into $\Gamma(P_{<p})$ that one could kill homology that contributes to the Betti numbers of a resolution. Never fear though, due to the fact that there will be other elements $p \in \psi^{-1}(q)$ where $\text{supp}(p) \neq \text{supp}(q)$ we will always be able to find the elements which contributed homology to Betti numbers in lower lattices in higher lattices.

Remark 4.1.2. A restatement of theorem 2.5.4 in this language is: If $P > Q$ in $\mathcal{L}(n)$ then $B_P \geqslant B_Q$

I provide the following proof for 4.1.2 which follows from the ideas introduced throughout this section.

Proof. Recall that

$$\beta_{i+2}(P) = \sum_{p \in P} \tilde{h}_{i-2}(\Gamma(P_{<p}); k).$$

Since, $\psi : P \to Q$ is a surjective map we can rewrite this sum as follows

$$\sum_{q \in Q} \sum_{p \in \psi^{-1}(q)} \tilde{h}_{i-2}(\Gamma(P_{<p}); k).$$

So it is enough to prove that for any point $q \in Q$,

$$\tilde{h}_{i-2}(\Gamma(Q_{<q}); k) \leqslant \sum_{p \in \psi^{-1}(q)} \tilde{h}_{i-2}(\Gamma(P_{<p}); k).$$

We begin by looking at $\psi^{-1}(q) = \{p_1, ..., p_s\} \subset P$ for some $q \in Q$. By

lemma 4.1.1 if $\tilde{h}_{i-2}(\Gamma(Q_{<q});k) = t$ then we can find $\{\sigma_1, ..., \sigma_t\} \subseteq \text{equiv}_Q(q)$ such that $\tilde{h}_{i-2}(\Gamma(Q_{<q}|_{\sigma_j});k) = 1$ for all $1 \leqslant j \leqslant t$.

By the discussion above, $\sigma_j \in \text{equiv}_P(p)$ for some $p \in \psi^{-1}(q)$. Moreover, we know that all subsets of σ_j are not in $\text{equiv}_Q(q)$ because σ_j corresponds to a basis element in the $\tilde{H}_{i-2}(\Gamma(Q_{<q});k)$. (i.e. enough subsets of σ_j need to be less than q because the boundary of the missing cell corresponding to σ_j needs to be present.) Thus since both $\Gamma(P_{<p})|_{\sigma_j}$ and $\Gamma(Q_{<q})|_{\sigma_j}$ have σ_j as their vertex set and faces which correspond to $J \subset \sigma_j$ such that $J \notin \text{equiv}_P(p)$ or $\text{equiv}_Q(q)$ respectively. We see that $\Gamma(P_{<p})|_{\sigma_j} = \Gamma(Q_{<q})|_{\sigma_j}$, thus $\tilde{h}_{i-2}(\Gamma(P_{<p})|_{\sigma_j};k) = 1$ for all j. □

The advantage of this proof, is that one can actually trace the homology classes which correspond to multigraded Betti numbers as one moves up chains in $\mathcal{L}(n)$.

4.2 Representing Finite Atomic Lattices

The proof provided above of the upper semi-continuity of Betti numbers in $\mathcal{L}(n)$ addresses the main difficulty of understanding the structure of $\mathcal{L}(n)$. The difficulty lies in the fact that it is nontrivial to see how the sets equiv_P change as P changes.

In other words, changing one set $\text{equiv}_P(m)$ potentially changes all of the other sets $\text{equiv}_P(m')$. There are two main methods that I use for keeping track of this information. One identifies a set of cones over each vertex of the $n-1$-simplex with each $P \in \mathcal{L}(n)$ which allows one to see the effects

of subtly changing $\text{equiv}_P(m)$. The other looks at the maximal elements in each set $\text{equiv}_P(m)$.

4.2.1 Cone Complexes

Define the following *cone complex* associated to a lattice $P \in \mathcal{L}(n)$ a follows:

- Define
$$I_i = \{\sigma | (\bigvee_{j \in \sigma} a_j) \vee a_i = (\bigvee_{j \in \sigma} a_j) \text{ and } i \notin \sigma\}$$
for each atom a_i.

- Define $C_i = \{\sigma^c | \sigma \in I_i\}$ where σ^c is the complement of σ in the set $\{1, \ldots, n\}$.

Example 4.2.1. In figure 4.1 we see a lattice P and its associated cone complex $\mathcal{C}_P$. Note that here the labels on P such as abc is shorthand for $a \vee b \vee c$ where $a, b, c \in \text{atoms}(P)$. In this example we have the following sets which consist of the cone complex:

$$I_a = \{bcd, cde, bcde\} \qquad \Rightarrow C_a = \{ae, ab, a\}$$

$$I_b = \{cde, acde\} \qquad \Rightarrow C_b = \{ab, b\}$$

$$I_c = \{ab, abd, abe, abde\} \quad \Rightarrow C_c = \{cde, ce, cd, c\}$$

$$I_d = \emptyset \qquad \Rightarrow C_d = \emptyset$$

$$I_e = \emptyset \qquad \Rightarrow C_e = \emptyset$$

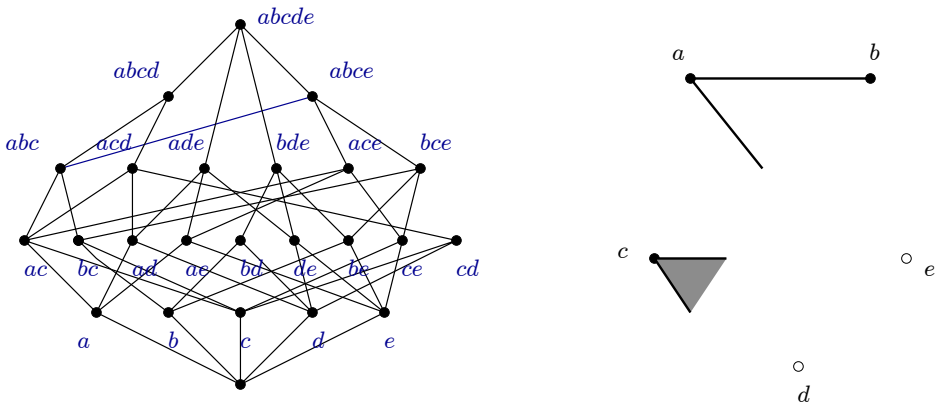


Figure 4.1: A lattice P and its cone complex $\mathcal{C}_P$

We will prove that each C_i is a cone over the vertex i, and the collection of the C_i's gives a description of the relations in P. It is important to note though that the faces appearing in C_i have maximum dimension $n-3$ due to the fact that faces of higher dimension would correspond to either $\hat{0}$, or atoms appearing in the set I_i which cannot happen as this P is an atomic lattice on n atoms.

Proposition 4.2.2. *Given a set $\mathcal{C}$ of n sets $C_i = \{F | F \subset [n], i \in F\}$, then $\mathcal{C}$ corresponds to a lattice $P \in \mathcal{L}(n)$ if and only if the following conditions are satisfied*

1. $|F| \leqslant n-2$ *for all* $F \in C_i$ *for all* i.

2. *For all* $F \in C_i$ *if* $G \subset F$ *and* $i \in G$ *then* $G \in C_i$.

3. *Let* $F \in C_i$ *and let* $G = F - \{i\}$. *If* $G \in C_j$ *then* $F \in C_j$.

Proof. Given $P \in \mathcal{L}(n)$ construct the sets in $\mathcal{C}_P$ as above. Now we must

show that these are cones over a vertex in the $(n-1)$-simplex and that they satisfy the 3 conditions.

That $|F| \leqslant n-2$ for all $F \in C_i$ is obvious as no atoms show up in any I_i.

To show condition (2), note that since $F \in C_i$ this means that

$$(\bigvee_{j \in F^c} a_j) \vee a_i = \bigvee_{j \in F^c} a_j.$$

Since $G \subset F$, we know that $F^c \subset G^c$ so if we want to check if

$$(\bigvee_{j \in G^c} a_j) \vee a_i = \bigvee_{j \in G^c} a_j$$

we can start with:

$$(\bigvee_{j \in G^c} a_j) = (\bigvee_{j \in G^c} a_j)$$

and decompose to

$$(\bigvee_{j \in F^c} a_j) \vee (\bigvee_{j \in G^c - F^c} a_j) = (\bigvee_{j \in F^c} a_j) \vee (\bigvee_{j \in G^c - F^c} a_j)$$

and then substituting we get

$$(\bigvee_{j \in F^c} a_j) \vee (\bigvee_{j \in G^c - F^c} a_j) \vee a_i = (\bigvee_{j \in F^c} a_j) \vee (\bigvee_{j \in G^c - F^c} a_j).$$

which proves the desired equality.

Note that this condition (2) is precisely the one that implies that each C_i is a cone over the vertex i.

Now, to show condition (3), our goal is to show

$$(\bigvee_{k\in F^c} a_k) \vee a_j = \bigvee_{k\in F^c} a_k.$$

If $F \in C_i$ then:

$$(\bigvee_{k\in F^c} a_k) \vee a_i = \bigvee_{k\in F^c} a_k \tag{4.2.1}$$

so wedging equation 4.2.1 with a_j on both sides yields

$$(\bigvee_{k\in F^c} a_k) \vee a_i \vee a_j = \bigvee_{k\in F^c} a_k \vee a_j. \tag{4.2.2}$$

And because $G \in C_j$:

$$(\bigvee_{k\in G^c} a_k) \vee a_j = \bigvee_{k\in G^c} a_k \tag{4.2.3}$$

so because $G^c = F^c \cup i$ equation 4.2.3 becomes

$$(\bigvee_{k\in F^c} a_k) \vee a_i \vee a_j = \bigvee_{k\in F^c} a_k \vee a_i. \tag{4.2.4}$$

Combining equations 4.2.2 and 4.2.4, we get that

$$(\bigvee_{k\in F^c} a_k) \vee a_j = \bigvee_{k\in F^c} a_k \vee a_i = \bigvee_{k\in F^c} a_k.$$

So $F \in C_j$ as needed.

So, we have shown that given P we can construct sets C_i which satisfy the necessary conditions. It remains to show that given such a collection of C_i's that one can always construct a finite atomic lattice P.

Given $\mathcal{C} = \{C_1, \ldots C_n\}$ satisfying the three conditions above, we construct P as follows. First take each C_i and complement each $F \in C_i$ in the set $\{1, \ldots, n\}$, call this set of complements I_i. Let $I = \bigcup I_i$ for all $1 \leqslant i \leqslant n$, then define P to be the set

$$\{p \in B_n | p \neq \bigvee_{j\in\sigma} a_j \text{ for all } \sigma \in I\}.$$

In other words, P is the complement of I in B_n. It remains to show that P with the induced ordering of B_n is a finite atomic lattice. Since F satisfies condition (1), we see that $|F^c| \geqslant 2$ for all $F \in C_i$, so every atom of B_n is in P as well as $\hat{0}$. Also, since no $F = \emptyset$ in any of the C_i, $\hat{1}$ is contained in P. So by theorem 2.1.1 it remains to show that P is either a meet-semilattice or a join-semilattice.

I will show that every two elements in P has a join. Let $p, q \in P$ the the join of these two elements will have support containing $\text{supp}(p) \cup \text{supp}(q)$. If $\text{supp}(p) \cup \text{supp}(q)$ is not in I then this is $\text{supp}(p \vee q)$. If $\text{supp}(p) \cup \text{supp}(q) \in I$

then there needs to be a unique smallest subset not in I containing it. If $\text{supp}(p) \cup \text{supp}(q) \in I$ then define $\text{supp}(p \vee q)$ to be the set

$$\text{supp}(p) \cup \text{supp}(q) \cup \{i \mid \text{supp}(p) \cup \text{supp}(q) \in I_i\}.$$

This is not in I since for each i if $\sigma \in I_i$ then $\sigma \cup \{i\} \notin I_i$. Moreover, this is clearly the unique smallest set satisfying this property.

□

Note that for small n one can easily count all of the possible cone complexes thus giving a nice way to enumerate all of the elements of $\mathcal{L}(n)$. The following table shows this enumeration for $n = 4$ where by summing over the last column one sees that $|\mathcal{L}(4)| = 545$.

vertices	**incident edges**	**Total**
0	0	1
1	3	4*8 = 32
2	5	6*32 = 192
3	6	4*64 = 256
4	6	64

If the definition of $\mathcal{C}_P$ reminds the reader of the process by which one finds the Alexander dual to a simplicial complex they are not mistaken as will be seen in the following proposition. I will call a cone complex *closed* if

for every $F \in C_i$ then if $j \in F$ then $F \in C_j$ as well. In other words, the union of all the cones is in fact a simplicial complex.

Proposition 4.2.3. *If the cone complex for P is closed then the cone complex is the Alexander dual to the Scarf complex associated to P.*

Proof. The Alexander dual to the Scarf complex is obtained by taking the complement (in the set of verticies) of each of the missing faces. So if we can show that for closed cone complexes, that the sets I_i consist of all of the missing faces of the Scarf complex containing the vertex i then we are done.

If $\mathcal{C}$ is cone complex of P and $\mathcal{C}$ is closed then

$$(\bigvee_{k \in F^c} a_k) \vee a_j = \bigvee_{k \in F^c} a_k$$

for all F and all j such that $F \in C_j$. So, F^c is in I_i for all $i \notin F^c$ and, every set containing F^c but not containing i is also in I_i. These are precisely the faces whose multidegrees are equal and so each I_i contains all of the faces missing from the Scarf complex which touch the vertex i. □

It is important to understand how the order relation on $\mathcal{L}(n)$ is understood in terms of these cone complexes. The next lemma addresses this.

Lemma 4.2.4. *$P \geqslant Q$ in $\mathcal{L}(n)$ implies that $C_{P,i} \subset C_{Q,i}$ for all $i \in \{1, \ldots, n\}$ where $C_{P,i} \in \mathcal{C}_P$ and $C_{Q,i} \in \mathcal{C}_Q$.*

Proof. First note that $C_{P,i} \subset C_{Q,i}$ is equivalent to saying that $I_{P,i} \subset I_{Q,i}$. So we will show that the later is true. If $P \geqslant Q$ then there exists a join

preserving map $f : P \to Q$ which is a bijection on atoms. Choose an element $\sigma \in I_{P,i}$ then since

$$(\bigvee_{j\in\sigma} a_j) \vee a_i = \bigvee_{j\in\sigma} a_j$$

we see that

$$\begin{aligned} f(\bigvee_{j\in\sigma} a_j) &= f((\bigvee_{j\in\sigma} a_j) \vee a_i) \\ &= f(\bigvee_{j\in\sigma} a_j) \vee f(a_i) \\ &= \bigvee_{j\in\sigma} f(a_j) \vee a_i. \end{aligned}$$

Thus, $\sigma \in I_{Q,i}$ as needed.

□

The join operation can be seen in this next lemma to be equivalent to appropriate intersections of the cone complexes. This is particularly nice because from Phan's thesis we understand meets of atomic lattices to be the intersections of their images in B_n. Joins however, cannot be understood as unions because the union operation is not closed as demonstrated in figure 4.2 since clearly $P \cup Q$ is not a lattice.

Lemma 4.2.5. *In $\mathcal{L}(n)$ the join of two elements, $P \vee Q$ corresponds to the cone complex $\{C_{P,i} \cap C_{Q,i} |$ for all $1 \leqslant i \leqslant n\}$.*

Proof. Clearly $C_{P,i} \cap C_{Q,i}$ is the largest set contained in both $C_{P,i}$ and $C_{Q,i}$

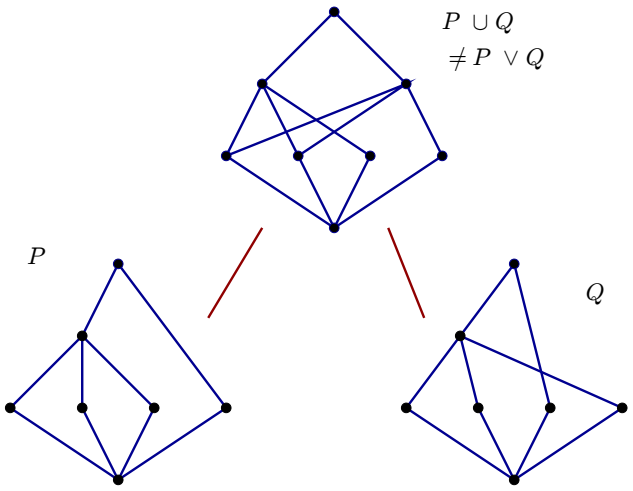


Figure 4.2: $P \cup Q$ is not the join of P and Q

thus by 4.2.4 the lattice T corresponding to that cone complex will be greater than both P and Q and there will be no lattice T' such that $T > T'$ and $T' > P$ and $T' > Q$. □

4.2.2 Description as sets closed under intersection

Alternatively, we can view each lattice as a set with certain properties. Let $\mathcal{S}$ be a set of subsets of $\{1, ..., n\}$ with no duplicates, closed under intersections, and containing the entire set, the empty set, and the sets $\{i\}$ for all $1 \leqslant i \leqslant n$. Then $\mathcal{S}$ corresponds to an element of $\mathcal{L}(n)$.

To see this consider an element $P \in \mathcal{L}(n)$ we define $\mathcal{S}_P$ to be the set of $\text{supp}(p)$ for all $p \in P$. Since P is a lattice both the empty set and the entire set are in $\mathcal{S}_P$. It remains to show that $\mathcal{S}_P$ is closed under intersections. If

$\text{supp}(p) \cap \text{supp}(p') \notin \mathcal{S}_P$ then

$$\bigvee_{j \in \text{supp}(p) \cap \text{supp}(p')} a_j$$

does not exist which cannot happen as P is a lattice.

Alternatively, given such as set $\mathcal{S}$ order the sets in $\mathcal{S}$ by inclusion, then we need to show that this is a finite atomic lattice. It obviously has a minimal element, a maximal element, and n atoms, so by proposition 2.1.1 we need to show it is either a meet-semilattice or a join-semilattice. Here the meet of two elements would be defined to be their intersection and since S is closed under intersections this is a meet-semilattice.

Lemma 4.2.6. *$P \geqslant Q$ in $\mathcal{L}(n)$ if and only if $\mathcal{S}_Q \subset \mathcal{S}_P$.*

Proof. If $P \geqslant Q$ then there exists a join preserving map $f : P \to Q$ which is a bijection on atoms. Clearly this means that f is surjective since if there were a $q \in Q$ such that $q \notin \text{im}\, f$ then this would imply that there were some subset of atoms in P which did not have a join. Since f is surjective we can consider the preimage of each element $q \in Q$. For all $p \in f^{-1}(q)$, $\text{supp}(p) \subseteq \text{supp}(q)$ since

$$f(\bigvee_{i \in \text{supp}(p)} a_i) = \bigvee_{i \in \text{supp}(p)} f(a_i) = \bigvee_{i \in \text{supp}(p)} a_i = q.$$

(Note: Since supp gives us subsets of the atoms, even though p and q live in different lattices, it makes sense to compare these sets.) Moreover, for the maximal element of $f^{-1}(q)$ there will be equality. Thus for every $q \in Q$,

$\text{supp}(q)$ is element of $\mathcal{S}_P$.

If $\mathcal{S}_Q \subset \mathcal{S}_P$ then define the following map $f : P \to Q$ to be the following map: let $s \in \mathcal{S}_Q$ be the smallest set containing $\text{supp}(p)$ then

$$f(p) = \bigvee_{i \in s} a_i.$$

Note that this set s will be unique since if there were two sets s and s' such that neither set was contained in the other and both contained $\text{supp}(p)$ then because $\mathcal{S}_Q$ is closed under intersections $s \cap s'$ will either be $\text{supp}(p)$ or a unique smallest set containing it. This map is obviously join preserving. □

This next lemma is equivalent to the method used to originally prove theorem 2.7.1 in [Pha06].

Lemma 4.2.7. *In $\mathcal{L}(n)$ the meet of two elements, $P \wedge Q$ corresponds to the set $\mathcal{S}_P \cap \mathcal{S}_Q$.*

Proof. Clearly $\mathcal{S}_P \cap \mathcal{S}_Q$ is the largest set contained in both $\mathcal{S}_Q$ and $\mathcal{S}_P$ thus by 4.2.6 the lattice T corresponding to $\mathcal{S}_P \cap \mathcal{S}_Q$ will be less than both P and Q and there will be no lattice T' such that $T' > T'$ and $P > T'$ and $Q > T'$. □

4.3 Structural properties of $\mathcal{L}(n)$

Using these two different ways of representing elements in $\mathcal{L}(n)$ we can begin to prove some structural results about the lattice. Having a clear sense of

the structure of this space is of use when considering questions relating to how the Betti numbers are changing along chains.

Up until this point, I have provided several ways of seeing that a lattice P is greater or equal to a lattice Q, but I have not provided a "nice" characterization that P minimally covers Q. The following proposition addresses this.

Proposition 4.3.1. *If $P \geqslant Q$ in $\mathcal{L}(n)$ then P covers Q if and only if $|P| = |Q| + 1$.*

Proof. We know that $P \geqslant Q$ which implies that $\mathcal{S}_Q \subset \mathcal{S}_P$. Since every set in $\mathcal{S}$ corresponds to an element in the associated lattice, this implies that $|P| \geqslant |Q|$. It remains to show that they differ by one element when P covers Q.

Suppose they differ by 2 elements, then $\mathcal{S}_P = \mathcal{S}_Q \cup \{\sigma, \beta\}$ where σ and β are subsets of $\{1, \ldots, n\}$ satisfying the conditions that $\sigma \cap \beta$, $\sigma \cap s$, and $\beta \cap s$ are either in $\mathcal{S}_Q$ or $\{\sigma, \beta\}$ for all $s \in \mathcal{S}_Q$. The argument is that in this case P cannot cover Q as there exists a lattice $T \neq P$ satisfying $P \geqslant T \geqslant Q$. Let $\mathcal{S}_T$ be one of the following

1. $\mathcal{S}_Q \cup \{\sigma \cap \beta\}$ if $\sigma \cap \beta \notin \mathcal{S}_Q$
2. $\mathcal{S}_T = \mathcal{S}_Q \cup \{\sigma\}$ if $\sigma \subset \beta$
3. $\mathcal{S}_T = \mathcal{S}_Q \cup \{\beta\}$ if $\beta \subset \sigma$
4. $\mathcal{S}_T = \mathcal{S}_Q \cup \{\sigma\}$ or $\mathcal{S}_T = \mathcal{S}_Q \cup \{\beta\}$ if σ and β are not subsets of each other.

In any of these cases, $T \geqslant Q$ and $|T| = |Q| + 1$.

□

The upshot of proposition 4.3.1 is the next nice result. It is easy to see that $\mathcal{L}(3) = B_3$, wheras $\mathcal{L}(4) \neq B_4$ (and the later is true for all $n \geqslant 4$ by proposition 4.3.3). However, one can ask, what if any are the nice properties of B_n that are retained by $\mathcal{L}(n)$. One answer is the following theorem.

Theorem 4.3.2. *$\mathcal{L}(n)$ is a graded lattice of rank $2^n - n - 2$*

Proof. The maximal element of $\mathcal{L}(n)$ is the lattice B_n and $|B_n| = 2^n$. The minimal element of $\mathcal{L}(n)$ is the unique lattice on n atoms where the atoms are also the coatoms, it has $n + 2$ elements. Then by 4.3.1 every chain in $\mathcal{L}(n)$ has length $2^n - (n + 2)$ and so it is graded of rank $2^n - n - 2$. □

It follows from theorem 4.3.2 that if $\mathcal{L}(n)$ is co-atomic then it will be isomorphic to B_n. With the following description of the meet-irreducibles it is easy to see that the only case where this happens is for $n = 3$.

Proposition 4.3.3. *The number of meet irreducibles in $\mathcal{L}(n)$ is*

$$n(2^{n-1} - n).$$

Proof. The meet-irreducibles in $\mathcal{L}(n)$ can be described best in of their cone complexes. A cone complex $\mathcal{C}$ is meet-irreducible in $\mathcal{L}(n)$ if $C_i = \emptyset$ for all $i \neq j$ and C_j consists of a only one face F and any faces $G \subset F$ (i.e. a

"simplex"). For each C_j there are

$$1 + \sum_{i=2}^{n-2} \binom{n}{i}$$

possible faces. Thus, the set of all such $\mathcal{C}$ is precisely

$$\sum_{i=1}^{n-2} i\binom{n}{i}. \tag{4.3.1}$$

To see that equation 4.3.1 equals the desired quantity, consider this specific instance of the binomial theorem

$$(1+t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i.$$

Taking derivatives we see the following

$$\begin{aligned} n(1+t)^{n-1} &= \sum_{i=1}^{n} i\binom{n}{i} t^{i-1} \\ &= n\binom{n}{n} t^{n-1} + (n-1)\binom{n}{n-1} t^{n-2} + \sum_{i=1}^{n-2} i\binom{n}{i} t^{i-1}. \end{aligned}$$

Thus rearranging we see that

$$n(1+t)^{n-1} - \binom{n}{n} t^{n-1} - (n-1)\binom{n}{n-1} t^{n-2} = \sum_{i=1}^{n-2} i\binom{n}{i} t^{i-1},$$

and so evaluating at $t = 1$ we get that

$$n2^{n-1} - n - n(n-1) = n(2^{n-1} - n) = \sum_{i=1}^{n-2} i\binom{n}{i}.$$

It remains to show that complexes $\mathcal{C}$ described above are in fact the meet-irreducibles. Let $\mathcal{C}$ be a cone complex satisfying the conditions above, and then let $\mathcal{C}'$ and $\mathcal{C}''$ be two cone complexes greater than $\mathcal{C}$, i.e. contained in $\mathcal{C}$. Then either, $\mathcal{C}'$ and $\mathcal{C}''$ lie in a chain or are uncomparable. If they lie in a chain, there is nothing to show. If they are uncomparable we must show that $\mathcal{C}$ is not their meet (or equivalently the greatest lower bound). We know that all C_i' and C_i'' are empty except for when $i = j$, that $C_j' \subset C_j$, $C_j'' \subset C_j$ and there exists $F \in C_j', \notin C_j''$ and a $G \in C_j'' \notin C_j'$. Since, C_j is a "simplex" we know that C_j' and C_j'' must differ by faces F and G that are contained in the maximal face of C_j. Note this implies that the maximal face is missing from both C_j' and C_j''. If this is the case then the largest cone complex containing C_j' and C_j'' is the union of these which is still missing the maximal face of C_j. Thus, the meet of $\mathcal{C}'$ and $\mathcal{C}''$ is not $\mathcal{C}$ and this is true for all cone complexes greater than $\mathcal{C}$ so it is a meet-irreducible. □

Note that this proof gives a concrete description of the meet-irreducibles of $\mathcal{L}(n)$. It should be noted then that using this one can easily figure out the minimal monomial coordinatization of $\mathcal{L}(n)$ which in theory could be used to enumerate all of the elements of $\mathcal{L}(n)$ by computing the LCM lattice of that monomial ideal.

Chapter 5

Deformation of exponents and Generic Monomial Ideals

Recall from section 2.4 that M_ϵ is the monomial ideal obtained by a deformation of the exponents of the generators of M by $\{\epsilon_1, \ldots, \epsilon_n\}$. It is noted, in [GPW99] that there is a join preserving map between the LCM lattices from M_ϵ to M for any monomial ideal M. In fact, for abstract monomial ideals we can realize all paths in $\mathcal{L}(n)$ as a deformation of exponents for some coordinatization, as seen in the following proposition.

Theorem 5.0.4. *If $P \geqslant Q$ in $\mathcal{L}(n)$ then there exists a coordinatization of Q such that via deformation of exponents one can obtain a coordinatization of P.*

Note that this proof makes use of the fact that we can represent any

deformation of exponents using integer vectors rather than working with real exponents.

Proof. First, label P with the greedy labeling from above. Then construct a labeling of Q as follows. Since $P \geqslant Q$ then there is a join preserving map $f : P \to Q$. To each element $q \in Q$ assign the monomial

$$\prod_{j \in I} x_j,$$

where $I = \{j \mid x_j \text{ divides } m_p \text{ for all } p \in f^{-1}(q)\}$

It remains to show that there exists ϵ_i for each of the n atoms, such that the monomial ideal obtained for P is a deformation of exponents for the monomial ideal obtained for Q (with these coordinatizations). We do this by considering chains in both P and Q and their relation to each other under the map f.

Let c_j be the chain in P which is labeled by the variable x_j under the greedy labeling. Note that we can write the monomial associated to an atom a_i as follows

$$\prod_{j} \prod_{p \in \lceil a_i \rceil_P^c \cap c_j} x_j,$$

where the subscript P indicates that both the order ideal and the complement are in P. Similarly with the coordinatization of Q given above we can think of the monomials for the lattice Q as

$$\prod_{j} \prod_{q \in \lceil a_i \rceil_Q^c \cap f(c_j)} x_j.$$

Given these descriptions of the monomials generators for each lattice makes it clear that for ϵ_i we want to define

$$\epsilon_{ij} = |\lceil a_i \rceil_P^c \cap c_j| - |\lceil a_i \rceil_Q^c \cap f(c_j)|.$$

Clearly, this give the desired deformation of exponents. □

Remark 5.0.5. This provides yet another proof that total Betti numbers increase as one moves up chains in $\mathcal{L}(n)$, since it is known that Betti numbers are upper-semi continuous under deformation of exponents.

Corollary 5.0.6. *Given a lattice $Q \in \mathcal{L}(n)$ there exists a coordinatization for which every element in $\lceil Q \rceil$ is the LCM lattice of a deformation of exponents of that coordinatization.*

Proof. Apply the coordinatization used to prove theorem 5.0.4 where Q is your given lattice and $P = B_n$. Use the same coordinatization given in the proof for every element $P' \in \lceil Q \rceil$. Now it remains to show that with these coordinatizations of P' and of Q that we can find a ϵ_i for each of the n atoms giving a deformation of exponents for the monomial ideal obtained for Q. Again for a variable x_j which appears only along one chain c_j we want to define ϵ_{ij} to be $|\lceil a_i \rceil_{P'}^c \cap c_j| - |\lceil a_i \rceil_Q^c \cap f(c_j)|$. □

This realization of the filter of Q as the "space of all deformations of exponents of Q" is of interest as it will hopefully give insight into the geometric model of deformation of exponents. By geometric model, I mean that given a monomial ideal in t variables with n generators one can view each possible

deformation as a vector in $\mathbb{R}^{nt}$. However, there are finitely many possible deformations each corresponding to any vector in a open convex polyhedral cone. Examples show that the correspondence between these cones and elements in the $\rceil Q \lceil$ is not one-to-one which suggests that $\mathcal{L}(n)$ is a better model for studying deformation of exponents. Understanding the structure of these fan consisting of these cones and how it relates to this filter will be the subject of further work.

Another goal, which will be addressed here, is to understand how to "minimally deform" exponents so that one does not increase Betti numbers. Specifically, recall the map

$$\phi : \mathcal{L}(n) \to \beta(n)$$

which takes a lattice L to the vector consisting of its total betti numbers. Given a lattice P in $\phi^{-1}(\mathbf{b})$ for some $\mathbf{b} \in \beta(n)$, I want to understand under what circumstances will a deformation of exponents force me to move to a lattice outside of $\phi^{-1}(\mathbf{b})$.

My approach is based on first understanding a coordinate free description of Scarf complexes and generic monomial ideals. Since deformation of exponents first appeared with the aim of deforming to generic monomial ideals where the resolution was known, this coordinate free description indicates where some "stopping" points are along chains in $\mathcal{L}(n)$. Then I intend to generalize some of these notions to account for ideals whose minimal resolution is not simplicial.

5.1 Simplicial cellular resolutions

Since the case where a monomial ideal has a minimal resolution supported by a simplicial complex is fairly well understood, I begin here by providing descriptions of what is known purely in terms of lattices.

Just as in chapter 4 I will continue to refer to the Betti numbers of a finite atomic lattice as opposed to the Betti numbers of a monomial ideal. Notice that since all monomial ideals with the same LCM lattice have isomorphic minimal resolutions this means that if a cell complex supports the minimal resolution of one ideal then it will support the minimal resolution for all possible coordinatizations of the LCM lattice of that ideal. Moreover, we can think of the multidegrees showing up in the resolution as simply the elements of the lattice P. With this idea, it is easy to define the Scarf complex of a monomial ideal M_P in terms of the lattice P,

$$\text{scarf}(P) = \Gamma(\{p \in P \,|\, |\,\text{equiv}_P(p)| = 1\}) \subset \Gamma(P).$$

Note that in this language, $\Gamma(P)$ is the Taylor Complex associated to P.

Recall from section 2.4 that if a given monomial ideal is generic or strongly generic then its minimal resolution is the Scarf complex. Note however, that there may be monomial ideals whose minimal resolution is the Scarf complex, yet the ideal is not generic. An obvious example of this phenomenon is if one takes a generic monomial ideal and polarizes to obtain a squarefree monomial ideal. It will have the same Scarf complex which supports the minimal resolution since LCM lattices are preserved under polarization. It is

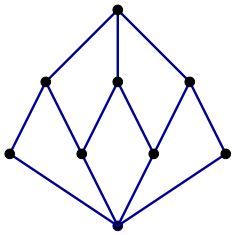


Figure 5.1: Lattice resolved by Scarf complex which has no generic coordinatization

rare however, for squarefree monomial ideals to be generic since all variables always appear with the same exponent.

Nontrivial examples exist though of monomial ideals whose minimal resolution is supported by the Scarf complex, but they are not generic or strongly generic. The following example of an abstract monomial ideal illustrates well an example of a monomial ideal whose minimal resolution is Scarf, but where there is no coordinatization which satisfies condition 2 of theorem 2.4.6.

Example 5.1.1. The lattice P in figure 5.1 is the augmented face lattice of a simplicial complex consisting of 4 vertices and 3 edges. Every point in P except for the minimal and maximal elements represents a multidegree that has a nonzero betti number. This is easy to see since for all of the atoms $\tilde{h}_{-1}(\Gamma(P_{<a_i}), k) = 1$ and for each element p covering an atom $\Gamma(P_{<p})$ consists of two vertices thus $\tilde{h}_0 = 1$. Thus, P will always be resolved by its Scarf complex.

I will demonstrate however, that every possible coordinatization of P fails condition 2 of theorem 2.4.6. In other words, that for every coordinatization a variable appears with the same non-zero exponent in either m_1 and m_2, m_2 and m_3, or in m_3 and m_4. Since the meet-irreducibles of P are precisely

$p_{12} = a_1 \vee a_2$, $p_{23} = a_2 \vee a_3$, $p_{34} = a_3 \vee a_4$, a_1, and a_4 any coordinatization must cover these. I will focus on just showing that for all coordinatizations there is a variable appearing with the same non-zero exponent for the pair m_1 and m_2. Note that the monomial m_1 is determined by the labelings found on $\{p_{23}, p_{34}, a_2, a_3, a_4\}$ and m_2 is determined by $\{p_{34}, a_1, a_3, a_4\}$. Moreover, p_{34} and a_4 must be labeled and if a_3 happens to be non-trivially labeled it also appears in both of these sets. Any variable appearing in the label on p_{34}, a_3, and a_4 cannot appear anywhere else, thus it must appear with the same exponent in both m_1 and m_2. So condition 2 can never be satisfied.

Observe that in the above example P is the face lattice of P together with a maximal element, I will refer to this as the *augmented face lattice of* P. An argument similar to the one in the example is used to prove proposition 5.14 in [Pha06]. I will restate this proposition here.

Proposition 5.1.2. *Every simplicial complex X not equal to the boundary of a simplex is the Scarf complex of some squarefree monomial ideal. If X is acyclic then X supports the minimal resolution of that ideal.*

The intention of the next proposition is to demonstrate that each of these strata have entire regions of lattices whose minimal resolution is the supported by the appropriate Scarf complex. This proof uses the ideas first presented in the proof of 4.1.2.

Proposition 5.1.3. *If P is minimally resolved by its Scarf complex, then every $Q \geqslant P$ in $\phi^{-1}(\mathbf{b}_P) \subset \mathcal{L}(n)$ is also resolved by its Scarf complex which will equal the Scarf complex of P.*

Proof. If P is minimally resolved by its Scarf complex, then $\beta_{i,p} \neq 0$ implies that $\beta_{i,p} = \tilde{h}_{i-2}(\Gamma(P_{<p}, k)) = 1$ and $|\operatorname{equiv}_P(p)| = 1$. So, we need to show the same is true for the appropriate elements $q \in Q$. Since $Q \geqslant P$ so there is a join preserving map $\psi : Q \to P$ and recall that

$$\operatorname{equiv}_P(p) = \bigcup_{q \in \psi^{-1}(p)} \operatorname{equiv}_Q(q).$$

Since any $p \in P$ contributing to a Betti number satisfies $|\operatorname{equiv}_P(p)| = 1$ we see that there must be only one $q \in \psi^{-1}(p)$. It remains to show that for this $q = \psi^{-1}(p)$ that $\tilde{h}_{i-2}(\Gamma(Q_{<q}, k)) = 1$.

To see this last fact, consider the fact that since P is resolved by its Scarf complex we can inductively apply the above argument to every single face of $\Gamma(P_{<p})$ to see that $\Gamma(Q_{<q})$ is precisely the same simplicial complex. Thus it has the same reduced homology.

Clearly no other elements of Q have nonzero Betti numbers since Q is assumed to be in the same strata of total Betti numbers as P. Thus, $\operatorname{scarf}(P) = \operatorname{scarf}(Q)$ and both are minimally resolved by their Scarf complexes.

□

Note that in the example 5.1.1, there is a lattice Q (figure 5.1) greater than the given lattice which has an isomorphic minimal resolution but that admits a generic coordinatization. The following theorem characterizes precisely when strongly generic coordinatizations exist.

Theorem 5.1.4. *If $P \in \mathcal{L}(n)$ is a graded finite atomic lattice whose chains*

have rank n, then P admits a strongly generic coordinatization.

Proof. Coordinatize P using the greedy coordinatization. Now, we just need to show that the resulting monomial ideal is strongly generic. In other words we need to show that for any variable x_j, if it appears in any two monomials m_{a_i} and m_{a_k} that it has a different exponent. By the definition of the monomials m_{a_i} and m_{a_k} this amounts to showing that the intersections of the complements of the filters $\lceil a_i \rceil$ and $\lceil a_k \rceil$ with the chain corresponding to x_j are different.

Let $c_j = \{\hat{0}, a_{1_j}, a_{1_j} \vee a_{2_j}, \ldots, a_{1_j} \vee a_{2_j} \vee \cdots \vee a_{n_j} = \hat{1}\}$ be the chain which is entirely labeled by the variable x_j. Since P is graded of rank n we know that there are $n+1$ elements in c_j and that the i-th element in the chain is the join of $i-1$ atoms of P. Every set $\lceil a_i \rceil$ intersects c_j at a different spot along the chain, so it is likewise with the complements of these filters. This guarantees that each variable x_j appears with a different exponent in each monomial generator. □

Note that having a LCM which is graded of rank n is a necessary condition. One can see this by looking at the deficit labeling for any strongly generic monomial ideal. All variables need to be labeled along a maximal chain and all meet-irreducibles must be covered so every meet irreducible needs to lie on a maximal chain. This demonstrates that if a given monomial ideal is not strongly generic, but its LCM lattice is graded of rank n, then depolarization rather than deformation of exponents is a better approach for finding the minimal resolution.

5.2 Nonsimplicial cellular resolutions

For monomial ideals whose minimal resolution is not supported on a simplicial complex less is known about how to "nicely" describe their minimal resolution. In this section, I try to generalize the results of the previous section to account for these types of ideals. Then the following propositions provide a generalization of Phan's proposition 5.1.2 and proposition 5.1.3.

Proposition 5.2.1. *Let X be a regular cell complex such that X is acyclic, and the augmented face poset of X, P_X is a finite atomic lattice on $|X^0| = n$ atoms. Then the minimal resolution of any coordinatization of P is supported on X.*

Proof. Observe that if P_X is the face lattice of X then labeling X with the monomials in any coordinatization of P_X as prescribed in section 2.4 simply puts the monomial at a point $p \in P_X$ on its corresponding face in X. Moreover, each face in X has a distinct multidegree labeling it. To show that the resolution of any coordinatization is supported on X we simply need to show that $X_{\leqslant p}$ is acyclic. This is true by construction though since $X_{\leqslant p}$ corresponds to the d-cell that p represents and its boundary.

Note also that $\Delta(P_{X_{<p}})$ is the barycentric subdivision of $X_{<p}$. In particular, we can also easily see that $\tilde{h}_i(\Delta(P_{X_{<p}}, k) = 1$ for $i = d-1$ where p corresponds to a d-cell in X since $X_{<p}$ is the boundary of that d-cell. Thus $\beta_{d+1,p} = 1$ for a d-cell F_p in X.

If P is the augmented face lattice of X (i.e. if the $\hat{1}$ element actually needs to be added in) then the above description applies to all $p \in P - \hat{1}$. For $p = \hat{1}$, we have by construction that $X_{\leqslant p} = X$ and since X was assumed

to be acyclic X still supports the resolution.

Moreover, only the multidegrees corresponding to each $p \in P$ can possibly have $\beta_{i,p}$ nonzero so since we've considered all such p's X supports the resolution of P. This resolution is minimal since $\beta_{i,p} = 1$ for only one i, i.e. no map has an integer as an entry of the matrix. □

Observe that when X is not acyclic all hope is not lost. If X itself has homology, then that will correspond to $\Gamma(P_{X_{<\hat{1}}})$ having the same homology. In particular if X is a triangulation of some CW-complex with homology [Vel08] gives a construction for how to find a minimal resolution of an ideal which is a coordinatization of its augmented face lattice.

Proposition 5.2.2. *Let P_X be as in proposition 5.2.1, then if Q in $\phi^{-1}(\mathbf{b}_{P_X}) \subset \mathcal{L}(n)$ satisfies the following two conditions:*

1. *Q covers P_X*

2. *$\beta_{i,q} = 1$ for $q = \max(\psi^{-1}(p))$ for any $p \in P_X$ where ψ is the join-preserving map from Q to P_X, and $\beta_{i,q} = 0$ otherwise.*

Then Q has a minimal resolution supported on X.

Proof. Since $Q \in \phi^{-1}(\mathbf{b}_{P_X})$ we know that the total Betti numbers of the minimal resolution of Q are the same as those of P_X. So, all that needs to be shown is that X supports a resolution of Q, and then since that resolution has the right total Betti numbers it must be minimal. Thus, I just need to show that $X_{\leqslant q}$ is acyclic for all $q \in Q$.

First recall from the proof of 4.1.2 that

$$\text{equiv}_{P_X}(p) = \bigcup_{q \in \psi^{-1}(p)} \text{equiv}_Q(q).$$

Moreover, $\text{supp}(p)$ is the maximal element in $\text{equiv}_{P_X}(p)$ when it is ordered by inclusion which implies $\text{supp}(p)$ is the maximal element in $\bigcup_{q \in \psi^{-1}(p)} \text{equiv}_Q(q)$. So we can observe that there are two types of elements $q \in Q$: one being where

$$q = \bigvee_{i \in \text{supp}(p)} a_i$$

for some $p \in P_X$ where the join is in Q; and the second being the $q \in Q$ that are not of the first type. Clearly the elements of type one are maximal in the appropriate $\phi^{-1}(p)$. Note also that since Q covers P_X there is precisely one element of type two in Q due to proposition 4.3.1.

Now we need to label X with the appropriate multidegrees. Notice that each face of X corresponding to $p \in P_X$ will be labeled with the appropriate $q \in Q$ of type one which is the maximal element in $\phi^{-1}(p)$. Moreover, no elements in Q of type two label any faces in X. Thus when we examine the complexes $X_{\leqslant q}$ for q of type one they will all be acyclic for the same reasons as in the proof of proposition 5.2.1.

It remains to show that for the one element $q \in Q$ of type two that $X_{\leqslant q}$ is acyclic. We know that since $\beta_{i,q} = 0$ for this element that $\Delta(Q_{<q})$ is acyclic. Moreover, since q is the only element in Q which does not correspond to a face in X, then $Q_{<q}$ is equal to the face lattice of $X_{<q}$. This shows that $X_{<q}$ is acyclic since $\Delta(Q_{<q})$ is homotopy equivalent to $X_{<q}$ by barycentric

subdivision. Finally, since q does not label a face of X, it is clear that $X_{<q} = X_{\leqslant q}$ thus concluding the proof.

□

In general, when a lattice $Q \geqslant P_X$ but still in the same Betti strata what happens can be quite subtle. The above proposition is a prototype for the type of theorem that I hope to prove in future work. I believe that both of the conditions which Q must satisfy can be relaxed. The goal of this is to give nice closed form descriptions of the maps in the minimal resolution for a larger class of ideals than those resolved by their Scarf complex.

Chapter 6

Connection to Maximal CM ideals

In [Fløy09], Fløystad defines the category of monomial ideals whose quotients are Cohen-Macaulay and defines maximal elements in this category. He then gives constructions which associate to certain regular cell complexes (trees, and some polytopes) maximal elements in this category whose minimal resolutions are supported on these cell complexes. This work is can be related to the work of Phan and the results found in this thesis since both are concerned with constructing monomial ideals with a specific cellular resolution.

Specifically, in [Fløy09] the setup is as follows. Fløystad defines the set $CM(n, c)$ to be ordered sets of n monomials generating a monomial ideal M such that the quotient ring is Cohen-Macaulay of codimension c. In actuality this is a category but we need not concern ourselves with this added

structure here. The set $CM_*(n,c)$ is the subset (subcategory) of $CM(n,c)$ consisting of monomial ideals which are squarefree and such that the sets $V_t = \{i \,|\, x_t \text{ divides } m_i\} \subseteq [n]$ are distinct.

Fløystad initially defines what it means for a monomial ideal to be *maximal* using the maps in his category $CM(n,c)$. The maps in this category though are heavily dependent on what coordinatization one has for a monomial ideal. Thus, I prefer to ignore this definition and rather use his characterization of the objects in these categories which identifies families $\mathcal{F}$ consisting of subsets of $[n]$ to objects in $CM(n,c)$. Essentially these sets $\mathcal{F}$ are meant to be the set of V_t's described above, but given a regular cell complex X one can construct sets $\mathcal{F}$ without having to start with an actual monomial ideal. This should remind the reader of the coordinatizations appearing in [Pha06] and chapter 3.

The following are actually propositions 1.7 and 1.10 in [Fløø9]. Since they are "if and only if" statements I will state them as definitions here to simplify language.

Definition 6.0.3. A family of subsets of $[n]$ $\mathcal{F}$, as described above, is *maximal* if it is reduced and is maximal among reduced associated families for the refinement order. A family $\mathcal{F}$ is *reduced* if it corresponds to an object in $CM_*(n,c)$ and consists of elements which are not the disjoint union of other elements in $\mathcal{F}$. The *refinement order* states that for two families of subsets $\mathcal{F} > \mathcal{G}$ if and only if $\mathcal{F}$ consists of refinements of elements of $\mathcal{G}$ together with additional subsets of $[n]$.

Definition 6.0.4. If X is a regular cell complex of dimension d. Define $CM_*(X)$ to be the subset of $CM_*(n,c)$ whose minimal resolution is sup-

ported on X.[1] Then a family $\mathcal{F}$ is an object in $CM_*(X)$ if the following conditions hold

1. No d of the subsets in $\mathcal{F}$ cover $[n]$.

2. Let W be a union of subsets $\mathcal{F}$. Then the restriction of X to W^c is acyclic.

3. For every pair $F \subsetneq G$ of faces of X, there is an S in $\mathcal{F}$ such that $S \cap F$ is empty but $S \cap G$ is not empty.

In some sense we can think of this last definition as the analog of proposition 3.2.1 in chapter 3. It tells us how to "label" a regular cell complex X so that we can construct an appropriate monomial ideal whose resolution is supported on X. In particular condition 1 shows that the corresponding ideal has codimension $\geqslant d+1$, condition 2 guarantees that X supports a cellular resolution and condition 3 says that this resolution is minimal.

If a family $\mathcal{F}$ satisfies the above definition then the following lemma from [Flø09] characterizes when it is maximal.

Lemma 6.0.5. *If a family of subsets $\mathcal{F}$ of $[n]$ corresponds to a maximal object in $CM_*(X)$, then for every $S \in \mathcal{F}$ the restriction of X to S is connected.*

[1]Note that for these more general statements about $CM_*(X)$ in [Flø09] X is never assumed to be anything more than a regular cell complex. It should be noted, however, that without the assumption that X is acyclic or that the augmented face poset of X is a lattice this set $CM_*(X)$ will often times be empty. Consider the example of the regular cell complex consisting of 2 vertices, 2 one-cells, and a single 2 cell. This is a regular cell complex, but since its augmented face poset is not a lattice it cannot support the minimal resolution of any monomial ideal. The fact that these sets could in fact be empty should explain any discrepancy between these constructions and the hypotheses in proposition 5.2.1

My observation is that Fløystad's families $\mathcal{F}$ which are used to describe ideals whose resolutions are supported on cell complexes X can be viewed as subsets of the LCM lattices of these monomial ideals. What follows is a description of the connections between the constructions appearing in [Fløø09] and in [Pha06].

6.1 Dictionary between labeling regular cell complexes and coordinatizing lattices

I begin by considering Fløystad's families $\mathcal{F}$ and what they correspond to in terms of the LCM lattice associated to the ideal that they represent. First I will disscuss the sets $V_t = \{i \,|\, x_t \text{ divides } m_i\}$. Let M be the squarefree monomial ideal in $CM(n,c)$ corresponding to a family $\mathcal{F} = \{V_1, .., V_s\}$. Then for a variable x_t if we consider the deficit labeling of $P = \text{LCM}(M)$ then there is a point $p \in P$ such that p is labeled with the variable x_t. In this case it is easy to see that $V_t = \lfloor p \rfloor^c \cap \operatorname{atoms} P$, since under Phan's construction those atoms correspond to precisely the monomials that x_t will divide.

Lemma 6.1.1. *If $\mathcal{M}$ is a labeling of P which gives a squarefree coordinatization of P such that none of the monomials in V have degree greater than 1 then the corresponding $\mathcal{F}$ is in $CM_*(n,c)$.*

Proof. Since the ideal associated to to $\mathcal{M}$ is squarefree we need only show that the corresponding sets V_t are distinct.

Since the sets V_t correspond to points $p \in P$ and the sets $\lfloor p \rfloor^c \cap \operatorname{atoms} P$ we can easily see that if there exists a $q \in P$ such that $\lfloor p \rfloor^c \cap \operatorname{atoms} P = \lfloor q \rfloor^c \cap \operatorname{atoms} P$ then $p = q$ since P is an atomic lattice. Thus, in order to have distinct V_t's associated to each $p \in P$ each monomial that labels a point $p \in P$ must have degree equal to 1. □

Next, I want to consider the condition in [Fløø9] for a set $\mathcal{F}$ to be reduced since that is a step towards understanding how maximal monomial ideals in Fløystad's sense correspond to maximal lattices in the strata of $\mathcal{L}(n)$.

Lemma 6.1.2. *If $\mathcal{M}$ is a coordinatization of P as in lemma 6.1.1 then $\mathcal{M}$ yields a monomial ideal whose associated family in $CM_*(n, c)$ is reduced if $q, r \in P$ and labeled such that V_q and V_r are two subsets in $\mathcal{F}$ that are disjoint then their meet is not labeled in P.*

Proof. First, consider what it means for two subsets in $\mathcal{F}$ to be disjoint. In other words, suppose there are elements $q, r \in P$ such that

$$(\lfloor r \rfloor^c \cap \operatorname{atoms} P) \cap (\lfloor q \rfloor^c \cap \operatorname{atoms} P) = \emptyset.$$

If this is true, then we can think of

$$q = (\bigvee_{i \in \lfloor r \rfloor^c \cap \operatorname{atoms} P} a_i) \vee s$$

and,

$$r = (\bigvee_{i \in \lfloor q \rfloor^c \cap \operatorname{atoms} P} a_i) \vee s.$$

So, the condition that no element of $\mathcal{F}$ is the disjoint union of two other elements in $\mathcal{F}$ is equivalent to saying that their meet s is not labeled in any coordinatization of P.

To see why s must not be labeled, recall that we want to avoid the situation where there exists a labeled $p \in P$ such that

$$\lfloor p \rfloor^c \cap \operatorname{atoms} P = (\lfloor q \rfloor^c \cap \operatorname{atoms} P) \cup (\lfloor r \rfloor^c \cap \operatorname{atoms} P).$$

This is equivalent to saying

$$\lfloor p \rfloor^c \cap \operatorname{atoms} P = (\lfloor q \rfloor^c \cup \lfloor r \rfloor^c) \cap \operatorname{atoms} P$$

and then by undoing the complements we see that the situation we want to avoid is

$$\lfloor p \rfloor \cap \operatorname{atoms} P = (\lfloor q \rfloor \cap \lfloor r \rfloor) \cap \operatorname{atoms} P,$$

which is precisely the s described above.

□

It remains to consider what being maximal in the refinement order means in this context. In essence this means that we want $\mathcal{F}$ to consist of as many subsets as possible while still satisfying the conditions of definition 6.0.4, lemma 6.0.5, and lemma 6.1.2. An algorithm for doing this is roughly as follows:

1. Label all of the meet-irreducibles, and compute their $V_p = \lfloor p \rfloor^c \cap$

atoms P.

2. Look at all of the pairs of the V_p's which are not disjoint. The meets of all of these pairs are candidates to be labeled. Compute their V_p's.

3. Check all of these candidates against definition 6.0.4, lemma 6.0.5, and lemma 6.1.2 and add them in if none of these conditions are violated.

Note however, that this algorithm neglects to address the issue of choosing the right P to coordinateize. In the case where X is a tree, I do have a description of the lattice P which is the LCM lattice of a maximal CM monomial ideal.

6.2 Codimension 2 Cohen-Macaulay monomial ideals

For Cohen-Macaulay monomial ideals of codimension two one can easily see using the Auslander-Buchsbaum formula that their projective dimension must be 2. In terms of cellular resolutions, this implies their resolutions are supported on trees. For this special case, Fløystad gives a very specific construction which associates to all trees T a maximal monomial ideal in $CM(T)$ given orientations of the edges of T.

His construction is as follows, assign to every edge e_i in T two variables x_i and y_i. Deleting the edge e_i produces two connected components of T call them $T_{i,1}$ and $T_{i,2}$. Then the monomial associated to any vertex $v \in T$

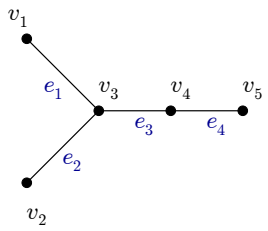


Figure 6.1: A tree T

is the following,

$$m_v = (\prod_{\{i \,|\, v \in T_{i,1}\}} x_i)(\prod_{\{i \,|\, v \in T_{i,2}\}} y_i).$$

Then the squarefree monomial ideal $M_T = (m_{v_1}, \ldots, m_{v_{n+1}})$ is maximal Cohen-Macaulay and its minimal resolution is supported on T.

Example 6.2.1. To the tree in figure 6.1 we would associate the following monomial ideal using Fløystad's construction. For each edge e_i the connected component $T_{i,1}$ will be the component with the smaller vertex indices and the component $T_{i,2}$ will be the one with the larger vertex indices. Then $M_T = (x_1y_2x_3x_4, y_1x_2x_3x_4, y_1y_2x_3x_4, y_1y_2y_3x_4, y_1y_2y_3y_4)$ in the ring $k[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$.

I intend to demonstrate that for an appropriate choice of finite atomic lattice P that this construction coincides with Phan's minimal squarefree coordinatization. Given a tree T with n edges and $n+1$ vertices then define P_T to be the set of all subtrees of T ordered by inclusion (note that by subtrees I am including the vertices and the empty set).

Lemma 6.2.2. *A poset P_T defined as above is an element of $\mathcal{L}(n+1)$.*

Proof. Per usual, I will show that P_T is a meet-semilattice with a maximal element and then by theorem 2.1.1 it will be a finite lattice. To show that P_T is a meet-semilattice, I need to show that every pair of elements $a, b \in P_T$ that there exists a meet or least upper bound. Since a and b are subtrees of T, define $a \wedge b$ to be the intersection of a and b. Since $\emptyset \in P_T$ and since $a \cap b$ will also be a subtree of T then clearly P_T is a meet-semilattice.

It remains to show that P_T is atomic with $n+1$ atoms. This follows from the fact that T has $n+1$ vertices and every subtree can be realized as an induced graph on a subset of the vertices. □

The following proposition demonstrates that any coordinatization of the lattice P_T defined above will yield a monomial ideal in $Mon(T)$, which is the set of monomial ideals whose resolution is supported on T.

Proposition 6.2.3. *The minimal resolution of P_T is supported on T.*

Proof. Here, we just need to show that $P_{T_{\leqslant p}}$ is acyclic for $p \in P_T$ by theorem 2.4.1. Clearly since P_T is the lattice where p corresponds to a subtree of P_T ordered by inclusion then $P_{T_{\leqslant p}}$ is simply the subtree corresponding to p. All of the subtrees are acyclic by virtue of being a tree. One needs only to check acyclicity for each $p \in P_T$ so T supports the minimal resolution of any coordinatization of P_T. □

Finally we see that Phan's minimal squarefree coordinatization of P_T always agrees with the ideals that Fløystad constructs.

Theorem 6.2.4. *Let $\mathcal{M}$ be the minimal squarefree coordinatization of P_T, then $M_{P_T,\mathcal{M}} \cong M_T$*

Proof. Recall that in the construction of M_T we assigned a variable to each subtree $T_{i,1}, T_{i,2}$ of T obtained by deleting an edge e_i of T. Then each component is assigned a variable and a vertex v is assigned a monomial which is the product of the variables corresponding to the trees $T_{i,j}$ containing v.

So first I need to show that the trees $T_{i,j}$ obtained by deleting edges are precisely the meet-irreducibles of P_T. Then I need to explain how to coordinateize P_T to obtain M_T.

Clearly the meet-irreducibles of P_T will be the subtrees of $T' \subset T$ that have only one subtree $T'' \subset T$ containing them which satisfy that the

$$|\{e_i \in T'\}| + 1 = |\{e_i \in T''\}| \tag{6.2.1}$$

where e_i are the edges of a tree.

If T' is obtained as above by deleting an edge e_i, (i.e. $T' = T_{i,1}$) then the only subtree T'' satisfying equation 6.2.1 is $T'' = T' \cup e_i$ (i.e. T'' is obtained by adding edge e_i to T'). Since the only other edges one could add in are in the other $T_{i,2}$ (or $T_{i,1}$ if we had started with $T_{i,2}$) then in order to add in one of those edges we would be forced to add in e_i as well thus violating equation 6.2.1. So the meet-irreducibles are precisely the subtrees $T_{i,j}$ obtained by deleting edges e_i.

As stated above, I want to use a minimal squarefree coordinatization of P_T but if I place my variables carefully it will be easy to see that $M_{P_T,\mathcal{M}} \cong M_T$. Recall that for M_T the variables x_i were assigned to the trees $T_{i,1}$ and y_i's were assigned to the trees $T_{i,2}$. Moreover, note that if $v \in T_{i,1}$ it is necessarily not in $T_{i,2}$ and vice versa. So in essence these trees $T_{i,1}, T_{i,2}$

are partitioning the vertices into two sets. Then the monomial label for the construction of M_T assigns to each vertex the product of the variables corresponding to the subtrees containing v. In lattice language the subtrees containing v will be in $\lceil a_v \rceil$ where a_v is the atom corresponding to the vertex v. For our coordinatization construction this is in some sense the opposite of what we want since we take the product over the complement of the filter. However since the complement of the filter consists precisely of the subtrees not containing v we make the following coordinatization.

Let $\mathcal{M}$ label P_T as follows. If $p \in P_T$ which is a meet-irreducible thus corresponding to a $T_{i,1}$ denote it as p_{i1} and label it with y_i, similarly if p corresponds to a $T_{i,2}$ denote it as p_{i2} label it with an x_i. Then it is easy to see that:

$$\begin{aligned} x(a_v) &= \prod_{p \in \lceil a_v \rceil^c} m_p \\ &= (\prod_{p_{i1} \in \lceil a_v \rceil^c} y_i)(\prod_{p_{i2} \in \lceil a_v \rceil^c} x_i) \\ &= (\prod_{\{i \,|\, v \in T_{i,2}\}} y_i)(\prod_{\{i \,|\, v \in T_{i,1}\}} x_i) \\ &= m_v \end{aligned}$$

□

It is easy to see that these lattices P_T are graded of rank n since every vertex will be contained in some subtree of T containing i edges where i ranges from 0 to n. This indicates that Fløystad's maximal ideals correspond

to "maximal" elements in the appropriate strata of of $\mathcal{L}(n+1)$ as is stated in the following conjecture.

Conjecture 1. *If X is a regular cell complex, then the LCM lattice P of a maximal monomial ideal $M \in CM_*(X)$ satisfies the property that if $Q > P$ in $\mathcal{L}(n)$ then the minimal resolution of Q has total Betti numbers greater than that of P.*

Appendix A

Reverse Search and enumeration of $\mathcal{L}(n)$ in Haskell

In [BT] Bayer and Taylor use the method of reverse search introduced in [AF96] and further explained in [NDM99] to search for multidegrees with high likelihood of having nonzero multigraded Betti numbers for a given monomial ideal.

In general reverse search works as follows. Let $S = (V, E)$ be a state space consisting of objects V to be enumerated with a directed edge set E, then following three conditions found in [NDM99] enable reverse search to enumerate S:

1. There is an adjacency operator or "oracle" $A : S \to 2^S$ which assigns to a state s an ordered set $A(s) = [s_1, \ldots, s_k]$ of its neighbors. Adjacency need not be symmetric, i.e. $s' \in A(s)$ does not imply $s \in A(s')$. The

pairs (s, s') with $s' \in A(s)$ define the set E of directed edges of S.

2. There is a gradient function $g : S \to S \cup \{nil\}$, where nil is a fictitious state not in S. A state s with $g(s) = nil$ is called a "sink" of g. To any state s, g assigns a unique successor $g(s)$ subject to two conditions:

   (a) For any state s that is not a sink, the pair $(g(s), s) \in E$, i.e. $s \in A(g(s))$.

   (b) The function g defines no cycles, i.e. $g(g(\dots g(s)\dots)) = s$ is impossible for all s.

3. It is possible to enumerate all of the sinks of g before exploring all of S.

In layman's terms we want to think of A as a "going-up rule" and g as a "going-down rule" or vice versa. Moreover we want one direction, i.e. going up, to have many options whereas we want the other direction, i.e. going down, to have only one direction. This plus the lack of cycles guaranteed by condition 2 allows us to form a directed tree consisting of paths that allow us to visit each element of V once and then make record of the visit.

In the case where we want to enumerate $\mathcal{L}(n)$ we want to use the presentation of each individual lattice in $\mathcal{L}(n)$ as a set of sets as discussed in section 4.2.2. First we fix an order on all subsets of $\{1, \dots, n\}$. Then our A operator will assign to each lattice $P \in \mathcal{L}(n)$ all of the lattices above it that can be obtained by adding in one extra element σ and the intersection of σ with all of the existing subsets in P. Our rule g deletes the maximal set in the above ordering from $P - \hat{1}$. The only sink of g is the minimal element

of $\mathcal{L}(n)$ so it is possible to efficiently enumerate it before exploring all of S. Clearly, $P \in A(g(P)$ so it remains to check that g does not define a cycle. Since g always deletes an element, it impossible that $g(g(\ldots g(P)\ldots)) = P$ because at no point are we adding the deleted elements back in.

Working with Dave Bayer, we have implemented this algorithm using Haskell. While we were completely motivated by the idea reverse search, our implementation could perhaps be best described as "implicit" reverse search. We say "implicit" is because we actually improved our operator A so that we did not need to check the values of g at each step. This reduces the problem to an ordinary tree search. The code shown below just focuses on the enumeration of $\mathcal{L}(5)$ but this code can easily be adapted to enumerate $\mathcal{L}(n)$ for other values of n. We have also written other functions which check conditions at each step of the enumeration. For instance we have verified proposition 4.3.3 for up to $n = 6$.

```
   module Main (main) where

  import Data.Bits (Bits, (.|.), (.&.), bit, shiftR, testBit)
  import Data.List ((\\), transpose, tails, foldl')
  import Data.Int (Int64)
  import Data.Word (Word32)
  import Control.Parallel (par)

-- choose bitfield size

  type Core = Word32

  oneN, twoN :: Int
  oneN = 5
  twoN = 2^oneN

  oneNs, twoNs :: [Int]
```

```
  oneNs = [0 .. (oneN-1)]
  twoNs = [0 .. (2^oneN-1)]

-- convert bits to boolean list, low bits first, with specified tail

  bits_boolsT :: Bits a => [Bool] -> a -> [Bool]
  bits_boolsT xt 0 = xt
  bits_boolsT xt w = testBit w 0 : bits_boolsT xt (shiftR w 1)

-- convert bits to minimal boolean list

  bits_bools :: Bits a => a -> [Bool]
  bits_bools = bits_boolsT []

-- convert bits to length n boolean list

  bits_boolsN :: Bits a => Int -> a -> [Bool]
  bits_boolsN n = take n . bits_boolsT (repeat False)

-- combine list using bitwise or

  bitOr :: Bits a => [a] -> a
  bitOr = foldl' (.|.) 0

-- filter list using boolean list

  boolFilter :: [Bool] -> [a] -> [a]
  boolFilter xs = map snd . filter fst . zip xs

-- filter list using binary bits

  bitFilter :: Bits a => a -> [b] -> [b]
  bitFilter = boolFilter . bits_bools

-- convert boolean list to bits, low bits first

  bools_bits :: Bits a => [Bool] -> a
  bools_bits xs = bitOr $ boolFilter xs $ map bit [0 .. ]

-- Seed type: lattice, list of subsets that can be added by reverse search
-- Mask type: data needed to delete an element from each subset

  type Seed = (Core, [Subset])
  type Mask = (Int, Core)
```

```
-- Subset type: bit for subset, Mask to intersect with subset complement

  data Subset = S { core :: Core, mask :: [Mask] }

-- masks for deleting elements from subsets

  shiftMasks :: [Mask]
  shiftMasks = zip (iterate (*2) 1) (map bools_bits masks)
    where bools = map (bits_boolsN oneN) twoNs
          masks = map (map not) $ transpose bools

-- treat Int as bitfield specifying subset in binary, return Subset

  subset :: Int -> Subset
  subset n = S (bit n) $
    boolFilter [ not $ elem m (bitFilter n oneNs) | m <- oneNs ] shiftMasks

-- base is a Core representing the minimal element of L(oneN)

  baseInts :: [Int]
  baseInts = [0, twoN-1] ++ map (2^) oneNs

  base :: Core
  base = foldr1 (.|.) [ bit n | n <- baseInts ]

-- list of atom (not base) subsets

  atomSubsets :: [Subset]
  atomSubsets = map subset $ twoNs \\ baseInts

-- shift and mask, to delete n from every subset

  poke :: Mask -> Core -> Core
  poke (r,m) w = (shiftR w r .|. w) .&. m

-- iterate poke to delete a list of elements from every subset

  pokes :: [Mask] -> Core -> Core
  pokes ms w = foldr ($) w $ map poke ms

-- test if Core is closed under intersection by Subset

  closed :: (Seed, Subset) -> Bool
  closed ((w,_),x) = w == w .|. pokes (mask x) w
```

```
-- find covering lattices using reverse search

  covers :: Seed -> [Seed]
  covers (w,xs) = map fst $ filter closed
    [ ((w .|. core x, xt), x) | (x:xt) <- tails xs ]

-- FG f g : apply f to sublists, combine results using g

  data FG a b = FG ([a] -> b) ([b] -> b)

-- iterate par on a list

  pars :: [a] -> b -> b
  pars [] y = y
  pars (x:xs) y = x `par` pars xs y

-- start with base, grows all of L(oneN)
-- argument is stack of lists of lattices that need to find covers

  search :: [[Seed]] -> [Core]
  search [] = []
  search ([]:yt) = search yt
  search ((x@(w,_):xt):yt) = w : (search $ covers x : xt : yt)

-- parallel search, using FG to combine results

  parSearch :: Int -> FG Core b -> b
  parSearch n (FG f g) = bins `pars` g bins
    where (lower, upper) = splitAt n atomSubsets
          bins = [ f $ search [[(x,upper)]] | x <- search [[(base,lower)]] ]

-- length64 is 64-bit length function

  length64 :: [a] -> Int64
  length64 = len 0
    where len n [] = n
          len n (_:xt) = len (n+1) xt

-- count elements of L(oneN)

  countFG :: FG Core Int64
  countFG = FG length64 sum

-- main
```

```
main :: IO ()
main = do print $ parSearch 10 countFG
```

To run the above code in parallel one would use the following command:

```
% ghc --make -Wall -Werror -threaded -O2 -o Lattices Lattices.hs
% time ./Lattices +RTS -N2
```

Which yields the following output:

```
702525

real  0m0.106s
user  0m0.189s
sys   0m0.009s
```

Appendix B

Posets package for Macaulay2

With Joesphine Yu and Gwyn Whieldon I began working on a package for Macaulay2 [GS] that introduces Posets as a data type. To define a poset one needs to input the set of elements of the poset and at least the minimal covering data of the relation. There is a function which computes from the minimal covering data a full matrix of all of the relations between elements. Additionally, this new package has a number of functions that allow the user to compute things of interest such as order ideals, filters, meets, joins, lcm lattices. It also can check if a given poset satisfies certain properties such as being a lattice. The interest in implementing this in Macaulay2 is due to the fact that it is then easier to move back and forth from the combinatorial data of a poset to algebraic objects such as ideals.

I include here the code of the first version of this package.

```
newPackage(
"Posets",
Version => "0.1",
Date => "April 2, 2009",
Authors =>
  {Name => "Sonja Mapes",
   Email => "mapes@math.columbia.edu",
   HomePage => "http://www.math.columbia.edu/~mapes/"},
  {Name => "Gwyn Whieldon",
   Email => "whieldon@math.cornell.edu",
   HomePage =>
     "http://www.math.cornell.edu/People/Grads/whieldon.html"},
  {Name => "Josephine Yu",
   Email => "jyu@math.mit.edu",
   HomePage => "http://www-math.mit.edu/~jyu/"}},
Headline => "Package for processing posets and order complexes",
DebuggingMode => true)

export {
     Poset,
     poset,
     DirectedGraph,
     directedGraph,
     allPairsShortestPath,
     transitiveClosure,
     RelationMatrix,
     compare,
     indexElement,
     OrderIdeal,
     Filter,
     Relations,
     GroundSet,
     Edges,
     PosetMeet,
     MeetExists,
     PosetJoin,
     JoinExists,
     isLattice,
     lcm,
     lcmLattice}

Poset = new Type of HashTable

poset = method()
poset(List,List) := (I,C) ->
```

```
     new Poset from {
 symbol GroundSet => I,
 symbol Relations => C,
       symbol RelationMatrix =>
      transitiveClosure(I,C),
 symbol cache => CacheTable}

-- in case you actually have M to begin with
poset(List,List,Matrix) := (I,C,M) ->
     new Poset from {
  symbol GroundSet => I,
  symbol Relations => C,
  symbol RelationMatrix => M,
  symbol cache => CacheTable}

DirectedGraph = new Type of HashTable

directedGraph = method()
directedGraph(List, List) := (I,C) ->
     new DirectedGraph from {
        symbol GroundSet => I,
        symbol Edges => C,
  symbol cache => CacheTable}

--------------


--inputs: (I,C), I is a List (ground set) and
--            C is a List of pairs of elements in I
--           OR DirectedGraph OR Poset
--output: a matrix whose rows and columns
--are indexed by I, where (i,j)
--entry is infinity (i.e. 1/0.)
--if (i,j) is not in C and 1 otherwise
--(i.e. tropicalization of the "usual"
--adjacency matrix)
--caveat: diagonal entries are 0
-- uses:  transitive closure

adjacencyMatrix = method()
adjacencyMatrix(List,List) := Matrix => (I, C) -> (
     M := mutableMatrix table(#I, #I, (i,j)->1/0.);
     ind := hashTable(
               apply(I, i-> i=> position(I,j-> j==i)));
     scan(C, e -> M_(ind#(e#0), ind#(e#1))= 1);
```

```
     scan(numrows M, i-> M_(i,i) = 0);
     matrix M)

adjacencyMatrix(DirectedGraph) := Matrix => (G) ->
 adjacencyMatrix(G.GroundSet,G.Edges)
adjacencyMatrix(Poset) := Matrix => (P) ->
 adjacencyMatrix(P.GroundSet,P.Relations)

--input:  adjacency matrix of a directed graph
--output:  a matrix whose (i,j) entry is the length of the
--    shortest path from i to j
--algorithm:  FloydWarshall algorithm for all pairs
-- shortest path

allPairsShortestPath = method()
allPairsShortestPath(Matrix) := Matrix => (A) -> (
     D := mutableMatrix(A);
     n := numrows D;
     scan(n, k->
       table(n,n,(i,j)-> D_(i,j) =
       min(D_(i,j), D_(i,k)+D_(k,j))));
     matrix D)
allPairsShortestPath(DirectedGraph) := Matrix =>
(G)-> allPairsShortestPath(adjacencyMatrix(G))



-- input:  a poset, and an element A from I
-- output:  the index of A in the ground set of P
-- usage:  compare, OrderIdeal
indexElement := (P,A) -> (
      sum apply(#P.GroundSet, i->
if P.GroundSet#i == A then i else 0))

-- input:  a list, potentially with nulls
-- output:  a list w/out nulls
-- usage:  OrderIdeal, Filter
nonnull :=(L) -> (
     select(L, i-> i =!= null))


-------------------------------------------------
--Transitive Closure and Element Inclusion
-------------------------------------------------
```

```
--input: (I,C).  I=List, ground set.  C=List, pairs
--output: matrix where 1 in (i,j) position
--             where i <= j, 0 otherwise
--uses: poset

transitiveClosure = method()
transitiveClosure(List,List) := List => (I,C)-> (
     A := adjacencyMatrix(I,C);
     D := mutableMatrix allPairsShortestPath(A);
     scan(numrows D, i-> D_(i,i) = 0);
     table(numrows D, numrows D, (i,j)->(
  if D_(i,j) ==1/0. then D_(i,j) = 0
  else D_(i,j) = 1;));
     matrix D)


-- input:  A poset, and two elements A and B from I
-- output: true if A<= B, false else
compare:= (P,A,B) -> (
     Aindex:=indexElement(P,A);
     Bindex:=indexElement(P,B);
          if P.RelationMatrix_Bindex_Aindex==0
          then false
          else true)



-------------------------------------------------
--Covering Relations
-------------------------------------------------

testcover=(P,A,B) -> (
     L:=poset(P.GroundSet,fullPosetRelation(P));
     k:=#L.GroundSet-2;
     if sum(nonnull(apply(k, i->
         if compare(L,A,(toList(set(L.GroundSet)-{A,B}))_i)
         ==true and
         compare(L,(toList(set(L.GroundSet)-{A,B}))_i,B)
         ==true
         then 1)))=!=0
       then C=C+set{(A,B)};
       C)

--input: A poset with any type of relation C
--           (minimal, maximal, etc.)
```

```
--output: The minimal relations defining our poset

coveringRelations:=(P) -> (
     C=set{};
     apply(#P.CRelations,i->
       testcover(P,P.CRelations#i#0,P.CRelations#i#1));
     toList(set(P.CRelations)-C))

--input: A poset with any type of relation C
--          (minimal, maximal, etc.)
--output:  A new poset P with the minimal relations

coveringRelationsPoset:=(P) -> (
     L=poset(P.GroundSet,coveringRelations(P)))

-------------------------------------------------
--Minimal Element Construction
-------------------------------------------------

minimalElementIndex:=(P)-> (
     M:=P.RelationMatrix;
     nonnull(apply(numcols(M), k->
        if (apply(numcols(M), j->
          (sum((apply(numrows(M),i->
            (transpose(M))_i))))_j))#k==1 then k)))

minimalElements:=(P) -> (
     L:=minimalElementIndex(P);
     apply(#L,i-> P.GroundSet#(L#i)))

PosetMinusMins:=(P)-> (
     L:=minimalElements(P);
     K:=fullPoset(P);
     N:=set{};
     S:=apply(#L, j->
        apply(#K.CRelations,i->
           (K.CRelations#i)#0===L#j));
     E:=sum set nonnull(apply(#K.CRelations,l->
         if member(true,set apply(#L,k->S#k#l))
         then N=N+set{K.CRelations#l}));
     C:=toList (set(K.CRelations)-N);
     I:=toList (set(K.GroundSet)-set(L));
     poset(I,C))
```

```
------------------------------------------------
--Order and Filter Ideals
------------------------------------------------

-- input: a poset, and an element from I
-- output: the order ideal of a, i.e. all elements in
--              the poset that are >= a

OrderIdeal= method()
OrderIdeal(Poset, Thing) := (P, a) -> (
     M:=P.RelationMatrix;
     aindex := indexElement (P,a);
     GreaterThana:= entries((transpose(M))_aindex);
     nonnull(apply(#GreaterThana, i->
         if GreaterThana_i == 1 then P.GroundSet#i)))



-- input: a poset, and an element from I
-- output:  the filter of a, i.e. all elements in
--               the poset that are <= a

Filter = method()
Filter(Poset, Thing) := (P,a) -> (
     M:=P.RelationMatrix;
     aindex := indexElement (P,a);
     LessThana:= entries M_aindex;
     nonnull(apply(#LessThana, i->
        if LessThana_i == 1 then P.GroundSet#i)))



-------------------------------------------------
--Joins, Meets, Lattices and Atoms
-------------------------------------------------
-- inputs: P, poset, and two elements of P.GroundSet
-- outputs:  the element of P.GroundSet that is the
--                 join of these or error
-- usage:  JoinExists used in isLattice

PosetJoin = method()
PosetJoin(Poset,Thing,Thing) := (P,a,b)  -> (
     OIa := OrderIdeal(P,a);
     OIb := OrderIdeal(P,b);
```

```
     upperBounds := toList (set(OIa)*set(OIb));
     if upperBounds == {}
       then (error "your elements do not share any upper bounds")
       else (M := P.RelationMatrix;
     heightUpperBounds :=
       flatten apply(upperBounds, element->
          sum entries M_{indexElement(P,element)});
     if #(select(heightUpperBounds, i->
           i== min heightUpperBounds)) > 1
     then error "join does not exist, least upper bound not unique"
     else(upperBounds_{position
          (heightUpperBounds, l ->
              l == min heightUpperBounds)})))


JoinExists = method()
JoinExists(Poset,Thing,Thing) := (P,a,b) -> (
     OIa := OrderIdeal(P,a);
     OIb := OrderIdeal(P,b);
     upperBounds := toList (set(OIa)*set(OIb));
     if upperBounds == {} then false
     else (M := P.RelationMatrix;
         heightUpperBounds :=
         flatten apply(upperBounds, element->
            sum entries M_{indexElement(P,element)});
      if #(select(heightUpperBounds, i->
            i== min heightUpperBounds)) > 1
         then false else true))


--inputs:  P a poset, and 2 elements of P.GroundSet
--outputs:  the element in P.GroundSet that is the
    meet of these, or error
-- usage:  MeetExits used in isLattice
PosetMeet = method()
PosetMeet(Poset,Thing,Thing) := (P,a,b) ->(
     Fa:= Filter(P,a);
     Fb:= Filter(P,b);
     lowerBounds:= toList (set(Fa)*set(Fb));
     if lowerBounds == {}
     then error "your elements do not share any lower bounds"
     else (M := P.RelationMatrix;
             heightLowerBounds :=
             flatten apply(lowerBounds, element->
               sum entries M_{indexElement(P,element)});
```

```
      if #(select(heightLowerBounds, i->
           i== max heightLowerBounds)) > 1
      then error "meet does not exist, greatest lower bound not unique"
      else(lowerBounds_{position
              (heightLowerBounds, l ->
                 l == max heightLowerBounds)}))))

MeetExists = method()
MeetExists(Poset, Thing, Thing) := (P,a,b) -> (
     Fa:= Filter(P,a);
     Fb:= Filter(P,b);
     lowerBounds:= toList (set(Fa)*set(Fb));
     if lowerBounds == {} then false else (
        M := P.RelationMatrix;
        heightLowerBounds :=
          flatten apply(lowerBounds, element->
             sum entries M_{indexElement(P,element)});
     if #(select(heightLowerBounds, i->
          i== max heightLowerBounds)) > 1
          then false else true ))


--inputs: a poset P
--output:  boolean value for whether or
--             not it is a lattice

isLattice = method()
isLattice(Poset) := (P) -> (
    checkJoins := unique flatten flatten
         apply(P.GroundSet, elt ->
           apply (P.GroundSet, elt2->
             JoinExists(P,elt, elt2)));
    checkMeets :=  unique flatten flatten
          apply(P.GroundSet, elt ->
            apply (P.GroundSet, elt2->
              MeetExists(P,elt, elt2) ));
    if member(false,
       set (flatten{checkJoins,checkMeets}) === true)
     then false else true )


-----------------------------------------------
-- LCM lattices
-----------------------------------------------
--input: a set of monomials
```

```
-- output: the lcm of those monomials
lcm = (L) -> (
    flatten entries gens intersect apply(L, i-> ideal (i)))

-- input:  generators of a monomial ideal
-- output: lcm lattice of that monomial ideal,
--              without the minimal element
-- potential problem:  subsets dies when a
--              set is too big (> 18)

lcmLattice = method()
lcmLattice(Ideal) := Poset => (I) -> (
   L := flatten entries gens I;
   subsetsL := flatten
      apply(#L, i-> subsets (L,i+1));
   Ground := unique flatten
      apply (subsetsL, r-> lcm(r));
   Rels := nonnull unique flatten
       apply (Ground, r->
          apply(Ground, s->
             if s%r == 0 then (r,s)));
   RelsMatrix :=  matrix
       apply (Ground, r->
          apply(Ground, s->
             if s%r == 0 then 1 else 0));
   P = poset (Ground, Rels, RelsMatrix);
   P)




beginDocumentation()

document { Key => Poset,
     }

---------------------------------
--Tests
---------------------------------

-- a lattice, B_3
TEST ///
I ={a,b,c,d,e,f,g,h};
C ={(a,b),(a,c),(a,d),(b,e),(b,f),(c,e),
  (c,g),(d,f),(d,g),(e,h),(f,h),(g,h)};
```

```
P=poset(I,C);
M = matrix {{1,1,1,1,1,1,1,1},
    {0,1,0,0,1,1,0,1},
    {0,0,1,0,1,0,1,1},
    {0,0,0,1,0,1,1,1},
    {0,0,0,0,1,0,0,1},
    {0,0,0,0,0,1,0,1},
    {0,0,0,0,0,0,1,1},
    {0,0,0,0,0,0,0,1}};
assert (entries P.RelationMatrix == entries M)
--G=directedGraph(I,C)
--A=adjacencyMatrix(I,C) -- not exported
--allPairsShortestPath(A) -- not exported
--adjacencyMatrix(G) -- not exported
--adjacencyMatrix(P) -- not exported
--transitiveClosure(I,C)
assert (PosetJoin(P,a,b) == {b})
assert (PosetJoin(P,b,d) == {f})
assert (PosetMeet(P,a,b) == {a})
assert (PosetMeet(P,f,g) == {d})
assert (OrderIdeal(P,a) == {a,b,c,d,e,f,g,h})
assert (OrderIdeal(P,b) == {b,e,f,h})
assert (Filter(P,a) == {a})
assert (Filter(P,g) == {a,c,d,g})
assert (isLattice(P))
///


-- two equivllaent non lattices with
-- different initial data
TEST ///
I1={a,b,c,d,e,f};
C1={(a,c),(a,d),(b,c),(b,d),(c,e),
  (d,e),(e,f)};
P1=poset(I1,C1);
--G1 = directedGraph(I1,C1)

-- Poset P1 with additional relations (a,e)
-- and (a,f) added
I2={a,b,c,d,e,f};
C2={(a,c),(a,d),(b,c),(b,d),(c,e),
  (d,e),(a,e),(a,f),(e,f)};
P2=poset(I2,C2);
```

```
assert (P1.RelationMatrix == P2.RelationMatrix)
assert (Filter(P1,b) == {b})
assert (Filter(P1,c) == {a,b,c})
assert (OrderIdeal (P1,b) == {b,c,d,e,f})
assert (isLattice (P1) == false)
///
```

Bibliography

[AF96] David Avis and Komei Fukuda. Reverse search for enumeration. *Discrete Appl. Math.*, 65(1-3):21–46, 1996. First International Colloquium on Graphs and Optimization (GOI), 1992 (Grimentz).

[BCP99] Dave Bayer, Hara Charalambous, and Sorin Popescu. Extremal Betti numbers and applications to monomial ideals. *J. Algebra*, 221(2):497–512, 1999.

[Bjö95] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier, Amsterdam, 1995.

[BPS98] Dave Bayer, Irena Peeva, and Bernd Sturmfels. Monomial resolutions. *Math. Res. Lett.*, 5(1-2):31–46, 1998.

[BS98] Dave Bayer and Bernd Sturmfels. Cellular resolutions of monomial modules. *J. Reine Angew. Math.*, 502:123–140, 1998.

[BT] David Bayer and Amelia Taylor. Reverse search for monomial ideals. *preprint*.

[BW02] E. Batzies and V. Welker. Discrete Morse theory for cellular resolutions. *J. Reine Angew. Math.*, 543:147–168, 2002.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Eis05] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[Fløø09] Gunnar Fløystad. Cellular resolutions of Cohen-Macaulay monomial ideals. *J. Commut. Algebra*, 1(1):57–89, 2009.

[GPW99] Vesselin Gasharov, Irena Peeva, and Volkmar Welker. The lcm-lattice in monomial resolutions. *Math. Res. Lett.*, 6(5-6):521–532, 1999.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Mas78] William S. Massey. *Homology and cohomology theory*. Marcel Dekker Inc., New York, 1978. An approach based on Alexander-Spanier cochains, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[MSY00] Ezra Miller, Bernd Sturmfels, and Kohji Yanagawa. Generic and cogeneric monomial ideals. *J. Symbolic Comput.*, 29(4-5):691–708, 2000. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).

[NDM99] J. Nievergelt, N. Deo, and A. Marzetta. Memory-eficient enumeration of constrained spanning trees. *Inform. Process. Lett.*, 72(1-2):47–53, 1999.

[NPS02] Isabella Novik, Alexander Postnikov, and Bernd Sturmfels. Syzygies of oriented matroids. *Duke Math. J.*, 111(2):287–317, 2002.

[Pha06] Jeffery Phan. *Properties of Monomial Ideals and their Free Resolutions*. PhD thesis, Columbia University, 2006.

[PV] Irena Peeva and Mauricio Velasco. Frames and degenerations of monomial ideals. *to appear in Trans. Amer. Math. Soc.*

[Sta97] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[Tay60] Diana Taylor. *Ideals generated by a monomial in an R-sequence*. PhD thesis, University of Chicago, 1960.

[Vel08] Mauricio Velasco. Minimal free resolutions that are not supported by a CW-complex. *J. Algebra*, 319(1):102–114, 2008.

[Yuz99] Sergey Yuzvinsky. Taylor and minimal resolutions of homogeneous polynomial ideals. *Math. Res. Lett.*, 6(5-6):779–793, 1999.